

# Modular anomaly equations in $\mathcal{N} = 2^*$ theories and their large- $N$ limit

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**ABSTRACT:** We propose a modular anomaly equation for the prepotential of the  $\mathcal{N} = 2^*$  super Yang-Mills theory on  $\mathbb{R}^4$  with gauge group  $U(N)$  in the presence of an  $\Omega$ -background. We then study the behavior of the prepotential in a large- $N$  limit, in which  $N$  goes to infinity with the gauge coupling constant kept fixed. In this regime instantons are not suppressed. We focus on two representative choices of gauge theory vacua, where the vacuum expectation values of the scalar fields are distributed either homogeneously or according to the Wigner semi-circle law. In both cases we derive an all-instanton exact formula for the prepotential. As an application, we show that the gauge theory partition function on  $S^4$  at large  $N$  localizes around a Wigner distribution for the vacuum expectation values leading to a very simple expression in which the instanton contribution becomes independent of the coupling constant.

**KEYWORDS:** Supersymmetric gauge theory, Extended Supersymmetry,  $1/N$  Expansion, D-branes

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**1 Introduction**

Holomorphic anomalies have made their first appearance in topological string theory in [1–3] and since then they have always received a lot of attention. In the context of field theories they were first studied for the  $\mathcal{N} = 2^*$  super Yang-Mills theory [4], that is a deformation of the maximally supersymmetric  $\mathcal{N} = 4$  gauge theory in which the adjoint hypermultiplet becomes massive. This theory can be regarded as an interpolation between the  $\mathcal{N} = 4$  model and the pure  $\mathcal{N} = 2$  super Yang-Mills theory to which it reduces by taking, respectively, the limit in which the hypermultiplet mass is sent to zero or decoupled. By giving a mass to the hypermultiplet, one breaks the  $SL(2, \mathbb{Z})$  duality invariance of the

original  $\mathcal{N} = 4$  theory, but a remnant of this symmetry manifests in the fact that the expansion coefficients of the  $\mathcal{N} = 2^*$  prepotential in the limit of small mass are almost modular forms of the bare gauge coupling. Quite remarkably, these coefficients satisfy a recursion relation [4] which translates into a partial differential equation for the prepotential itself. Such an equation, sometimes called modular anomaly equation, can be regarded as the gauge theory counterpart of the holomorphic anomaly equation satisfied by the topological string amplitudes, and can be used as a very efficient tool for a fast computation of the full prepotential order by order in the mass.

With the introduction of the so-called  $\Omega$ -background [5–7] and its interpretation in terms of Neveu-Schwarz/Neveu-Schwarz or Ramond/Ramond field strengths of closed string theory [8–11], the connection between the gauge theory prepotential and the topological string amplitudes has become more clear, and recently there has been a renewed interest both in computing such amplitudes [12–16] and in adapting the previous works to this new scenario [17–21]. In particular, in [19, 21] the  $\mathcal{N} = 2^*$  theory with gauge group  $SU(2)$  has been studied in a generic  $\Omega$ -background characterized by two independent parameters,  $\epsilon_1$  and  $\epsilon_2$ , and its prepotential, including all its non-perturbative instanton corrections, has been computed in a small mass expansion. Moreover, it has been shown that the resulting quantum prepotential obeys a modular anomaly equation with a term proportional to  $\epsilon_1\epsilon_2$ , which generalizes the previously known equation of the underformed theory [4].

The purpose of this paper is to extend these ideas to gauge theories with an arbitrary number  $N$  of colors, and later study the large- $N$  limit. The large- $N$  limit of the  $\mathcal{N} = 2^*$  theories with gauge group  $U(N)$  has been recently analyzed in [22–24] using localization methods [25], and in [26–28] using holography. In all these papers the  $\mathcal{N} = 2^*$  theory is defined on the sphere  $S^4$ , and the large- $N$  limit is taken by keeping fixed the 't Hooft coupling  $\lambda = Ng^2$ . In this limit, instanton configurations, which are weighted by the usual factor  $q = \exp(-8\pi^2/g^2) = \exp(-8\pi^2 N/\lambda)$ , are exponentially suppressed and play no role.<sup>1</sup>

Here, instead we consider the  $\mathcal{N} = 2^*$   $U(N)$  theory on  $\mathbb{R}^4$  and take the large- $N$  limit by keeping fixed the Yang-Mills coupling  $g^2$ . In this regime, sometimes called the very strongly coupled large- $N$  limit [31, 32], the weight  $q$  remains *finite* so that instantons cannot be discarded a priori and their effects have to be consistently included in the picture. Moreover, being on  $\mathbb{R}^4$ , the gauge theory should be supplemented by boundary conditions specifying how the fields behave at infinity. Focusing on the Coulomb branch characterized by the  $N$  vacuum expectation values  $a_i$ 's for the scalar fields in the vector multiplet, we study how the prepotential behaves in the large- $N$  limit for different choices of vacuum expectation values distributions. In particular we will consider a uniform distribution and the Wigner semi-circle distribution, which both allow to perform explicit calculations and checks. In both cases we find that all instanton sectors contribute to the prepotential in

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<sup>1</sup>Actually, before reaching this conclusion, one should also make sure that the integration over the instanton moduli space does not overcome the exponential suppression of the instanton weight, thus leading to an instanton-induced large- $N$  phase transition [29]. In [30] it has been checked that this phenomenon does not occur in the  $\mathcal{N} = 2^*$  theory under consideration and that instantons remain exponentially suppressed in the 't Hooft large- $N$  limit.

this large- $N$  limit and that the sum of all such contributions can be explicitly performed yielding a simple expression for the non-perturbative contribution to the prepotential at large  $N$ . It turns out that this contribution is subleading in  $N$  with respect to the classical and perturbative terms, and has some interesting properties.

Finally, we use these results to study the gauge theory prepotential on  $\mathbb{S}^4$ . In this case, the partition function can be written as  $Z_{\mathbb{S}^4} = \int da |Z_{\mathbb{R}^4}|^2$  where the integral is over the vacuum expectation values  $a_i$  of the scalar field in the vector multiplet and  $Z_{\mathbb{R}^4}$  is related to the gauge theory prepotential in the  $\Omega$  background on  $\mathbb{R}^4$  via  $\mathcal{F} = -\epsilon_1 \epsilon_2 \log Z_{\mathbb{R}^4}$ . The large- $N$  limit of  $Z_{\mathbb{S}^4}$  is evaluated using saddle-point techniques and the exact results obtained for the prepotential on  $\mathbb{R}^4$ . In particular we show that the integral over  $a$  localizes around a vacuum characterized by a Wigner distribution of the expectation values with a critical length leading to a remarkably simple expression in which the instanton contribution to the prepotential becomes independent of the coupling constant and hence negligible.

The plan of the paper is as follows. In section 2 we propose a holomorphic anomaly equation for the prepotential of the  $\mathcal{N} = 2^*$  theory with gauge group  $U(N)$  in a generic  $\Omega$ -background. We display exact formulae for the first few terms in the small mass expansion of the prepotential and check our results against those based on localization. In sections 3 and 4, we analyze the large- $N$  limit for the two representative choices of gauge theory vacua where the scalar eigenvalues are distributed either homogeneously or following the Wigner semi-circle law, and after resumming over all instanton numbers we obtain the non-perturbative prepotential at large  $N$ . In section 5 we present our conclusions and the results for the gauge theory partition function on  $\mathbb{S}^4$ . Finally, in the appendices we collect several technical details which are useful to reproduce the calculations presented in the main text.

## 2 The prepotential of the $\mathcal{N} = 2^*$ theory

The  $\mathcal{N} = 2^*$  theory is a massive deformation of the  $\mathcal{N} = 4$  super Yang-Mills theory describing the interactions of a  $\mathcal{N} = 2$  gauge vector multiplet with a massive  $\mathcal{N} = 2$  hypermultiplet in the adjoint representation. In the following we consider the  $\mathcal{N} = 2^*$  theory with gauge group  $U(N)$  and mass  $m$  in an  $\Omega$ -background [6, 7] parameterized by  $\epsilon_1$  and  $\epsilon_2$  which, for later convenience, we combine in

$$\epsilon \equiv \epsilon_1 + \epsilon_2 \quad \text{and} \quad h \equiv \sqrt{\epsilon_1 \epsilon_2} . \tag{2.1}$$

In the vacuum where the scalar field  $\Phi$  of the vector multiplet has expectation value

$$\langle \Phi \rangle = \text{diag} (a_1, \dots, a_N) , \tag{2.2}$$

the prepotential  $\mathcal{F}$  can be written as

$$\mathcal{F} = \mathcal{F}^{\text{class}} + \mathcal{F}^{\text{quant}} \tag{2.3}$$

where the classical term is simply

$$\mathcal{F}^{\text{class}} = \pi i \tau \sum_{i=1}^N a_i^2 , \tag{2.4}$$

with

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2} \tag{2.5}$$

being the (complexified) gauge coupling constant.

The quantum prepotential is the sum of a perturbative 1-loop term and a non-perturbative part due to instantons:

$$\mathcal{F}^{\text{quant}} = \mathcal{F}^{1\text{-loop}} + \mathcal{F}^{\text{inst}} . \tag{2.6}$$

### 2.1 The one-loop prepotential

The 1-loop piece for the  $\mathcal{N} = 2^*$  theory is given by [7, 17, 19]

$$\mathcal{F}^{1\text{-loop}} = h^2 \sum_{i \neq j} \left[ \log \Gamma_2(a_{ij}) - \log \Gamma_2(a_{ij} + m + \epsilon) \right] \tag{2.7}$$

where  $a_{ij} = a_i - a_j$  and  $\Gamma_2$  is the Barnes double  $\Gamma$ -function (see appendix A for our notations and conventions). Expanding for small values of  $m$ ,  $h$  and  $\epsilon$ , one gets

$$\mathcal{F}^{1\text{-loop}} = \sum_{n=1}^{\infty} f_n^{1\text{-loop}} \tag{2.8}$$

where the  $f_n^{1\text{-loop}}$ 's are homogeneous polynomials of degree  $2n$  in  $(m, h, \epsilon)$ . The first few of them are

$$f_1^{1\text{-loop}} = \frac{M^2}{4} \sum_{i \neq j} \log \left( \frac{a_{ij}}{\Lambda} \right)^2 , \tag{2.9a}$$

$$f_2^{1\text{-loop}} = -\frac{M^2(M^2 + h^2)}{24} C_2 , \tag{2.9b}$$

$$f_3^{1\text{-loop}} = -\frac{M^2(M^2 + h^2)(2M^2 - \epsilon^2 + 3h^2)}{240} C_4 . \tag{2.9c}$$

Here we have defined<sup>2</sup>

$$M^2 = m^2 - \frac{\epsilon^2}{4} , \tag{2.10}$$

and introduced the sums

$$C_n = \sum_{i \neq j} \frac{1}{(a_{ij})^n} , \tag{2.11}$$

and an arbitrary scale  $\Lambda$  in the logarithmic term. The coefficients  $f_n^{1\text{-loop}}$ 's for higher  $n$  have more complicated expressions but their dependence on the vacuum expectation values  $a_i$ 's is entirely through the sums  $C_n$ . For a vanishing  $\epsilon$ -background, (2.7) reduces to<sup>3</sup>

$$\begin{aligned} \mathcal{F}^{1\text{-loop}} &= \frac{1}{4} \sum_{i \neq j} \left[ -a_{ij}^2 \log \left( \frac{a_{ij}}{\Lambda} \right)^2 + (a_{ij} + m)^2 \log \left( \frac{a_{ij} + m}{\Lambda} \right)^2 + 3a_{ij}^2 - 3(a_{ij} + m)^2 \right] \\ &= \frac{m^2}{4} \sum_{i \neq j} \log \left( \frac{a_{ij}}{\Lambda} \right)^2 - \sum_{n=2}^{\infty} \frac{m^{2n}}{4n(n-1)(2n-1)} C_{2n-2} , \end{aligned} \tag{2.12}$$

<sup>2</sup>Note that the  $\mathcal{N} = 4$  theory is obtained by setting  $M$ , and not  $m$ , to zero, see also [25, 33].

<sup>3</sup>Notice that all terms with odd powers of  $m$  and hence of  $a_i$  vanish when summed over  $i \neq j$ .

so that, in this case, we simply have

$$f_n^{1\text{-loop}} = -\frac{m^{2n}}{4n(n-1)(2n-1)} C_{2n-2} \tag{2.13}$$

for any  $n > 1$ , while  $f_1^{1\text{-loop}}$  is given by (2.9a) with  $M^2$  replaced by  $m^2$ .

## 2.2 The instanton prepotential

The instanton part  $\mathcal{F}^{\text{inst}}$  can be computed order by order in the instanton counting parameter

$$q = e^{2\pi i r} \tag{2.14}$$

using localization methods [5–7]. In this framework, the instanton prepotential is viewed as the free energy

$$\mathcal{F}^{\text{inst}} = -\epsilon_1 \epsilon_2 \log Z^{\text{inst}} = \sum_{k=1}^{\infty} q^k \mathcal{F}^{(k)} \tag{2.15}$$

of a statistical system with a (gran canonical) partition function  $Z^{\text{inst}} = 1 + \sum_{k=1}^{\infty} q^k Z_k$ , counting the number of excitations in the instanton moduli space.

Referring for example to [34–36] for details, the  $k$ -instanton partition function of the  $\mathcal{N} = 2^*$  theory is given by

$$\begin{aligned} Z_k = & (-1)^k \oint \prod_I^k \frac{d\chi_I}{2\pi i} \prod_{I,J}^k \left[ \frac{\chi'_{IJ}(\chi_{IJ} + \epsilon)}{(\chi_{IJ} + \epsilon_1)(\chi_{IJ} + \epsilon_2)} \frac{(\chi_{IJ} + m + \frac{\epsilon_1 - \epsilon_2}{2})(\chi_{IJ} + m - \frac{\epsilon_1 - \epsilon_2}{2})}{(\chi_{IJ} + m + \frac{\epsilon}{2})(\chi_{IJ} + m - \frac{\epsilon}{2})} \right] \\ & \times \prod_{I=1}^k \prod_{i=1}^N \left[ \frac{-(\chi_I - a_i)^2 + m^2}{-(\chi_I - a_i)^2 + \frac{\epsilon^2}{4}} \right] \end{aligned} \tag{2.16}$$

where  $\chi_{IJ} = \chi_I - \chi_J$ , with  $\chi_I$  being the instanton moduli that, in the  $U(k)$  theory, are the analogue of the scalar vacuum expectation values  $a_i$ ,  $\chi'_{IJ} = \chi_{IJ}$  when  $I \neq J$  and  $\chi'_{IJ} = 1$  when  $I = J$ . Finally, the  $\chi$ -integrals are computed on contours around the poles in the upper half-plane after taking  $\text{Im } \epsilon_1 \gg \text{Im } \epsilon_2 > 0$  [5]. In particular, at  $k = 1$  the instanton prepotential is

$$\begin{aligned} \mathcal{F}^{(1)} &= -\epsilon_1 \epsilon_2 Z_1 \\ &= -\left(m + \frac{\epsilon}{2} - \epsilon_1\right) \left(m + \frac{\epsilon}{2} - \epsilon_2\right) \sum_{i=1}^N \prod_{j \neq i} \frac{(a_{ij} + m + \frac{\epsilon}{2})(a_{ji} + m - \frac{\epsilon}{2})}{a_{ij}(a_{ji} - \epsilon)}. \end{aligned} \tag{2.17}$$

The localization techniques allow to push the calculation to higher instanton numbers without major problems, but the resulting explicit expressions for the  $\mathcal{F}^{(k)}$ 's quickly become rather cumbersome. However, if we expand the instanton prepotential for small values of  $m$ ,  $h$  and  $\epsilon$  as in (2.8), *i.e.*

$$\mathcal{F}^{\text{inst}} = \sum_{n=1}^{\infty} f_n^{\text{inst}}, \tag{2.18}$$

then it is possible to write the homogeneous polynomials  $f_n^{\text{inst}}$  in a reasonably compact way. For example, up to three instantons we find

$$f_1^{\text{inst}} = -N(M^2 + h^2) \left( q + \frac{3}{2} q^2 + \frac{4}{3} q^3 + \dots \right), \tag{2.19a}$$

$$f_2^{\text{inst}} = M^2(M^2 + h^2)(q + 3q^2 + 4q^3 + \dots)C_2, \tag{2.19b}$$

$$f_3^{\text{inst}} = M^2(M^2 + h^2) \left[ \epsilon^2 q - 3(2M^2 - 3\epsilon^2 + 3h^2)q^2 - 4(8M^2 - 7\epsilon^2 + 12h^2)q^3 + \dots \right] C_4 - \frac{1}{2}M^4(M^2 + h^2)(q + 6q^2 + 12q^3 + \dots)C_{22}. \tag{2.19c}$$

Of course there are no obstructions in obtaining the instanton expansions of the higher  $f_n^{\text{inst}}$ 's, but we do not write them here since their explicit expressions will not be needed in the following. A few comments are in order. First, we notice that there are no non-perturbative corrections to the logarithmic term (2.9a), as expected, and that  $f_1^{\text{inst}}$  is independent of the vacuum expectation values  $a_i$ 's and that its  $k$ -th instanton coefficient is  $\sigma_1(k)/k$  where  $\sigma_1(k)$  is the sum of the divisors of  $k$ . Second, we see that new structures, which were not present in the 1-loop results (2.9), start appearing in the non-perturbative sector; for example  $f_3^{\text{inst}}$  contains the triple sum

$$C_{nm} = \sum_{i \neq j \neq k \neq i} \frac{1}{(a_{ij})^n (a_{ik})^m} \tag{2.20}$$

with  $n = m = 2$ . More and more structures appear in the higher  $f_n^{\text{inst}}$ 's. For instance, the exact 1-instanton contribution (2.17) in the undeformed case ( $\epsilon_1, \epsilon_2 \rightarrow 0$ ) can be written as<sup>4</sup>

$$\mathcal{F}^{(1)} = -m^2 \sum_{i=1}^N \prod_{j \neq i} \left( 1 - \frac{m^2}{a_{ij}^2} \right) = \sum_{n=1}^N \frac{(-1)^n m^{2n}}{(n-1)!} C_{\underbrace{22 \dots 2}_{n-1}}, \tag{2.21}$$

where

$$C_{\underbrace{22 \dots 2}_{n-1}} = \sum_{[i_1, i_2, \dots, i_n]} \frac{1}{(a_{i_1 i_2})^2 (a_{i_1 i_3})^2 \dots (a_{i_1 i_n})^2}, \tag{2.22}$$

and the symbol  $[i_1, i_2, \dots, i_n]$  denotes a sum over  $n$  non-coinciding positive integers (see appendix B for details).

### 2.3 The exact prepotential

The sum of the perturbative and non-perturbative contributions (2.9) and (2.19) can be recast in a more compact and suggestive way. In fact, setting

$$f_n = f_n^{1\text{-loop}} + f_n^{\text{inst}}, \tag{2.23}$$

and building on previous results obtained both in the  $U(N)$  undeformed  $\mathcal{N} = 2^*$  theory [4] and in the  $SU(2)$  deformed one [17–21], we expect that the  $f_n$ 's (with  $n > 1$ ) are almost

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<sup>4</sup>Here we use the shorthand  $0! = 1$ .

modular forms of weight  $(2n - 2)$  that can be expressed as homogeneous polynomials in the Eisenstein series  $E_{2n}$  (see appendix C for our conventions). Indeed we have

$$f_1 = \frac{M^2}{4} \sum_{i \neq j} \log \left( \frac{a_{ij}}{\Lambda} \right)^2 + N(M^2 + h^2) \log \widehat{\eta}(q), \tag{2.24a}$$

$$f_2 = -\frac{M^2(M^2 + h^2)}{24} E_2 C_2, \tag{2.24b}$$

$$f_3 = -\frac{M^2(M^2 + h^2)}{288} \left[ (2M^2 + 3h^2)E_2^2 + \frac{1}{5}(2M^2 - 6\epsilon^2 + 3h^2)E_4 \right] C_4 + \frac{M^4(M^2 + h^2)}{576} (E_2^2 - E_4) C_{22}. \tag{2.24c}$$

In (2.24a) we have used the identity

$$\sum_{k=1}^{\infty} \frac{\sigma_1(k)}{k} q^k = -\log \widehat{\eta}(q) \tag{2.25}$$

to rewrite the instanton contribution of  $f_1$  in terms of the Dedekind  $\eta$ -function, and we have defined  $\widehat{\eta}(q) = q^{-1/24}\eta(q)$ . Expanding the modular functions for small values of  $q$ , one can easily check that both the perturbative part (2.9) and the first few instanton corrections (2.19) are correctly reproduced by (2.24).

## 2.4 Recursion relation

The explicit expressions (2.24) allow us to verify the following non-linear relation

$$\partial_{E_2} f_n = -\frac{1}{24} \sum_{m=1}^{n-1} \vec{\nabla} f_m \cdot \vec{\nabla} f_{n-m} + \frac{h^2}{24} \Delta f_{n-1} \tag{2.26}$$

for  $n = 1, 2, 3$ , where  $\vec{\nabla} = (\partial_{a_1}, \partial_{a_2}, \dots, \partial_{a_N})$  and  $\Delta = \sum_i \partial_{a_i}^2$ . To prove this relation one has to use various identities, like for example

$$C_{11} = 0, \quad C_{31} = -\frac{1}{4} C_{22}, \tag{2.27}$$

which can be easily checked as discussed in appendix B.

Eq. (2.26) is a generalization to  $U(N)$  of the recursion relation found in [19, 21] for the  $\epsilon$ -deformed  $SU(2)$  theory, and a generalization in presence of the  $\Omega$ -background of the relation found in [4] for the undeformed  $U(N)$  theory.<sup>5</sup> As discussed in those references, the recursion relation obeyed by the prepotential coefficients can be regarded as a direct consequence of the modular anomaly equation which, in turn, encodes the same information as the holomorphic anomaly equation of the topological string amplitudes [1–3] and its generalizations [17–21]. Indeed, the holomorphic anomaly equation implies an anomalous modular behavior of the prepotential coefficients with respect to  $\tau$  which can only occur through the Eisenstein series  $E_2$ .

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<sup>5</sup>The results reported in [4] contain two minor misprints: one in the recursion equation for the  $f_n$ 's and one regarding a factor of 1/4 in the last term of  $f_4$ .



We can therefore view (2.26) as a distinctive property of the  $\mathcal{N} = 2^* \text{U}(N)$  theory which is valid for any  $n$ , and use it to derive recursively the various polynomials  $f_n$  for  $n > 3$ . To this purpose one has to choose a basis for the sum structures that appear for high values of  $n$  and fix the integration constants. We do this by choosing to write everything only in terms of sums involving even inverse powers of  $a_{ij}, a_{ik}, \dots$ , which is always possible due to the existence of algebraic identities like those displayed in (2.27) or in appendix B, and fix the  $E_2$  independent terms by comparing with the perturbative and the first-few instanton expressions. Proceeding in this way, we have computed  $f_n$  up to  $n = 6$ . The result for  $f_4$  is given in appendix D (see (D.1)) including its full dependence on the  $\Omega$ -background parameters; in the same appendix (see (D.2) and (D.3)), we also write  $f_5$  and  $f_6$  for the undeformed  $\mathcal{N} = 2^* \text{U}(N)$  theory. We have checked explicitly that the expansion of these expressions in powers of  $q$  up to  $q^3$ , i.e. up to instanton number 3, agrees with a direct computation by means of localization à la Nekrasov. This is a highly non-trivial consistency check on our proposal: many tens of terms perfectly match, leading to a very robust test of the recursion relation (2.26).

Finally, we observe that the recursion relation (2.26) implies the following differential equation for the quantum prepotential:

$$\partial_{E_2} \mathcal{F}^{\text{quant}} = -\frac{1}{24} \vec{\nabla} \mathcal{F}^{\text{quant}} \cdot \vec{\nabla} \mathcal{F}^{\text{quant}} + \frac{h^2}{24} \Delta \mathcal{F}^{\text{quant}} \tag{2.28}$$

which is the homogeneous Kardar-Parisi-Zhang equation in  $N$  space dimensions. By introducing the quantum partition function

$$\mathcal{Z}^{\text{quant}} = \exp\left(-\frac{\mathcal{F}^{\text{quant}}}{h^2}\right), \tag{2.29}$$

we can map the above non-linear equation into a (parabolic) linear one [3], namely

$$\partial_{E_2} \mathcal{Z}^{\text{quant}} - \frac{h^2}{24} \Delta \mathcal{Z}^{\text{quant}} = 0, \tag{2.30}$$

which is the heat equation in  $N$  space dimensions.

### 3 Large- $N$ limit with a uniform distribution

We now discuss the large- $N$  limit of the  $\mathcal{N} = 2^*$  theory, focusing in particular on its non-perturbative sector. The first regime we consider is a naive semi-classical configuration [37] in which every charged multiplet has a mass of order  $N^0$  or greater. This is possible if we choose the eigenvalues  $a_i$  in such a way that the minimum difference among them is  $O(N^0)$ . A representative choice in this regime is

$$a_i = v \left( i - \frac{N+1}{2} \right) \quad \text{for } i = 1, \dots, N, \tag{3.1}$$

where  $v$  is a constant scale carrying the physical dimension of a mass. Such a distribution has been considered several times in the literature and its properties have been discussed in various contexts, see for example [37, 38].

If  $v$  is constant, the highest eigenvalue is  $O(N)$  and the others are equally spaced in an interval  $[-\mu, \mu]$  where  $\mu$  is  $O(N)$ . From (3.1) we see that  $\sum_i a_i = 0$  so that this distribution actually applies to  $SU(N)$  gauge groups. Moreover, it is easy to check that the classical prepotential (2.4) grows as  $N^3$  when it is evaluated with (3.1).

It is also interesting to observe that among all possible distributions of  $a_i$  in a range of size  $O(N)$ , the uniform distribution (3.1) seems to be the one which minimizes the various sums  $C_n, C_{nm}$  and so on, which we introduced in the previous section. We have checked this property numerically in several examples. We therefore think that it is worth investigating the large- $N$  behavior of the  $\mathcal{N} = 2^*$   $SU(N)$  theory with the uniform distribution (3.1) which, given its simplicity, will allow us also to obtain quite explicit results.

### 3.1 The pure $SU(N)$ theory

We begin by considering the pure  $\mathcal{N} = 2$  super Yang-Mills theory with gauge group  $SU(N)$ , whose large- $N$  behavior was studied long time ago in [37]. In the pure theory a scale  $\Lambda$  is dynamically generated and the  $k$ -th instanton sector contributes at order  $\Lambda^{2Nk}$ . One then typically studies 1-instanton effects to understand if, and when, they can survive in the large- $N$  limit and signal the breakdown of the semi-classical perturbative regime dominated by the 1-loop term.

Considering the uniform distribution (3.1) and exploiting the explicit form of the Seiberg-Witten curve for the pure theory, the 1-instanton contributions to the prepotential at large  $N$  were estimated in [37] to be of order

$$q \mathcal{F}^{(1)} \Big|_{\text{pure } SU(N)} \sim \left( \frac{\Lambda}{Nv} \right)^{2N}, \tag{3.2}$$

This shows that instanton contributions in this regime are strongly suppressed, not just because of their dependence  $\Lambda^{2N}$  on the dynamical scale  $\Lambda$  but also by an extra  $N^{-2N}$  dependence. Thus instantons do not contribute unless we take  $v$  of order  $\Lambda/N$ . For this choice the range of values of the  $a_i$ 's in (3.1) stays finite inside a region of length  $\Lambda$ , and the differences  $a_{ij}$  (related to the  $W$ -boson masses) vanish at large  $N$ .

The result (3.2) can be retrieved from the explicit form of the  $\mathcal{N} = 2^*$  prepotential given in the previous section by taking a *decoupling limit* in which the mass of the adjoint hypermultiplet diverges and the gauge coupling vanishes keeping finite the product  $m^{2N} q$ , namely

$$m \rightarrow \infty, \quad q \rightarrow 0, \quad \text{with } \Lambda^{2N} \equiv (-1)^N m^{2N} q \text{ finite.} \tag{3.3}$$

This check makes us confident in deriving later the large- $N$  limit of instanton contributions in the  $\mathcal{N} = 2^*$  theory, where a derivation based on the Seiberg-Witten curve is not available.

In the decoupling limit (3.3) only the terms with the highest mass power in (2.21) survive, and thus the 1-instanton contribution to the prepotential of the pure  $SU(N)$  theory is

$$q \mathcal{F}^{(1)} \Big|_{\text{pure } SU(N)} \sim \frac{\Lambda^{2N}}{(N-1)!} C_{\underbrace{22 \dots 2}_{N-1}}. \tag{3.4}$$

One can check that this expression exactly agrees with the explicit 1-instanton prepotential obtained from the Seiberg-Witten curve (see for example [39]).

Exploiting the results of appendix B.2 we can now evaluate the behavior of the above expression in the large- $N$  limit with the uniform distribution (3.1). In particular, from (B.35) we see that

$$C_{\underbrace{22\dots 2}_{N-1}} \underset{N \rightarrow \infty}{\sim} \frac{2N(N-1)!}{(2N)!} \left(\frac{2\pi}{v}\right)^{2N-2}. \quad (3.5)$$

Plugging this result into (3.4), using the Stirling approximation and absorbing all numerical constants into  $\Lambda$ , the estimate (3.2) is reproduced. In the following we generalize this analysis to the  $\mathcal{N} = 2^*$  theory and evaluate the large- $N$  limit at all-instanton orders.

### 3.2 The $\mathcal{N} = 2^*$ theory

If we do not take the decoupling limit (3.3) and consider the full  $\mathcal{N} = 2^*$  theory, then all mass structures contribute to the prepotential and the complete expression of the latter is necessary to find the large- $N$  behavior of the theory. Recalling that the prepotential coefficients  $f_n$ 's depend on the expectation values  $a_i$  through the sums introduced in (B.1), we can use the results of appendix B.2 to estimate how the  $\mathcal{N} = 2^*$  prepotential behaves in the large- $N$  limit with the uniform distribution (3.1). Basically what we have to do is to plug (B.15), (B.31) and (B.37) into (2.24) (where now we take  $h = \epsilon = 0$ ) and into (D.1) – (D.3). This amounts, for instance, to replace the various sums according to

$$(C_2, C_4, C_{22}, \dots) \underset{N \rightarrow \infty}{\sim} N \left( \frac{2\zeta(2)}{v^2}, \frac{2\zeta(4)}{v^4}, \frac{4\zeta(2)^2 - 2\zeta(4)}{v^4}, \dots \right). \quad (3.6)$$

Doing this, we find that numerous remarkable cancellations take place: indeed, in  $f_3, \dots, f_6$  all terms involving  $E_2$  cancel and we remain with the following very simple result:

$$\begin{aligned} f_2 &\underset{N \rightarrow \infty}{\sim} -N \frac{m^4}{v^2} \frac{\zeta(2)}{12} E_2, \\ f_3 &\underset{N \rightarrow \infty}{\sim} -N \frac{m^6}{v^4} \frac{\zeta(4)}{60} E_4, \\ f_4 &\underset{N \rightarrow \infty}{\sim} -N \frac{m^8}{v^6} \frac{\zeta(6)}{168} E_6, \\ f_5 &\underset{N \rightarrow \infty}{\sim} -N \frac{m^{10}}{v^8} \frac{\zeta(8)}{360} E_8, \\ f_6 &\underset{N \rightarrow \infty}{\sim} -N \frac{m^{12}}{v^{10}} \frac{\zeta(10)}{660} E_{10}. \end{aligned} \quad (3.7)$$

It is very tempting to generalize these formulæ to any  $n > 1$  by writing

$$f_n \underset{N \rightarrow \infty}{\sim} -N \frac{m^{2n}}{v^{2n-2}} \frac{\zeta(2n-2)}{2n(n-1)(2n-1)} E_{2n-2}. \quad (3.8)$$

This extrapolation is actually not at all obvious, even if we assume that for  $n > 2$  all contributions containing  $E_2$  factors cancel as they do up to  $n = 6$ , and that the result is

a modular form of weight  $2n - 2$ . Indeed, for  $n = 3, \dots, 6$  (and for  $n = 8$ ) the space of modular forms of weight  $2n - 2$  is one-dimensional, and if we assume no  $E_2$  dependence except in  $f_2$ , then the  $f_n$ 's have to be proportional to  $E_{2n-2}$  as we have found in (3.7). However, for higher values of  $n$ , the corresponding space of modular forms has dimension greater than one; for instance, for  $n = 7$ ,  $E_4^3$  and  $E_6^2$  represent two independent modular forms of weight 12 and in principle any linear combination of them could appear in  $f_7$ . Thus, in general it is not at all guaranteed that the result should be simply proportional to the particular modular form  $E_{2n-2}$ . Nonetheless, we will present strong evidence in favor of the conjecture (3.8) by showing a perfect match of its predictions at the perturbative and 1-instanton level with the large- $N$  limit of the exact perturbative and 1-instanton results presented in section 2. This check is actually conclusive only for the values of  $n$  such that the space of modular forms of weight  $2n - 2$  is two-dimensional; yet, also in the general case, the agreement that we find at this level is already very far from trivial.

Let us now give some details. Up to order  $q$ , i.e. up to 1-instanton, the Eisenstein series  $E_{2n-2}$  are given by (see appendix C and in particular (C.8))

$$E_{2n-2} = 1 + \frac{2}{\zeta(3-2n)} q + \dots \tag{3.9}$$

for any  $n > 1$ . Inserting this into (3.8), we easily see that our conjecture predicts a perturbative part for  $f_n$  given by

$$f_n^{1\text{-loop}} \underset{N \rightarrow \infty}{\sim} -N \frac{m^{2n}}{v^{2n-2}} \frac{\zeta(2n-2)}{2n(n-1)(2n-1)}, \tag{3.10}$$

and a 1-instanton part that can be expressed as

$$f_n^{(1)} \underset{N \rightarrow \infty}{\sim} N \frac{m^{2n}}{v^{2n-2}} \frac{\zeta(2n-2)}{(n+1)(2n+1)\zeta(1-2n)}, \tag{3.11}$$

On the other hand,  $f_n^{1\text{-loop}}$  and  $f_n^{(1)}$  can be extracted from the *exact* expressions of the perturbative and 1-instanton prepotential presented in section 2. The exact 1-loop contribution is given in (2.13) and, after taking into account the uniform large- $N$  behavior (B.31) of the sums  $C_n$ , we easily see that it precisely agrees with the prediction (3.10) that follows from the ansatz (3.8).

The 1-instanton coefficients  $f_n^{(1)}$  can be extracted from (2.21) after estimating the sum  $C_{22\dots 2}$  in the large- $N$  limit. This can be done using (B.32) and the coefficients  $\alpha_n$  given in table (B.33). Then, for any  $n > 1$  we find

$$f_n^{(1)} = (-1)^n m^{2n} \frac{1}{(n-1)!} \underbrace{C_{22\dots 2}}_{n-1} \underset{N \rightarrow \infty}{\sim} (-1)^n N \frac{m^{2n}}{v^{2n-2}} \frac{\alpha_{n-1} \zeta(2n-2)}{(n-1)!}. \tag{3.12}$$

Comparing (3.12) with (3.11) we find perfect agreement.<sup>6</sup> Thus we can conclude that our ansatz is indeed correct at least up to  $n = 9$ . However, given the very non-trivial fractions that appear, it is more than natural to extend the validity of (3.8) to arbitrary  $n > 1$ .

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<sup>6</sup>The computation of the  $\alpha_n$ 's can be straightforwardly extended to arbitrary  $n$ , finding all the times agreement with (3.11); it is therefore very natural to conjecture that the two formulæ coincide and that  $\alpha_n$  is given by (B.34) for all  $n$ 's.

Exploiting this result, we then find that the large- $N$  behavior of the instanton part of the prepotential in the  $\mathcal{N} = 2^*$  theory is

$$\mathcal{F}^{\text{inst}} \underset{N \rightarrow \infty}{\sim} N m^2 \log \hat{\eta}(q) - N m^2 \sum_{n=2}^{\infty} \left(\frac{m}{v}\right)^{2n-2} \frac{\zeta(2n-2)}{n(2n+1)(2n+2)} (E_{2n-2} - 1). \quad (3.13)$$

Here all instanton sectors have been taken into account, and thus (3.13) is exact in  $q$ . Differently from the pure  $\text{SU}(N)$  theory discussed in section 3.1, we see that in this case the dependence on  $N$  is just in the overall prefactor. The instanton prepotential scales as  $N$  but now, unlike in the pure  $\mathcal{N} = 2$  case, the eigenvalues are distributed in a region growing as  $N$ , while the masses of gauge bosons are kept finite in the limit. This instanton contribution is subleading with respect to the classical term that scales as  $N^3$ , but it gives the dominant contribution to terms of order  $m^4$  or higher in the prepotential.

Formula (3.13) is exact in  $q$  and perturbative in  $m$ . Alternatively, this formula can be rewritten in a form where the exact  $m$ -dependence is displayed order by order in  $q$ . To this aim, one can use the expansion (C.8) of the Eisenstein series and perform the sum over  $n$ . After some easy algebra, one finds the simple result

$$\mathcal{F}^{\text{inst}} \underset{N \rightarrow \infty}{\sim} \frac{N v^2}{2\pi^2} \sum_{k=1}^{\infty} \left[ \cos\left(\frac{2\pi k m}{v}\right) - 1 \right] \frac{1}{k^3} \frac{q^k}{1 - q^k}. \quad (3.14)$$

Since all powers of the hypermultiplet mass have been resummed, this formula is exact in  $m$ . It is interesting to observe that  $\mathcal{F}^{\text{inst}}$  is periodic, i.e.  $m \sim m + v$  and that it is always of order of  $N$  except for  $m = v$  where all instanton contributions exactly cancel.<sup>7</sup>

We conclude this section by observing that when an  $\Omega$ -background is turned on, the remarkable cancellations of all terms containing  $E_2$  that led to (3.8), do not occur any longer. However, in the Nekrasov-Shatashvili limit  $h \rightarrow 0$  with a non-vanishing  $\epsilon$ , we find

$$\begin{aligned} f_2 \underset{N \rightarrow \infty}{\sim} -N \frac{M^4}{v^2} \frac{\zeta(2)}{12} E_2, \\ f_3 \underset{N \rightarrow \infty}{\sim} -N \frac{M^6}{v^4} \frac{\zeta(4)}{60} \left(1 - \frac{\epsilon^2}{2M^2}\right) E_4, \\ f_4 \underset{N \rightarrow \infty}{\sim} -N \frac{M^8}{v^6} \frac{\zeta(6)}{168} \left(1 - \frac{4}{3} \frac{\epsilon^2}{M^2} + \frac{2}{3} \frac{\epsilon^4}{M^4}\right) E_6 \end{aligned} \quad (3.15)$$

where  $M$  is defined in (2.10). This seems to suggest that also in the Nekrasov-Shatashvili limit the same pattern of the undeformed theory occurs, with  $f_n$  proportional to  $E_{2n-2}$ ; however our current results are too limited to allow for a reliable extrapolation at this stage.

#### 4 Large- $N$ limit with the Wigner distribution

In this section we study the large- $N$  behavior of the  $\mathcal{N} = 2^*$  theory assuming that the eigenvalues  $a_i$  are real and distributed in an interval  $[-\mu, +\mu]$  with the Wigner semi-circle

<sup>7</sup>This cancellation can also be seen directly from (2.16) noticing that the poles in the integrand always contain the term  $\chi_I = a_i$  for some  $i$ , which leads to a zero eigenvalue in the numerator.

law

$$\rho(x) = \frac{2}{\pi\mu^2} \sqrt{\mu^2 - x^2}, \quad (4.1)$$

where the prefactor has been chosen in order to have unit normalization:

$$\int_{-\mu}^{+\mu} dx \rho(x) = 1. \quad (4.2)$$

#### 4.1 The extremal distribution

The Wigner distribution is interesting for several reasons. First, one can show that it extremizes the exact prepotential of the  $\mathcal{N} = 2^*$  gauge theory in the large- $N$  limit. To see this, we start from the exact formulæ (2.24) plus the classical term (2.4) (with the Yang-Mills  $\theta$ -angle set to zero) and vary them with respect to  $a_i$ . One gets

$$\begin{aligned} \frac{\delta\mathcal{F}}{\delta a_i} = & -\frac{8\pi^2}{g^2} a_i + M^2 C_1(i) + \frac{M^2(M^2 + h^2)}{6} E_2 C_3(i) \\ & + \frac{M^2(M^2 + h^2)}{36} \left[ (2M^2 + 3h^2)E_2^2 + \frac{1}{5}(2M^2 - 6\epsilon^2 + 3h^2)E_4 \right] C_5(i) \\ & - \frac{M^4(M^2 + h^2)}{144} (E_2^2 - E_4) [C_{23}(i) + C_{1320}(i)] + \dots \end{aligned} \quad (4.3)$$

where the sums  $C_n(i)$ ,  $C_{nm}(i)$ ,  $C_{1mnp}(i)$  are defined in (B.2) and the dots stand for higher order terms which have a similar structure.

In the large- $N$  limit when the eigenvalues  $a_i$ 's have a distribution  $\rho$ , sums over the discrete index  $i$  are replaced by integrals over a continuous variable  $x$  according to

$$\sum_{i=1}^N f(a_i) \rightarrow N \oint_{-\mu}^{+\mu} dx \rho(x) f(x), \quad (4.4)$$

where the symbol  $\oint$  denotes that a suitable prescription has to be used in order to remove the divergent contributions that arise when two eigenvalues get close leading to singularities along the integration path. As discussed in appendix B.3, we adopt the so-called Hadamard regularization (see for instance [40] and references therein) that is a generalization of the usual Cauchy Principal Value prescription particularly suited for our case, where singularities of higher order appear in the various sums. Applying this prescription to the Wigner semi-circle law (4.1), in appendix B.3 we show that

$$C_n(i) \rightarrow \begin{cases} +\frac{2N}{\mu^2} x & \text{for } n = 1, \\ -\frac{2N}{\mu^2} & \text{for } n = 2, \\ 0 & \text{for } n > 2. \end{cases} \quad (4.5)$$

Using this result and observing that, for instance  $C_{nm}(i) = C_n(i)C_m(i) - C_{n+m}(i)$ , we easily see that all terms containing sums  $C_{n_1 n_2 \dots}(i), \dots$  with at least one index  $n_i \geq 3$

vanish. Thus, only the first two terms in (4.3) survive and lead to

$$\frac{\delta \mathcal{F}}{\delta a_i} \rightarrow -\frac{8\pi^2}{g^2} x + M^2 \frac{2N}{\mu^2} x \tag{4.6}$$

which vanishes if

$$\mu^2 = \frac{Ng^2}{4\pi^2} M^2 . \tag{4.7}$$

Thus, the Wigner semi-circle law (4.1) with a mass scale  $\mu$  given as in (4.7) is a stationary configuration of the full prepotential of the  $\mathcal{N} = 2^*$  gauge theory in the large- $N$  limit. This same result (but restricted to  $\mathcal{F}^{\text{pert}}$ ) has been recently obtained in [26, 27] where the large- $N$  behavior of the  $\mathcal{N} = 2^*$  theory on  $\mathbb{S}^4$  and its Wilson loops have been studied using localization techniques [22–24]. As suggested in [25], in this context the sphere  $\mathbb{S}^4$  can be mimicked by considering an  $\Omega$  background whose parameters  $\epsilon_1$  and  $\epsilon_2$  are equal and related to the sphere radius  $R$ . In our notations this means taking

$$\epsilon^2 = 4h^2 = -\frac{4}{R^2} . \tag{4.8}$$

Recalling the definition (2.10), we can therefore rewrite (4.7) as

$$\mu^2 = \frac{Ng^2}{4\pi^2 R^2} (m^2 R^2 + 1) \tag{4.9}$$

which exactly agrees with the findings of [26, 27]. This is also consistent with the analysis based on the supergravity solution dual to the  $\mathcal{N} = 2^*$  theory on  $\mathbb{S}^4$  recently found in [28] using holography. Notice that by taking the limit  $R \rightarrow \infty$ , or equivalently  $\epsilon, h \rightarrow 0$ , we recover the flat space description and can easily reproduce the derivation of the Wigner distribution law originally presented in [41] starting from the so-called Pilch-Warner supergravity solution [42]. We also observe that all these results imply that the interval  $[-\mu, +\mu]$  in which the scalar eigenvalues are distributed has a width which grows as  $\sqrt{N}$ , differently from the uniform distribution of the previous section whose width was growing linearly with  $N$ .

## 4.2 The large N limit

We now want to evaluate the large- $N$  limit of the prepotential  $\mathcal{F}$  with the Wigner semi-circle law (4.1). To do so we need to estimate how the various sums behave at large  $N$ . As it is clear from (4.4), a sum over a single index produces a factor of  $N$ ; analogously multiple sums over  $m$  indices become proportional to  $N^m$ . Therefore, the large- $N$  limit is dominated by the terms with the largest number of indices summed over. From the explicit expressions of the prepotential coefficients  $f_n$  presented in section 2, we easily realize that those are the contributions containing the sums

$$C_{\underbrace{22\dots 2}_{n-1}} \tag{4.10}$$

already encountered in (2.22). These sums involve  $n$  summed indices and therefore are proportional<sup>8</sup> to  $N^n$ . Actually, using the Hadamard regularization prescription described in appendix B.3, one can check that all sums except (4.10) vanish when evaluated with the Wigner distributions, so they can be discarded in the large- $N$  limit.

Thus, at large  $N$  the  $f_n$ 's reduce to

$$\begin{aligned}
 f_1 &= \frac{M^2}{4} \sum_{i \neq j} \log \left( \frac{a_{ij}}{\Lambda} \right)^2 + N (M^2 + h^2) \log \widehat{\eta}(q), \\
 f_2 &= -\frac{M^2(M^2 + h^2)}{24} E_2 C_2, \\
 f_3 &= \dots + \frac{M^4(M^2 + h^2)}{576} (E_2^2 - E_4) C_{22}, \\
 f_4 &= \dots - \frac{M^6(M^2 + h^2)}{10368} (E_2^3 - 3E_2E_4 + 2E_6) C_{222}, \\
 f_5 &= \dots + \frac{M^8(M^2 + h^2)}{165888} (E_2^4 - 6E_2^2E_4 + 8E_2E_6 - 3E_8) C_{2222}, \\
 f_6 &= \dots - \frac{M^{10}(M^2 + h^2)}{2488320} (E_2^5 - 10E_2^3E_4 + 20E_2^2E_6 - 15E_2E_8 + 4E_{10}) C_{22222}.
 \end{aligned} \tag{4.11}$$

Using the properties of the Eisenstein series collected in appendix C, one can show that all the terms with  $n > 1$  can be nicely written as

$$f_n = \dots - (-1)^n \frac{M^{2n-2}(M^2 + h^2)}{24(n-1)!} D^{n-2} E_2 \underbrace{C_{22\dots 2}}_{n-1} \tag{4.12}$$

where  $D = q d/dq$  is the logarithmic derivative with respect to the instanton weight. It is natural to think that such a formula is valid for any  $n > 1$ . On the other hand, the large- $N$  behavior of the sums appearing above has been evaluated for the Wigner distribution in (B.59), which we rewrite here for convenience:

$$\underbrace{C_{22\dots 2}}_{n-1} \rightarrow (-1)^{n-1} N^n \frac{2^{n-1}}{\mu^{2n-2}}. \tag{4.13}$$

Collecting (4.11), (4.12) and (4.13), the large- $N$  limit of the instanton prepotential can be written as

$$\mathcal{F}^{\text{inst}} \underset{N \rightarrow \infty}{\sim} N (M^2 + h^2) \left[ \log \widehat{\eta}(q) + \sum_{n=1}^{\infty} \frac{1}{24 n!} \left( \frac{2M^2 N}{\mu^2} \right)^n D^{n-1} (E_2 - 1) \right], \tag{4.14}$$

We now use the definitions of the Eisenstein series given in appendix C to write

$$D^{n-1} (E_2 - 1) = -24 \sum_{k=1}^{\infty} k^{n-1} \sigma_1(k) q^k; \tag{4.15}$$

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<sup>8</sup>To check this, take into account that, according to their definitions (B.1), the quantities  $C_{n_1 n_2 \dots n_m}$  involve a sum over  $m + 1$  indices, while  $C_{1 n_1 n_2 \dots n_m}$ ,  $C_{2 n_1 n_2 \dots n_m}$ ,  $\dots$  are sums over  $m$  indices only.



then, combining with (2.25), the sum over  $n$  in (4.14) can be performed leading to the remarkably simple formula

$$\mathcal{F}^{\text{inst}} \underset{N \rightarrow \infty}{\sim} N (M^2 + h^2) \log \widehat{\eta}(q_{\text{eff}}) \tag{4.16}$$

where

$$q_{\text{eff}} = e^{2\pi i \tau + \frac{2M^2 N}{\mu^2}} = e^{-8\pi^2/g^2 + \frac{2M^2 N}{\mu^2} + i\theta} . \tag{4.17}$$

Eq. (4.16) is exact in  $q$ , in the  $\Omega$ -background parameters  $\epsilon, h$  and in the mass  $m$ . Interestingly, the instanton prepotential depends on  $\mu$  only through the shift (4.17) in the gauge coupling constant! The finiteness of the effective gauge coupling  $q_{\text{eff}}$  requires that  $\mu$  grows like  $\sqrt{N}M$  in the large- $N$  limit. With this scaling, the instanton prepotential is of order  $N$  and thus is subleading with respect to the classical and perturbative parts which, with the Wigner distribution, are  $O(N^2)$ . Like in the uniform distribution considered in the previous section, also in this case there is no extra suppression in  $N$  of the type that occurs for example in the pure  $SU(N)$  theory (see (3.2)). If we take the large- $N$  limit by keeping fixed the 't Hooft coupling, all instantons are exponentially suppressed at large  $N$  and negligible, but if instead we keep fixed the effective Yang-Mills coupling, then the instanton sectors, which are weighted by powers of  $q_{\text{eff}}$ , cannot be discarded.

## 5 Conclusions

We have generalized the modular anomaly equation of [19, 21] to  $\mathcal{N} = 2^*$  theories with an arbitrary number of colors in a generic  $\Omega$ -background, and used it to explicitly compute the coefficients of the prepotential in a small-mass expansion, including all non-perturbative corrections. Quite remarkably, these coefficients can be expressed as polynomials of quasi-modular functions of the gauge coupling, or better of the instanton weight  $q$ . These expressions are  $q$ -exact and thus in principle could be used to analyze the behavior of the prepotential in the strong coupling regime where  $q \rightarrow 1$ .

We have then studied the large- $N$  limit of these coefficients by taking  $N \rightarrow \infty$  with the Yang-Mills coupling  $g^2$  kept fixed. Differently from the usual 't Hooft limit in which  $q$  is exponentially suppressed at large  $N$ , in our case  $q$  remains finite so that instantons can not be discarded a priori. In this scenario it is therefore important to study the non-perturbative sector of the theory at large- $N$  and to take its effects into account. In particular, we have investigated how the prepotential coefficients behave in this large- $N$  limit for two different distributions of scalar eigenvalues: a uniform distribution and the Wigner semi-circle law.

In the case of the homogeneous distribution, we have provided two alternative formulae for the large- $N$  prepotential: one, perturbative in the mass with coefficients given in terms of modular functions (see (3.13)), and the other where the exact dependence on the mass is displayed order by order in  $q$  (see (3.14)).

In the case of the Wigner distribution, the large- $N$  limit of the prepotential leads to the remarkable simple formula (4.16), which is exact both in  $q$  and in  $m$ , also in presence

of an  $\Omega$ -background. It is interesting to notice that by taking the limit  $M \rightarrow 0$  in this formula, one finds the  $\mathcal{N} = 4$  instanton partition function [34], namely

$$Z^{\text{inst}} = e^{-\frac{\mathcal{F}^{\text{inst}}}{h^2}} = \widehat{\eta}(q)^{-N} = \prod_{n=1}^{\infty} (1 - q^n)^{-N}, \quad (5.1)$$

in agreement with the general expectation that in the massless limit the  $\mathcal{N} = 2^*$  theory becomes the  $\mathcal{N} = 4$  super Yang-Mills theory. Remarkably, the combined effects of turning on a mass and of selecting the Wigner distribution amount to the simple replacements

$$q \rightarrow q_{\text{eff}} = e^{2\pi i \tau + \frac{2M^2 N}{\mu^2}}, \quad h^2 \rightarrow h^2 + M^2 \quad (5.2)$$

in the  $\mathcal{N} = 4$  prepotential to obtain the  $\mathcal{N} = 2^*$  expression.

We conclude by observing that our results can be applied also to the computation of the partition function for a gauge theory on  $S^4$ . In fact, as it follows from (4.3) and (4.7), the Wigner distribution extremizes the prepotential also in presence of non-perturbative contributions and therefore it gives the leading contribution in a saddle-point approximation of the prepotential at large  $N$ . Moreover, the critical value  $\mu^2 = \frac{g^2 N M^2}{4\pi^2}$  for which the prepotential is extremized, is such that the typical instanton suppression factor  $\exp(-8\pi^2/g^2)$  exactly balances an identical contribution coming from the  $\mu$ -dependent term, leaving a constant effective coupling  $q_{\text{eff}}$ . Hence in this case instantons contribute with a  $g$ -independent term of order  $M^2$ , and the relevant prepotential of the gauge theory on  $S^4$  at large  $N$  comes entirely from the classical and 1-loop terms. These two contributions reduce to

$$\mathcal{F} = -\frac{4\pi^2}{g^2} \sum_{i=1}^N a_i^2 + \frac{M^2}{2} \sum_{i \neq j} \log\left(\frac{a_{ij}}{\Lambda}\right) + \dots \underset{N \rightarrow \infty}{\sim} \frac{M^2 N^2}{4} \log\left(\frac{g^2 N M^2}{\widehat{\Lambda}^2}\right) + \dots \quad (5.3)$$

where dots stand for subleading terms and all  $g$ -independent factors of order  $M^2$  have been reabsorbed by redefining  $\Lambda \rightarrow \widehat{\Lambda}$ . Taking the third derivative of  $\mathcal{F}$  with respect to the mass  $m$  and using (2.10), we obtain

$$\frac{d^3 \mathcal{F}}{dm^3} \underset{N \rightarrow \infty}{\sim} N^2 \frac{m(m^2 - \frac{3}{4}\epsilon^2)}{(m^2 - \frac{1}{4}\epsilon^2)^2} \quad (5.4)$$

which, considering our conventions, is in perfect agreement with the results of [22–24] obtained using localization, and of [28] obtained from holography.

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## A The $\Gamma_2$ -function

The Barnes double  $\Gamma$ -function is defined as

$$\begin{aligned} \log \Gamma_2(x|\epsilon_1, \epsilon_2) &= \frac{d}{ds} \left( \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} \frac{t^s e^{-xt}}{(1 - e^{-\epsilon_1 t})(1 - e^{-\epsilon_2 t})} \right) \Big|_{s=0} \\ &= \log \left( \frac{x}{\Lambda} \right)^2 \left( -\frac{1}{4} b_0 x^2 + \frac{1}{2} b_1 x - \frac{b_2}{4} \right) \\ &\quad + \left( \frac{3}{4} b_0 x^2 - b_1 x \right) + \sum_{n=3}^{\infty} \frac{b_n x^{2-n}}{n(n-1)(n-2)} \end{aligned} \quad (\text{A.1})$$

where the coefficients  $b_n$  are given by

$$\frac{1}{(1 - e^{-\epsilon_1 t})(1 - e^{-\epsilon_2 t})} = \sum_{n=0}^{\infty} \frac{b_n}{n!} t^{n-2}. \quad (\text{A.2})$$

The first few of them are

$$b_0 = \frac{1}{\epsilon_1 \epsilon_2} = \frac{1}{h^2}, \quad b_1 = \frac{\epsilon_1 + \epsilon_2}{2\epsilon_1 \epsilon_2} = \frac{\epsilon}{2h^2}, \quad b_2 = \frac{\epsilon_1^2 + 3\epsilon_1 \epsilon_2 + \epsilon_2^2}{6\epsilon_1 \epsilon_2} = \frac{\epsilon^2 + h^2}{6h^2}. \quad (\text{A.3})$$

## B Useful formulæ for sums and their large- $N$ behavior

Here we give the definitions of the sum structures appearing in the prepotential coefficients  $f_n$ 's and study some of their properties. We have

$$\begin{aligned} C_{n_1, n_2, \dots, n_m} &= \sum_{[i_1, i_2, \dots, i_{m+1}]} \frac{1}{(a_{i_1 i_2})^{n_1}} \frac{1}{(a_{i_1 i_3})^{n_2}} \cdots \frac{1}{(a_{i_1 i_{m+1}})^{n_m}} \\ C1_{n_1, n_2, \dots, n_m} &= \sum_{[i_1, i_2, \dots, i_m]} \frac{1}{(a_{i_1 i_2})^{n_1}} \frac{1}{(a_{i_2 i_3})^{n_2}} \cdots \frac{1}{(a_{i_m i_1})^{n_m}} \\ C2_{n_1, n_2, \dots, n_m} &= \sum_{[i_1, i_2, \dots, i_m]} \frac{1}{(a_{i_1 i_2})^{n_1}} \frac{1}{(a_{i_2 i_3})^{n_2}} \frac{1}{(a_{i_2 i_4})^{n_3}} \frac{1}{(a_{i_4 i_5})^{n_4}} \cdots \frac{1}{(a_{i_m i_1})^{n_m}} \\ C3_{n_1, n_2, \dots, n_m} &= \sum_{[i_1, i_2, \dots, i_m]} \frac{1}{(a_{i_1 i_2})^{n_1}} \frac{1}{(a_{i_2 i_3})^{n_2}} \frac{1}{(a_{i_2 i_4})^{n_3}} \frac{1}{(a_{i_2 i_5})^{n_4}} \frac{1}{(a_{i_5 i_6})^{n_5}} \cdots \frac{1}{(a_{i_m i_1})^{n_m}} \end{aligned} \quad (\text{B.1})$$

where the symbol  $[i, j, k, \dots]$  denotes the sum over the positive integers with  $i \neq j \neq k \dots$ . It is also convenient to consider the analogous structures but without summing on one color index, namely

$$\begin{aligned}
 C_{n_1, n_2, \dots, n_m}(i) &= \sum_{[j_1, j_2, \dots, j_m] \neq i} \frac{1}{(a_{ij_1})^{n_1}} \frac{1}{(a_{ij_2})^{n_2}} \dots \frac{1}{(a_{ij_m})^{n_m}} \\
 C^1_{n_1, n_2, \dots, n_m}(i) &= \sum_{[j_1, j_2, \dots, j_{m-1}] \neq i} \frac{1}{(a_{ij_1})^{n_1}} \frac{1}{(a_{j_1 j_2})^{n_2}} \dots \frac{1}{(a_{j_{m-1} i})^{n_m}}
 \end{aligned}
 \tag{B.2}$$

and so and so forth. Actually, not all these sums are independent of each other, since there exist various algebraic identities among them which we are going to discuss.

### B.1 Identities

First of all, from the definition (B.1) it is straightforward to see that

$$C_n = 0 \quad \text{for } n \text{ odd}, \tag{B.3}$$

and that

$$C_{11} = 0 \tag{B.4}$$

as a consequence of the relation

$$\frac{1}{a_{ij}a_{ik}} + \frac{1}{a_{ji}a_{jk}} + \frac{1}{a_{ki}a_{kj}} = 0 \tag{B.5}$$

which holds for any  $i, j$  and  $k$ . Multiplying (B.5) by  $1/(a_{ij})^2$  and then summing over  $[i, j, k]$ , we easily find

$$2C_{31} - C_{1211} = 0; \tag{B.6}$$

likewise, multiplying (B.5) by  $1/(a_{ij}a_{ik})$  and then summing over  $[i, j, k]$  we find

$$C_{22} + 2C_{1211} = 0. \tag{B.7}$$

These two relations together imply

$$C_{22} + 4C_{31} = 0 \tag{B.8}$$

which is used in the recursion relation to write  $f_3$  as in (2.24).

This method can be easily generalized to derive many other identities. For example, multiplying (B.5) by  $1/[(a_{ij})^{n-1}(a_{jk}^{m-1})(a_{ki}^{p-1})]$  and summing over the indices, we get

$$C_{1_{n,m-1,p}} + C_{1_{n,m,p-1}} + C_{1_{n-1,m,p}} = 0. \tag{B.9}$$

Setting  $n = 3, m = p = 2$  we have

$$C_{1_{312}} + C_{1_{321}} + C_{1_{222}} = 0; \tag{B.10}$$

on the other hand setting  $n = m = 3, p = 1$  and observing that  $C_{1_{330}} = -C_{33}$ , we have

$$C_{1_{321}} - C_{33} + C_{1_{231}} = 0. \tag{B.11}$$

Combining these last two relations and taking into account the cyclic properties  $C1_{nmp} = C1_{mpn} = C1_{pnm}$ , we obtain

$$C_{33} = -C_{1222} \tag{B.12}$$

which is the identity needed to write  $f_4$  as in (D.1). Repeatedly using (B.9), one can check that

$$\begin{aligned} C_{1_npp} &= 0 \quad \text{for } n \text{ odd}, \\ C_{1422} &= 2C_{62} + 4C_{71}, \\ C_{1622} &= 2C_{82} + 4C_{91}, \\ C_{1442} &= C_{64} + 2C_{73} + 4C_{82} + 8C_{91}. \end{aligned} \tag{B.13}$$

These relations are needed to cast  $f_5$  and  $f_6$  in the form presented in (D.2) and (D.3).

Generalizing further these manipulations, we can prove a whole set of identities involving sums with more indices. Those which are useful to check our explicit results are

$$\begin{aligned} C_{12222} - 2C_{22222} &= -4C_{23122} = 2C_{332}, \\ C_{12222} - 8C_{22222} &= 4C_{332} - 4C_{62} + 4\sum_{i=1}^N C_3(i)C_{1320}(i), \\ \frac{3}{2}C_{14222} - C_{22224} - \frac{1}{2}C_{22422} - \frac{1}{2}C_{24222} &= C_{532}, \\ \frac{3}{2}C_{14222} - \frac{1}{2}C_{22224} - C_{22422} + \frac{1}{2}C_{24222} &= -\frac{1}{2}C_{433}, \\ C_{14222} - C_{22422} + \frac{1}{2}C_{24222} &= \frac{1}{2}C_{442}, \\ \frac{5}{2}C_{14222} - \frac{5}{2}C_{22422} &= \frac{1}{4}C_{1442} - C_{1622} + \sum_{i=1}^N C_3(i)C_{1322}(i). \end{aligned} \tag{B.14}$$

## B.2 Large- $N$ limit with a uniform distribution

We now estimate the sums (B.1) in the large- $N$  limit with the distribution (3.1). The dependence on the scale  $v$  is trivially fixed by dimensional arguments and so in the following we simply use dimensionless quantities denoted with small  $c$ 's as opposed to capital  $C$ 's used for the sums (B.1). Thus we set

$$c_{n_1, n_2, \dots, n_m} \equiv v^{n_1 + n_2 + \dots + n_m} C_{n_1, n_2, \dots, n_m}, \tag{B.15}$$

and similarly for the other types of sums.

To proceed, it is convenient to consider the quantities  $C_n(i)$  introduced in (B.2), or better their dimensionless counterparts which in this case become

$$c_n(i) \equiv v^n C_n(i) = \sum_{j < i} \frac{1}{(i-j)^n} + \sum_{j > i} \frac{1}{(i-j)^n}. \tag{B.16}$$

Changing summation index, these quantities can be expressed in terms of (generalized) harmonic numbers:

$$c_n(i) = \sum_{s=1}^{i-1} \frac{1}{s^n} + (-1)^n \sum_{s=1}^{N-i} \frac{1}{s^n} = H(i-1, n) + (-1)^n H(N-i, n). \tag{B.17}$$

In turn, the harmonic numbers can be expressed in terms of the Riemann and Hurwitz  $\zeta$ -functions as follows

$$H(M, n) = \sum_{s=1}^M \frac{1}{s^n} = \sum_{s=1}^{\infty} \frac{1}{s^n} - \sum_{s=M+1}^{\infty} \frac{1}{s^n} = \zeta(n) - \zeta(n, M+1), \quad (\text{B.18})$$

so that

$$c_n(i) = (1 + (-1)^n)\zeta(n) - \zeta(n, i) - (-1)^n\zeta(n, N-i+1). \quad (\text{B.19})$$

The asymptotic of the Hurwitz  $\zeta$ -function

$$\zeta(n, M+1) \underset{M \rightarrow \infty}{\sim} \frac{1}{(n-1)M^{n-1}} - \frac{1}{2M^n} + \frac{n}{12M^{n+1}} + \dots \quad (\text{B.20})$$

implies that, for  $n > 1$ , the  $i$ -dependent parts in (B.19) are limited and rapidly approach zero for either  $i$  growing toward  $N$  or decreasing toward 1. These parts will not contribute when (products of) the  $c_n(i)$ 's are summed over  $i$ , thus greatly simplifying the evaluation of many sums.

Let us give some details starting from  $c_n$ . By symmetry we have  $c_n = 0$  when  $n$  is odd. For  $n$  even, instead, using (B.19) we find

$$c_n = \sum_{i=1}^N c_n(i) = 2N\zeta(n) - 2 \sum_{i=1}^N \zeta(n, i). \quad (\text{B.21})$$

As argued above, the sum over the  $i$ -dependent parts is subleading in the large- $N$  limit; in fact one can easily show that

$$\lim_{N \rightarrow \infty} \sum_{i=1}^N \zeta(n, i) = \zeta(n-1). \quad (\text{B.22})$$

We thus conclude that

$$c_n \underset{N \rightarrow \infty}{\sim} \begin{cases} 2N\zeta(n) & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \quad (\text{B.23})$$

Let us now consider  $c_{nm}$ . If  $n+m$  is odd,  $C_{nm}$  vanishes when evaluated on the uniform distribution (3.1). We can thus restrict to the case when  $n+m$  is even. For any choice of the  $a_i$ 's, one can show the following algebraic identity

$$\sum_{i=1}^N C_n(i) C_m(i) = C_{nm} + C_{n+m}, \quad (\text{B.24})$$

from which we deduce that

$$c_{nm} = \sum_{i=1}^N c_n(i) c_m(i) - c_{n+m}. \quad (\text{B.25})$$

If we substitute (B.19) in this formula, we see that again all terms in the sum involving the  $i$ -dependent parts give subleading contributions in the large- $N$  limit. Thus, using (B.23) we obtain

$$c_{nm} \underset{N \rightarrow \infty}{\sim} \begin{cases} 4N\zeta(n)\zeta(m) - 2N\zeta(n+m) & \text{for } n, m \text{ even,} \\ -2N\zeta(n+m) & \text{for } n, m \text{ odd.} \end{cases} \quad (\text{B.26})$$

The only exception is when  $n = m = 1$  since in this case the subleading terms in  $c_1(i)$  can give rise to an  $O(N)$  contribution in the sum. In fact one can show that

$$\sum_{i=1}^N (c_1(i))^2 = \sum_{i=1}^N (H(i-1, 1) - H(N-i, 1))^2 \underset{N \rightarrow \infty}{\sim} 2N\zeta(2); \quad (\text{B.27})$$

thus using (B.23), (B.24) and (B.25), we obtain

$$c_{11}(i) \underset{N \rightarrow \infty}{\sim} 0 \quad (\text{B.28})$$

consistently with the fact that  $C_{11}$  identically vanishes. As a further check we notice that from (B.26) we have

$$\begin{aligned} c_{31} \underset{N \rightarrow \infty}{\sim} -2N\zeta(4) &= -\frac{\pi^4}{45}N, \\ c_{22} \underset{N \rightarrow \infty}{\sim} 4N\zeta(2)^2 - 2N\zeta(4) &= \frac{4\pi^4}{45}N, \end{aligned} \quad (\text{B.29})$$

so that  $c_{22} + 4c_{31} = 0$ , in perfect agreement with the identity (B.8) valid for all  $N$ 's.

It is clear that in a similar way one can deduce the limiting behavior of all  $c_{n_1 n_2 \dots n_m}$ 's. For instance, one can start from the identity

$$\sum_{i=1}^N C_n(i)C_m(i)C_p(i) = C_{nmp} + C_{(n+m)p} + C_{(n+p)m} + C_{(p+m)n} + C_{n+m+p} \quad (\text{B.30})$$

and use it to express  $c_{nmp}$  in terms of the already evaluated sums  $c_n$  and  $c_{nm}$ , and of  $\sum_i c_n(i)c_m(i)c_p(i)$ , in which only the  $i$ -independent parts of (B.19) contribute. Other types of sums, like for instance  $C_{1n_1 n_2 \dots}$  or  $C_{2n_1 n_2 \dots}$ , are related to the  $C_{n_1 n_2 \dots}$  by identities like those presented in subsection B.1, and thus can also be evaluated in this way.

We now collect the large- $N$  behavior of all sums that appear in the prepotential coefficients used in the main text or given in appendix D, obtained via the procedure just described. We have

$$\begin{aligned} c_n &\underset{N \rightarrow \infty}{\sim} 2N\zeta(n), \\ c_{n_1 n_2} &\underset{N \rightarrow \infty}{\sim} N [4\zeta(n_1)\zeta(n_2) - 2\zeta(n_1 + n_2)], \\ c_{n_1 n_2 n_3} &\underset{N \rightarrow \infty}{\sim} N [8\zeta(n_1)\zeta(n_2)\zeta(n_3) - 4\zeta(n_1 + n_2)\zeta(n_3) - 4\zeta(n_1 + n_3)\zeta(n_2) \\ &\quad - 4\zeta(n_2 + n_3)\zeta(n_1) + 2\zeta(n_1 + n_2 + n_3)], \\ c_{n_1 n_2 n_3 n_4} &\underset{N \rightarrow \infty}{\sim} N [16\zeta(n_1)\zeta(n_2)\zeta(n_3)\zeta(n_4) - 8\zeta(n_1 + n_2)\zeta(n_3)\zeta(n_4) \\ &\quad - 8\zeta(n_1 + n_3)\zeta(n_2)\zeta(n_4) - 8\zeta(n_1 + n_4)\zeta(n_2)\zeta(n_3) \\ &\quad - 8\zeta(n_2 + n_3)\zeta(n_1)\zeta(n_4) - 8\zeta(n_2 + n_4)\zeta(n_1)\zeta(n_3) \\ &\quad - 8\zeta(n_3 + n_4)\zeta(n_1)\zeta(n_2) + 8\zeta(n_1 + n_2 + n_3)\zeta(n_4) \\ &\quad + 8\zeta(n_1 + n_2 + n_3)\zeta(n_4) + 8\zeta(n_1 + n_2 + n_3)\zeta(n_4) \\ &\quad + 8\zeta(n_1 + n_2 + n_3)\zeta(n_4) + 4\zeta(n_1 + n_2)\zeta(n_3 + n_4) \end{aligned} \quad (\text{B.31})$$

$$+ 4\zeta(n_1 + n_3)\zeta(n_2 + n_4) + 4\zeta(n_1 + n_4)\zeta(n_2 + n_3) - 12\zeta(n_1 + n_2 + n_3 + n_4)]$$

where all the  $n_i$ 's are even. Recursively using these relations, we can obtain the large- $N$  behavior of the sums with more indices. In particular using the above rules, we can show that

$$\underbrace{c_{22\dots 2}}_n \underset{N \rightarrow \infty}{\sim} N \alpha_n \zeta(2n) \tag{B.32}$$

where the numerical coefficients  $\alpha_n$  are given in the following table:

$n$	1	2	3	4	5	6	7	8	9
$\alpha_n$	2	8	18	32	48	$\frac{43200}{691}$	72	$\frac{268800}{3617}$	$\frac{3048192}{43867}$

(B.33)

Based on these values, and in view of the 1-instanton checks performed in section 3.2, we may infer that these coefficients are actually given by

$$\alpha_n = (-1)^n \frac{(n-1)!}{(n+1)(2n+1)\zeta(1-2n)} . \tag{B.34}$$

Inserting this into (B.32), we find

$$\underbrace{c_{22\dots 2}}_n \underset{N \rightarrow \infty}{\sim} (-1)^n \frac{N(n-1)!}{(n+1)(2n+1)} \frac{\zeta(2n)}{\zeta(1-2n)} = \frac{2N(2\pi)^{2n}n!}{(2n+2)!} \tag{B.35}$$

where the last equality follows from the relations

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \quad \zeta(1-2n) = -\frac{1}{2n} B_{2n} \tag{B.36}$$

with  $B_{2n}$  being the Bernoulli numbers.

Finally, using (B.31) and the identities (B.12), (B.13) and (B.14), we find

$$\begin{aligned} c_{1_{222}} \underset{N \rightarrow \infty}{\sim} 2N\zeta(6), & \quad c_{1_{422}} \underset{N \rightarrow \infty}{\sim} \frac{4}{3}N\zeta(8), & \quad c_{1_{622}} \underset{N \rightarrow \infty}{\sim} \frac{6}{5}N\zeta(10), \\ c_{1_{442}} \underset{N \rightarrow \infty}{\sim} \frac{4}{5}N\zeta(10), & \quad c_{1_{2222}} \underset{N \rightarrow \infty}{\sim} \frac{8}{3}N\zeta(8), & \quad c_{2_{2222}} \underset{N \rightarrow \infty}{\sim} 4N\zeta(8), \\ c_{1_{4222}} \underset{N \rightarrow \infty}{\sim} \frac{8}{5}N\zeta(10), & \quad c_{2_{4222}} \underset{N \rightarrow \infty}{\sim} \frac{12}{5}N\zeta(10), & \quad c_{2_{2422}} \underset{N \rightarrow \infty}{\sim} 2N\zeta(10), \\ c_{2_{2224}} \underset{N \rightarrow \infty}{\sim} \frac{14}{5}N\zeta(10), & \quad c_{2_{22222}} \underset{N \rightarrow \infty}{\sim} \frac{24}{5}N\zeta(10), & \quad c_{3_{22222}} \underset{N \rightarrow \infty}{\sim} \frac{32}{5}N\zeta(10). \end{aligned} \tag{B.37}$$



### B.3 Large- $N$ limit with the Wigner distribution

We now estimate the behavior of the sums (B.1) in the large- $N$  limit using the Wigner semi-circle distribution (4.1) with a scale given by (4.7).

Let us first consider the simple sums  $C_n(i)$  defined in (B.2). It is easy to realize that they can all be obtained from the following generating function

$$C(i) = \sum_{j \neq i} \frac{1}{a_i - a_j - \eta} = \sum_{n=1}^{\infty} C_n(i) \eta^{n-1}. \quad (\text{B.38})$$

In the large- $N$  limit with the Wigner distribution this sum gets replaced according to

$$C(i) \rightarrow C(x) = N \int_{-\mu}^{+\mu} dy \frac{\rho(y)}{x - y - \eta} = \frac{2N}{\pi\mu^2} \int_{-\mu}^{\mu} dy \frac{\sqrt{\mu^2 - y^2}}{x - y - \eta}, \quad (\text{B.39})$$

where the symbol  $\int$  means that the integral has to be evaluated using the Cauchy Principal Value prescription if  $-\mu \leq x - \eta \leq \mu$ . In the following, for simplicity we will often set  $\mu = 1$  since the correct  $\mu$ -dependence can always be recovered using simple dimensional arguments. We then have

$$\int_{-1}^{+1} dy \frac{\sqrt{1 - y^2}}{x - y - \eta} = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2} \left[ I_+(\epsilon) + I_-(\epsilon) \right] \quad (\text{B.40})$$

where

$$I_{\pm}(\epsilon) = \int_{-1}^{+1} dy \frac{\sqrt{1 - y^2}}{x - y - \eta \pm i\epsilon} = i\pi \sqrt{1 - (x - \eta \pm i\epsilon)^2} + \pi(x - \eta \pm i\epsilon). \quad (\text{B.41})$$

These integrals have been computed using the techniques for integrals of multivalued functions and the residue theorem. Inserting this result into (B.40) and reinstating the  $\mu$ -dependence, we get

$$C(x) = \frac{2N}{\mu^2} (x - \eta), \quad (\text{B.42})$$

which, after expanding in powers of  $\eta$ , yields

$$C_n(x) = \begin{cases} +\frac{2N}{\mu^2} x & \text{for } n = 1, \\ -\frac{2N}{\mu^2} & \text{for } n = 2, \\ 0 & \text{for } n > 2. \end{cases} \quad (\text{B.43})$$

Notice that this is equivalent to define

$$C_n(x) = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} \left[ N \int_{-\mu}^{+\mu} dy \frac{\rho(y)}{x - y} \right] \equiv \frac{2N}{\pi\mu^2} \int_{-\mu}^{\mu} dy \frac{\sqrt{\mu^2 - y^2}}{(x - y)^n}, \quad (\text{B.44})$$

where the symbol  $\int$  denotes the so-called Hadamard Finite Part prescription [40]. This prescription is a generalization of the Cauchy Principal Value which is suitable when the

integrand exhibits high order singularities. Essentially, the Hadamard Finite Part prescription amounts to minimally subtract all divergences from the integrand and retain just the contributions coming from the residue at infinity. If the integrand has a simple pole on the integration path ( $n = 1$  in the above formula), the Hadamard prescription reduces to the Cauchy Principal Value, and if the integrand is regular it computes the standard integral. For this reason from now on we always use the symbol  $\rlap{-}\int$  even if sometimes it may be redundant.

Exploiting this result, the large- $N$  limit of the sums  $C_n$  is then given by

$$C_n = \sum_{i=1}^N C_n(i) \rightarrow N \rlap{-}\int_{-\mu}^{\mu} dx \rho(x) C_n(x) = \begin{cases} -\frac{2N^2}{\mu^2} & \text{for } n = 2, \\ 0 & \text{for } n \neq 2. \end{cases} \quad (\text{B.45})$$

Let us now consider the sums  $C_{nm}$ . Take for example the case  $n = m = 2$ , corresponding to

$$C_{22} = \sum_{i \neq j_1 \neq j_2} \frac{1}{(a_i - a_{j_1})^2} \frac{1}{(a_i - a_{j_2})^2}. \quad (\text{B.46})$$

In the large- $N$  limit, one integration variable (the one corresponding to the index  $i$ ) appears in the integrand differently from those corresponding to  $j_1$  and  $j_2$  and thus the result may depend from the chosen order of integration. If, after setting  $\mu = 1$ , we write

$$C_{22} \rightarrow N^3 \rlap{-}\int_{-1}^{+1} dx \rho(x) \rlap{-}\int_{-1}^{+1} dy_1 \frac{\rho(y_1)}{(x - y_1)^2} \rlap{-}\int_{-1}^{+1} dy_2 \frac{\rho(y_2)}{(x - y_2)^2}, \quad (\text{B.47})$$

and use (B.44), we get

$$C_{22} \rightarrow 4N^3. \quad (\text{B.48})$$

The same result is obtained with the choice

$$C_{22} \rightarrow N^3 \rlap{-}\int_{-1}^{+1} dy_1 \rho(y_1) \rlap{-}\int_{-1}^{+1} dx \frac{\rho(x)}{(x - y_1)^2} \rlap{-}\int_{-1}^{+1} dy_2 \frac{\rho(y_2)}{(x - y_2)^2} = 4N^3. \quad (\text{B.49})$$

In both cases, nothing changes if we swap the order of integration over  $y_1$  and  $y_2$ . However, if instead we write

$$C_{22} \rightarrow N^3 \rlap{-}\int_{-1}^{+1} dy_1 \rho(y_1) \rlap{-}\int_{-1}^{+1} dy_2 \rho(y_2) \rlap{-}\int_{-1}^{+1} dx \frac{\rho(x)}{(x - y_1)^2 (x - y_2)^2}, \quad (\text{B.50})$$

we immediately get

$$C_{22} \rightarrow 0 \quad (\text{B.51})$$

since the Hadamard Finite Part of the integral over  $x$  vanishes. Again, we get the same result if we invert the order of integration on  $y_1$  and  $y_2$ .

To overcome this problem and treat all integration variables on the same footing as one does with the discrete indices in the original finite- $N$  sum (B.46), we have to symmetrize over all possible orderings of integration. Introducing a simplified notation where

$$\rlap{-}\int_{x_1 x_2 \dots x_n} \text{ stands for } \rlap{-}\int_{-1}^1 dx_1 \rho(x_1) \rlap{-}\int_{-1}^1 dx_2 \rho(x_2) \dots \rlap{-}\int_{-1}^1 dx_n \rho(x_n), \quad (\text{B.52})$$

we denote the symmetrized integration as

$$\oint_{\{x_1 x_2 \dots x_n\}} \equiv \mathcal{N}_n \sum_{P \in S_n} \oint_{x_{P(1)} x_{P(2)} \dots x_{P(n)}} \quad (\text{B.53})$$

where  $\mathcal{N}_n$  is a suitable normalization coefficient to be fixed and  $P$  is a permutation.

Evaluating  $C_{22}$  with this symmetrized prescription and reinstating the  $\mu$ -dependence, we get

$$C_{22} \rightarrow \oint_{\{x y_1 y_2\}} \frac{1}{(x - y_1)^2 (x - y_2)^2} = \mathcal{N}_3 \frac{16N^3}{\mu^4} . \quad (\text{B.54})$$

Indeed, as we have seen before, there are four different integration orderings each of which yields a contribution proportional  $4N^3$  and two orderings which do not contribute. This same procedure can be straightforwardly applied to other sums; for example we find

$$C_{31} \rightarrow \oint_{\{x y_1 y_2\}} \frac{1}{(x - y_1)^3 (x - y_2)} = -\mathcal{N}_3 \frac{4N^3}{\mu^4} . \quad (\text{B.55})$$

In this case, out of the six different integration orderings, only one of them gives a non-vanishing contribution proportional to  $-4N^3$ . Notice that the results (B.54) and (B.55) are perfectly consistent with the identity  $4C_{31} + C_{22} = 0$ , which is valid for any  $N$  (see (B.8)). Likewise, one can check that

$$C_{11} \rightarrow \oint_{\{x y_1 y_2\}} \frac{1}{(x - y_1)(x - y_2)} = 0 \quad (\text{B.56})$$

as it should be in view of (B.4). It is worth stressing that in this case all six different orderings give non-vanishing contributions to the integral, and it is only after summing all of them that we obtain the correct result. We can apply this prescription to evaluate the large- $N$  behavior of all types of sums defined in (B.1). In all cases one can verify that the results are always consistent with the relations derived in appendix B.1; these non-trivial checks put the entire procedure on a very solid basis.

To fix the normalization we can exploit the algebraic identity

$$C_{22} = \sum_{i=1}^N C_2(i) C_2(i) - \sum_{i=1}^N C_4(i) , \quad (\text{B.57})$$

which in the large- $N$  limit becomes

$$C_{22} \rightarrow N \oint_{-\mu}^{+\mu} dx \rho(x) [C_2(x) C_2(x) - C_4(x)] = \frac{4N^3}{\mu^4} \quad (\text{B.58})$$

where in the last step we used (B.44). Comparing with (B.54), we can fix the normalization factor:  $\mathcal{N}_3 = 1/4$ .

The generalization of this procedure to other sums is straightforward. In particular, for the sums (4.10), that are relevant to study the large- $N$  limit of the  $\mathcal{N} = 2^*$  prepotential, we find

$$\underbrace{C_{22\dots 2}}_{n-1} \rightarrow (-1)^{n-1} N^n \frac{2^{n-1}}{\mu^{2n-2}} . \quad (\text{B.59})$$

### C Eisenstein series and their modular properties

The Eisenstein series  $E_{2n}$  are holomorphic functions of  $\tau$  defined as

$$E_{2n} = \frac{1}{2\zeta(2n)} \sum_{m,n \in \mathbb{Z}^2 \setminus \{0,0\}} \frac{1}{(m + n\tau)^{2n}}. \quad (\text{C.1})$$

For  $n > 1$ , they are modular forms of degree  $2n$ : under an  $\text{SL}(2, \mathbb{Z})$  transformation

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d} \quad \text{with } a, b, c, d \in \mathbb{Z} \quad \text{and } ad - bc = 1 \quad (\text{C.2})$$

one has

$$E_{2n}(\tau') = (c\tau + d)^{2n} E_{2n}(\tau). \quad (\text{C.3})$$

For  $n = 1$ , the  $E_2$  series is instead quasi-modular:

$$E_2(\tau') = (c\tau + d)^2 E_2(\tau) + \frac{6}{i\pi} c(c\tau + d). \quad (\text{C.4})$$

All the modular forms of degree  $2n > 6$  can be expressed in terms of  $E_4$  and  $E_6$ ; in particular, this is true for the Eisenstein series:

$$\begin{aligned} E_8 &= E_4^2, \\ E_{10} &= E_4 E_6, \\ 691 E_{12} &= 441 E_4^3 + 250 E_6^2, \\ &\dots \quad \dots \end{aligned} \quad (\text{C.5})$$

Quasi-modular forms of higher degree can be expressed as polynomials in  $E_2$ ,  $E_4$  and  $E_6$ .

The Eisenstein series admit a Fourier expansion which, in terms of  $q = \exp(2\pi i\tau)$ , takes the form

$$E_{2n} = 1 + \frac{2}{\zeta(1-2n)} \sum_{k=1}^{\infty} \sigma_{2n-1}(k) q^k, \quad (\text{C.6})$$

where  $\sigma_p(k)$  is the sum of the  $p$ -th powers of the divisors of  $k$ . In particular, this amounts to

$$\begin{aligned} E_2 &= 1 - 24 \sum_{k=1}^{\infty} \sigma_1(k) q^k = 1 - 24q - 72q^2 - 96q^3 + \dots, \\ E_4 &= 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k) q^k = 1 + 240q + 2160q^2 + 6720q^3 + \dots, \\ E_6 &= 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k) q^k = 1 - 504q - 16632q^2 - 122976q^3 + \dots \end{aligned} \quad (\text{C.7})$$

The Fourier expansion (C.6) can also be rewritten as

$$E_{2n} = 1 + \frac{2}{\zeta(1-2n)} \sum_{k=1}^{\infty} k^{2n-1} \frac{q^k}{1-q^k} = 1 + \frac{(-1)^n (2\pi)^{2n}}{(2n-1)! \zeta(2n)} \sum_{k=1}^{\infty} k^{2n-1} \frac{q^k}{1-q^k}, \quad (\text{C.8})$$

where in the second step we used the relations (B.36).

The  $E_2$  series is related to Dedekind's eta function

$$\eta(q) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k). \quad (\text{C.9})$$

Indeed, taking the logarithm of this definition we get

$$\log \left( \frac{\eta(q)}{q^{1/24}} \right) \equiv \log \hat{\eta}(q) = \sum_{r=1}^{\infty} \log(1 - q^r) = - \sum_{k=1}^{\infty} \frac{\sigma_1(k)}{k} q^k. \quad (\text{C.10})$$

If we apply now to this relation the derivative operator  $D = q \frac{d}{dq}$ , we get

$$D \log \hat{\eta} = - \sum_{k=1}^{\infty} \sigma_1(k) q^k = \frac{E_2 - 1}{24}. \quad (\text{C.11})$$

Applying repeatedly the operator  $D$  to this last expression, we find

$$D^{n-1}(E_2 - 1) = -24 \sum_{k=1}^{\infty} k^{n-1} \sigma_1(k) q^k. \quad (\text{C.12})$$

Finally, we also have

$$\begin{aligned} DE_2 &= \frac{1}{12} (E_2^2 - E_4), \\ DE_4 &= \frac{1}{3} (E_2 E_4 - E_6), \\ DE_6 &= \frac{1}{2} (E_2 E_6 - E_4^2). \end{aligned} \quad (\text{C.13})$$

## D Explicit expressions for $f_4$ , $f_5$ and $f_6$

Exploiting the recursion relation (2.26) and following the procedure outlined in the main text, we have computed the coefficients  $f_4$ ,  $f_5$  and  $f_6$  of the prepotential. Their explicit expressions are

$$\begin{aligned} f_4 &= - \frac{M^2(M^2 + h^2)}{2592} \left[ (5M^4 + 17M^2h^2 + 15h^4) E_2^3 \right. \\ &\quad + \frac{3}{80} (64M^4 + 256M^2h^2 + 240h^4 - 192M^2\epsilon^2 - 480\epsilon^2h^2 + 15\epsilon^4) E_2 E_4 \\ &\quad \left. + \frac{1}{35} (11M^4 + 59M^2h^2 + 60h^4 - 108M^2\epsilon^2 - 270\epsilon^2h^2 + 45\epsilon^4) E_6 \right] C_6 \\ &\quad + \frac{M^4(M^2 + h^2)}{1728} \left[ (2M^2 + 3h^2) E_2^3 - \frac{3}{5} (2M^2 + 4\epsilon^2 + 3h^2) E_2 E_4 \right. \\ &\quad \left. - \frac{2}{5} (2M^2 - 6\epsilon^2 + 3h^2) E_6 \right] C_{42} \\ &\quad - \frac{M^4(M^2 + h^2)}{10368} \left[ (M^2 + 4h^2) E_2^3 - \frac{3}{5} (M^2 + 12\epsilon^2 + 4h^2) E_2 E_4 \right. \\ &\quad \left. - \frac{2}{5} (M^2 - 18\epsilon^2 + 4h^2) E_6 \right] C_{1222} \\ &\quad - \frac{M^6(M^2 + h^2)}{10368} (E_2^3 - 3E_2 E_4 + 2E_6) C_{222}, \end{aligned} \quad (\text{D.1})$$

and, in the limit  $\epsilon, h \rightarrow 0$ ,

$$\begin{aligned}
 f_5 = & -\frac{m^{10}}{10368} \left( 7E_2^4 + \frac{28}{5}E_2^2E_4 + \frac{44}{35}E_2E_6 + \frac{19}{35}E_8 \right) C_8 \\
 & + \frac{m^{10}}{10368} \left( 5E_2^4 - E_2^2E_4 - \frac{18}{7}E_2E_6 - \frac{10}{7}E_8 \right) \left( C_{62} + \frac{13}{30}C_{1422} \right) \\
 & - \frac{m^{10}}{6912} (E_2^4 - 2E_2^2E_4 + E_8) \left( C_{422} + \frac{7}{36}C_{1422} + \frac{5}{24}C_{12222} - \frac{1}{6}C_{22222} \right) \\
 & + \frac{m^{10}}{165888} (E_2^4 - 6E_2^2E_4 + 8E_2E_6 - 3E_8) C_{2222} , \tag{D.2}
 \end{aligned}$$

and

$$\begin{aligned}
 f_6 = & -\frac{m^{12}}{25920} \left( 7E_2^5 + 8E_2^3E_4 + \frac{33}{14}E_2^2E_6 + \frac{68}{35}E_2E_8 + \frac{37}{110}E_{10} \right) C_{10} \\
 & + \frac{m^{12}}{31104} \left( 7E_2^5 + \frac{7}{5}E_2^3E_4 - \frac{17}{5}E_2^2E_6 - \frac{18}{5}E_2E_8 - \frac{7}{5}E_{10} \right) \\
 & \quad \left( C_{82} + \frac{17}{56}C_{1622} + \frac{71}{112}C_{1442} \right) \\
 & - \frac{m^{12}}{62208} \left( 5E_2^5 - \frac{13}{2}E_2^3E_4 - \frac{7}{2}E_2^2E_6 + \frac{3}{2}E_2E_8 + \frac{7}{2}E_{10} \right) \\
 & \quad \left( C_{622} + \frac{11}{30}C_{1622} + \frac{43}{180}C_{1442} + \frac{109}{45}C_{14222} + \frac{64}{45}C_{24222} - \frac{24}{15}C_{22422} - \frac{5}{9}C_{22224} \right) \\
 & + \frac{m^{12}}{62208} (E_2^5 - 4E_2^3E_4 + 2E_2^2E_6 + 3E_2E_8 - 2E_{10}) \\
 & \quad \left( C_{4222} + \frac{1}{12}C_{1622} - \frac{13}{72}C_{1442} + \frac{11}{18}C_{14222} - \frac{1}{4}C_{122222} \right. \\
 & \quad \quad \left. + \frac{31}{36}C_{24222} - \frac{1}{2}C_{22422} - \frac{5}{18}C_{22224} + \frac{5}{8}C_{22222} - \frac{1}{4}C_{32222} \right) \\
 & - \frac{m^{12}}{2488320} (E_2^5 - 10E_2^3E_4 + 20E_2^2E_6 - 15E_2E_8 + 4E_{10}) C_{22222} . \tag{D.3}
 \end{aligned}$$

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