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$N = \frac{1}{2}$ deformations of chiral superspaces from new quantum Poincaré and Euclidean superalgebras

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ABSTRACT: We present a large class of supersymmetric classical r -matrices, describing the supertwist deformations of Poincaré and Euclidean superalgebras. We consider in detail new family of four supertwists of $N = 1$ Poincaré superalgebra and provide as well their Euclidean counterpart. The proposed supertwists are better adjusted to the description of deformed $D = 4$ Euclidean supersymmetries with independent left-chiral and right-chiral supercharges. They lead to new quantum superspaces, obtained by the superextension of twist deformations of spacetime providing Lie-algebraic noncommutativity of space-time coordinates. In the Hopf-algebraic Euclidean SUSY framework the considered supertwist deformations provide an alternative to the $N = \frac{1}{2}$ SUSY Seiberg's star product deformation scheme.

KEYWORDS: Supersymmetry Breaking, Quantum Groups, Superspaces, Non-Commutative Geometry

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1 Introduction

Basic theories of fundamental interactions (string theory, M -theory) are supersymmetric and the framework of quantum deformations for relativistic systems should be supersymmetrically extended. In this paper we shall describe the deformations of $D = 4$ Poincaré and Euclidean supersymmetries and provide new models of deformed chiral superspace, realized by twist quantization procedure.

There were considered two ways of embedding the noncommutative (super)space algebras into the (super)symmetry framework, which were further used for the formulation of deformed dynamical theories:

- (a) One postulates as a basic notion the noncommutative structures of deformed (super)space coordinates, and subsequently one defines corresponding star products representing the multiplication of deformed (super)fields. In such a scheme one keeps unchanged the standard (super)Poincaré symmetries, and noncommutativity is interpreted as introducing the breaking of standard relativistic (super)symmetries (see e.g. [1–3] for non-SUSY and [4–8] for SUSY case). In such a framework the simplest $N = \frac{1}{2}$ supersymmetric deformation was proposed by Seiberg [4].
- (b) One introduces the star product describing the noncommutative structure of the (super)space as derived from quantum-deformed Poincaré–Hopf (super)algebra. If the quantum symmetry is generated by a twist factor, it provides explicit definition of the star multiplication [9]–[14]. In such framework the primary notion defining the choice of deformation is given by quantum Poincaré–Hopf (super)algebra. In particular in the case of triangular deformation, only the coalgebra is modified by the twist factor

F via a similarity transformation.¹ In such formulation the algebraic relations describing noncommutativity structure of Minkowski (super)space are by construction covariant under the transformations of quantum (super)Poincaré group. We stress that if the deformation of standard relativistic (super)symmetry is obtained by (super)twist factor then the whole deformation is located in the coalgebra sector and the classical Lie (super) algebras describing spacetime (super)symmetry are not modified.

In this paper we shall employ the second approach with primary notion of quantum Hopf-algebraic (super)symmetries. For chosen supertwists which should be generated from classical supersymmetric r -matrices we derive star-multiplication rules as well the noncommutativity relations for (super)space coordinates. Firstly we shall study the classification of superextensions of the classical r -matrices respectively for the Poincaré Lie algebra and its Euclidean counterpart. It appears that due to different reality conditions for Lorentz and $\mathfrak{o}(4)$ spinors the superextensions are consistent either for Minkowski or for Euclidean metric.

Let us recall firstly the twist deformation of relativistic symmetries. If we provide a twist function $F \in U(\mathcal{P}(3,1)) \otimes U(\mathcal{P}(3,1))$ (where $\mathcal{P}(3,1)$ is Poincaré Lie algebra) both the deformed Poincaré symmetries and quantum deformations of spacetime coordinates are uniquely determined. The basic example of a twisted Poincaré deformation is provided by the canonical (Moyal-Weyl) twist [9]–[12] which preserves the constant values of the commutator of noncommutative Minkowski coordinates (for examples of other Poincaré twists providing more general covariant noncommutative space-times see e.g. [17]). We add that twisting of Poincaré symmetries was used for obtaining the quantum covariant formulation of noncommutative field theories (see e.g. [9, 10, 18, 19]) as well as of noncommutative gravity (see e.g. [20]).

Analogously, in supersymmetric relativistic theories the supertwist function F with values in graded tensor product $U(\mathcal{P}(3,1|1)) \otimes U(\mathcal{P}(3,1|1))$ (where $\mathcal{P}(3,1|1)$ is the $N = 1$ Poincaré Lie superalgebra) defines the deformed Poincaré–Hopf supersymmetries as well as covariant quantum deformation of superspace. The technique of twisted Poincaré supersymmetry extending to SUSY theories the results of [11, 12] has been already studied for Minkowski (see e.g. [21, 22]) as well as for Euclidean (see e.g. [23]–[26]) supersymmetry.²

Following the classification of $D = 4$ Poincaré deformations by the classical r -matrices [33] we shall study their Euclidean counterpart and further the supersymmetric extensions of Poincaré and Euclidean cases. The aim and novelty of our approach is:

¹If we permit the twists satisfying the cocycle condition modified by nontrivial co-associator Φ , it was argued by Drinfeld ([15]; see also [16]) that any quantum deformation of Poincaré (super)algebra can be represented by (super)twist in the framework of quasi-Hopf algebras. Here we shall consider standard framework of quantum groups with twists satisfying standard two-cocycle condition.

²We observe that in Minkowski case there were often used in field theoretic applications nonstandard twists which are not spanned by the generators of Poincaré superalgebra (see e.g. [27]–[30]). In particular there was employed a twist factor defined as function of the odd covariant derivatives in the superspace (see e.g. [31, 32]) which do anticommute with supercharges and extend the basis of Poincaré superalgebra by graded Abelian algebra. We shall restrict in this paper to standard supertwists of Drinfeld type, depending only on the Poincaré superalgebra generators.

- (i) to show that in view of the known list for classical Poincaré r -matrices presented in [33] one can find corresponding list of $D = 4$ Euclidean r -matrices,
- (ii) to consider the consistency of superextensions of classical $\mathfrak{o}(3,1)$ and $\mathfrak{o}(4)$ r -matrices with reality conditions defining respectively the Poincaré and Euclidean superalgebras,
- (iii) to supersymmetrize twist deformations which provide the Lie-algebraic deformations of the space-time coordinates,
- (iv) to demonstrate that in the framework of twisted Euclidean superalgebras one can obtain also $N = \frac{1}{2}$ superspace deformation proposed by Seiberg [4] however with the deformed superspace algebra covariant under quantum Euclidean supersymmetries.

The plan of the paper is the following. In section 2 we shall consider $N = 1$ Poincaré and Euclidean superalgebras and describe Minkowski and Euclidean reality structures based respectively on conjugation and pseudoconjugation in a fermionic sector. In section 3 we firstly introduce Euclidean counterpart of $D = 4$ Poincaré classical r -matrices and then we classify their corresponding supersymmetric Poincaré and Euclidean classical r -matrices. Further, in section 4 we recall the standard Moyal-Weyl twist deformation of a space-time and present new four Euclidean supertwists which provide in bosonic sector the Lie-algebraic deformations of the spacetime. In section 5 we shall present in Euclidean case the corresponding $N = \frac{1}{2}$ SUSY deformation of Euclidean chiral superspace. We get the first examples of deformations of Euclidean superspace coordinates containing the Lie-algebraic deformation in its even space-time sector. In our case due to twist deformation only the half of odd coalgebra relations are deformed and in alternative deformation scheme of Seiberg [4] only the algebraic sector is changed, with modified anticommutativity of antichiral supercharges. In section 6 we shall present conclusion and outlook.

2 $D = 4$ Poincaré and Euclidean superalgebras

Real Poincaré (Euclidean) Lie algebra $\mathcal{P}(3,1) = \mathfrak{o}(3,1) \ltimes \mathbf{P}$ ($\mathcal{E}(4) = \mathfrak{o}(4) \ltimes \mathbf{P}$) is generated by the Poincaré (Euclidean) fourmomenta³ $P_\mu \in \mathbf{P}$ ($\mu = 0, 1, 2, 3$), and the six Lorentz (Euclidean) rotations $L_{\mu\nu} \in \mathfrak{o}(3,1)$ ($L_{\mu\nu} \in \mathfrak{o}(4)$) ($\mu, \nu = 0, 1, 2, 3$) satisfying the standard relations:

$$\begin{aligned}
 [L_{\mu\nu}, L_{\lambda\rho}] &= i(g_{\nu\lambda} L_{\mu\rho} - g_{\nu\rho} L_{\mu\lambda} + g_{\mu\rho} L_{\nu\lambda} - g_{\mu\lambda} L_{\nu\rho}), & L_{\mu\nu} &= -L_{\nu\mu}, \\
 [L_{\mu\nu}, P_\rho] &= i(g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu), & [P_\mu, P_\nu] &= 0,
 \end{aligned}
 \tag{2.1}$$

where the metric $g_{\mu\nu}$ is given by $(g_{\mu\nu}) = (g_{\mu\nu}^{\mathcal{P}}) = \text{diag}(1, -1, -1, -1)$ for the Poincaré case ($(g_{\mu\nu}) = (g_{\mu\nu}^{\mathcal{E}}) = \text{diag}(-1, -1, -1, -1)$ for the Euclidean case) and the reality conditions imposed:

$$L_{\mu\nu}^* = L_{\mu\nu}, \quad P_\mu^* = P_\mu.
 \tag{2.2}$$

³In order to shorten our presentation we denote in the same way the tensorial indexes of Poincaré and Euclidean generators. In conventional notation our Euclidean tensorial index “0” is denoted as the index “4”.

Here the $*$ -antiinvolution is represented by Hermitian conjugation.⁴

$N = 1$ Poincaré (Euclidean) superalgebra $\mathcal{P}(3,1|1)$ ($\mathcal{E}(4|1)$) is generated by the Poincaré (Euclidean) algebra $\mathcal{P}(3,1)$ ($\mathcal{E}(4)$) and the four complex spinor generators Q_α and $\bar{Q}_{\dot{\alpha}}$ ($\alpha = 1, 2; \dot{\alpha} = \dot{1}, \dot{2}$) satisfying the following relations:⁵

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, & \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, \\ [L_{\mu\nu}, Q_\alpha] &= -(\sigma_{\mu\nu})_\alpha^\beta Q_\beta, & [L_{\mu\nu}, \bar{Q}_{\dot{\alpha}}] &= \bar{Q}_{\dot{\beta}} (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}}, \\ [P_\mu, Q_\alpha] &= 0, & [P_\mu, \bar{Q}_{\dot{\alpha}}] &= 0, \end{aligned} \tag{2.3}$$

where by using the ordinary Pauli matrices σ^i ($i = 1, 2, 3$) one sets: $\sigma^\mu = (\mathbb{1}_2, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{1}_2, -\sigma^i)$ for the Poincaré case and $\sigma^\mu = (i\mathbb{1}_2, \sigma^i)$ and $\bar{\sigma}^\mu = (i\mathbb{1}_2, -\sigma^i)$ for the Euclidean case. Following typical spinorial notation one reads $(\sigma^\mu)_{\alpha\dot{\beta}}$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}$; the matrices $\sigma_{\mu\nu} = \frac{i}{4}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu)$ and $\bar{\sigma}_{\mu\nu} = \frac{i}{4}(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu)$ are the generators of $D = 4$ Lorentz and Euclidean algebras given in two fundamental two-dimensional spinorial representations. Moreover the antiinvolutions ($*$) in (2.2) are lifted from Poincaré and Euclidean Lie algebras to their superextensions as follows:

$$Q_\alpha^* = \bar{Q}_{\dot{\alpha}}, \quad \bar{Q}_{\dot{\alpha}}^* = Q_\alpha \quad \text{for } \mathcal{P}(3,1|1), \tag{2.4}$$

$$Q_\alpha^* = \varepsilon_{\alpha\beta} Q_\beta, \quad \bar{Q}_{\dot{\alpha}}^* = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{Q}_{\dot{\beta}} \quad \text{for } \mathcal{E}(4|1), \tag{2.5}$$

where $\varepsilon_{\alpha\alpha} = \varepsilon_{\dot{\alpha}\dot{\alpha}} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = -\varepsilon_{\dot{1}\dot{2}} = \varepsilon_{\dot{2}\dot{1}} = 1$. It should be noted that antiinvolution ($*$) in (2.4) is the antilinear antiautomorphism of second order (conjugation) $(x^*)^* = x$ for $\forall x \in \mathcal{P}(3,1|1)$ and it reduces by half the number of independent Poincaré supercharges. The constraints (2.4) together with (2.2) define the reality condition for $N = 1$ Poincaré superalgebra. The star operation ($*$) in the fermionic sector of the supercharges (2.3) for the Euclidean superalgebra, due to the relation $(\varepsilon_{\alpha\beta})^2 = (\varepsilon_{\dot{\alpha}\dot{\beta}})^2 = -\mathbb{1}_2$, is the antilinear antiautomorphism of fourth order called pseudoconjugation (for fermionic generators $(Q^*)^* = -Q$) and two pairs of Euclidean supercharges Q_α and $\bar{Q}_{\dot{\alpha}}$ should be treated as independent. In particular, the pseudoconjugation (2.5) can not be implemented by Hermitian conjugation in Hilbert space.

We will also use below $\mathfrak{so}(3)$ physical basis in the Lie algebras $\mathfrak{o}(3,1)$ and $\mathfrak{o}(4)$. Namely, we put

$$M_i := \epsilon_{ijk} L_{jk}, \quad N_i := L_{0i} \quad (i, j = 1, 2, 3). \tag{2.6}$$

⁴Physical Poincaré $\mathcal{P}(3,1)$ and Euclidean $\mathcal{E}(4)$ Lie algebras can be defined as two real forms of complex inhomogeneous algebra $IO(4; \mathbb{C}) = O(4; \mathbb{C}) \ltimes \mathbf{P}_{\mathbb{C}}$ where $\mathbf{P}_{\mathbb{C}}$ denotes complex four-translations.

⁵Again in order to have a compact presentation we did adjust the notation in Euclidean superalgebra case to the standard formulation of $N = 1$ Poincaré superalgebra. In Euclidean case due to the spinorial covering $O(4) = SU_L(2) \otimes SU_R(2)$ the supercharges Q_α and $\bar{Q}_{\dot{\alpha}}$ are two independent $SU_L(2)$ and $SU_R(2)$ spinors. In a transparent notation the Euclidean supercharges $(Q_\alpha, \bar{Q}_{\dot{\alpha}})$ can be denoted as in [34] by $(Q_\alpha, \bar{Q}_{\dot{\alpha}} = Q^{\dot{\alpha}})$.

In the terms of these elements the defining relations (2.1), (2.3) for $\mathcal{P}(3, 1|1)$ and $\mathcal{E}(4|1)$ take the form for the bosonic sector

$$\begin{aligned}
 [M_i, M_j] &= i\varepsilon_{ijk}M_k, & [M_i, N_j] &= i\varepsilon_{ijk}N_k, & [N_i, N_j] &= \xi i\varepsilon_{ijk}M_k, \\
 [M_i, P_j] &= i\varepsilon_{ijk}P_k, & [M_i, P_0] &= 0, & [N_i, P_j] &= -i\delta_{ij}P_0, \\
 [N_i, P_0] &= \xi iP_i, & [P_\mu, P_\nu] &= 0,
 \end{aligned}
 \tag{2.7}$$

where now parameter ξ distinguishes the Euclidean (Poincaré) cases $\xi = 1$ ($\xi = -1$) and for fermionic sector

$$\begin{aligned}
 \{Q_\alpha, Q_\beta\} &= \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, & \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu, \\
 [M_i, Q_\alpha] &= -\frac{1}{2}(\sigma^i)_\alpha^\beta Q_\beta, & [N_i, Q_\alpha] &= \chi \frac{1}{2}(\sigma^i)_\alpha^\beta Q_\beta, \\
 [M_i, \bar{Q}_{\dot{\alpha}}] &= \frac{1}{2}\bar{Q}_{\dot{\beta}}(\sigma^i)^\beta_\alpha, & [N_i, \bar{Q}_{\dot{\alpha}}] &= \chi \frac{1}{2}\bar{Q}_{\dot{\beta}}(\sigma^i)^\beta_\alpha, \\
 [P_\mu, Q_\alpha] &= [P_\mu, \bar{Q}_{\dot{\alpha}}] = 0,
 \end{aligned}
 \tag{2.8}$$

where $\chi = 1$ for the super-Euclidean case and $\chi = -i$ for the super-Poincaré case. It is well-known that we can pass in the relations (2.7), (2.8) from the super-Poincaré to the super-Euclidean case by the replacement

$$P_i \rightarrow P_i, \quad P_0 \rightarrow iP_0, \quad M_i \rightarrow M_i, \quad N_i \rightarrow iN_i.
 \tag{2.9}$$

The replacement (2.9) can be justified by the change $x_0 \rightarrow ix_0$ of physical real time (Poincare case) into the purely imaginary Euclidean time.

3 Classical r -matrices of $N = 1$ Poincaré and Euclidean superalgebras

In this paper we will use for construction of covariant deformations of the super-Minkowski and super-Euclidean space-time the quantum deformations obtained by twist factors of the corresponding superalgebras. Such quantum deformations are classified by the classical supersymmetric r -matrices. Since the considered superalgebras contain the Poincaré and Euclidean Lie algebras as subalgebras we shall firstly consider the classical r -matrices for these Lie algebras.

(1) Non-supersymmetric case. For the Poincaré algebra the classical r -matrices were almost completely classified already some time ago by S. Zakrzewski in [35] for the Lorentz algebra and in [33] for the Poincaré algebra. We shall briefly remind these results.

It was shown in [33] that each classical r -matrix, $r \in \mathcal{P}(3, 1) \wedge \mathcal{P}(3, 1)$, has a decomposition

$$r = a + b + c,
 \tag{3.1}$$

c	b	a	#	\mathcal{N}
$\gamma h' \wedge h$	0	$\alpha P_+ \wedge P_- + \tilde{\alpha} P_1 \wedge P_2$	2	1
$\gamma e'_+ \wedge e_+$	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge h'$	0	1	2
	$\beta_1 b_{P_+}$	$\alpha P_+ \wedge P_1$	1	3
	$\gamma \beta_1 (P_1 \wedge e_+ + P_2 \wedge e'_+)$	$P_+ \wedge (\alpha_1 P_1 + \alpha_2 P_2) - \gamma \beta_1^2 P_1 \wedge P_2$	2	4
$\gamma (h \wedge e_+ - h' \wedge e'_+)$ $+ \gamma_1 e'_+ \wedge e_+$	0	0	1	5
$\gamma h \wedge e_+$	$\beta_1 b_{P_2} + \beta_2 P_2 \wedge e_+$	0	1	6
0	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge h'$	0	1	7
	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge e_+$	0	1	8
	$P_1 \wedge (\beta_1 e_+ + \beta_2 e'_+) + \beta_1 P_+ \wedge (h + \sigma e_+)$, $\sigma = 0, \pm 1$	$\alpha P_+ \wedge P_2$	2	9
	$\beta_1 (P_1 \wedge e'_+ + P_+ \wedge e_+)$	$\alpha_1 P_- \wedge P_1 + \alpha_2 P_+ \wedge P_2$	2	10
	$\beta_1 P_2 \wedge e_+$	$\alpha_1 P_+ \wedge P_1 + \alpha_2 P_- \wedge P_2$	1	11
	$\beta_1 P_+ \wedge e_+$	$P_- \wedge (\alpha P_+ + \alpha_1 P_1 + \alpha_2 P_2) + \tilde{\alpha} P_+ \wedge P_2$	3	12
	$\beta_1 P_0 \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	2	13
	$\beta_1 P_3 \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	2	14
	$\beta_1 P_+ \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	1	15
	$\beta_1 P_1 \wedge h$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	2	16
	$\beta_1 P_+ \wedge h$	$\alpha P_1 \wedge P_2 + \alpha_1 P_+ \wedge P_1$	1	17
	$P_+ \wedge (\beta_1 h + \beta_2 h')$	$\alpha_1 P_1 \wedge P_2$	1	18
	0	$\alpha_1 P_1 \wedge P_+$	0	19
		$\alpha_1 P_1 \wedge P_2$	0	20
		$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	1	21

Table 1. Classification of classical r -matrices for $D = 4$ Poincaré algebra (see [33]).

where $a \in \mathbf{P} \wedge \mathbf{P}$, $b \in \mathbf{P} \wedge \mathfrak{o}(3, 1)$, $c \in \mathfrak{o}(3, 1) \wedge \mathfrak{o}(3, 1)$. The terms a , b , c of r satisfy the following relations:

$$\begin{aligned}
 [[c, c]] &= 0, \\
 [[b, c]] &= 0, \\
 2[[a, c]] + [[b, b]] &= t\Omega \quad (t \in \mathbb{R}, \Omega \neq 0), \\
 [[a, b]] &= 0,
 \end{aligned} \tag{3.2}$$

where $[[\cdot, \cdot]]$ means the Schouten bracket, and Ω is \mathfrak{g} -invariant element, $\Omega \in (\wedge^3 \mathfrak{g})_{\mathfrak{g}}$ ($\mathfrak{g} = \mathcal{P}(3, 1)$). A complete list of the classical r -matrices was found for the case $c \neq 0$ and as well for the case $c = 0$, $t = 0$; classification of the r -matrices for the case $c = 0$, $t \neq 0$

is still not complete. The results of [33] are presented in table 1, where $P_{\pm} = P_0 \pm P_3$ and b_{P_+}, b_{P_2} are given by the expressions:

$$\begin{aligned} b_{P_+} &= P_1 \wedge e_+ - P_2 \wedge e'_+ + P_+ \wedge h, \\ b_{P_2} &= 2P_1 \wedge h' + P_- \wedge e'_+ - P_+ \wedge e'_-. \end{aligned} \tag{3.3}$$

The generators e_{\pm}, h, e'_{\pm}, h' in table 1 describe the canonical (mathematical) basis of the Lorentz Lie algebra $\mathfrak{o}(3, 1)$ which is obtained by realification of $\mathfrak{sl}(2, \mathbb{C})$ (see [33, 36]) and satisfy the following non-vanishing commutation relations:

$$\begin{aligned} [h, e_{\pm}] &= \pm e_{\pm}, & [e_+, e_-] &= 2h, \\ [h, e'_{\pm}] &= \pm e'_{\pm}, & [h', e_{\pm}] &= \pm e'_{\pm}, & [e_{\pm}, e'_{\mp}] &= \pm 2h', \\ [h', e'_{\pm}] &= \mp e_{\pm}, & [e'_+, e'_-] &= -2h. \end{aligned} \tag{3.4}$$

Table 1 lists 21 cases labeled by the number \mathcal{N} in the last column. The forth column (labeled by #) indicates a maximal number of *independent* parameters defining deformations. This number is in all cases smaller than the number of parameters actually used in the table 1. Following [37], we introduced an additional parameter γ in the component c (in the cases $\mathcal{N} = 2, \dots, 6$), a parameter β_1 in the component b (in the cases $\mathcal{N} = 7, \dots, 18$) and a parameter α_1 in the component a (in the cases $\mathcal{N} = 19, 20, 21$).⁶ The maximal numbers of independent parameters can be calculated using of automorphisms of the Poincaré algebra $\mathcal{P}(3, 1)$ (for details see [33]).

Important point in our consideration is the property that the relations (3.4) can describe Lorentz $\mathfrak{o}(3, 1)$ as well as the Euclidean $\mathfrak{o}(4)$ algebra.

The canonical generators h, h', e_{\pm}, e'_{\pm} are related with the physical generators (2.6), (2.7) of the Lorentz and Euclidean algebra by the relations:

$$\begin{aligned} h &= \chi N_3, & e_{\pm} &= (\chi N_1 \pm iM_2), \\ h' &= iM_3, & e'_{\pm} &= (iM_1 \mp \chi N_2). \end{aligned} \tag{3.5}$$

where as before $\chi = -i$ for Lorentz algebra $\mathfrak{o}(3, 1)$, and $\chi = 1$ for Euclidean $\mathfrak{o}(4)$. It follows from (3.5) that in Poincaré case the canonical basis is anti-Hermitian, i.e.

$$x^* = -x \quad (\forall x \in \{e_{\pm}, h, e'_{\pm}, h'\}), \tag{3.6}$$

but the generators (3.5) for the Euclidean case have different reality properties with respect to the conjugation (2.2) in $\mathfrak{o}(4)$, namely

$$e_{\pm}^* = e_{\mp}, \quad h^* = h, \quad e'_{\pm}{}^* = -e'_{\mp}, \quad h'^* = -h', \tag{3.7}$$

If we introduce in the Poincaré case all the four-momenta generators P_1, P_2, P_{\pm} anti-Hermitian (purely imaginary), we see that all the classical r -matrices with real deformation

⁶In the original paper by S. Zakrzewski [33] all these additional parameters are equal to 1, therefore we should assume in table 1 that they are not equal to zero.

parameters in table 1 for the Poincaré Lie algebra are Hermitian ($r^* = r$).

It should be noted that for all (but $\mathcal{N} = 6, 12$) classical r -matrices⁷ of table 1 corresponding quantum deformations of $\mathcal{P}(3, 1)$, described by twists, were constructed in [37].

It appears however that table 1 can be used as well for Euclidean Lie algebra because all r -matrices of table 1 are the classical r -matrices for complexified $\mathcal{E}(4)$. Indeed, if we replace in the formulas (3.5) generators M_i and N_i of $\mathfrak{o}(3, 1)$ by the generators of $\mathfrak{o}(4)$ and put $\chi = 1$ then the new generators will satisfy the same relations (3.4). Moreover the commutation relations of the rotation generators with the four momenta P_μ also remain unchanged if we replace $P_\pm = P_0 \pm P_3$ for Lorentzian metric by $P_\pm = iP_0 \pm P_3$ for (compare for (2.9)) Euclidean one. Therefore all the r -matrices of table 1 will satisfy the classical Yang-Baxter equation in Euclidean case however due to complex values of P_\pm and the reality constraints (3.7) (complex e_\pm and e'_\pm) we observe that only six classical r -matrices from table 1 with $\mathcal{N} = 1, 13, 14, 16, 20, 21$ describe in Euclidean case the Hermitian classical r -matrices. Additional two Euclidean real classical r -matrices can be described by the formulae from table 1 with $\mathcal{N} = 15$ and $\mathcal{N} = 19$ provided we keep the formula $P_\pm = P_0 \pm P_3$ as well in Euclidean case.

(2) Supersymmetric case. Firstly we observe that all classical $\mathfrak{o}(3, 1)$ ($\mathfrak{o}(4)$) r -matrices satisfying homogeneous YBE are as well the r -matrices for the Poincaré (Euclidean) super-algebra and the corresponding twists of the bosonic subalgebra can be used for the deformation of full superalgebra. Further we shall be interested in the supersymmetric extension of classical r -matrices of the super-Poincaré algebra $\mathcal{P}(3, 1|1)$, which contain the supercharges Q_α ($\alpha = 1, 2$) and $\bar{Q}_{\dot{\alpha}}$ ($\dot{\alpha} = \dot{1}, \dot{2}$) and will consider also their Euclidean counterparts. It should be noted that there is not known any classification of such r -matrices in spirit of the classification done by S. Zakrzewski. It turns out however that by explicit calculations it is possible to extend the Zakrzewski's classification to the Poincaré and Euclidean superalgebras by an addition of terms depending on supercharges. The superextensions of Zakrzewski's r -matrices before taking into account the real structures (2.4), (2.5) are presented in table 2.

These supersymmetric r -matrices can be presented as a sum of subordinated r -matrices which are of super-Abelian and super-Jordanian types. The subordination enables us to construct a correct sequence of quantizations and to obtain the corresponding twists describing the quantum deformations. These twists are in general case complex super-extensions of the twists obtained in [37].

Let us now select out of the supersymmetric r -matrices r_{susy} from table 2

$$r_{susy} = r + s = a + b + c + s$$

the ones that are self-conjugate respectively under the Poincaré conjugation (see (2.2) and (2.4)) and Euclidean pseudo-conjugation (see (2.2) and (2.5)).

(i) Real classical super-Poincaré r -matrices. Following [33], all classical Poincaré r -matrices from table 1 are real (Hermitian). It appears however that only seven out of 21

⁷It should be noted that in the paper [33] there is a misprint for the case $\mathcal{N} = 12$ which is corrected here.

c	b	a	s	\mathcal{N}
$\gamma h' \wedge h$	0	$\alpha P_+ \wedge P_- + \tilde{\alpha} P_1 \wedge P_2$	$\eta Q_2 \wedge Q_1$	1
$\gamma e'_+ \wedge e_+$	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge h'$	0	$\beta_1 \bar{Q}_1 \wedge Q_1$	2
	$\beta_1 b_{P_+}$	$\alpha P_+ \wedge P_1$	$\beta_1 \bar{Q}_1 \wedge Q_1 + \eta Q_1 \wedge Q_1$	3
	$\gamma \beta_1 (P_1 \wedge e_+ + P_2 \wedge e'_+)$	$P_+ \wedge (\alpha_1 P_1 + \alpha_2 P_2)$ $-\gamma \beta_1^2 P_1 \wedge P_2$	$\eta Q_1 \wedge Q_1$	4
$\gamma(h \wedge e_+ - h' \wedge e'_+) + \gamma_1 e'_+ \wedge e_+$	0	0	0	5
$\gamma h \wedge e_+$	$\beta_1 b_{P_2} + \beta_2 P_2 \wedge e_+$	0	$i\beta_1 (Q_1 + \bar{Q}_1) \wedge (Q_2 - \bar{Q}_2)$	6
0	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge h'$	0	$\beta_1 \bar{Q}_1 \wedge Q_1$	7
	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge e_+$	0	$\beta_1 \bar{Q}_1 \wedge Q_1 + \eta Q_1 \wedge Q_1$	8
	$P_1 \wedge (\beta_1 e_+ + \beta_2 e'_+)$	$\alpha P_+ \wedge P_2$	$\beta_1 \bar{Q}_1 \wedge Q_1 + \eta Q_1 \wedge Q_1$	9
	$\beta_1 P_+ \wedge (h + \chi e_+), \chi = 0, \pm 1$			
	$\beta_1 (P_1 \wedge e'_+ + P_+ \wedge e_+)$	$\alpha_1 P_- \wedge P_1 + \alpha_2 P_+ \wedge P_2$	$\eta Q_1 \wedge Q_1$	10
	$\beta_1 P_2 \wedge e_+$	$\alpha_1 P_+ \wedge P_1 + \alpha_2 P_- \wedge P_2$	$\eta Q_1 \wedge Q_1$	11
	$\beta_1 P_+ \wedge e_+$	$P_- \wedge (\alpha P_+ + \alpha_1 P_1) + \tilde{\alpha} P_+ \wedge P_2$	$\eta Q_1 \wedge Q_1$	12
	$\beta_1 P_0 \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	$\eta Q_2 \wedge Q_1$	13
	$\beta_1 P_3 \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	$\eta Q_2 \wedge Q_1$	14
	$\beta_1 P_+ \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	$\eta Q_2 \wedge Q_1$	15
	$\beta_1 P_1 \wedge h$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	$\eta Q_2 \wedge Q_1$	16
	$\beta_1 P_+ \wedge h$	$\alpha_2 P_1 \wedge P_2 + \alpha_1 P_+ \wedge P_1$	$\beta_1 \bar{Q}_1 \wedge Q_1 + \eta Q_1 \wedge Q_1$	17
	$P_+ \wedge (\beta_1 h + \beta_2 h')$	$\alpha_2 P_1 \wedge P_2$	$\eta Q_2 \wedge Q_1$	18
	0	$\alpha_1 P_1 \wedge P_+$	$\eta^{\alpha\beta} Q_\alpha \wedge Q_\beta$	19
		$\alpha_2 P_1 \wedge P_2$	$\eta^{\alpha\beta} Q_\alpha \wedge Q_\beta$	20
		$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	$\eta^{\alpha\beta} Q_\alpha \wedge Q_\beta$	21

Table 2. Supersymmetric extension of the classical r -matrices from table 1.

cases are real after supersymmetrization, namely

$$\begin{aligned}
 & 1). \mathcal{N} = 2, \quad 2). \mathcal{N} = 3 \text{ for } \eta = 0, \quad 3). \mathcal{N} = 6, \\
 & 4). \mathcal{N} = 7, \quad 5). \mathcal{N} = 8, 9 \text{ for } \eta = 0, \quad 6). \mathcal{N} = 17 \text{ for } \eta = 0.
 \end{aligned}
 \tag{3.8}$$

(ii) Real (self-conjugate) classical super-Euclidean r -matrices. Let us observe firstly that only 8 out of 21 cases ($\mathcal{N} = 1, 13 - 16, 19 - 21$) listed in the table 1 describe real classical Euclidean r -matrices. Out of them the following ones provide self-pseudoconjugate super-Euclidean r -matrices:

$$\begin{aligned}
 & 1). \mathcal{N} = 1, \quad 2). \mathcal{N} = 13 - 16 \text{ (with } P_+ = P_0 + P_3 \text{ for } \mathcal{N} = 15), \\
 & 3). \mathcal{N} = 19 - 21 \text{ (with } P_+ = P_0 + P_3 \text{ for } \mathcal{N} = 19).
 \end{aligned}
 \tag{3.9}$$

The cases $\mathcal{N} = 13 - 16$ will be considered separately more in details below. The cases $\mathcal{N} = 19 - 21$ describe three basic superextension of the canonical (Moyal-Weyl) deformations which were considered by several authors [21]–[23].

Finally we would like to mention that our classification technique of Euclidean r -matrices and corresponding supersymmetric r -matrices has one limitation: it is obtained by

the "Euclideanization" of the Zakrzewski table 1 for Poincaré r -matrices. It is however quite possible that there are Euclidean r -matrices with self-pseudoconjugate supersymmetric extension which are not corresponding to the r -matrices presented in table 1.

4 Twist deformations of (super)space-time

(1) Twisted deformations of Poincaré and Euclidean algebras and deformed Minkowski and Euclidean space-times. Let us consider firstly non-supersymmetric cases. At the beginning of this section the Poincaré and Euclidean cases will be simultaneously analyzed and therefore we introduce the following unified denotations: deformed Poincaré algebra and Minkowski space-time are denoted by $U_\kappa(\mathcal{A}_1) := U_\kappa(\mathcal{P}(3,1))$ and $U_\kappa(V_1) := U_\kappa(\mathcal{M}(3,1))$; analogously deformed Euclidean algebra and Euclidean space-time are denoted by $U_\kappa(\mathcal{A}_2) := U_\kappa(\mathcal{E}(4))$ and $U_\kappa(V_2) := U_\kappa(E(4))$, where κ describes the mass-like deformation parameter.

Most of the deformed space-times considered in the literature can be described by constant and linear values of the commutator of quantum space-time variables $\hat{x}^\mu \in V_i$ ($\mu = 0, 1, 2, 3$; $i = 1, 2$):

$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{i}{\kappa^2} \Theta^{\mu\nu}(\kappa\hat{x}) = \frac{i}{\kappa^2} \left(\vartheta^{\mu\nu} + \kappa \vartheta_\lambda^{\mu\nu} \hat{x}^\lambda + \dots \right) \quad (4.1)$$

where the tensor $\Theta^{\mu\nu}(\kappa\hat{x})$ is determined by the dimensionless set of constant parameter $\{\vartheta^{\mu\nu}\}$, $\{\vartheta_\lambda^{\mu\nu}\}$, etc. A large class of deformations (4.1) can be interpreted as covariant under twisted \mathcal{A}_i symmetries. If $\Theta^{\mu\nu}(\kappa\hat{x}) = \Theta^{\mu\nu}(0) \equiv \vartheta^{\mu\nu}$ (a case of the simplest canonical or Moyal-Weyl deformation) the corresponding Abelian twist is the following [9]–[12]

$$F_0 = \exp\left(\frac{i}{2\kappa^2} \vartheta^{\mu\nu} P_\mu \wedge P_\nu\right). \quad (4.2)$$

The quantum-deformed Hopf algebra $\hat{U}_\kappa(\mathcal{A}_i)$ acts on the enveloping algebra $U_\kappa(V_i)$ of the corresponding quantum-deformed spacetime (see (4.1)) as its representation (Hopf-algebra module). If $\hat{g} \in U_\kappa(\mathcal{A}_i)$ and $\hat{x}, \hat{y} \in U_\kappa(V_i)$ the Hopf algebraic action has the property

$$\hat{g} \triangleright (\hat{x}\hat{y}) = (\hat{g}_{(1)} \triangleright \hat{x})(\hat{g}_{(2)} \triangleright \hat{y}), \quad (4.3)$$

where we use the standard Sweedler's notation $\Delta(\hat{g}) = \hat{g}_{(1)} \otimes \hat{g}_{(2)}$. The deformation of classical Hopf algebra $\hat{U}(\mathcal{A}_i)$ by twist F does not modify the algebraic sector described by (2.1) but it changes the primitive coproduct $\Delta_0(\hat{g}) = \hat{g} \otimes 1 + 1 \otimes \hat{g}$ ($\hat{g} \in \mathcal{A}_i$) and antipode $S_0(\hat{g}) = -\hat{g}$ as follows [15]:

$$\Delta^{(F)}(\hat{g}) = F\Delta_0(\hat{g})F^{-1}, \quad S_F(\hat{g}) = uS_0(\hat{g})u^{-1}, \quad (4.4)$$

where $u = F_{(1)}S_0(F_{(2)})$ for $F = F_{(1)} \otimes F_{(2)}$.

If the algebra (4.1) is associated with twisted \mathcal{A}_i -symmetries defined by a twist F the multiplication of noncommutative coordinates \hat{x}^μ can be isomorphically represented by suitable star multiplication of the commuting classical coordinates

$$f(\hat{x})\varphi(\hat{x}) \xrightarrow{W} f(x) \star \varphi(x) \quad \Rightarrow \quad \hat{x}^\mu \hat{x}^\nu \xrightarrow{W} x^\mu \star x^\nu, \quad (4.5)$$

where W denotes the Weyl map, defined for the twist factor F by the formula (see [9]–[13])

$$\begin{aligned}
 f(x) \star \varphi(x) &= f(x) \star_F \varphi(x) := m\left(F^{-1} \triangleright (f(x) \otimes \varphi(x))\right) \Rightarrow \\
 x^\mu \star x^\nu &= x^\mu \star_F x^\nu = m\left(F^{-1} \triangleright (x^\mu \otimes x^\nu)\right),
 \end{aligned}
 \tag{4.6}$$

If we insert $F = F_0$ (see (4.2)) one obtains the relation (4.1) with $\Theta^{\mu\nu}(\kappa\hat{x}) = \vartheta^{\mu\nu}$

$$[x^\mu, x^\nu]_{\star_{F_0}} \equiv x^\mu \star_{F_0} x^\nu - x^\nu \star_{F_0} x^\mu = \frac{i}{\kappa^2} \vartheta^{\mu\nu}.
 \tag{4.7}$$

The last relation is quantum-covariant, i.e. one can show using (4.3) and (4.4) that

$$\hat{g} \triangleright \left([x^\mu, x^\nu]_{\star_{F_0}} - \frac{i}{\kappa^2} \vartheta^{\mu\nu} \right) = 0
 \tag{4.8}$$

where $\hat{g} \in U_\kappa(\mathcal{A}_i)$ and the action of symmetry generators is obtained in the \star -product realization.

In this paper we shall consider other family of Abelian Poincaré (Euclidean) twists which are described by the following general formula

$$\mathcal{F} = F_1 F_0 = \exp\left(\frac{i}{2\kappa} \vartheta^{\mu\nu\lambda} L_{\mu\nu} \wedge P_\lambda\right) \exp\left(\frac{i}{2\kappa^2} \vartheta^{\mu\nu} P_\mu \wedge P_\nu\right).
 \tag{4.9}$$

It follows from the paper [37] (see table 1 $\mathcal{N} = 13 - 16$) there are four inequivalent (i.e. modulo a Poincaré transformations) Abelian Poincaré twists of this type which in terms of the generators (2.6) are defined as follows:

$$\begin{aligned}
 \mathcal{F}_a &= F_{1a} F_0 = \exp(i\beta M_3 \wedge P_0) F_0, \\
 \mathcal{F}_b &= F_{1b} F_0 = \exp(i\beta M_3 \wedge P_3) F_0, \\
 \mathcal{F}_c &= F_{1c} F_0 = \exp(i\beta M_3 \wedge P_+) F_0, \\
 \mathcal{F}_d &= F_{1d} F_0 = \exp(i\beta N_3 \wedge P_1) F_0,
 \end{aligned}
 \tag{4.10}$$

where $P_+ := P_0 + P_3$ and the fourmomentum-dependent factor F_0 (the same for all four formulas) is given by⁸

$$F_0 = \exp(i\alpha_1 P_3 \wedge P_0 + i\alpha_2 P_2 \wedge P_1).
 \tag{4.11}$$

(2) The examples of twisted deformations of Euclidean superspaces. It follows from section 3 that four classical r -matrices (see table 2, $\mathcal{N} = 13 - 16$) as well as the corresponding twists (4.10) are valid for both cases of Poincaré and Euclidean symmetries (with the formula $P_+ = P_0 + P_3$ valid also in Euclidean case). In this paper we shall describe more in detail the supersymmetric extension of four twists (4.10) (other examples of super-Poincaré were considered in [36]). We shall provide the algebraic structure of the corresponding deformed chiral superspaces.

⁸It corresponds to particular choice of the parameter $\vartheta^{\mu\nu}$ in (4.2).

The r -matrices generating of the twists (4.10) are given by the following formulas:

$$r_w = r_{1w} + r_0 \quad (w = a, b, c, d), \quad (4.12)$$

where the indices $w = (a, b, c, d)$ label four twists F_w ,

$$\begin{aligned} r_{1a} &= \beta P_0 \wedge M_3, & r_{1b} &= \beta P_3 \wedge M_3, \\ r_{1c} &= \beta P_+ \wedge M_3, & r_{1d} &= \beta P_1 \wedge N_3, \end{aligned} \quad (4.13)$$

$$r_0 = \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2 .$$

From table 2 we get the following formulae for the supersymmetric extension r_{ws} of the classical r -matrices r_w and the corresponding supertwists \mathcal{F}_{ws} :⁹

$$r_{ws} = r_w + r_s = r_{1w} + r_0 + r_s \quad (w = a, b, c, d), \quad (4.14)$$

$$\mathcal{F}_{ws} = F_{1w} F_0 F_s = \exp\left(\frac{i}{\kappa} \tilde{r}_{1w}\right) \exp\left(\frac{i}{\kappa^2} \tilde{r}_0\right) \exp\left(\frac{i}{\kappa} \tilde{r}_s\right), \quad (4.15)$$

where $\tilde{r}_w = -r_w$ and the term $\tilde{r}_s = r_s$ describing supersymmetric extension is given as follows

$$r_s = \eta Q_2 \wedge Q_1 = \eta(Q_2 \otimes Q_1 + Q_1 \otimes Q_2) \quad (4.16)$$

and η is a deformation parameter.

The super-term F_s in (4.15) for any value of η is not invariant under the involution (2.4) characterizing the Poincaré superalgebra, and leads to the co-product, which does not satisfy the condition

$$(\Delta^{(F)}(g))^* = \Delta^{(F)}(g^*) . \quad (4.17)$$

Consequently the deformed chiral (Q_α) and antichiral ($\bar{Q}_{\dot{\alpha}}$) supercharges will not be complex conjugate to each other. Fortunately, if we consider the Euclidean case and corresponding pseudoconjugation (2.5) then for real η we obtain the following invariance relation for r -matrices r_s (as well as r_{ws})

$$r_s^* = \eta Q_2^* \wedge Q_1^* = r_s \quad (4.18)$$

expressing its the pseudoreality property. Furthermore the super-twist \mathcal{F}_{ws} will be (pseudo)unitary

$$(\mathcal{F}_{ws})^* = \mathcal{F}_{ws}^{-1}, \quad (4.19)$$

where the pseudoconjugation (2.5) is used. It can be shown that the new deformed co-product $\Delta^{(F)} := \mathcal{F}_{ws} \Delta_0 \mathcal{F}_{ws}^{-1}$ satisfies the relation (4.17). We see therefore that if we discard the possibility of doubling of the number of supercharges in the deformation procedure the twist deformations generated by the super-twists (4.15) should be only applied in the Euclidean case (not for Poincaré superalgebra).

⁹It should be stressed that the order of factors in the formula (4.15) is important, because the Abelian r -matrices r_{1w} and $r_{1w} + r_s$ are subordinate. One say that r_2 is subordinated to r_1 , $r_1 \succ r_2$, if $[x \otimes 1 + 1 \otimes x, r_1] = 0$ ($\forall x \in \text{Sup}(r_2)$), where $\text{Sup}(r_2)$ is a support of r_2 (see [37] for details).

5 Twist-deformed Euclidean chiral superspace

In order to obtain the \star -products (and also \star -commutators) of the superspace coordinates with help of the formula (4.6) (where the twist F is replaced by supertwists (4.15)) we need a realization of the $N = 1$ super-Euclidean algebra, $\mathfrak{o}(4|1)$, in terms of linear differential operators on superspace. In accordance with [38] there are in Minkowski as well as in Euclidean superspace three differential superspace realizations: non-chiral, chiral and antichiral. The chiral supersymmetric covariant derivatives, anticommuting with the supercharges ($\{Q_\alpha, D_\beta\} = \{Q_\alpha, D_{\dot{\beta}}\} = \{\bar{Q}_{\dot{\alpha}}, D_\beta\} = \{\bar{Q}_{\dot{\alpha}}, D_{\dot{\beta}}\} = 0$) are defined as follows

$$D_\alpha = \partial_\alpha - 2(\sigma^\mu \bar{\theta})_\alpha \partial_\mu, \quad \bar{D}_{\dot{\alpha}} = i\partial_{\dot{\alpha}}. \quad (5.1)$$

If we impose the chirality condition

$$\bar{D}_{\dot{\alpha}} \Phi(x, \theta, \bar{\theta}) = 0 \quad (5.2)$$

one gets the realization on Grassmann-holomorphic variables θ^α ($\alpha = 1, 2$) which enters into Euclidean $N = 1$ holomorphic chiral superfields (see e.g. [34]). We obtain the following chiral differential realization of Euclidean superalgebra generators by adjusting to Euclidean case so-called Mandelstam realization [39, 40]

$$\begin{aligned} P_\mu &= -i\partial_\mu, & M_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2}(\theta\sigma_{\mu\nu})^\alpha \partial_\alpha, \\ Q_\alpha &= i\partial_\alpha, & \bar{Q}_{\dot{\alpha}} &= 2(\theta\sigma^\mu)_{\dot{\alpha}} \partial_\mu. \end{aligned} \quad (5.3)$$

Inserting the realization (5.3) into the formulae (4.15), and using the formula (4.6) extended to supertwist deformations of chiral superfields

$$\Phi(x, \theta) \star_{\mathcal{F}} \Psi(x, \theta) := m\left(\mathcal{F}^{-1} \triangleright (\Phi(x, \theta) \otimes \Psi(x, \theta))\right), \quad (5.4)$$

(the action \triangleright is described by the realization (5.3)) we obtain the following set of deformed superspace relations:

(1). Deformation of Euclidean space-time. We use the notation

$$[x^\mu, x^\nu]_{\star_{\mathcal{F}_{ws}}} := x^\mu \star_{\mathcal{F}_{ws}} x^\nu - x^\nu \star_{\mathcal{F}_{ws}} x^\mu. \quad (5.5)$$

One can show that the general structure of this deformed commutator has the form

$$[x^\mu, x^\nu]_{\star_{\mathcal{F}_{ws}}} = [x^\mu, x^\nu]_{\star_{F_{1w}}} + [x^\mu, x^\nu]_{\star_{F_0}} + [x^\mu, x^\nu]_{\star_{F_s}} \quad (5.6)$$

for $w = a, b, c, d$. This property follows from the relations $P_\mu P_\nu \triangleright x^\lambda = 0$, $(Q^\alpha)^2 = 0$. The explicit calculations yield to following results:

$$[x^\mu, x^\nu]_{\star_{F_s}} = 0, \quad (5.7)$$

$$[x^\mu, x^\nu]_{\star_{F_0}} = \frac{2i}{\kappa^2} (\alpha_1 \delta_3^{[\mu} \delta_0^{\nu]} + \alpha_2 \delta_2^{[\mu} \delta_1^{\nu]}), \quad (5.8)$$

$$[x^\mu, x^\nu]_{\star_{F_{1a}}} = \frac{2i\beta}{\kappa} (\delta_2^{[\mu} \delta_0^{\nu]} x^1 + \delta_0^{[\mu} \delta_1^{\nu]} x^2), \quad (5.9)$$

$$[x^\mu, x^\nu]_{\star_{F_{1b}}} = \frac{2i\beta}{\kappa} (\delta_2^{[\mu} \delta_3^{\nu]} x^1 + \delta_3^{[\mu} \delta_1^{\nu]} x^2), \quad (5.10)$$

$$[x^\mu, x^\nu]_{\star_{F_{1c}}} = [x^\mu, x^\nu]_{\star_{F_{1a}}} + [x^\mu, x^\nu]_{\star_{F_{1b}}}, \quad (5.11)$$

$$[x^\mu, x^\nu]_{\star_{F_{1d}}} = \frac{2i\beta}{\kappa} (\delta_1^{[\mu} \delta_3^{\nu]} x^0 + \delta_1^{[\mu} \delta_0^{\nu]} x^3). \quad (5.12)$$

It should be pointed out that only if we use in (5.4) the Mandelstam chiral realization (5.3) the commutators of spacetime coordinates do not depend on the parameter η and second Grassmann spinor $\bar{\theta}^{\dot{\alpha}}$. If we consider the usual chiral realization which is obtained by introducing the standard chiral fields as follows: $\tilde{\Phi}(x, \theta, \bar{\theta}) = \Phi(x - i\bar{\theta}\sigma\theta, \theta)$, the space-time commutators will depend also on bilinear product $\bar{\theta}\sigma\theta$ of Grassmann variables $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$. Further we observe that

- (i) (5.7) describes a particular choice of the canonical deformation;
- (ii) the relations (5.9)–(5.12) provide particular examples of the Lie-algebraic deformations of spacetime coordinates;
- (iii) from the relation (5.10) follows that after deformation the time coordinate remains commutative but other relations (5.9) and (5.12) describe the examples with quantum noncommutative time coordinate.

(2). Deformation of Grassmann sector. The \star -product of the chiral Grassmann variables is the same for all four deformations described by the twists \mathcal{F}_{ws} ($w = a, b, c, d$), and we obtain the following result ($\{\theta^\alpha, \theta^\beta\}_{\star_{\mathcal{F}_{ws}}} = \theta^\alpha \star_{\mathcal{F}_{ws}} \theta^\beta + \theta^\beta \star_{\mathcal{F}_{ws}} \theta^\alpha$):

$$\{\theta^\alpha, \theta^\beta\}_{\star_{\mathcal{F}_{ws}}} = \{\theta^\alpha, \theta^\beta\}_{\star_{\mathcal{F}_s}} = -\frac{2\eta}{\kappa} \delta_1^{(\alpha} \delta_2^{\beta)}. \quad (5.13)$$

The relation (5.13) leads in the sector of Grassmann variables to the choice of \star -product postulated by Seiberg [4] which was introduced however as an assumption without any link with the notation of quantum-deformed space-time symmetries.

(3). Deformation of mixed spacetime-Grassmann sector. The \star -product of Grassmann and spacetime coordinates depends on choice of the twist \mathcal{F}_{ws} ($w = a, b, c, d$) in the following way:

$$[x^\mu, \theta^\alpha]_{\star_{\mathcal{F}_{ws}}} = [x^\mu, \theta^\alpha]_{\star_{F_{1w}}} \quad (w = a, b, c, d), \quad (5.14)$$

where

$$\begin{aligned} [x^\mu, \theta^\alpha]_{\star_{F_{1a}}} &= \frac{\beta}{\kappa} \delta_0^\mu (\theta\sigma_{12})^\alpha, & [x^\mu, \theta^\alpha]_{\star_{F_{1b}}} &= \frac{\beta}{\kappa} \delta_3^\mu (\theta\sigma_{12})^\alpha, \\ [x^\mu, \theta^\alpha]_{\star_{F_{1c}}} &= \frac{\beta}{\kappa} (\delta_0^\mu + \delta_3^\mu) (\theta\sigma_{12})^\alpha, & [x^\mu, \theta^\alpha]_{\star_{F_{1d}}} &= \frac{\beta}{\kappa} \delta_1^\mu (\theta\sigma_{03})^\alpha, \end{aligned} \quad (5.15)$$

We see from the relations (5.13)–(5.15) that only the chiral superspace coordinates θ^α are deformed, and the antichiral described by $\bar{\theta}^{\dot{\alpha}}$ -sector remains unchanged i.e. one obtained $N = \frac{1}{2}$ SUSY deformation see [4, 5, 7, 8]. Such form of deformation is consistent only in Euclidean framework, where the left-chiral and right-chiral coordinates can be deformed independently.

We recall that first $N = \frac{1}{2}$ deformation of Euclidean supersymmetries was described by Seiberg in [4] without any use of quantum supersymmetries. The primary deformation in [4] (and followed by other authors [5, 7, 8]) is introduced by ansatz modifying anticommutator of half of Grassmann variables

$$\{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = 0 \implies \{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} = \eta C_{\dot{\alpha}\dot{\beta}}$$

with other anti-commutators ($\{\theta_\alpha, \theta_\beta\} = \{\bar{\theta}_{\dot{\alpha}}, \theta_\beta\} = 0$) left unchanged. Inserting such deformed Grassmann variables into the superspace realizations (5.3) we obtain the following deformed anticommutators of antichiral supercharges

$$\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = \eta \delta_{\dot{\alpha}\dot{\beta}} \square \equiv \eta \delta_{\dot{\alpha}\dot{\beta}} \delta^{\mu\nu} \partial_\mu \partial_\nu$$

with unchanged anticommutators $\{Q_\alpha, Q_\beta\}$ and $\{\bar{Q}_{\dot{\alpha}}, Q_\beta\}$. In such a framework we obtain the $N = \frac{1}{2}$ deformation of superspace superalgebra which breaks the standard $D = 4$ Euclidean supersymmetry.

For getting the modified Grassmann variables as in formula (5.13) it is sufficient to consider the simplest canonical supertwist, described by the supersymmetric r -matrices $\mathcal{N} = 19 - 21$ in table 2. Indeed such twist were considered for such purpose in [24, 25] and [44] and they lead to the noncommutativity of spacetime described by a constant matrix $\vartheta^{\mu\nu}$. The novelty of our results here is the use of supersymmetric r -matrices with $\mathcal{N} = 13 - 16$, which leads to Lie-algebraic deformations of the spacetime sector, i.e. non-vanishing parameters $\vartheta_\rho^{\mu\nu}$ in formula (4.1).

6 Conclusions

In this presentation we employed the formulae for the $D = 4$ Poincaré classical r -matrices, obtained by Zakrzewski [33], and considered the corresponding $D = 4$ Euclidean classical r -matrices. It appears that in Euclidean case due to the reality condition (3.6) only some of the r -matrices from table 1 can be used as the real Euclidean classical r -matrices. Subsequently we did show also how to supersymmetrize the $D = 4$ Poincaré and Euclidean r -matrices (see table 2) and considered the restrictions imposed by the Poincaré and Euclidean reality conditions. Further for four chosen supersymmetric r -matrices we constructed corresponding supertwists and described respective quantum deformations of Euclidean superspace.

We made an important step in the task of providing the complete classification of Hopf-algebraic quantum deformations of $D = 4$ relativistic supersymmetries and their Euclidean counterparts. We recall however that some of the considered supertwists violate the reality condition in Minkowski superspace, but they are consistent with the reality structure of Euclidean superalgebra, which is described by the covariance under the pseudoconjugation. We recall also that the known example of $D = 4$ quantum deformations of

supersymmetries with Lie-algebraic deformed spacetime, the κ -deformation [41–43] does not belong to the considered class of triangular quantum superalgebras, because they can not be generated by Drinfeld twist.¹⁰ It should be added that besides the structure of twisted Poincaré and Euclidean superalgebras one can also consider deformations of a dual Poincaré [42, 43] as well as $D = 4$ Euclidean [44] supergroups.

In this presentation we considered explicitly new examples of Hopf-algebraic framework of twist-deformed $D = 4$ supersymmetries and described corresponding new quantum deformations of superspace. We stress that the considered class of deformed superspaces is new with the Minkowski quantum algebra satisfying Lie-algebraic relations (see (5.8)–(5.12)). The next step in our studies will be the introduction of superfields on twist-deformed superspaces and the construction of new deformed field-theoretical SUSY models with new quantum supersymmetries.

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¹⁰The exact description of κ -deformed symmetries by twisting is only possible if we go beyond the framework of standard quantum groups [16].

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