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New supersymmetric higher-derivative couplings: full $N = 2$ superspace does not count!

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Abstract: An extended class of $N = 2$ locally supersymmetric invariants with higher-derivative couplings based on full superspace integrals, is constructed. These invariants may depend on unrestricted chiral supermultiplets, on vector supermultiplets and on the Weyl supermultiplet. Supersymmetry is realized off-shell. A non-renormalization theorem is proven according to which none of these invariants can contribute to the entropy and electric charges of BPS black holes. Some of these invariants may be relevant for topological string deformations.

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Contents

1 Introduction

Supersymmetric invariants with higher-derivative couplings play a role in many applications. Here we will be dealing with $N = 2$ supergravity, where the first higher-derivative couplings that were considered involve the square of the Weyl tensor coupled to vector supermultiplets [1]. This particular class of invariants is based on an integration over a chiral subspace of $N = 2$ superspace. It is relevant for the topological string [2, 3], and furthermore, it has important implications for BPS black hole entropy [4]. Another class of invariants for vector multiplets that involve terms quartic in the field strengths, was derived in terms of $N = 1$ superfields, both for the abelian [5] and for the non-abelian case [6]. Unlike the previous class, this one is based on an integral over full superspace. It yields important contributions to the effective action of $N = 2$ supersymmetric gauge theories (for some additional references, see e.g., [7–10]). Only recently, a related class of locally supersymmetric higher-derivative couplings was considered in [11]. Those couplings which involve both the Weyl tensor and higher-order coupling of the vector field strengths, were conjectured to describe certain deformations of the topological string partition function.

This paper deals with an explicit construction of this rather large class of invariant couplings based on full superspace integrals. They are coupled to conformal supergravity and are realized off-shell. This feature greatly facilitates their construction, which is based on previous work on $N = 2$ supergravity (in particular, on [12, 13]). The general procedure underlying this construction will be presented, and, as an explicit example, many of the bosonic terms of the supergravity-coupled invariants that contain F^4-, R^2F^2-, and R^4-terms, will be discussed. Here F denotes the abelian vector multiplet field strengths and R the Riemann tensor.

One of the motivations for this work is to study the possible contribution of these new couplings to the entropy and the electric charges of BPS black holes. As it turns out we can derive a 'non-renormalization' theorem according to which these contributions vanish. This result is not entirely unexpected, in view of the fact that there was already a good agreement for the subleading contributions to the BPS entropy obtained from microstate counting and from supergravity, in which the new couplings had so far not been incorporated. Hence the existence of the non-renormalization theorem offers a partial explanation for this agreement.

This paper is organized as follows. Section 2 presents the superconformal transformations of chiral supermultiplets in a conformal supergravity background as well as a number of related issues. Section 3 describes the general strategy for the construction of the higher-derivative couplings, based on the use of the so-called 'kinetic supermultiplet', which can be constructed from an anti-chiral supermultiplet of zero Weyl weight. The components of this multiplet are given in considerable detail, fully taking into account the presence of the superconformal background. The construction of the bosonic terms of the higher-derivative couplings is presented in section 4, together with explicit examples based on a class of Lagrangians that involves terms such as F^4, R^2F^2 and R^4. A non-renormalization theorem pertaining to the entropy and the electric charges of BPS black holes is proven in section 5. Some concluding remarks are presented in section 6.

In view of future applications and for the convenience of the reader we have added four appendices, A, B, C, and D, containing basic results on the superconformal multiplet calculus. Many of these results have appeared at various places in the literature, but we have updated them in uniform notation. While some of them may not have been overly relevant in the past, they are now required in the context of the new invariant couplings.

2 Chiral multiplets

Chiral superfields in flat $N = 2$ superspace were first discussed in [14]. Subsequently they were derived in a conformal supergravity background [12, 13]. The latter result was formulated in components and the same approach is followed in this paper, although it is convenient to make use of superfield notions at the same time. Chiral multiplets are complex and $N = 2$ superspace is based on four chiral and four anti-chiral anticommuting coordinates, θ^i and θ_i, so that a scalar chiral multiplet contains two times 2^4 field components. These multiplets carry a Weyl weight w and a chiral U(1) weight c, which is opposite to the Weyl weight, i.e. $c = -w$. The weights indicate how the lowest-θ component of the superfield scales under Weyl and chiral U(1) transformations. Anti-chiral multiplets can be obtained from chiral ones by complex conjugation, so that anti-chiral multiplets will have equal Weyl and chiral weights, hence $w = c$.

	Chiral multiplet					
field	A	Ψ_i	B_{ij}	F_{ab}^-	Λ_i	C
w	w	$w+\frac{1}{2}$	$w+1$	$w+1$	$w+\frac{3}{2}$	$w+2$
c	$-w$	$-w+\frac{1}{2}$	$-w+1$	$-w+1$	$-w+\frac{3}{2}$	$-w+2$
γ_5		$+$			$+$	

Table 1. Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the chiral multiplet component fields.

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The components of a generic scalar chiral multiplet are a complex scalar A, a Majorana doublet spinor Ψ_i, a complex symmetric scalar B_{ij}, an anti-selfdual tensor F_{ab}^-, a Majorana doublet spinor Λ_i, and a complex scalar C. The assignment of their Weyl and chiral weights is shown in table 1. The Q- and S-supersymmetry transformations for a scalar chiral multiplet of weight w, are as follows,[1]

$$\begin{aligned}
\delta A =&\, \bar\epsilon^i\Psi_i\,,\\
\delta\Psi_i =&\, 2\,\not{D}A\epsilon_i + B_{ij}\,\epsilon^j + \frac{1}{2}\gamma^{ab}F_{ab}^-\,\varepsilon_{ij}\epsilon^j + 2\,wA\,\eta_i\,,\\
\delta B_{ij} =&\, 2\,\bar\epsilon_{(i}\not{D}\Psi_{j)} - 2\,\bar\epsilon^k\Lambda_{(i}\,\varepsilon_{j)k} + 2(1-w)\,\bar\eta_{(i}\Psi_{j)}\,,\\
\delta F_{ab}^- =&\, \frac{1}{2}\varepsilon^{ij}\,\bar\epsilon_i\not{D}\gamma_{ab}\Psi_j + \frac{1}{2}\bar\epsilon^i\gamma_{ab}\Lambda_i - \frac{1}{2}(1+w)\,\varepsilon^{ij}\,\bar\eta_i\gamma_{ab}\Psi_j\,,\\
\delta\Lambda_i =& -\frac{1}{2}\gamma^{ab}\not{D}F_{ab}^-\epsilon_i - \not{D}B_{ij}\varepsilon^{jk}\epsilon_k + C\varepsilon_{ij}\,\epsilon^j + \frac{1}{4}\big(\not{D}A\,\gamma^{ab}T_{abij} + w\,A\,\not{D}\gamma^{ab}T_{abij}\big)\varepsilon^{jk}\epsilon_k\\
&- 3\,\gamma_a\varepsilon^{jk}\epsilon_k\,\bar\chi_{[i}\gamma^a\Psi_{j]} - (1+w)\,B_{ij}\varepsilon^{jk}\,\eta_k + \frac{1}{2}(1-w)\,\gamma^{ab}\,F_{ab}^-\eta_i\,,\\
\delta C =& -2\,\varepsilon^{ij}\bar\epsilon_i\not{D}\Lambda_j - 6\,\bar\epsilon_i\chi_j\,\varepsilon^{ik}\varepsilon^{jl}B_{kl}\\
&- \frac{1}{4}\varepsilon^{ij}\varepsilon^{kl}\big((w-1)\,\bar\epsilon_i\gamma^{ab}\not{D}T_{abjk}\Psi_l + \bar\epsilon_i\gamma^{ab}T_{abjk}\not{D}\Psi_l\big) + 2\,w\varepsilon^{ij}\,\bar\eta_i\Lambda_j\,.
\end{aligned}\tag{2.1}$$

The spinors ϵ^i and η_i are the positive chirality spinorial parameters associated with Q- and S-supersymmetry. The corresponding negative chirality parameters are denoted by ϵ_i and η^i. We note that hermitian conjugation is always accompanied by raising or lowering of the SU(2) indices.

The transformation rules (2.1) are linear in the chiral multiplet fields, and contain also other fields associated with the conformal supergravity background, such as the self-dual tensor field T_{abij} and the spinor χ^i. Other conformal supergravity fields are contained in the superconformal derivatives D_μ. The superconformal multiplet of fields is described in more detail in appendix A.

Products of chiral superfields constitute again a chiral superfield, whose Weyl weight is equal to the sum of the Weyl weights of the separate multiplets. Also functions of chiral superfields may describe chiral superfields, assuming that they can be assigned a proper

[1] Observe that [12, 13] employ different conventions, in particular for (anti)symmetrization. Here (anti)symmetrization is always applied with unit strength.

	Vector multiplet			
field	X	Ω_i	W_μ	Y_{ij}
w	1	$\frac{3}{2}$	0	2
c	-1	$-\frac{1}{2}$	0	0
γ_5		$+$		

Table 2. Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the vector multiplet component fields.

Weyl weight. For instance, homogeneous functions of chiral superfields of the same Weyl weight w define a chiral supermultiplet whose Weyl weight equals the product of w times the degree of homogeneity. The relevant formulae are presented in appendix C.

Chiral multiplets of $w = 1$ are special, because they are reducible [12, 14]. Some details about these multiplets are given in appendix D. For a scalar chiral multiplet with $w = 1$ the tensor $F^-_{ab} + F^+_{ab}$ is subject to a Bianchi identity, which can be solved in terms of a vector gauge field. The reduced scalar chiral multiplet thus describes the covariant fields and field strength of a *vector multiplet*, which encompasses $8+8$ bosonic and fermionic components. Table 2 summarizes the Weyl and chiral weights of the various fields belonging to the vector multiplet: a complex scalar X, a Majorana doublet spinor Ω_i, a vector gauge field W_μ, and a triplet of auxiliary fields Y_{ij}. There also exists an anti-selfdual tensor version of the chiral multiplet with $w = 1$ that is reducible. This multiplet is the so-called *Weyl supermultiplet*, which contains all the covariant fields and curvatures of $N = 2$ conformal supergravity. It contains $24 + 24$ bosonic and fermionic degrees of freedom. Both vector supermultiplets and the Weyl multiplet play a central role in this paper.

Another special chiral multiplet is the so-called 'kinetic' multiplet, which has Weyl weight $w = 2$. This multiplet is constructed from an anti-chiral multiplet with $w = 0$. It will be discussed in detail in the next section.

Finally, scalar chiral multiplets with $w = 2$ lead to superconformal actions when including a conformal supergravity background. Their highest component C has Weyl weight 4, and chiral weight 0. To define an action that is invariant under local superconformal transformations one makes use of a density formula,

$$\begin{aligned} e^{-1}\mathcal{L} = {} & C - \varepsilon^{ij}\,\bar\psi_{\mu i}\gamma^\mu\Lambda_j - \frac{1}{8}\bar\psi_{\mu i}T_{ab\,jk}\gamma^{ab}\gamma^\mu\Psi_l\,\varepsilon^{ij}\varepsilon^{kl} - \frac{1}{16}A(T_{ab\,ij}\varepsilon^{ij})^2 \\ & - \frac{1}{2}\bar\psi_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}\,B_{kl}\;\varepsilon^{ik}\varepsilon^{jl} + \varepsilon^{ij}\bar\psi_{\mu i}\psi_{\nu j}\Big(F^{-\mu\nu} - \frac{1}{2}A\,T^{\mu\nu}{}_{kl}\,\varepsilon^{kl}\Big) \\ & - \frac{1}{2}\varepsilon^{ij}\varepsilon^{kl}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\psi_{\nu j}(\bar\psi_{\rho k}\gamma_\sigma\Psi_l + \bar\psi_{\rho k}\psi_{\sigma j}\,A)\,. \end{aligned} \tag{2.2}$$

3 The kinetic chiral multiplet

The term 'kinetic' multiplet was first used in the context of the $N = 1$ tensor calculus [15], because this is the chiral multiplet that enables the construction of the kinetic terms, conventionally described by a real superspace integral, in terms of a chiral superspace integral. In flat $N=1$ superspace, this construction is simply effected by the conversion,

$$\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\; \Phi\, \bar\Phi' \approx \int \mathrm{d}^2\theta\; \Phi\, \mathbb{T}(\bar\Phi')\,, \tag{3.1}$$

up to space-time boundary terms. Here Φ and Φ' are two chiral superfields and $\bar\Phi'$ is the anti-chiral field obtained from Φ' by complex conjugation. The kinetic multiplet equals $\mathbb{T}(\bar\Phi') = \bar D^2\bar\Phi'$, where $\bar D$ denotes the supercovariant $\bar\theta$-derivative. Obviously the kinetic multiplet contains terms linear and quadratic in space-time derivatives, so that, upon identifying Φ and Φ', the right-hand side of (3.1) does indeed give rise to the kinetic terms of an $N=1$ chiral multiplet.

In [13] a corresponding kinetic multiplet was identified for $N=2$ supersymmetry, which now involves four rather than two covariant $\bar\theta$-derivatives, i.e. $\mathbb{T}(\bar\Phi) \propto \bar D^4\bar\Phi$. As a result, $\mathbb{T}(\bar\Phi)$ contains now up to four space-time derivatives, so that the expression

$$\int \mathrm{d}^4\theta\, \mathrm{d}^4\bar\theta\; \Phi\, \bar\Phi' \approx \int \mathrm{d}^4\theta\, \Phi\, \mathbb{T}(\bar\Phi')\,, \tag{3.2}$$

does not correspond to a kinetic term, but to a higher-order derivative coupling. Furthermore, for $N=2$ supersymmetry one has the option of expressing the chiral multiplets in terms of (products of) reduced chiral multiplets. In that case, expressions such as (3.2) will correspond to higher-derivative couplings of vector multiplets. Since we are considering the kinetic multiplets in a conformal supergravity background, their Weyl weight is relevant. Both in $N=1,2$ supergravity the kinetic multiplet carries Weyl weight $w=2$. The conversion starts from a $w=1$ chiral multiplet for $N=1$ and from a $w=0$ chiral multiplet for $N=2$ supersymmetry, respectively.

To demonstrate this in more detail, consider an anti-chiral $N=2$ supermultiplet in the presence of the superconformal background. Its supersymmetry transformations follow from taking the complex conjugate of (2.1). Precisely for $w=0$ we note that the field $\bar C$ is invariant under S-supersymmetry and transforms under Q-supersymmetry as the lowest component of a chiral supermultiplet with $w=2$. This observation proves that we are dealing with a $w=2$ chiral supermultiplet, as is also confirmed by the weight assignments specified in table 1. What remains is to identify the various components of this multiplet in terms of the underlying $w=0$ multiplet. This can be done by applying successive Q-supersymmetry transformations on $\bar C$, something that requires rather tedious calculations in the presence of a superconformal background.

Denoting the components of $\mathbb{T}(\bar\Phi_{w=0})$ by $(A,\Psi,B,F^-,\Lambda,C)|_{\mathbb{T}(\bar\Phi)}$, while (A,Ψ,B,F^-,Λ,C) will denote the components of the original $w=0$ chiral multiplet, we have established the following relation,

$$\begin{aligned}
A|_{\mathbb{T}(\bar\Phi)} =&\, \bar C\,,\\
\Psi_i|_{\mathbb{T}(\bar\Phi)} =&\, -2\,\varepsilon_{ij}\,\not{D}\Lambda^j - 6\;\varepsilon_{ik}\varepsilon_{jl}\chi^j B^{kl} - \frac{1}{4}\varepsilon_{ij}\varepsilon_{kl}\,\gamma^{ab}T_{ab}{}^{jk}\,\overset{\leftrightarrow}{\not{D}}\,\Psi^l\,,\\
B_{ij}|_{\mathbb{T}(\bar\Phi)} =&\, -2\,\varepsilon_{ik}\varepsilon_{jl}\big(\Box_{\mathrm{c}} + 3\,D\big)B^{kl} - 2\,F^+_{ab}\,R(\mathcal{V})^{ab\,k}{}_i\,\varepsilon_{jk} - 6\,\varepsilon_{k(i}\,\bar\chi_{j)}\Lambda^k + 3\,\varepsilon_{ik}\varepsilon_{jl}\bar\Psi^{(k}\not{D}\chi^{l)}\,,
\end{aligned}$$

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$$
\begin{aligned}
F^-_{ab}|_{\mathbb{T}(\bar\Phi)} =& -\left(\delta_a{}^{[c}\delta_b{}^{d]} - \frac{1}{2}\varepsilon_{ab}{}^{cd}\right)\\
&\times \left[4\, D_c D^e F^+_{ed} + (D^e \bar A\, D_c T_{de}{}^{ij} + D_c \bar A\, D^e T_{ed}{}^{ij})\varepsilon_{ij}\right]\\
&+ \Box_{\rm c} \bar A\, T_{ab}{}^{ij}\varepsilon_{ij} - R(\mathcal{V})^-{}_{ab}{}^i{}_k\, B^{jk}\,\varepsilon_{ij} + \frac{1}{8} T_{ab}{}^{ij}\, T_{cdij} F^{+cd} - \varepsilon_{kl}\,\bar\Psi^k \overleftrightarrow{\not{D}}\, R(Q)_{ab}{}^l\\
&- \frac{9}{4}\varepsilon_{ij}\,\bar\Psi^i\gamma^c\gamma_{ab} D_c\chi^j + 3\,\varepsilon_{ij}\bar\chi^i\gamma_{ab}\not{D}\Psi^j + \frac{3}{8} T_{ab}{}^{ij}\varepsilon_{ij}\,\bar\chi_k\Psi^k\,,\\
\Lambda_i|_{\mathbb{T}(\bar\Phi)} =& \,2\,\Box_{\rm c}\not{D}\Psi^j\varepsilon_{ij} + \frac{1}{4}\gamma^c\gamma_{ab}(2\, D_c T^{ab}{}_{ij}\,\Lambda^j + T^{ab}{}_{ij}\, D_c\Lambda^j)\\
&- \frac{1}{2}\varepsilon_{ij}\left(R(\mathcal{V})_{ab}{}^j{}_k + 2{\rm i}\, R(A)_{ab}\delta^j{}_k\right)\gamma^c\gamma^{ab} D_c\Psi^k\\
&+ \frac{1}{2}\,\varepsilon_{ij}\left(3\, D_b D - 4{\rm i} D^a R(A)_{ab} + \frac{1}{4} T_{bc}{}^{ij}\,\overleftrightarrow{D}_a\, T^{ac}{}_{ij}\right)\gamma^b\Psi^j\\
&- 2\, F^{+ab}\,\not{D} R(Q)_{abi} + 6\,\varepsilon_{ij} D\,\not{D}\Psi^j + 3\,\varepsilon_{ij}\left(\not{D}\chi_k\, B^{kj} + \not{D}\bar A\,\not{D}\chi^j\right)\\
&+ \frac{3}{2}\left(2\,\not{D} B^{kj}\varepsilon_{ij} + \not{D} F^+_{ab}\gamma^{ab}\,\delta^k_i + \frac{1}{4}\varepsilon_{mn} T_{ab}{}^{mn}\,\gamma^{ab}\,\not{D}\bar A\,\delta_i{}^k\right)\chi_k\\
&+ \frac{9}{4}\,(\bar\chi^l\gamma_a\chi_l)\,\varepsilon_{ij}\gamma^a\Psi^j - \frac{9}{2}\,(\bar\chi_i\gamma_a\chi^k)\,\varepsilon_{kl}\gamma^a\Psi^l\,,\\
C|_{\mathbb{T}(\bar\Phi)} =& \,4(\Box_{\rm c} + 3\, D)\Box_{\rm c}\bar A - \frac{1}{2} D_a\left(T^{ab}{}_{ij}\, T_{cb}{}^{ij}\right) D^c\bar A + \frac{1}{16}(T_{abij}\varepsilon^{ij})^2\bar C\\
&+ D_a\left(\varepsilon^{ij} D^a T_{bcij}\, F^{+bc} + 4\,\varepsilon^{ij} T^{ab}{}_{ij}\, D^c F^+_{cb} - T_{bc}{}^{ij}\, T^{ac}{}_{ij}\, D^b\bar A\right)\\
&+ \left(6\, D_b D - 8{\rm i} D^a R(A)_{ab}\right) D^b\bar A + \cdots\,,
\end{aligned}
\tag{3.3}
$$

where in the last expression we suppressed terms quadratic in the covariant fermion fields. Obviously terms involving the fermionic gauge fields, $\psi_\mu{}^i$ and $\phi_\mu{}^i$, are already contained in the superconformal derivatives. Observe that the right-hand side of these expressions is always linear in the conjugate components of the $w = 0$ chiral multiplet, i.e. in $(\bar A, \Psi^i, B^{ij}, F^+_{ab}, \Lambda^i, \bar C)$. As an extra test of the correctness of (3.3) we verified that these expressions satisfy the correct transformation behaviour under S-supersymmetry. This test cannot be performed on the last component $C|_{\mathbb{T}(\bar\Phi)}$, because we refrained from collecting the fermionic contributions. As an extra check we have therefore verified that the bosonic terms of $C|_{\mathbb{T}(\bar\Phi)}$ are invariant under special conformal boosts.

The definition of the superconformal D'Alembertian $\Box_{\rm c}$, defined by the contraction of two superconformal derivatives D_a, as well as multiple superconformal derivatives in general, may require further comment. Therefore we have presented some relevant material in appendix B. Below we give the most non-trivial transformation rules under special conformal boosts that are needed in this paper,

$$
\begin{aligned}
\delta_{\rm K}\Box_{\rm c}\Box_{\rm c} A =& \,-2\,\Lambda_{\rm K}^a\left([D_a, D_b] D^b + D^b [D_a, D_b]\right) A\\
=& \,\frac{1}{4}\Lambda_{\rm K}^a\, T_{ac}{}^{ij}\, T^{bc}{}_{ij}\, D_b A - 3\,\Lambda_{\rm K}{}^a D\, D_a A - 2\,\Lambda_{\rm K}{}^a\, D^b(\bar R(Q)_{ba}\Psi_i)\\
&- \frac{3}{4}\Lambda_{\rm K}{}^a\,\bar\chi_i T_{ab}{}^{ij}\gamma^b\Psi_j + \frac{3}{4}\Psi_i\not{\Lambda}_{\rm K}\not{D}\chi^i\,,
\end{aligned}
$$

$$\delta_{\rm K}\Box_{\rm c}\not{D}\Psi_i = \not{\Lambda}_{\rm K}\Big[\frac{1}{4}\big(R(\mathcal{V})_{ab}{}^j{}_i + 2{\rm i}R(A)_{ab}\,\delta^j{}_i\big)\gamma^{ab}\Psi_j - \frac{3}{2}D\,\Psi_i\Big]$$
$$+\not{\Lambda}_{\rm K}\Big[\frac{3}{2}B_{ij}\,\chi^j - \varepsilon_{ij}F^{-\,ab}\,R(Q)^j_{ab} - \frac{3}{4}\varepsilon_{ij}F^-_{ab}\,\gamma^{ab}\chi^j\Big]\,, \tag{3.4}$$

These results follow from (B.6), upon making use of the relevant curvatures.

4 Invariant higher-derivative couplings

Using the results of the previous section one can construct a large variety of superconformal invariants for chiral multiplets with higher-derivative couplings. For unrestricted chiral supermultiplets one cannot write down Lagrangians that are at most quadratic in derivatives, so they usually play a role as composite fields that are expressed in terms of reduced chiral multiplets, such as the vector multiplets and the Weyl multiplet. The construction of the higher-order Lagrangians therefore proceeds in two steps. First one constructs the Lagrangian in terms of unrestricted chiral multiplets of the appropriate Weyl weights, and subsequently one expresses the unrestricted supermultiplets in terms of reduced supermultiplets. In these expressions it is natural to introduce a variety of arbitrary homogeneous functions.

The invariants are expressed as chiral superspace integrals, because all possible anti-chiral fields are contained in the kinetic multiplets that we have introduced in section 3. A simple example of this approach was already exhibited in (3.2). The fact that these invariants are actually based on full superspace integrals implies that they must vanish whenever all the chiral (or, alternatively, all the anti-chiral) fields are put equal to a constant. In the chiral formulation of the integral, this phenomenon is reflected in the fact that the kinetic multiplet of a *constant* anti-chiral multiplet vanishes. This result can easily be deduced from (3.3). Invariants can be substantially more complicated than (3.2). The integrand does not have to be linear in a kinetic multiplet, and can depend on a function of kinetic multiplets. One can also consider 'nested' situations, where a kinetic multiplet is constructed starting from an expression of superfields among which there are other kinetic multiplets, thus leading to even higher multiple derivatives.

The above approach is a constructive one and in general it will be hard to classify all these invariant couplings, say, in terms of a limited number of functions, as is often possible for supersymmetric theories. For definiteness, we henceforth restrict attention to invariants proportional to a single kinetic multiplet, as given in (3.2). In that case, expressing the composite chiral multiplets in terms of vector multiplets, one obtains the supergravity-coupled invariants corresponding to the actions derived in [5, 6] in the abelian limit, which contain F^4-couplings. By including the Weyl multiplet, one also obtains R^2F^2- and R^4-couplings. The R^2F^2-couplings will in principle overlap with part of a subclass of invariants discussed recently in [11] in connection with certain deformations of the topological string partition function. These couplings are encoded in terms of a single function of holomorphic and anti-holomorphic fields. In a rigid supersymmetry background these actions exhibit Kähler geometry with this function playing the role of a Kähler potential, just as happens in $N = 1$ supersymmetric actions for non-linear sigma models. As we will demonstrate below, this feature survives in the presence of the superconformal background. Other examples of higher-derivative couplings based on more than a single kinetic multiplet will be discussed in section 6.

Hence we start by writing down the bosonic terms of the Lagrangian (3.2). It is convenient to first note the following relation,

$$
\begin{aligned}
C|_{\mathbb{T}(\bar\Phi)} = {}& \frac{1}{16}(T_{abij}\varepsilon^{ij})^2\bar C + 4\left(\mathcal{D}^\mu\mathcal{D}_\mu\right)^2\bar A \\
& - 8\,\mathcal{D}^\mu\Big[\big(R_\mu{}^a(\omega,e) - \frac{1}{3}R(\omega,e)\,e_\mu{}^a - D\,e_\mu{}^a + \mathrm{i}R(A)_\mu{}^a\big)\mathcal{D}_a\bar A\Big] \\
& + \mathcal{D}_\mu\big[\varepsilon^{ij}\mathcal{D}^\mu T_{bcij}\,F^{+bc} + 4\,\varepsilon^{ij}T^{\mu b}{}_{ij}\,\mathcal{D}^c F^+_{cb} - 2\,T_{bc}{}^{ij}\,T^{\mu c}{}_{ij}\,\mathcal{D}^b\bar A\big] \\
& + \cdots\,,
\end{aligned}
\tag{4.1}
$$

where we suppressed all fermionic contributions. In deriving this result we made use of (A.6). Subsequently we derive the bosonic part of the Lagrangian corresponding to (3.2), making use of the density formula (2.2) and of the product rule (C.1),

$$
\begin{aligned}
e^{-1}\mathcal{L} = {}& 4\,\mathcal{D}^2 A\,\mathcal{D}^2\bar A + 8\,\mathcal{D}^\mu A\left[R_\mu{}^a(\omega,e) - \frac{1}{3}R(\omega,e)\,e_\mu{}^a\right]\mathcal{D}_a\bar A + C\,\bar C \\
& - \mathcal{D}^\mu B_{ij}\,\mathcal{D}_\mu B^{ij} + \left(\frac{1}{6}R(\omega,e) + 2\,D\right)B_{ij}B^{ij} \\
& - \big[\varepsilon^{ik}\,B_{ij}\,F^{+\mu\nu}\,R(\mathcal{V})_{\mu\nu}{}^j{}_k + \varepsilon_{ik}\,B^{ij}\,F^{-\mu\nu}\,R(\mathcal{V})_{\mu\nu j}{}^k\big] \\
& - 8\,D\,\mathcal{D}^\mu A\,\mathcal{D}_\mu\bar A + \big(8\,\mathrm{i}R(A)_{\mu\nu} + 2\,T_\mu{}^{cij}\,T_{\nu cij}\big)\mathcal{D}^\mu A\,\mathcal{D}^\nu\bar A \\
& - \big[\varepsilon^{ij}\mathcal{D}^\mu T_{bcij}\mathcal{D}_\mu A\,F^{+bc} + \varepsilon_{ij}\mathcal{D}^\mu T_{bc}{}^{ij}\mathcal{D}_\mu\bar A\,F^{-bc}\big] \\
& - 4\big[\varepsilon^{ij}T^{\mu b}{}_{ij}\,\mathcal{D}_\mu A\,\mathcal{D}^c F^+_{cb} + \varepsilon_{ij}T^{\mu bij}\,\mathcal{D}_\mu\bar A\,\mathcal{D}^c F^-_{cb}\big] \\
& + 8\,\mathcal{D}_a F^{-ab}\,\mathcal{D}^c F^+{}_{cb} + 4\,F^{-ac}\,F^+{}_{bc}\,R(\omega,e)_a{}^b + \frac{1}{4}T_{ab}{}^{ij}\,T_{cdij}F^{-ab}F^{+cd}\,.
\end{aligned}
\tag{4.2}
$$

Note that we suppressed the prime on the second chiral multiplet indicated in (3.2). In general, however, we will not always identify the two multiplets, so that the complex conjugated components in the above formula do not have to correspond to the same supermultiplet. However, upon making this identification, the above Lagrangian is manifestly real, which provides an additional check on the correctness of our result. The reason is that the corresponding lowest-order Lagrangian (3.2) is also real in that case (up to total derivatives that we have also suppressed in deriving the above result). Note also that the Lagrangian (4.2) vanishes whenever either one of the multiplets is equal to a constant, thus confirming the analysis presented at the beginning of this section.

We will now use the above results to write down the extension to local supersymmetry of the class of vector multiplet Lagrangians constructed in [5, 6]. Just as above we concentrate on the purely bosonic terms. The extension follows by writing the $w = 0$ chiral multiplets Φ and Φ' as composite multiplets expressed in terms of vector multiplets.

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In (3.2), and correspondingly in (4.2), one thus performs the following substitutions,

$$\Phi \to f(\Phi^I)\,, \qquad \bar\Phi' \to \bar g(\bar\Phi^I)\,, \tag{4.3}$$

where Φ^I denote the (reduced) chiral multiplets associated with vector multiplets. Upon expanding Φ and $\bar\Phi'$ in terms of the vector supermultiplets, making use of the material presented in appendices C and D, one obtains powers of the vector multiplet components multiplied by derivatives of $f(X)$ and $\bar g(\bar X)$, where the X^I denote the complex scalars of the vector multiplets. Homogeneity implies that $X^I f_I(X) = 0 = \bar X^I \bar g_{\bar I}(\bar X)$, where f_I and $\bar g_{\bar I}$ denote the first derivatives of the two functions with respect to X^I and $\bar X^I$, respectively. Here we recall that the expression (4.2) vanishes whenever $f(X)$ or $\bar g(\bar X)$ are constant. As noted previously, the origin of this phenomenon can be traced back to the fact that the full superspace integral of a chiral or an anti-chiral field vanishes (up to total derivatives). Therefore the Lagrangian will depend exclusively on mixed holomorphic/anti-holomorphic derivatives of the product function $f(X)\,\bar g(\bar X)$. By summing over an arbitrary set of pairs of functions $f^{(n)}(X)\,\bar g^{(n)}(\bar X)$, we can further extend this function to a general function $\mathcal{H}(X,\bar X)$ that is separately homogeneous of zeroth degree in X and $\bar X$. Because $\mathcal{H}(X,\bar X)$ is only defined up to a purely holomorphic or anti-holomorphic function, it is thus subject to Kähler transformations,

$$\mathcal{H}(X,\bar X) \to \mathcal{H}(X,\bar X) + \Lambda(X) + \bar\Lambda(\bar X)\,. \tag{4.4}$$

Hence $\mathcal{H}(X,\bar X)$ can be regarded as a Kähler potential, which may be taken real (so that $\bar\Lambda(\bar X) = [\Lambda(X)]^*$).

Carrying out the various substitutions leads directly to the following bosonic contribution to the supersymmetric Lagrangian (for convenience, we assume $\mathcal{H}$ to be real, unless stated otherwise),

$$\begin{aligned}
e^{-1}\mathcal{L} = {}& \mathcal{H}_{IJ\bar K\bar L}\Big[\frac{1}{4}\Big(F_{ab}^{-I}\,F^{-ab\,J} - \frac{1}{2}Y_{ij}{}^I\,Y^{ijJ}\Big)\Big(F_{ab}^{+K}\,F^{+ab\,L} - \frac{1}{2}Y^{ijK}\,Y_{ij}{}^L\Big) \\
&\qquad + 4\,\mathcal{D}_aX^I\,\mathcal{D}_b\bar X^K\Big(\mathcal{D}^aX^J\,\mathcal{D}^b\bar X^L + 2\,F^{-\,ac\,J}\,F^{+b}{}_c{}^L - \frac{1}{4}\eta^{ab}\,Y_{ij}^J\,Y^{L\,ij}\Big)\Big] \\
& + \Big\{\mathcal{H}_{IJ\bar K}\Big[4\,\mathcal{D}_aX^I\,\mathcal{D}^aX^J\,\mathcal{D}^2\bar X^K \\
&\qquad - \Big(F^{-ab\,I}\,F_{ab}^{-\,J} - \frac{1}{2}Y_{ij}^I\,Y^{Jij}\Big)\Big(\Box_{\rm c}X^K + \frac{1}{8}F_{ab}^{-\,K}\,T^{abij}\varepsilon_{ij}\Big) \\
&\qquad + 8\,\mathcal{D}^aX^I F_{ab}^{-\,J}\Big(\mathcal{D}_cF^{+\,cb\,K} - \frac{1}{2}\mathcal{D}_c\bar X^K T^{ij\,cb}\varepsilon_{ij}\Big) - \mathcal{D}_aX^I\,Y_{ij}^J\,\mathcal{D}^aY^{K\,ij}\Big] + \text{h.c.}\Big\} \\
& + \mathcal{H}_{I\bar J}\Big[4\Big(\Box_{\rm c}\bar X^I + \frac{1}{8}F_{ab}^{+\,I}\,T^{ab}{}_{ij}\varepsilon^{ij}\Big)\Big(\Box_{\rm c}X^J + \frac{1}{8}F_{ab}^{-\,J}\,T^{abij}\varepsilon_{ij}\Big) + 4\,\mathcal{D}^2X^I\,\mathcal{D}^2\bar X^J \\
&\qquad + 8\,\mathcal{D}_aF^{-\,abI}\,\mathcal{D}_cF^{+c}{}_b{}^J - \mathcal{D}_aY_{ij}{}^I\,\mathcal{D}^aY^{ij\,J} + \frac{1}{4}T_{ab}{}^{ij}\,T_{cdij}\,F^{-ab\,I}F^{+cd\,J} \\
&\qquad + \Big(\frac{1}{6}R(\omega,e) + 2\,D\Big)Y_{ij}{}^I\,Y^{ij\,J} + 4\,F^{-ac\,I}\,F^{+}{}_{bc}{}^J\,R(\omega,e)_a{}^b
\end{aligned}$$

$$
\begin{aligned}
&+8\Big(R^{\mu\nu}(\omega,e)-\frac{1}{3}g^{\mu\nu}R(\omega,e)+\frac{1}{4}T^{\mu}{}_{b}{}^{ij}\,T^{\nu b}{}_{ij}+\mathrm{i}R(A)^{\mu\nu}-g^{\mu\nu}D\Big)\mathcal{D}_\mu X^I\,\mathcal{D}_\nu\bar{X}^J\\
&-\big[\mathcal{D}_c\bar{X}^J\big(\mathcal{D}^cT_{ab}{}^{ij}\,F^{-I\,ab}+4\,T^{ij\,cb}\,\mathcal{D}^aF_{ab}^{-I}\big)\varepsilon_{ij}+[\text{h.c.};I\leftrightarrow J]\big]\\
&-\big[\varepsilon^{ik}\,Y_{ij}{}^I\,F^{+ab\,J}\,R(\mathcal{V})_{ab}{}^j{}_k+[\text{h.c.};I\leftrightarrow J]\big]\Big]\,,
\end{aligned}
\tag{4.5}
$$

where (we suppress fermionic contributions),

$$
\begin{aligned}
F_{ab}^{-I}&=\Big(\delta_{ab}{}^{cd}-\frac{1}{2}\varepsilon_{ab}{}^{cd}\Big)e_c{}^\mu e_d{}^\nu\,\partial_{[\mu}W_{\nu]}{}^I-\frac{1}{4}\,\bar{X}^I\,T_{ab}{}^{ij}\varepsilon_{ij}\,,\\
\Box_{\mathrm{c}}X^I&=\mathcal{D}^2X^I+\Big(\frac{1}{6}R(\omega,e)+D\Big)\,X^I\,.
\end{aligned}
\tag{4.6}
$$

In view of the Kähler equivalence transformations (4.4), the mixed derivative $\mathcal{H}_{I\bar{J}}$ can be identified as a Kähler metric. Hence we have the following results for the metric, connection, and the curvature of the corresponding Kähler space,

$$
\begin{aligned}
g_{I\bar{J}}&=\mathcal{H}_{I\bar{J}}\,,\\
\Gamma^I{}_{JK}&=g^{I\bar{L}}\,\mathcal{H}_{JK\bar{L}}\,,\\
R_{I\bar{J}K\bar{L}}&=\mathcal{H}_{IK\bar{J}\bar{L}}-g_{M\bar{N}}\,\Gamma^M{}_{IK}\,\Gamma^{\bar{N}}{}_{\bar{J}\bar{L}}\,.
\end{aligned}
\tag{4.7}
$$

The Lagrangian (4.5) can then be written in a Kähler covariant form,

$$
\begin{aligned}
e^{-1}\mathcal{L}=R_{I\bar{K}J\bar{L}}\Big[&\frac{1}{4}\Big(F_{ab}^{-I}\,F^{-ab\,J}-\frac{1}{2}Y_{ij}{}^I\,Y^{ijJ}\Big)\Big(F_{ab}^{+K}\,F^{+ab\,L}-\frac{1}{2}Y^{ijK}\,Y_{ij}{}^L\Big)\\
&+4\,\mathcal{D}_aX^I\,\mathcal{D}_b\bar{X}^K\Big(\mathcal{D}^aX^J\,\mathcal{D}^b\bar{X}^L+2\,F^{-\,ac\,J}\,F^{+b}{}_c{}^L-\frac{1}{4}\eta^{ab}\,Y_{ij}^J\,Y^{L\,ij}\Big)\Big]\\
+g_{I\bar{J}}\Big[&4\Big(\Box_{\mathrm{c}}\bar{X}^I+\frac{1}{8}F_{ab}^{+I}\,T^{ab}{}_{ij}\varepsilon^{ij}-\frac{1}{4}\Gamma^I{}_{KL}(F_{ab}^{-K}\,F^{-ab\,L}-\frac{1}{2}Y^{ijK}\,Y_{ij}{}^L)\Big)\\
&\times\Big(\Box_{\mathrm{c}}X^J+\frac{1}{8}F_{ab}^{-J}\,T^{abij}\varepsilon_{ij}-\frac{1}{4}\Gamma^{\bar{J}}{}_{\bar{K}\bar{L}}(F_{ab}^{+K}\,F^{+ab\,L}-\frac{1}{2}Y^{ijK}\,Y_{ij}{}^L)\Big)\\
&+4\,(\mathcal{D}^2X^I+\Gamma^I{}_{KL}\,\mathcal{D}_bX^K\,\mathcal{D}^bX^L)\,(\mathcal{D}^2\bar{X}^J+\Gamma^{\bar{J}}{}_{\bar{K}\bar{L}}\,\mathcal{D}_b\bar{X}^K\,\mathcal{D}^b\bar{X}^L)\\
&+8\,(\mathcal{D}_aF^{-\,abI}+\Gamma^I{}_{KL}\,\mathcal{D}_aX^K\,F^{-\,abL})\,(\mathcal{D}_cF^{+c}{}_b{}^J+\Gamma^{\bar{J}}{}_{\bar{K}\bar{L}}\,\mathcal{D}_c\bar{X}^K\,F^{+c}{}_b{}^L)\\
&-(\mathcal{D}_aY_{ij}{}^I+\Gamma^I{}_{KL}\,\mathcal{D}_bX^K\,Y_{ij}{}^L)\,(\mathcal{D}^aY^{J\,ij}+\Gamma^{\bar{J}}{}_{\bar{K}\bar{L}}\,\mathcal{D}_b\bar{X}^K\,Y^{ijL})\\
&+\frac{1}{4}T_{ab}{}^{ij}\,T_{cdij}\,F^{-ab\,I}F^{+cd\,J}\\
&+\Big(\frac{1}{6}R(\omega,e)+2\,D\Big)Y_{ij}{}^I\,Y^{ij\,J}+4\,F^{-ac\,I}\,F^{+}{}_{bc}{}^J\,R(\omega,e)_a{}^b\\
&+8\Big(R^{\mu\nu}(\omega,e)-\frac{1}{3}g^{\mu\nu}R(\omega,e)+\frac{1}{4}T^{\mu}{}_b{}^{ij}\,T^{\nu b}{}_{ij}+\mathrm{i}R(A)^{\mu\nu}-g^{\mu\nu}D\Big)\mathcal{D}_\mu X^I\,\mathcal{D}_\nu\bar{X}^J
\end{aligned}
$$

$$
\begin{aligned}
&- \big[\mathcal{D}_c \bar X^J \big(\mathcal{D}^c T_{ab}{}^{ij}\, F^{-\,I\,ab} + 4\, T^{ij\,cb}\, (\mathcal{D}^a F^{-\,I}_{ab} + \Gamma^I{}_{KL} \mathcal{D}^a X^K F^{-\,L}_{ab})\big)\varepsilon_{ij} \\
&+ [\text{h.c.}; I \leftrightarrow J]\big] \\
&- \big[\varepsilon^{ik}\, Y_{ij}{}^I\, F^{+ab\,J}\, R(\mathcal{V})_{ab}{}^j{}_k + [\text{h.c.}; I \leftrightarrow J]\big]\Bigg]\,.
\end{aligned} \tag{4.8}
$$

The covariantizations in the various combinations can be understood systematically by rewriting the chiral multiplet components of the vector multiplets such that they are covariant with respect to the complex reparametrizations of the Kähler space (in the limit where the fermions are suppressed). An easy way to appreciate these covariantizations is by reorganizing the expansion of a composite chiral multiplet into vector multiplets according to (C.2) by replacing the ordinary derivatives of the function $\mathcal{G}$ by covariant derivatives.

The Lagrangians (4.5) and/or (4.8) can also be used in the context of rigidly supersymmetric theories upon suppressing all the superconformal fields. The resulting Lagrangian is then superconformally invariant in flat Minkowski space. This invariance can be further reduced to ordinary Poincaré supersymmetry by replacing one of the vector multiplets by a constant.

As an extension of the previous results we return to (4.2), and consider composite chiral multiplets that depend on both vector multiplets and on the Weyl multiplet. Hence we replace (4.3) by

$$
\Phi \to f(\Phi^I, W^2)\,, \qquad \bar\Phi' \to \bar g(\bar\Phi^I, \bar W^2)\,, \tag{4.9}
$$

where W^2 refers to the square of the Weyl multiplet. The components of this reduced chiral multiplet are given in (D.6). Upon expanding these functions and substituting the results into (4.2), one obtains a Lagrangian that contains R^4-, R^2F^2- and F^4-terms. All terms are proportional to mixed holomorphic/anti-holomorphic derivatives of a function $\mathcal{H}(X, T^2, \bar X, \bar T^2)$, where $T^2 = (T_{ab}{}^{ij}\varepsilon_{ij})^2$ and $\bar T^2 = (T_{abij}\varepsilon^{ij})^2$, and where $\mathcal{H}$ is constructed from pairs of products of functions $f(X, T^2)$ and $\bar g(\bar X, \bar T^2)$. The fact that the composite multiplets have $w = 0$ implies a modified homogeneity property,

$$
X^I \mathcal{H}_I(X, T^2, \bar X, \bar T^2) + 2\, T^2 \mathcal{H}_{T^2}(X, T^2, \bar X, \bar T^2) = 0\,, \tag{4.10}
$$

and likewise for the anti-holomorphic derivatives.

The Lagrangian consists of the Lagrangian (4.5) plus a large number of terms that involve multiple derivatives of $\mathcal{H}$ with respect to T^2, $\bar T^2$, X^I and $\bar X^I$. Below we concentrate on terms proportional to multiple derivatives of $\mathcal{H}$ with respect to only T^2 and $\bar T^2$. Among others those contain contributions of fourth order in $\mathcal{R}(M)$, whose leading contribution is equal to the Weyl tensor,

$$
\begin{aligned}
&(64)^{-2}\, e^{-1}\mathcal{L} = \\
&\quad 4\, \mathcal{H}_{T^2T^2\bar T^2\bar T^2}\, T^{abij}\varepsilon_{ij}\, T^{cdkl}\varepsilon_{kl}\, T^{ef}{}_{mn}\varepsilon^{mn}\, T^{gh}{}_{pq}\varepsilon^{pq} \\
&\qquad \times \Big[\mathcal{R}(M)_{aba'b'}\mathcal{R}(M)_{cd}{}^{a'b'} + \frac{1}{2} R(\mathcal{V})_{ab}{}^i{}_j\, R(\mathcal{V})_{cd}{}^j{}_i\Big] \\
&\qquad \times \Big[\mathcal{R}(M)_{efe'f'}\mathcal{R}(M)_{gh}{}^{e'f'} + \frac{1}{2} R(\mathcal{V})_{ef}{}^i{}_j\, R(\mathcal{V})_{gh}{}^j{}_i\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+2\Big\{\mathcal{H}_{T^2T^2\bar{T}^2}\,T^{abij}\varepsilon_{ij}\,T^{cdkl}\varepsilon_{kl}\\
&\quad\times\Big[\mathcal{R}(M)_{aba'b'}\mathcal{R}(M)_{cd}{}^{a'b'}+\frac{1}{2}R(\mathcal{V})_{ab}{}^{i}{}_{j}\,R(\mathcal{V})_{cd}{}^{j}{}_{i}\Big]\\
&\quad\times\Big[\mathcal{R}(M)^{+}_{efgh}\mathcal{R}(M)^{+efgh}+\frac{1}{2}R(\mathcal{V})^{+}_{ef}{}^{i}{}_{j}\,R(\mathcal{V})^{+efj}{}_{i}-\frac{1}{2}T^{ef}{}_{mn}D_eD^hT_{hf}{}^{mn}\Big]+[\text{h.c.}]\Big\}\\
&+\mathcal{H}_{T^2\bar{T}^2}\Big\{\big|\mathcal{R}(M)^{+}_{abcd}\mathcal{R}(M)^{+abcd}+\frac{1}{2}R(\mathcal{V})^{+}_{ab}{}^{i}{}_{j}\,R(\mathcal{V})^{+abj}{}_{i}-\frac{1}{2}T^{ab}{}_{mn}D_aD^eT_{eb}{}^{mn}\big|^2+\cdots\Big\}.
\end{aligned}
\tag{4.11}
$$

Besides the terms quartic in $\mathcal{R}(M)$ we have retained some of the terms that come with them as part of the basic building blocks that emerge in the calculation (similar blocks appear in (4.5)). Besides giving a little more information in this way, this has the advantage that the origin of the various term will be easier to track down.

In addition to the above terms there are mixed terms which lead to explicit contributions from the vector multiplets (i.e. beyond the X and $\bar{X}$ dependence in the function $\mathcal{H}$). Those include, for instance, terms proportional to $[\mathcal{R}(M)]^2$ times the product of two vector multiplet field strengths, $F_{\mu\nu}{}^{I}$. We will not exhibit those terms here (they can in principle be deduced from (4.2) along the same lines as for the previous contributions). Some of these terms will be shown in the equation below.

A special case, which is worth mentioning in view of the work of [11], corresponds to functions $\mathcal{H}(X,T^2,\bar{X})$ that do not depend on $\bar{T}^2$. Hence the function $\mathcal{H}$ is not real. Again we do not present all the terms, but we give all the terms that contain $\mathcal{R}(M)$ (with some completions), with the exception of terms proportional to derivatives of X^I and $T_{ab}{}^{ij}$ or their complex conjugates,

$$
\begin{aligned}
&(64)^{-1}e^{-1}\mathcal{L}=\\
&\mathcal{H}_{T^2T^2\bar{K}\bar{L}}\Big\{T^{abij}\varepsilon_{ij}\,T^{cdkl}\varepsilon_{kl}\,\Big[\mathcal{R}(M)_{aba'b'}\mathcal{R}(M)_{cd}{}^{a'b'}+\frac{1}{2}R(\mathcal{V})_{ab}{}^{i}{}_{j}\,R(\mathcal{V})_{cd}{}^{j}{}_{i}\Big]\\
&\qquad\times\Big[F^{+K}_{ef}F^{+efL}-\frac{1}{2}Y^{mnK}Y_{mn}{}^{L}\Big]+\cdots\Big\}\\
&-4\,\mathcal{H}_{T^2T^2\bar{K}}\Big\{T^{abij}\varepsilon_{ij}\,T^{cdkl}\varepsilon_{kl}\,\Big[\mathcal{R}(M)_{aba'b'}\mathcal{R}(M)_{cd}{}^{a'b'}+\frac{1}{2}R(\mathcal{V})_{ab}{}^{i}{}_{j}\,R(\mathcal{V})_{cd}{}^{j}{}_{i}\Big]\\
&\qquad\times\Big[\Box_{\mathrm{c}}X^K+\frac{1}{8}F_{ef}{}^{K}T^{efij}\varepsilon_{ij}\Big]+\cdots\Big\}\\
&+\frac{1}{2}\mathcal{H}_{T^2I\bar{K}}\Big\{T^{cdlm}\varepsilon_{lm}\Big[F^{-I}_{ab}\mathcal{R}(M)_{cd}{}^{ab}-\frac{1}{2}Y^{ijI}\varepsilon_{ki}\,R(\mathcal{V})_{cd}{}^{k}{}_{j}\Big]\\
&\qquad\times\Big[\Box_{\mathrm{c}}X^K+\frac{1}{8}F_{ef}{}^{K}T^{efij}\varepsilon_{ij}\Big]+\cdots\Big\}\\
&-\frac{1}{8}\mathcal{H}_{T^2I\bar{K}\bar{L}}\Big\{T^{cdlm}\varepsilon_{lm}\Big[F_{ab}{}^{I}\mathcal{R}(M)_{cd}{}^{ab}-\frac{1}{2}Y^{ijI}\varepsilon_{ki}\,R(\mathcal{V})_{cd}{}^{k}{}_{j}\Big]\\
&\qquad\times\Big[F^{+I}_{ab}F^{+abJ}-\frac{1}{2}Y^{ijK}Y_{ij}{}^{L}\Big]+\cdots\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\mathcal{H}_{T^2\bar K\bar L}\Big\{\Big[\mathcal{R}(M)^-_{cdef}\,\mathcal{R}(M)^{-cdef}+\frac{1}{2}R(\mathcal{V})^{-}_{cd}{}^{i}{}_{j}\,R(\mathcal{V})^{-cdj}{}_{i}-\frac{1}{2}T^{cdmn}D_cD^eT_{edmn}\Big]\\
&\qquad\qquad\times\Big[F^{+I}_{ab}F^{+abJ}-\frac{1}{2}Y^{ijK}Y_{ij}{}^{L}\Big]+\cdots\Big\}\\
&-2\,\mathcal{H}_{T^2\bar K}\Big\{\Big[\mathcal{R}(M)^-_{abcd}\,\mathcal{R}(M)^{-abcd}+\frac{1}{2}R(\mathcal{V})^{-}_{ab}{}^{i}{}_{j}\,R(\mathcal{V})^{-abj}{}_{i}-\frac{1}{2}T^{abmn}D_aD^cT_{cbmn}\Big]\\
&\qquad\qquad\times\Big[\Box_{\rm c}X^K+\frac{1}{8}F_{ef}{}^{K}T^{efij}\varepsilon_{ij}\Big]\\
&\qquad+T_{cd}{}^{ij}\varepsilon_{ij}\,\mathcal{R}(M)^{cdab}\times\\
&\qquad\qquad\times\Big[\frac{1}{32}T_{ab}{}^{kl}T_{efkl}\,F^{efK}+\frac{1}{2}F^{+K}_{eb}\,R(\omega,e)_a{}^{e}-\frac{1}{8}\varepsilon_{km}\,Y^{klK}R(\mathcal{V})_{abl}{}^{m}\Big]\\
&\qquad+T_{cd}{}^{ij}\varepsilon_{ij}\,\mathcal{D}_a\mathcal{R}(M)^{cdab}\,\mathcal{D}^eF^{+K}_{eb}+\cdots\Big\}\,.
\end{aligned}
\tag{4.12}
$$

5 A non-renormalization theorem for BPS black hole entropy

The results of this paper can be used in the study of black holes. Based on any linear combination of the various $N=2$ locally supersymmetric Lagrangians, one can evaluate the corresponding expressions for the Wald entropy and the electric charges in terms of the values of the fields taken at the black hole horizon. In the case of BPS black holes, the horizon values of the fields are highly restricted due to full supersymmetry enhancement at the horizon, and therefore the resulting expressions for the entropy and the charges will simplify. To explore this one must determine the possible supersymmetric field configurations, preferably in an off-shell formulation so that the results do not depend on the specific Lagrangian. This has already been done in [16], which provided a generalization of the attractor equations found in [17–19]. So far, generic chiral supermultiplets were not considered, but it is convenient to do so as well. As it will turn out, it suffices to restrict oneself to chiral multiplets of Weyl weight $w=0$, for which results are rather straightforward to obtain.

The first relevant observation is that a constant chiral superfield (i.e. a supermultiplet with constant A and all other components vanishing) is only supersymmetric provided it has $w=0$. In fact there exist no other supersymmetric values of the chiral superfield. All this can be derived directly from the transformation rules (2.1). The second observation is that the kinetic multiplet constructed from a $w=0$ anti-chiral multiplet, vanishes when the latter multiplet is equal to a constant. This follows by inspection of (3.3). These two observations prove immediately that any invariant proportional to a kinetic multiplet, must vanish for supersymmetric field configurations. This fact can immediately be verified from (4.2), because when the fields A and $\bar A'$ are constant and all other chiral multiplet component fields are vanishing, the expression (4.2) indeed vanishes.

The above result is interesting in its own right, but we are also interested in the first-order variation of the action induced by a change of some of the fields, evaluated for a supersymmetric background. Given the fact that all the invariants discussed in this paper will contain at least one kinetic multiplet, we thus consider

$$\delta\mathcal{L} \propto \int \mathrm{d}^4\theta \left[\delta\Phi\,\mathbb{T}(\bar{\Phi}') + \Phi\,\delta\mathbb{T}(\bar{\Phi}')\right], \tag{5.1}$$

where Φ and Φ' are composite chiral fields, which are themselves expressed in various chiral fields, including possible kinetic multiplets. They are not necessarily uniquely defined, and it is also possible to consider linear combinations of such terms. Since we will be evaluating the variation at supersymmetric values of the fields, the first term in (5.1) vanishes, because the kinetic multiplet vanishes, whereas the second term can be evaluated for constant Φ.

However, rather than continuing in this way, we may simply return to (4.2) and consider its variation. Observe that each term is proportional to a product of one component of Φ and another one of $\bar{\Phi}'$ (we remind the reader that in (4.2) we suppressed the prime for notational clarity). All these components will be equal to zero in a supersymmetric background, with the exception of A and $\bar{A}'$, which will take constant values. However, only space-time derivatives of A and $\bar{A}'$ appear, and those will vanish as well. In other words, (4.2) is always quadratic in quantities that are vanishing in the supersymmetry limit. Hence any first-order variation of any Lagrangian of this type must necessarily vanish in a supersymmetric background!

The above result suffices to derive a non-renormalization theorem for electric charges and the Wald entropy [20–22] for BPS black holes. The reason is that these quantities are always expressed in terms of first-order derivatives of the Lagrangian with respect to certain fields, such as the abelian field strengths or the Riemann tensor, or possible derivatives thereof. This concludes the proof of the non-renormalization theorem.

As we already mentioned in section 1, the existence of this non-renormalization theorem is a welcome result. So far good agreement has been established for BPS black hole entropy evaluated on the basis of supergravity and of microstate counting, suggesting that other invariants in supergravity should contribute only marginally, or perhaps not at all, at the subleading level. The result of this section lends support to this idea. Nevertheless the possible existence of alternative supersymmetric invariants that do not belong to the class of invariants discussed in this paper, cannot be excluded at this stage.

6 Concluding remarks

In this paper we studied a large class of $N = 2$ superconformal invariants involving higher-derivative couplings, based on full superspace integrals. For a special subclass we have presented explicit results for some of the bosonic terms. This is the subclass that contains only a single kinetic multiplet.

As indicated already, there are further options. The most obvious one is to include more kinetic multiplets, based on various composite chiral and anti-chiral multiplets with suitable Weyl weights,

$$\int \mathrm{d}^4\theta\; \Phi_0\,\mathbb{T}(\bar{\Phi}_1)\cdots\mathbb{T}(\bar{\Phi}_n)\,, \tag{6.1}$$

where $\bar{\Phi}_1,\ldots\bar{\Phi}_n$ are anti-chiral superfields of zero weight and Φ_0 is a chiral superfield of weight $w = -2(n-1)$. This leads to actions that contain four space-time derivatives.

However, when treating the chiral multiplets as composites of reduced chiral multiplets, one obtains invariants with terms of $2(1+n)$ powers of field strengths and/or explicit derivatives, i.e., $R^{2m}F^{2p}\mathcal{D}^{2(n+1-m-p)}$. The case of $n=1$ has been dealt with in considerable detail in section 4. The expression of the composite chiral multiplets in terms of the reduced ones allows again for the presence of functions $\mathcal{H}^{(n)}$ which are subject to a generalized version of the Kähler transformations noted in section 4.

As alluded to before, one can also consider nested situations where the kinetic multiplet is constructed from a combination of (anti)chiral fields that include again other kinetic multiplets. In this way one constructs multiplets with multiple derivatives of arbitrary power. We are then led to introduce quantities of the type,

$$\mathbb{T}^{(2)} = \mathbb{T}(\bar\Phi_2\,\mathbb{T}(\Phi_1))\,,\quad \mathbb{T}^{(3)} = \mathbb{T}(\bar\Phi_3\,\mathbb{T}(\Phi_2\,\mathbb{T}(\bar\Phi_1)))\,,\ \ldots,\ \mathbb{T}^{(n)} = \mathbb{T}(\bar\Phi_n\,\mathbb{T}^{(n-1)})\,, \tag{6.2}$$

which can be part of any superspace integrand, on the same footing as the kinetic multiplets in (6.1). Here Φ_1 has $w=0$ and $\Phi_2,\Phi_3,\cdots$ have $w=-2$. This extends the number of invariants to all possible combinations of the form

$$\int \mathrm{d}^4\theta\,\Phi_0\,\mathbb{T}^{(n_1)}\,\mathbb{T}^{(n_2)}\cdots\,\mathbb{T}^{(n_k)}\,, \tag{6.3}$$

where Φ_0 has $w=-2(k-1)$ and where we assume $n_k\geq 1$ with $\mathbb{T}(\bar\Phi_1)\equiv\mathbb{T}^{(1)}$. When expressing all the chiral multiplets in terms of reduced ones, then one can show that the maximal number of derivatives of the invariants (6.3) is equal to $2(1+\sum_k n_k)$.

These types of invariants are not necessarily independent in the sense that there can be linear combinations that are equal to a total derivative. For example, at the six-derivative level, one has

$$\int \mathrm{d}^4\theta\,\Phi_0\,\mathbb{T}(\bar\Phi_2\,\mathbb{T}(\Phi_1)) \approx \int \mathrm{d}^4\bar\theta\,\bar\Phi_2\mathbb{T}(\Phi_0)\,\mathbb{T}(\Phi_1)\,, \tag{6.4}$$

up to total derivatives. Nevertheless it is clear that we are dealing with an infinite hierarchy of higher-derivative invariants.

Of course, a relevant question is whether the invariant couplings presented in this paper exhaust the possible higher-derivative invariants. Most likely, this will not be the case. From the perspective of BPS black holes the question would then remain whether these conjectured couplings could still contribute to the entropy and electric charges.

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	Weyl multiplet											parameters	
field	$e_\mu{}^a$	$\psi_\mu{}^i$	b_μ	A_μ	$\mathcal{V}_\mu{}^i{}_j$	$T_{ab}{}^{ij}$	χ^i	D	ω_μ^{ab}	$f_\mu{}^a$	$\phi_\mu{}^i$	ϵ^i	η^i
w	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
c	0	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
γ_5		$+$					$+$				$-$	$+$	$-$

Table 3. Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the Weyl multiplet component fields and the supersymmetry transformation parameters.

A Superconformal calculus

Throughout this paper we use Pauli-Källén conventions and follow the notation used e.g. in [16]. Space-time and Lorentz indices are denoted by $\mu, \nu, \ldots$, and $a, b, \ldots$, respectively; SU(2)-indices are denoted by $i, j, \ldots$. As mentioned already in footnote 1, (anti-)symmetrizations are always defined with unit strength.

In this appendix we present the transformation rules of the superconformal fields and their relation to the superconformal algebra, as well as their covariant quantities contained in the so-called Weyl supermultiplet. The superconformal algebra comprises the generators of the general-coordinate, local Lorentz, dilatation, special conformal, chiral SU(2) and U(1), supersymmetry (Q) and special supersymmetry (S) transformations. The gauge fields associated with general-coordinate transformations ($e_\mu{}^a$), dilatations (b_μ), chiral symmetry ($\mathcal{V}_\mu{}^i{}_j$ and A_μ) and Q-supersymmetry ($\psi_\mu{}^i$) are independent fields. The remaining gauge fields associated with the Lorentz ($\omega_\mu{}^{ab}$), special conformal ($f_\mu{}^a$) and S-supersymmetry transformations ($\phi_\mu{}^i$) are dependent fields. They are composite objects, which depend on the independent fields of the multiplet [13, 23, 24]. The corresponding supercovariant curvatures and covariant fields are contained in a tensor chiral multiplet, which comprises $24+24$ off-shell degrees of freedom. In addition to the independent superconformal gauge fields, it contains three other fields: a Majorana spinor doublet χ^i, a scalar D, and a selfdual Lorentz tensor T_{abij}, which is anti-symmetric in $[ab]$ and $[ij]$. The Weyl and chiral weights have been collected in table 3.

Under Q-supersymmetry, S-supersymmetry and special conformal transformations the independent fields of the Weyl multiplet transform as follows,

$$
\begin{aligned}
\delta e_\mu{}^a &= \bar\epsilon^i\,\gamma^a\psi_{\mu i} + \bar\epsilon_i\,\gamma^a\psi_\mu{}^i\,, \\
\delta\psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i - \frac{1}{8}T_{ab}{}^{ij}\gamma^{ab}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i \\
\delta b_\mu &= \frac{1}{2}\bar\epsilon^i\phi_{\mu i} - \frac{3}{4}\bar\epsilon^i\gamma_\mu\chi_i - \frac{1}{2}\bar\eta^i\psi_{\mu i} + \text{h.c.} + \Lambda_K^a e_{\mu a}\,, \\
\delta A_\mu &= \frac{1}{2}\mathrm{i}\bar\epsilon^i\phi_{\mu i} + \frac{3}{4}\mathrm{i}\bar\epsilon^i\gamma_\mu\,\chi_i + \frac{1}{2}\mathrm{i}\bar\eta^i\psi_{\mu i} + \text{h.c.}\,, \\
\delta\mathcal{V}_\mu{}^i{}_j &= 2\,\bar\epsilon_j\phi_\mu{}^i - 3\bar\epsilon_j\gamma_\mu\,\chi^i + 2\bar\eta_j\,\psi_\mu{}^i - (\text{h.c.}\ ;\ \text{traceless})\,, \\
\delta T_{ab}{}^{ij} &= 8\,\bar\epsilon^{[i}R(Q)_{ab}{}^{j]}\quad,
\end{aligned}
$$

$$
\begin{aligned}
\delta\chi^i &= -\frac{1}{12}\gamma^{ab}\,\not{D}T_{ab}{}^{ij}\,\epsilon_j + \frac{1}{6}R(\mathcal{V})_{\mu\nu}{}^i{}_j\gamma^{\mu\nu}\epsilon^j - \frac{1}{3}\mathrm{i}R_{\mu\nu}(A)\gamma^{\mu\nu}\epsilon^i + D\epsilon^i + \frac{1}{12}\gamma_{ab}T^{abij}\eta_j\,,\\
\delta D &= \bar\epsilon^i\,\not{D}\chi_i + \bar\epsilon_i\,\not{D}\chi^i\,.
\end{aligned}
\tag{A.1}
$$

Here ϵ^i and ϵ_i denote the spinorial parameters of Q-supersymmetry, η^i and η_i those of S-supersymmetry, and $\Lambda_K{}^a$ is the transformation parameter for special conformal boosts. The full superconformally covariant derivative is denoted by D_μ, while $\mathcal{D}_\mu$ denotes a covariant derivative with respect to Lorentz, dilatation, chiral U(1), and SU(2) transformations,

$$
\mathcal{D}_\mu\epsilon^i = \Big(\partial_\mu - \frac{1}{4}\omega_\mu{}^{cd}\,\gamma_{cd} + \frac{1}{2}\,b_\mu + \frac{1}{2}\mathrm{i}\,A_\mu\Big)\epsilon^i + \frac{1}{2}\,\mathcal{V}_\mu{}^i{}_j\,\epsilon^j\,. \tag{A.2}
$$

The covariant curvatures of the various gauge symmetries take the following form,

$$
\begin{aligned}
R(P)_{\mu\nu}{}^a &= 2\,\partial_{[\mu}\,e_{\nu]}{}^a + 2\,b_{[\mu}\,e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\,e_{\nu]b} - \frac{1}{2}(\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]i} + \text{h.c.})\,,\\
R(Q)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^i - \gamma_{[\mu}\phi_{\nu]}{}^i - \frac{1}{8}\,T^{abij}\,\gamma_{ab}\,\gamma_{[\mu}\psi_{\nu]j}\,,\\
R(A)_{\mu\nu} &= 2\,\partial_{[\mu}A_{\nu]} - \mathrm{i}\left(\frac{1}{2}\bar\psi_{[\mu}{}^i\phi_{\nu]i} + \frac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \text{h.c.}\right)\,,\\
R(\mathcal{V})_{\mu\nu}{}^i{}_j &= 2\,\partial_{[\mu}\mathcal{V}_{\nu]}{}^i{}_j + \mathcal{V}_{[\mu}{}^i{}_k\,\mathcal{V}_{\nu]}{}^k{}_j + 2(\bar\psi_{[\mu}{}^i\,\phi_{\nu]j} - \bar\psi_{[\mu j}\,\phi_{\nu]}{}^i) - 3(\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_j - \bar\psi_{[\mu j}\gamma_{\nu]}\chi^i)\\
&\quad - \delta_j{}^i(\bar\psi_{[\mu}{}^k\,\phi_{\nu]k} - \bar\psi_{[\mu k}\,\phi_{\nu]}{}^k) + \frac{3}{2}\delta_j{}^i(\bar\psi_{[\mu}{}^k\gamma_{\nu]}\chi_k - \bar\psi_{[\mu k}\gamma_{\nu]}\chi^k)\,,\\
R(M)_{\mu\nu}{}^{ab} &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 4f_{[\mu}{}^{[a}e_{\nu]}{}^{b]} + \frac{1}{2}(\bar\psi_{[\mu}{}^i\,\gamma^{ab}\,\phi_{\nu]i} + \text{h.c.})\\
&\quad + \left(\frac{1}{4}\bar\psi_\mu{}^i\,\psi_\nu{}^j\,T^{ab}{}_{ij} - \frac{3}{4}\bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,\gamma^{ab}\chi_i - \bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,R(Q)^{ab}{}_i + \text{h.c.}\right),\\
R(D)_{\mu\nu} &= 2\,\partial_{[\mu}b_{\nu]} - 2f_{[\mu}{}^ae_{\nu]a} - \frac{1}{2}\bar\psi_{[\mu}{}^i\phi_{\nu]i} + \frac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \frac{1}{2}\bar\psi_{[\mu i}\phi_{\nu]}{}^i + \frac{3}{4}\bar\psi_{[\mu i}\gamma_{\nu]}\chi^i\,,\\
R(S)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\phi_{\nu]}{}^i - 2f_{[\mu}{}^a\gamma_a\psi_{\nu]}{}^i - \frac{1}{8}\not{D}T_{ab}{}^{ij}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]\,j} - \frac{3}{2}\gamma_a\psi_{[\mu}{}^i\,\bar\psi_{\nu]}{}^j\gamma^a\chi_j\\
&\quad + \frac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^j + \frac{1}{2}\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^i\,,\\
R(K)_{\mu\nu}{}^a &= 2\,\mathcal{D}_{[\mu}f_{\nu]}{}^a - \frac{1}{4}(\bar\phi_{[\mu}{}^i\gamma^a\phi_{\nu]i} + \bar\phi_{[\mu i}\gamma^a\phi_{\nu]}{}^i)\\
&\quad + \frac{1}{4}\Big(\bar\psi_\mu{}^iD_bT^{ba}{}_{ij}\psi_\nu{}^j - 3\,e_{[\mu}{}^a\psi_{\nu]}{}^i\not{D}\chi_i\\
&\qquad + \frac{3}{2}D\,\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]j} - 4\,\bar\psi_{[\mu}{}^i\gamma_{\nu]}D_bR(Q)^{ba}{}_i + \text{h.c.}\Big)\,.
\end{aligned}
\tag{A.3}
$$

There are three conventional constraints (which have already been incorporated in (A.3),

$$
\begin{aligned}
R(P)_{\mu\nu}{}^a &= 0\,,\\
\gamma^\mu R(Q)_{\mu\nu}{}^i + \frac{3}{2}\gamma_\nu\chi^i &= 0\,,\\
e^\nu{}_b\,R(M)_{\mu\nu a}{}^b - \mathrm{i}\tilde R(A)_{\mu a} + \frac{1}{8}T_{abij}T_\mu{}^{bij} - \frac{3}{2}D\,e_{\mu a} &= 0\,,
\end{aligned}
\tag{A.4}
$$

which are S-supersymmetry invariant. They determine the fields $\omega_\mu{}^{ab}$, $\phi_\mu{}^i$ and $f_\mu{}^a$ as follows,

$$
\begin{aligned}
\omega_\mu^{ab} =& -2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\sigma e_\nu{}^c - 2e_\mu{}^{[a}e^{b]\nu}b_\nu \\
& -\frac{1}{4}(2\bar{\psi}_\mu^i\gamma^{[a}\psi_i^{b]} + \bar{\psi}^{ai}\gamma_\mu\psi_i^b + \text{h.c.})\,, \\
\phi_\mu{}^i =& \frac{1}{2}\left(\gamma^{\rho\sigma}\gamma_\mu - \frac{1}{3}\gamma_\mu\gamma^{\rho\sigma}\right)\left(\mathcal{D}_\rho\psi_\sigma{}^i - \frac{1}{16}T^{abij}\gamma_{ab}\gamma_\rho\psi_{\sigma j} + \frac{1}{4}\gamma_{\rho\sigma}\chi^i\right)\,, \\
f_\mu{}^\mu =& \frac{1}{6}R(\omega,e) - D - \left(\frac{1}{12}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu{}^i\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i} - \frac{1}{12}\bar{\psi}_\mu{}^i\psi_\nu{}^jT^{\mu\nu}{}_{ij} - \frac{1}{4}\bar{\psi}_\mu{}^i\gamma^\mu\chi_i + \text{h.c.}\right)\,.
\end{aligned}
\tag{A.5}
$$

We will also need the bosonic part of the expression for the uncontracted connection $f_\mu{}^a$,

$$
f_\mu{}^a = \frac{1}{2}R(\omega,e)_\mu{}^a - \frac{1}{4}\left(D + \frac{1}{3}R(\omega,e)\right)e_\mu{}^a - \frac{1}{2}\mathrm{i}\tilde{R}(A)_\mu{}^a + \frac{1}{16}T_{\mu b}{}^{ij}T^{ab}{}_{ij}\,, \tag{A.6}
$$

where $R(\omega,e)_\mu{}^a = R(\omega)_{\mu\nu}{}^{ab}e_b{}^\nu$ is the non-symmetric Ricci tensor, and $R(\omega,e)$ the corresponding Ricci scalar. The curvature $R(\omega)_{\mu\nu}{}^{ab}$ is associated with the spin connection field $\omega_\mu{}^{ab}$, given in (A.5).

The transformations of $\omega_\mu{}^{ab}$, $\phi_\mu{}^i$ and $f_\mu{}^a$ are induced by the constraints (A.4). We present their Q- and S-supersymmetry variations, as well as the transformations under conformal boosts, below,

$$
\begin{aligned}
\delta\omega_\mu{}^{ab} =& -\frac{1}{2}\bar{\epsilon}^i\gamma^{ab}\phi_{\mu i} - \frac{1}{2}\bar{\epsilon}^i\psi_\mu{}^j\,T^{ab}{}_{ij} + \frac{3}{4}\bar{\epsilon}^i\gamma_\mu\gamma^{ab}\chi_i \\
& + \bar{\epsilon}^i\gamma_\mu\,R^{ab}{}_i(Q) - \frac{1}{2}\bar{\eta}^i\gamma^{ab}\psi_{\mu i} + \text{h.c.} + 2\,\Lambda_{\mathrm{K}}{}^{[a}e_\mu{}^{b]}\,, \\
\delta\phi_\mu{}^i =& -2\,f_\mu{}^a\gamma_a\epsilon^i + \frac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_\mu\epsilon^j + \frac{1}{2}\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_\mu\epsilon^i - \frac{1}{8}\not{D}T^{ab\,ij}\gamma_{ab}\gamma_\mu\epsilon_j \\
& + \frac{3}{2}[(\bar{\chi}_j\gamma^a\epsilon^j)\gamma_a\psi_\mu{}^i - (\bar{\chi}_j\gamma^a\psi_\mu{}^j)\gamma_a\epsilon^i] + 2\,\mathcal{D}_\mu\eta^i + \Lambda_{\mathrm{K}}{}^a\gamma_a\psi_\mu{}^i\,, \\
\delta f_\mu{}^a =& -\frac{1}{2}\bar{\epsilon}^i\psi_\mu{}^i\,D_bT^{ba}{}_{ij} - \frac{3}{4}e_\mu{}^a\bar{\epsilon}^i\not{D}\chi_i - \frac{3}{4}\bar{\epsilon}^i\gamma^a\psi_{\mu i}\,D \\
& + \bar{\epsilon}^i\gamma_\mu\,D_bR^{ba}{}_i(Q) + \frac{1}{2}\bar{\eta}^i\gamma^a\phi_{\mu i} + \text{h.c.} + \mathcal{D}_\mu\Lambda_{\mathrm{K}}{}^a\,.
\end{aligned}
\tag{A.7}
$$

The transformations under S-supersymmetry and conformal boosts reflect the structure of the underlying $\mathrm{SU}(2,2|2)$ gauge algebra. The presence of curvature constraints and of the non-gauge fields T_{abij}, χ^i and D induce deformations of the Q-supersymmetry algebra, as is manifest in the above results, in particular in (A.3) and (A.7).

Combining the conventional constraints (A.4) with the various Bianchi identities one derives that not all the curvatures are independent. For instance,

$$
\begin{aligned}
\varepsilon^{abcd}D_b\mathcal{R}(M)_{cd}{}^{ef} =& \,2\,\varepsilon^{abc[e}\,R(K)_{bc}{}^{f]} + \frac{9}{2}\,\eta^{a[e}\bar{\chi}^i\gamma^{f]}\chi_i \\
& + \frac{1}{2}\left[3\bar{\chi}^i\gamma^aR(Q)^{efi} + \frac{1}{8}D^b(T^{ab}{}_{ij}T^{efij}) - \text{h.c.}\right]\,.
\end{aligned}
\tag{A.8}
$$

Furthermore it is convenient to modify two of the curvatures by including suitable covariant terms,

$$\mathcal{R}(M)_{ab}{}^{cd} = R(M)_{ab}{}^{cd} + \frac{1}{16}\big(T_{abij}\,T^{cdij} + T_{ab}{}^{ij}\,T^{cd}{}_{ij}\big)\,,$$
$$\mathcal{R}(S)_{ab}{}^{i} = R(S)_{ab}{}^{i} + \frac{3}{4}T_{ab}{}^{ij}\chi_j\,. \tag{A.9}$$

where we observe that $\gamma^{ab}\big(\mathcal{R}(S)-R(S)\big)_{ab}{}^{i}=0$. The modified curvature $\mathcal{R}(M)_{ab}{}^{cd}$ satisfies the following relations,

$$\mathcal{R}(M)_{\mu\nu}{}^{ab}\,e^{\nu}{}_{b} = \mathrm{i}\tilde{R}(A)_{\mu\nu}e^{\nu a} + \frac{3}{2}D\,e_{\mu}{}^{a}\,,$$
$$\frac{1}{4}\varepsilon_{ab}{}^{ef}\,\varepsilon^{cd}{}_{gh}\,\mathcal{R}(M)_{ef}{}^{gh} = \mathcal{R}(M)_{ab}{}^{cd}\,,$$
$$\varepsilon_{cdea}\,\mathcal{R}(M)^{cd\,e}{}_{b} = \varepsilon_{becd}\,\mathcal{R}(M)_{a}{}^{e\,cd} = 2\tilde{R}(D)_{ab} = 2\mathrm{i}R(A)_{ab}\,. \tag{A.10}$$

The first of these relations corresponds to the third constraint given in (A.4), while the remaining equations follow from combining the curvature constraints with the Bianchi identities. Note that the modified curvature does not satisfy the pair exchange property; instead we have,

$$\mathcal{R}(M)_{ab}{}^{cd} = \mathcal{R}(M)^{cd}{}_{ab} + 4\,\mathrm{i}\,\delta^{[c}_{[a}\,\tilde{R}(A)_{b]}{}^{d]}\,. \tag{A.11}$$

We now turn to the fermionic constraint given in (A.4) and its consequences for the modified curvature defined in (A.9). First we note that the constraint on $R(Q)_{\mu\nu}{}^{i}$ implies that this curvature is anti-selfdual, as follows from contracting the constraint with $\gamma^{\nu}\,\gamma_{ab}$,

$$\tilde{R}(Q)_{\mu\nu}{}^{i} = -R(Q)_{\mu\nu}{}^{i}\,. \tag{A.12}$$

Furthermore, combination of the Bianchi identity and the constraint on $R(Q)_{\mu\nu i}$ yields the following condition on the modified curvature $\mathcal{R}(S)_{ab}{}^{i}$,

$$\gamma^{a}\tilde{\mathcal{R}}(S)_{ab}{}^{i} = 2\,D^{a}\tilde{R}(Q)_{ab}{}^{i} = -2\,D^{a}R(Q)_{ab}{}^{i}\,. \tag{A.13}$$

This identity (upon contraction with $\gamma^{b}\gamma_{cd}$) leads to the following identity on the anti-selfdual part of $\mathcal{R}(S)_{ab}{}^{i}$,

$$\mathcal{R}(S)_{ab}{}^{i} - \tilde{\mathcal{R}}(S)_{ab}{}^{i} = 2\,\not{D}\bigg(R(Q)_{ab}{}^{i} + \frac{3}{4}\gamma_{ab}\chi^{i}\bigg)\,. \tag{A.14}$$

Finally we note the following useful identities for products of (anti)selfdual tensors,

$$G^{\pm}_{[a[c}\,H^{\pm}_{d]b]} = \pm\,\frac{1}{8}G^{\pm}_{ef}\,H^{\pm ef}\,\varepsilon_{abcd} - \frac{1}{4}(G^{\pm}_{ab}\,H^{\pm}_{cd} + G^{\pm}_{cd}\,H^{\pm}_{ab})\,,$$
$$G^{\pm}_{ab}\,H^{\mp cd} + G^{\pm cd}\,H^{\mp}_{ab} = 4\delta^{[c}_{[a}G^{\pm}_{b]e}\,H^{\mp d]e}\,,$$
$$\frac{1}{2}\varepsilon^{abcd}\,G^{\pm e}_{[c}\,H^{\pm}_{d]e} = \pm\,G^{\pm[a}{}_{e}\,H^{\pm b]e}\,,$$
$$G^{\pm ac}\,H^{\pm b}_{c} + G^{\pm bc}\,H^{\pm a}_{c} = -\,\frac{1}{2}\eta^{ab}\,G^{\pm cd}\,H^{\pm}_{cd}\,,$$
$$G^{\pm ac}\,H^{\mp b}_{c} = G^{\pm bc}\,H^{\mp a}_{c}\,, \qquad G^{\pm ab}\,H^{\mp}_{ab} = 0\,. \tag{A.15}$$

B Covariantization under special superconformal boosts

In principle covariant (multiple) derivatives are defined by the standard procedure by adding gauge fields to absorb all symmetry variations proportional to derivatives of the transformation parameters. In this procedure the gauge field $f_\mu{}^a$ associated with the conformal boosts (parametrized by $\Lambda_{\rm K}{}^a$) appears somewhat indirectly, because the only other fields that transform under the conformal boosts are the gauge fields b_μ, $\omega_\mu{}^{ab}$ and $\phi_\mu{}^i$. Therefore supercovariant derivatives of fields that are themselves invariant, will transform under these K-transformations, and usually these variations take a relatively simple form. We give some examples for a scalar field ϕ, a spinor field ψ, and a tensor field t_{ab}, each of Weyl weight w,

$$\begin{aligned}
\delta_{\rm K} D_a\phi &= -\,w\,\Lambda_{{\rm K}a}\phi\,,\\
\delta_{\rm K} D_a t_{bc} &= -\,w\,\Lambda_{{\rm K}a} t_{bc} + 2\,t_{a[b}\Lambda_{{\rm K}c]} - 2\,\eta_{a[b}t_{c]d}\,\Lambda_{\rm K}{}^d\,,\\
\delta_{\rm K} D_a\psi &= \Big[-\,w\,\Lambda_{{\rm K}a} + \frac{1}{2}\Lambda_{\rm K}{}^b\gamma_{ab}\Big]\psi\,.
\end{aligned}\tag{B.1}$$

These transformation rules simplify for certain contractions, such as in $D^a t_{ab}$ or $\not{D}\psi$,

$$\begin{aligned}
\delta_{\rm K} D^a t_{ab} &= (2-w)\Lambda_{\rm K}{}^a t_{ab}\,,\\
\delta_{\rm K} D_{[a}t_{bc]} &= (2-w)\Lambda_{{\rm K}[a}t_{bc]}\,,\\
\delta_{\rm K}\not{D}\psi &= \Big(\frac{3}{2}-w\Big)\not{\Lambda}_{\rm K}\,\psi\,,
\end{aligned}\tag{B.2}$$

showing, for instance, that the Dirac operator on a spinor field of weight $w=\frac{3}{2}$ is invariant.

Applying an extra covariant derivative we explicitly indicate the presence of the K-connection field $f_\mu{}^a$,

$$\begin{aligned}
D_\mu D_a\phi &= \mathcal{D}_\mu D_a\phi + w f_{\mu a}\,\phi\,,\\
D_\mu D^a t_{ab} &= \mathcal{D}_\mu D^a t_{ab} + (w-2) f_\mu{}^a\, t_{ab}\,,\\
D_\mu \not{D}\psi &= \mathcal{D}_\mu\not{D}\psi + \Big(w-\frac{3}{2}\Big) f_\mu{}^a\gamma_a\psi\,,
\end{aligned}\tag{B.3}$$

where $\mathcal{D}_\mu$ denotes the covariant derivative without including the field $f_\mu{}^a$. Under K-transformations these multiple derivatives transform as,

$$\begin{aligned}
\delta_{\rm K} D_\mu D_a\phi &= -\,(w+1)\big[\Lambda_{{\rm K}\mu}D_a + \Lambda_{{\rm K}a}D_\mu\big]\phi + e_{\mu a}\Lambda_{\rm K}{}^b D_b\phi\,,\\
\delta_{\rm K} D_\mu D^a t_{ab} &= -\,(w+1)\Lambda_{{\rm K}\mu}D^a t_{ab} - \Lambda_{{\rm K}b}D^a t_{a\mu} + e_{\mu b}\Lambda_{\rm K}{}^c D^a t_{ac} + (2-w)\Lambda_{\rm K}{}^a D_\mu t_{ab}\,,\\
\delta_{\rm K} D_\mu\not{D}\psi &= \Big[-\,(w+1)\Lambda_{{\rm K}\mu} + \frac{1}{2}\Lambda_{\rm K}{}^a\gamma_{\mu a}\Big]\not{D}\psi + \Big(\frac{3}{2}-w\Big)\not{\Lambda}_{\rm K} D_\mu\psi\,.
\end{aligned}\tag{B.4}$$

Contracting the first equation with $e^{a\mu}$ shows that the conformal D'Alembertian transforms under K-transformations as $\delta_{\rm K}\Box_{\rm c}\phi = -2(w-1)\Lambda_{\rm K}{}^a D_a\phi$, which vanishes for $w=1$.

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This pattern repeats itself when considering even higher derivatives. We present the following results,

$$
\begin{aligned}
D_\mu \Box_{\rm c}\phi &= \mathcal{D}_\mu \Box_{\rm c}\phi + 2(w-1) f_\mu{}^a D_a\phi\,,\\
\Box_{\rm c}\Box_{\rm c}\phi &= \mathcal{D}_\mu D^\mu \Box_{\rm c}\phi + (w+2) f_\mu{}^\mu \Box_{\rm c}\phi + 2(w-1) f_{\mu a} D^\mu D^a\phi\,,\\
\Box_{\rm c} \not{D}\psi &= \mathcal{D}_\mu D^\mu \not{D}\psi + \Big[(w+1) f_\mu{}^\mu - \frac{1}{2} f_{\mu a}\gamma^{\mu a}\Big]\not{D}\psi + \Big(w-\frac{3}{2}\Big) f_{\mu a}\gamma^a D^\mu\psi\,,
\end{aligned}
\tag{B.5}
$$

and,

$$
\begin{aligned}
\delta_{\rm K}\Box_{\rm c}\Box_{\rm c}\phi =&\, -2(w-1)\Lambda_{\rm K}{}^a \Box_{\rm c} D_a\phi - 2(w+1)\Lambda_{\rm K}{}^a D_a \Box_{\rm c}\phi\\
=&\, -2w\Lambda_{\rm K}{}^a \big[\Box_{\rm c} D_a\phi + D_a\Box_{\rm c}\big]\phi + 2\Lambda_{\rm K}{}^a\big[\Box_{\rm c} D_a - D_a\Box_{\rm c}\big]\phi\ ,\\
\delta_{\rm K}\Box_{\rm c}\not{D}\psi =&\, -(2w-1)\Lambda_{\rm K}{}^a D_a\not{D}\psi - \frac{1}{2}\not{\Lambda}_{\rm K}\big[(2w-1)\Box_{\rm c} + [\not{D},\not{D}]\big]\psi\,.
\end{aligned}
\tag{B.6}
$$

For future use we have evaluated the previous two variations for the fields A and Ψ_i, which have weights $w = 0, \frac{1}{2}$, respectively. In this case all the terms cubic and quadratic in derivatives in (B.6) appear with a certain degree of anti-symmetry, such that they become proportional to curvatures. Upon substituting the results for the various curvatures, one obtains (3.4).

C Multiplication of chiral multiplets

In this appendix we summarize the product rules for two chiral supermultiplets and the Taylor expansion for functions of these multiplets. In the local supersymmetry setting, we will usually be dealing with homogeneous functions of chiral multiplets with equal Weyl weight so that a scaling weight under Weyl transformations can be assigned to the function.

The product of two chiral multiplets, specified by the component fields $\big(A,\Psi_i,B_{ij},F_{ab}^-,\Lambda_i,C\big)$ and $\big(a,\psi_i,b_{ij},f_{ab}^-,\lambda_i,c\big)$, respectively, leads to the following decomposition,

$$
\begin{aligned}
&\big(A,\Psi_i,B_{ij},F_{ab}^-,\Lambda_i,C\big)\otimes\big(a,\psi_i,b_{ij},f_{ab}^-,\lambda_i,c\big) =\\
&\qquad\Big(A\,a\,,\ A\,\psi_i + a\,\Psi_i, A\,b_{ij} + a\,B_{ij} - \bar\Psi_{(i}\psi_{j)}\,,\\
&\qquad\quad A\,f_{ab}^- + a\,F_{ab}^- - \frac{1}{4}\varepsilon^{ij}\bar\Psi_i\gamma_{ab}\psi_j\,,\\
&\qquad\quad A\,\lambda_i + a\,\Lambda_i - \frac{1}{2}\varepsilon^{kl}(B_{ik}\,\psi_l + b_{ik}\,\Psi_l) - \frac{1}{4}(F_{ab}^-\gamma^{ab}\psi_i + f_{ab}^-\gamma^{ab}\Psi_i)\,,\\
&\qquad\quad A\,c + a\,C - \frac{1}{2}\varepsilon^{ik}\varepsilon^{jl}B_{ij}\,b_{kl} + F_{ab}^-\,f^{-ab} + \varepsilon^{ij}(\bar\Psi_i\lambda_j + \bar\psi_i\Lambda_j)\Big)\,.
\end{aligned}
\tag{C.1}
$$

A function $\mathcal{G}(\Phi)$ of chiral superfields Φ^I defines a chiral superfield, whose component fields take the following form,

$$
\begin{aligned}
A|_{\mathcal{G}} &= \mathcal{G}(A)\,, \\
\Psi_i|_{\mathcal{G}} &= \mathcal{G}(A)_I\, \Psi_i{}^I\,, \\
B_{ij}|_{\mathcal{G}} &= \mathcal{G}(A)_I\, B_{ij}{}^I - \frac{1}{2}\mathcal{G}(A)_{IJ}\, \bar\Psi_{(i}{}^I \Psi_{j)}{}^J\,, \\
F^-_{ab}|_{\mathcal{G}} &= \mathcal{G}(A)_I\, F^{-I}_{ab} - \frac{1}{8}\mathcal{G}(A)_{IJ}\, \varepsilon^{ij} \bar\Psi_i{}^I \gamma_{ab} \Psi_j{}^J\,, \\
\Lambda_i|_{\mathcal{G}} &= \mathcal{G}(A)_I\, \Lambda_i{}^I - \frac{1}{2}\mathcal{G}(A)_{IJ} \Big[B_{ij}{}^I \varepsilon^{jk} \Psi_k{}^J + \frac{1}{2} F^{-I}_{ab} \gamma^{ab} \Psi_k{}^J \Big] \\
&\quad + \frac{1}{48}\mathcal{G}(A)_{IJK}\, \gamma^{ab} \Psi_i{}^I\, \varepsilon^{jk} \bar\Psi_j{}^J \gamma_{ab} \Psi_k{}^K\,, \\
C|_{\mathcal{G}} &= \mathcal{G}(A)_I\, C^I - \frac{1}{4}\mathcal{G}(A)_{IJ} \big[B_{ij}{}^I B_{kl}{}^J\, \varepsilon^{ik} \varepsilon^{jl} - 2\, F^{-I}_{ab} F^{-abJ} + 4\, \varepsilon^{ik} \bar\Lambda_i{}^I \Psi_j{}^J \big]\,, \\
&\quad + \frac{1}{4}\mathcal{G}(A)_{IJK} \Big[\varepsilon^{ik} \varepsilon^{jl} B_{ij}{}^I \Psi_k{}^J \Psi_l{}^K - \frac{1}{2} \varepsilon^{kl} \bar\Psi_k{}^I F^{-J}_{ab} \gamma^{ab} \Psi_l{}^K \Big] \\
&\quad + \frac{1}{192}\mathcal{G}(A)_{IJKL}\, \varepsilon^{ij} \bar\Psi_i{}^I \gamma_{ab} \Psi_j{}^J\, \varepsilon^{kl} \bar\Psi_k{}^K \gamma_{ab} \Psi_l{}^L\,.
\end{aligned}
\tag{C.2}
$$

This result follows straightforwardly from expanding the superfield expression in powers of the fermionic coordinates.

D Reduced chiral multiplets

Chiral multiplets can be consistently reduced by imposing a reality constraint. This usually requires specific values for the Weyl and chiral weights. The two cases that are relevant are the vector multiplet, which arises upon reduction from a scalar chiral multiplet, and the Weyl multiplet, which is a reduced anti-selfdual chiral tensor multiplet. Both reduced multiplets require weight $w = 1$.

We will denote the components of the $w = 1$ multiplet that describes the vector multiplet by $(A, \Psi, B, F^-, \Lambda, C)|_{\text{vector}}$. The constraint for a scalar chiral supermultiplet reads, $\varepsilon^{ij}\, \bar D_i \gamma_{ab} D_j \Phi = [\varepsilon^{ij}\, \bar D_i \gamma_{ab} D_j \Phi]^*$, which implies that $C|_{\text{vector}}$ and $\Lambda_i|_{\text{vector}}$ are expressed in terms of the lower components of the multiplet, and imposes a reality constraint on $B|_{\text{vector}}$ and a Bianchi identity on $F^-|_{\text{vector}}$ [12–14]. The latter implies that $F^-|_{\text{vector}}$ can be expressed in terms of a gauge field W_μ. This feature is not affected by the presence of the superconformal background field.

Denoting the independent components of the vector multiplet by (X, Ω, Y, F^-), the identification with the chiral multiplet components is as follows,

$$
\begin{aligned}
A|_{\text{vector}} &= X\,, \\
\Psi_i|_{\text{vector}} &= \Omega_i\,, \\
B_{ij}|_{\text{vector}} &= Y_{ij} = \varepsilon_{ik}\varepsilon_{jl} Y^{kl}\,,
\end{aligned}
$$

$$
\begin{aligned}
F^-_{ab}|_{\text{vector}} &= \left(\delta_{ab}{}^{cd} - \frac{1}{2}\varepsilon_{ab}{}^{cd}\right) e_c{}^\mu e_d{}^\nu\, \partial_{[\mu} W_{\nu]} \\
&\quad + \frac{1}{4}\big[\bar\psi_\rho{}^i \gamma_{ab}\gamma^\rho \Omega^j + \bar X\,\bar\psi_\rho{}^i \gamma^{\rho\sigma}\gamma_{ab}\psi_\sigma{}^j - \bar X\, T_{ab}{}^{ij}\big]\varepsilon_{ij}\,, \\
\Lambda_i|_{\text{vector}} &= -\,\varepsilon_{ij} \not{D}\Omega^j \\
C|_{\text{vector}} &= -\,2\,\Box_{\rm c}\bar X - \frac{1}{4} F^+_{ab}\, T^{ab}{}_{ij}\varepsilon^{ij} - 3\,\bar\chi_i \Omega^i\,.
\end{aligned}
\tag{D.1}
$$

The Bianchi identity on F_{ab} can be written as,

$$
D^b\left(F^+_{ab} - F^-_{ab} + \frac{1}{4} X T_{abij}\varepsilon^{ij} - \frac{1}{4}\bar X T_{ab}{}^{ij}\varepsilon_{ij}\right) + \frac{3}{4}\left(\bar\chi_i\gamma_a\Omega_j\varepsilon^{ij} - \bar\chi^i\gamma_a\Omega^j\varepsilon_{ij}\right) = 0\,, \tag{D.2}
$$

and the reality constraint on Y_{ij} is included in (D.1).

The Q- and S-supersymmetry transformations for the vector multiplet take the form,

$$
\begin{aligned}
\delta X &= \bar\epsilon^i \Omega_i\,, \\
\delta\Omega_i &= 2\not{D} X \epsilon_i + \frac{1}{2}\varepsilon_{ij} F_{\mu\nu}\gamma^{\mu\nu}\epsilon^j + Y_{ij}\epsilon^j + 2X\eta_i\,, \\
\delta W_\mu &= \varepsilon^{ij}\bar\epsilon_i(\gamma_\mu\Omega_j + 2\,\psi_{\mu j} X) + \varepsilon_{ij}\bar\epsilon^i(\gamma_\mu\Omega^j + 2\,\psi_\mu{}^j \bar X)\,, \\
\delta Y_{ij} &= 2\,\bar\epsilon_{(i}\not{D}\Omega_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar\epsilon^{(k}\not{D}\Omega^{l)}\,,
\end{aligned}
\tag{D.3}
$$

and, for $w = 1$, are in clear correspondence with the supersymmetry transformations of generic scalar chiral multiplets given in (2.1).

Subsequently we turn to the Weyl multiplet, which is a chiral anti-selfdual tensor multiplet subject to $\bar D_i \gamma^{ab} D_j\, \Phi_{ab}{}^{ij} = [\bar D_i \gamma^{ab} D_j\, \Phi_{ab}{}^{ij}]^*$. Its chiral superfield components take the following form,

$$
\begin{aligned}
A_{ab}|_W &= T_{ab}{}^{ij}\varepsilon_{ij}\,, \\
\Psi_{abi}|_W &= 8\,\varepsilon_{ij} R(Q)^j_{ab}\,, \\
B_{abij}|_W &= -\,8\,\varepsilon_{k(i} R(\mathcal{V})^-_{ab}{}^k{}_{j)}\,, \\
\left(F^-_{ab}\right)^{cd}|_W &= -\,8\,\mathcal{R}(M)^-_{ab}{}^{cd}\,, \\
\Lambda_{abi}|_W &= 8\left(\mathcal{R}(S)^-_{abi} + \frac{3}{4}\gamma_{ab}\not{D}\chi_i\right)\,, \\
C_{ab}|_W &= 4 D_{[a}\, D^c T_{b]c\,ij}\varepsilon^{ij} - \text{dual}\,.
\end{aligned}
\tag{D.4}
$$

We give the Q- and S-supersymmetry variations for the first few components,

$$
\begin{aligned}
\delta T_{ab}{}^{ij} &= 8\,\bar\epsilon^{[i} R(Q)_{ab}{}^{j]}\,, \\
\delta R(Q)_{ab}{}^i &= -\,\frac{1}{2}\not{D} T_{ab}{}^{ij}\,\epsilon_j + R(\mathcal{V})^-{}_{ab}{}^i{}_j\,\epsilon^j - \frac{1}{2}\mathcal{R}(M)_{ab}{}^{cd}\,\gamma_{cd}\epsilon^i + \frac{1}{8} T_{cd}{}^{ij}\,\gamma^{cd}\gamma_{ab}\,\eta_j\,, \\
\delta R(\mathcal{V})^-{}_{ab}{}^i{}_j &= 2\bar\epsilon_j \not{D} R(Q)_{ab}{}^i - 2\bar\epsilon^i\left(\mathcal{R}(S)^-_{abj} + \frac{3}{4}\gamma_{ab}\not{D}\chi_j\right) \\
&\quad + \bar\eta_j(2R(Q)_{ab}{}^i + 3\gamma_{ab}\chi^i) - (\text{traceless})\,,
\end{aligned}
$$

$$\delta\mathcal{R}(M)_{ab}^{-cd} = \frac{1}{2}\bar{\epsilon}_i \not{D}\gamma^{cd} R(Q)_{ab}{}^i - \frac{1}{2}\bar{\epsilon}^i\gamma^{cd}\Big(\mathcal{R}(S)^-_{abi} + \frac{3}{4}\gamma_{ab}\not{D}\chi_i\Big)$$
$$- \bar{\eta}_i\gamma_{ab}R(Q)^{cdi} - \frac{1}{2}\bar{\eta}_i\gamma^{cd}R(Q)_{ab}{}^i - \frac{3}{4}\bar{\eta}_i\gamma_{ab}\gamma^{cd}\chi^i\,. \tag{D.5}$$

A scalar chiral multiplet with $w = 2$ is obtained by squaring the Weyl multiplet. The various scalar chiral multiplet components are given by,

$$\begin{aligned}
A|_{W^2} &= (T_{ab}{}^{ij}\varepsilon_{ij})^2\,,\\
\Psi_i|_{W^2} &= 16\,\varepsilon_{ij}R(Q)^j_{ab}\,T^{klab}\,\varepsilon_{kl}\,,\\
B_{ij}|_{W^2} &= -16\,\varepsilon_{k(i}R(\mathcal{V})^k{}_{j)ab}\,T^{lmab}\,\varepsilon_{lm} - 64\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{R}(Q)_{ab}{}^k\,R(Q)^{l\,ab}\,,\\
F^{-ab}|_{W^2} &= -16\,\mathcal{R}(M)_{cd}{}^{ab}\,T^{klcd}\,\varepsilon_{kl} - 16\,\varepsilon_{ij}\,\bar{R}(Q)^i_{cd}\gamma^{ab}R(Q)^{cd\,j}\,,\\
\Lambda_i|_{W^2} &= 32\,\varepsilon_{ij}\,\gamma^{ab}R(Q)^j_{cd}\,\mathcal{R}(M)^{cd}{}_{ab} + 16\,(\mathcal{R}(S)_{ab\,i} + 3\gamma_{[a}D_{b]}\chi_i)\,T^{klab}\,\varepsilon_{kl}\\
&\quad - 64\,R(\mathcal{V})_{ab}{}^k{}_i\,\varepsilon_{kl}\,R(Q)^{ab\,l}\,,\\
C|_{W^2} &= 64\,\mathcal{R}(M)^{-cd}{}_{ab}\,\mathcal{R}(M)^{-ab}_{cd} + 32\,R(\mathcal{V})^{-ab\,k}{}_l\,R(\mathcal{V})^{-l}_{ab\,k}\\
&\quad - 32\,T^{ab\,ij}\,D_a\,D^cT_{cb\,ij} + 128\,\bar{\mathcal{R}}(S)^{ab}{}_i\,R(Q)_{ab}{}^i + 384\,\bar{R}(Q)^{ab\,i}\gamma_a D_b\chi_i\,.
\end{aligned} \tag{D.6}$$

These components can straightforwardly be substituted in the expression for the higher-derivative couplings.

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