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Spectral analysis of the integral operator arising from the beam deflection problem on elastic foundation II: eigenvalues

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Abstract

We analyze the eigenstructure of the integral operator $\mathcal{K}_{l,\alpha,k}$ which arise naturally from the beam deflection equation on linear elastic foundation with finite beam. We show that $\mathcal{K}_{l,\alpha,k}$ has countably infinite number of positive eigenvalues approaching 0 as the limit, and give explicit upper and lower bounds on each of them. Consequently, we obtain explicit upper and lower bounds on the L^2-norm of the operator $\mathcal{K}_{l,\alpha,k}$. We also present precise approximations of the eigenvalues as they approach the limit 0, which describes the almost regular structure of the spectrum of $\mathcal{K}_{l,\alpha,k}$. Additionally, we analyze the dependence of the eigenvalues, including the L^2-norm of $\mathcal{K}_{l,\alpha,k}$, on the intrinsic length $L = 2l\alpha$ of the beam, and show that each eigenvalue is continuous and strictly increasing with respect to L. In particular, we show that the respective limits of each eigenvalue as L goes to 0 and infinity are 0 and $1/k$, where k is the linear spring constant of the given elastic foundation. Using Newton?s method, we also compute explicitly numerical values of the eigenvalues, including the L^2-norm of $\mathcal{K}_{l,\alpha,k}$, corresponding to various values of L.

MSC: 34L15; 47G10; 74K10

Keywords: beam; deflection; elastic foundation; integral operator; eigenvalue; L^2-norm

1 Introduction

We consider the linear integral operator $\mathcal{K}_{l,\alpha,k}$, defined by

$$\mathcal{K}_{l,\alpha,k}[u](x) := \int_{-l}^{l} K\big(|x-\xi|\big)u(\xi)\,d\xi$$

for complex functions u on the real interval $[-l,l]$, $l > 0$. Here, the function $K(\cdot)$ is

$$K(y) := \frac{\alpha}{2k}\exp\left(-\frac{\alpha}{\sqrt{2}}y\right)\sin\left(\frac{\alpha}{\sqrt{2}}y+\frac{\pi}{4}\right)$$

for a constant $k > 0$ and $\alpha := \sqrt[4]{k/(EI)}$. The function K arises naturally as the Green?s function of the following linear ordinary differential equation:

$$EI\frac{d^4u(x)}{dx^4} + k \cdot u(x) = w(x) \qquad (1)$$

with the boundary condition $\lim_{x\to\pm\infty} u(x) = \lim_{x\to\pm\infty} u'(x) = 0$, whose closed form solution [1] is

$$u(x) = \int_{-\infty}^{\infty} K(|x-\xi|)w(\xi)\,d\xi = \lim_{l\to\infty} \mathcal{K}_{l,\alpha,k}[u].$$

According to the classical Euler beam theory, (1) is the governing equation for the vertical deflection $u(x)$ of a linear-shaped beam resting horizontally on an elastic foundation, where the beam is subject to the downward load distribution $w(x)$ applied vertically on the beam. $k > 0$ is the linear spring constant of the elastic foundation, so that $k \cdot u(x)$ is the spring force distribution by the elastic foundation. The constants E and I are the Young?s modulus and the mass moment of inertia, respectively, so that EI is the flexural rigidity of the beam. Historically, the beam deflection problem has been one of the cornerstones of mechanical engineering [2–11].

Recently, Choi and Jang [12] obtained existence and uniqueness result for the solution of the following nonlinear and nonuniform equation which generalizes (1):

$$EI\frac{d^4u(x)}{dx^4} + f\bigl(u(x),x\bigr) = w(x).$$

It turned out to be crucial in their work to analyze the integral operator defined by

$$\mathcal{K}[u](x) := \int_{-\infty}^{\infty} K\bigl(|x-\xi|\bigr)u(\xi)\,d\xi. \tag{2}$$

However, (2) is for *infinitely long* beams, while beams with finite lengths are important in practice. To deal with finite beams, we need to analyze the integral operator $\mathcal{K}_{l,\alpha,k}$, instead of $\mathcal{K}$. With this motivation, Choi [13, 14] performed an analysis of the eigenstructure of $\mathcal{K}_{l,\alpha,k}$ as a linear operator on the Hilbert space $L^2[-l,l]$ of the square-integrable complex functions on $[-l,l]$. It was shown that all the eigenvalues of $\mathcal{K}_{l,\alpha,k}$ are contained in the real interval $(0,1/k)$, and hence $\mathcal{K}_{l,\alpha,k}$ is positive and contractive in dimension-free sense.

In this paper, we analyze concretely the structure of the eigenvalues of $\mathcal{K}_{l,\alpha,k}$ *inside* the interval $(0,1/k)$. Note that $\mathcal{K}_{l,\alpha,k}$ is in the important class of compact, self-adjoint operators, of whose eigenstructures the following general property is well known.

Proposition 1 ([15]) *Let X be a nontrivial real or complex inner-product space, and let $\mathcal{T}$ be a compact self-adjoint operator from X to X. Then the eigenvalues of $\mathcal{T}$ are real, and the number of them is at most countably infinite. Moreover, the eigenvalues, denoted by $\lambda_1, \lambda_2, \lambda_3, \ldots$, can be ordered such that*

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \cdots > 0,$$

and the L^2-norm $\|\mathcal{T}\| := \|\mathcal{T}\|_2$ of $\mathcal{T}$ is $|\lambda_1|$.

For the operator $\mathcal{K}_{l,\alpha,k}$, we will prove the results below.

Theorem 1

(a) *The spectrum of the operator $\mathcal{K}_{l,\alpha,k}$ is of the form*

$$\left\{\frac{\mu_n}{k}\;\middle|\; n = 1,2,3,\ldots\right\} \cup \left\{\frac{\nu_n}{k}\;\middle|\; n = 1,2,3,\ldots\right\},$$

where μ_n and ν_n depend only on $L := 2l\alpha$, and, for $n = 1, 2, 3, \ldots$,

$$\frac{1}{1+\{h^{-1}(2\pi n+\frac{\pi}{2})\}^4} < \nu_n < \frac{1}{1+\{h^{-1}(2\pi n)\}^4} < \mu_n < \frac{1}{1+\{h^{-1}(2\pi n-\frac{\pi}{2})\}^4}.$$

(b) $\mu_n \sim \nu_n \sim n^{-4}$, *and*

$$\frac{1}{1+\{h^{-1}(2\pi n-\frac{\pi}{2})\}^4} - \mu_n \sim \nu_n - \frac{1}{1+\{h^{-1}(2\pi n+\frac{\pi}{2})\}^4} \sim n^{-5}e^{-2\pi n},$$

$$\frac{1}{1+\frac{1}{L^4}(2\pi(n-1)-\frac{\pi}{2})^4} - \mu_n \sim \frac{1}{1+\frac{1}{L^4}(2\pi(n-1)+\frac{\pi}{2})^4} - \nu_n \sim n^{-6}.$$

Here, the function h, parametrized by $L = 2l\alpha$, is strictly increasing, one-to-one and onto from $[0,\infty)$ to $[0,\infty)$. See Section 3 for its definition and properties. See also Section 2 for the definition of the notation $\sim$, which denotes ?asymptotically same order?. Thus $1 > \mu_1 > \nu_1 > \mu_2 > \nu_2 > \cdots > \cdots \searrow 0$, and the eigenvalues of $\mathcal{K}_{l,\alpha,k}$ are ordered as

$$\mu_1/k > \nu_1/k > \mu_2/k > \nu_2/k > \cdots \searrow 0.$$

In fact, the asymptotic approximation in Theorem 1(b) gives a quite precise description of the distribution of the eigenvalues of $\mathcal{K}_{l,\alpha,k}$ as $n \to \infty$.

Theorem 1 also gives explicit upper and lower bounds on each of these eigenvalues. Among these eigenvalues, the largest one, μ_1/k, is of special importance, since it is precisely the L^2-norm $\|\mathcal{K}_{l,\alpha,k}\|$ of the operator $\mathcal{K}_{l,\alpha,k}$ by Proposition 1. In consequence, we obtain the following explicit upper and lower bounds on the L^2-norm $\|\mathcal{K}_{l,\alpha,k}\| = \mu_1/k$ of the operator $\mathcal{K}_{l,\alpha,k}$:

$$0 < \frac{1}{k[1+\{h^{-1}(2\pi)\}^4]} < \|\mathcal{K}_{l,\alpha,k}\| < \frac{1}{k[1+\{h^{-1}(\frac{3\pi}{2})\}^4]} < \frac{1}{k}.$$

We can actually compute numerical values of μ_n and ν_n with Newton?s method on the equation (25) in Section 3. See Section 6 for further details.

Each of the quantities μ_n and ν_n changes only when L changes. For example, if L remains fixed, then they do not change even if k changes. In fact, $L = 2l\alpha = 2l\sqrt[4]{k/(EI)}$ is dimensionless and hence can be regarded as the *dimension-free* or *intrinsic* length of the beam. Similarly, the dimensionless quantities μ_n and ν_n can also be regarded as *dimension-free* or *intrinsic* eigenvalues of $\mathcal{K}_{l,\alpha,k}$, which depend only on L. Especially, the dimensionless $\mu_1 = k \cdot \|\mathcal{K}_{l,\alpha,k}\|$ is the *dimension-free* or *intrinsic* L^2-norm of $\mathcal{K}_{l,\alpha,k}$.

We also analyze the behavior of the eigenvalues of $\mathcal{K}_{l,\alpha,k}$ with respect to the intrinsic length L of the beam.

Theorem 2 *Each eigenvalue λ of $\mathcal{K}_{l,\alpha,k}$ in Theorem* 1 *is continuous and strictly increasing with respect to L, and* $\lim_{L\to 0} \lambda = 0$, $\lim_{L\to\infty} \lambda = 1/k$.

Thus each of the intrinsic eigenvalues μ_n and ν_n is continuous and strictly increasing with respect to L, and $\lim_{L\to 0} \mu_n = \lim_{L\to 0} \nu_n = 0$, $\lim_{L\to\infty} \mu_n = \lim_{L\to\infty} \nu_n = 1$ for $n = 1, 2, 3, \ldots$ Table 1, which results from the numerical computation in Section 6, illustrates the dependence of μ_n and ν_n on L in Theorem 2. In particular, the norm $\|\mathcal{K}_{l,\alpha,k}\| =$

Table 1 Numerical values of $\mu_1 = k\|\mathcal{K}_{l,\alpha,k}\|$, ν_1, μ_2, ν_2 corresponding to various $L = 2l\alpha$

L	μ_1	ν_1	μ_2	ν_2
10^{-2}	0.003535504526434	0.000000029355791	0.000000000019880	0.000000000002624
10^{-1}	0.035326704321880	0.000028406573449	0.000000190403618	0.000000025815905
1	0.331681981441542	0.020235634105536	0.001302361278230	0.000221108040807
2	0.578350951060946	0.109509249925520	0.014548864439394	0.003014813082734
3	0.737796746567301	0.249144755528815	0.052681487593071	0.013049474696160
4	0.835237998797342	0.400500295380442	0.119710823211630	0.035118466933057
5	0.894054175695477	0.537478928105431	0.209949500302561	0.072359812095134
6	0.929940126283050	0.649631031236143	0.312512968129316	0.125219441432141
7	0.952321667263849	0.736387662150921	0.416408511420210	0.191399578520264
8	0.966653810417898	0.801474122928057	0.513537323059282	0.266679190778082
9	0.976084258929463	0.849614047989366	0.599392090820732	0.346127057405707
10	0.982453999322008	0.885083551582694	0.672409494807652	0.425184184899229
10^2	0.999995523152271	0.999965988373225	0.999869326766519	0.999643102015955

μ_1/k is continuous and strictly increasing as a function of L, and $\lim_{L\to 0}\|\mathcal{K}_{l,\alpha,k}\| = 0$, $\lim_{L\to\infty}\|\mathcal{K}_{l,\alpha,k}\| = 1/k$.

The rest of the paper is organized as follows. In Section 2, basic preliminaries and notations used in this paper are given. In Section 3, we derive a characteristic equation for the eigenvalues of $\mathcal{K}_{l,\alpha,k}$, and transform it into a relatively manageable form (25). Theorems 1 and 2 are proved in Sections 4 and 5, respectively. In Section 6, examples of numerical computation of the eigenvalues of $\mathcal{K}_{l,\alpha,k}$ are given.

2 Preliminaries

Let $f(t)$, $g(t)$ be positive functions on $[0,\infty)$. We will use the notation $f(t)\sim g(t)$, meaning that $f(t)$ and $g(t)$ are of the same order asymptotically as $t\to\infty$, if there exists $T > 0$ such that $m \le f(t)/g(t) \le M$ for every $t > T$ for some constants $0 < m \le M < \infty$. We also use similar notation for positive sequences. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be positive sequences. Then we denote $a_n \sim b_n$ if there exists $N > 0$ such that $m \le a_n/b_n \le M$ for every $n > N$ for some constants $0 < m \le M < \infty$. Note that $f(t)\sim g(t)$ if $0 < \lim_{t\to\infty} f(t)/g(t) < \infty$, and $a_n \sim b_n$ if $0 < \lim_{n\to\infty} a_n/b_n < \infty$.

For $l > 0$, let $L^2[-l,l]$ be the space of all square-integrable complex functions on the interval $[-l,l]$, which is a Hilbert space with the usual inner product

$$\langle u,v\rangle = \int_{-l}^{l} u(x)\overline{v(x)}\,dx, \quad u,v\in L^2[-l,l].$$

The L^2-norm $\|\mathcal{T}\|_2$, denoted also by $\|\mathcal{T}\|$, of a linear operator $\mathcal{T}$ from $L^2[-l,l]$ to $L^2[-l,l]$, is

$$\|\mathcal{T}\| := \|\mathcal{T}\|_2 = \sup_{0\ne u\in L^2[-l,l]} \frac{\|\mathcal{T}[u]\|}{\|u\|},$$

where $\|u\| := \|u\|_2 = \sqrt{\langle u,u\rangle}$. For $n = 0,1,2,\ldots$, let $C^n[-l,l]$ be the space of all n-times differentiable complex functions on $[-l,l]$. Note that $C^0[-l,l] := C[-l,l]$ is the space of all continuous complex functions on $[-l,l]$.

One of the main tools for our analysis is the following necessary and sufficient condition for being an eigenfunction of $\mathcal{K}_{l,\alpha,k}$.

Proposition 2 (Lemma 2.5 in [13]) *Let $u \in L^2[-l,l]$. Then $\mathcal{K}_{l,\alpha,k}[u] = \lambda u$ for some $\lambda \in \mathbb{C}$, if and only if $u \in C^4[-l,l]$, and u is a solution to the following fourth-order linear boundary value problem*:

$$\lambda u^{(4)} + \left(\lambda - \frac{1}{k}\right)\alpha^4 u = 0, \tag{3}$$

$$u^{(3)}(l) + \sqrt{2}\alpha u''(l) + \alpha^2 u'(l) = 0, \tag{4}$$

$$u^{(3)}(-l) - \sqrt{2}\alpha u''(-l) + \alpha^2 u'(-l) = 0, \tag{5}$$

$$u^{(3)}(l) - \alpha^2 u'(l) - \sqrt{2}\alpha^3 u(l) = 0, \tag{6}$$

$$u^{(3)}(-l) - \alpha^2 u'(-l) + \sqrt{2}\alpha^3 u(-l) = 0. \tag{7}$$

Using Proposition 2, the following property of $\mathcal{K}_{l,\alpha,k}$ was shown in [14].

Proposition 3 (Theorem 1 in [14]) *All the eigenvalues of $\mathcal{K}_{l,\alpha,k}$ are in the real interval* $(0, 1/k)$.

3 Characteristic equation for the eigenvalues of $\mathcal{K}_{l,\alpha,k}$

It is well known [15] that an operator of the type $\mathcal{K}_{l,\alpha,k}$ is self-adjoint. Since the eigenvalues of a self-adjoint operator are real, and the eigenspace corresponding to each eigenvalue is spanned by real eigenfunctions, it is sufficient to consider only real eigenfunctions and eigenvalues.

As noted in [13], the solution space of the differential equation (3) changes qualitatively according to the sign of the quantity $1 - 1/(\lambda k)$, and we have the following three possibilities:

(I) $1 - 1/(\lambda k) = 0$: $\lambda = 1/k$,
(II) $1 - 1/(\lambda k) > 0$: $\lambda < 0$ or $\lambda > 1/k$,
(III) $1 - 1/(\lambda k) < 0$: $0 < \lambda < 1/k$.

It was shown in [13] and [14] that there are no eigenvalues in the cases (I) and (II) (Proposition 3). We will investigate the remaining case (III). So we assume $1 - 1/(\lambda k) < 0$, or equivalently, $0 < \lambda < 1/k$ for the rest of the paper.

We introduce the variable κ defined by

$$\kappa := \sqrt[4]{\frac{1}{\lambda k} - 1} > 0, \tag{8}$$

which simplifies (3) to

$$u^{(4)} - \kappa^4 \alpha^4 u = 0. \tag{9}$$

Note that (8) gives a one-to-one correspondence between κ in $(0, \infty)$ and λ in $(0, 1/k)$ for any fixed $k > 0$.

3.1 Derivation of characteristic equation

Suppose $0 < \lambda < 1/k$ is an eigenvalue of $\mathcal{K}_{l,\alpha,k}$, and u is a nonzero eigenfunction corresponding to λ. By Proposition 2, u should satisfy the differential equation (3), and hence

(9). The general (real) solution of (9) is

$$u(x) = A\mathrm{e}(x) + B\mathrm{e}(-x) + C\mathrm{c}(x) + D\mathrm{s}(x), \quad A, B, C, D \in \mathbb{R},$$

where we denote

$$\mathrm{e}(x) := \exp(\kappa\alpha x), \qquad \mathrm{c}(x) := \cos(\kappa\alpha x), \qquad \mathrm{s}(x) := \sin(\kappa\alpha x).$$

So we have

$$\begin{aligned}
u'(x) &= \kappa\alpha\bigl\{A\mathrm{e}(x) - B\mathrm{e}(-x) - C\mathrm{s}(x) + D\mathrm{c}(x)\bigr\},\\
u''(x) &= (\kappa\alpha)^2\bigl\{A\mathrm{e}(x) + B\mathrm{e}(-x) - C\mathrm{c}(x) - D\mathrm{s}(x)\bigr\},\\
u^{(3)}(x) &= (\kappa\alpha)^3\bigl\{A\mathrm{e}(x) - B\mathrm{e}(-x) + C\mathrm{s}(x) - D\mathrm{c}(x)\bigr\},
\end{aligned}$$

and hence

$$\begin{aligned}
&u^{(3)}(x) \pm \sqrt{2}\alpha u''(x) + \alpha^2 u'(x)\\
&\quad = \kappa\alpha^3\bigl[\bigl(\kappa^2 \pm \sqrt{2}\kappa + 1\bigr)\mathrm{e}(x)\cdot A - \bigl(\kappa^2 \mp \sqrt{2}\kappa + 1\bigr)\mathrm{e}(-x)\cdot B\\
&\qquad + \bigl\{\mp\sqrt{2}\kappa\,\mathrm{c}(x) + \bigl(\kappa^2 - 1\bigr)\mathrm{s}(x)\bigr\}\cdot C\\
&\qquad - \bigl\{\bigl(\kappa^2 - 1\bigr)\mathrm{c}(x) \pm \sqrt{2}\kappa\,\mathrm{s}(x)\bigr\}\cdot D\bigr], \qquad (10)\\
&u^{(3)}(x) - \alpha^2 u'(x) \mp \sqrt{2}\alpha^3 u(x)\\
&\quad = \alpha^3\bigl[\bigl(\kappa^3 - \kappa \mp \sqrt{2}\bigr)\mathrm{e}(x)\cdot A - \bigl(\kappa^3 - \kappa \pm \sqrt{2}\bigr)\mathrm{e}(-x)\cdot B\\
&\qquad + \bigl\{\mp\sqrt{2}\mathrm{c}(x) + \bigl(\kappa^3 + \kappa\bigr)\mathrm{s}(x)\bigr\}\cdot C\\
&\qquad - \bigl\{\bigl(\kappa^3 + \kappa\bigr)\mathrm{c}(x) \pm \sqrt{2}\mathrm{s}(x)\bigr\}\cdot D\bigr]. \qquad (11)
\end{aligned}$$

Using (10) and (11), the boundary conditions (4), (5), (6), (7) in Proposition 2, respectively, become

$$\begin{aligned}
0 &= \bigl(\kappa^2 + \sqrt{2}\kappa + 1\bigr)\mathrm{e}(l)\cdot A - \bigl(\kappa^2 - \sqrt{2}\kappa + 1\bigr)\mathrm{e}(-l)\cdot B\\
&\quad + \bigl\{-\sqrt{2}\kappa\,\mathrm{c}(l) + \bigl(\kappa^2 - 1\bigr)\mathrm{s}(l)\bigr\}\cdot C + \bigl\{-\bigl(\kappa^2 - 1\bigr)\mathrm{c}(l) - \sqrt{2}\kappa\,\mathrm{s}(l)\bigr\}\cdot D,\\
0 &= \bigl(\kappa^2 - \sqrt{2}\kappa + 1\bigr)\mathrm{e}(-l)\cdot A - \bigl(\kappa^2 + \sqrt{2}\kappa + 1\bigr)\mathrm{e}(l)\cdot B\\
&\quad + \bigl\{\sqrt{2}\kappa\,\mathrm{c}(l) - \bigl(\kappa^2 - 1\bigr)\mathrm{s}(l)\bigr\}\cdot C + \bigl\{-\bigl(\kappa^2 - 1\bigr)\mathrm{c}(l) - \sqrt{2}\kappa\,\mathrm{s}(l)\bigr\}\cdot D,\\
0 &= \bigl(\kappa^3 - \kappa - \sqrt{2}\bigr)\mathrm{e}(l)\cdot A - \bigl(\kappa^3 - \kappa + \sqrt{2}\bigr)\mathrm{e}(-l)\cdot B\\
&\quad + \bigl\{-\sqrt{2}\mathrm{c}(l) + \bigl(\kappa^3 + \kappa\bigr)\mathrm{s}(l)\bigr\}\cdot C + \bigl\{-\bigl(\kappa^3 + \kappa\bigr)\mathrm{c}(l) - \sqrt{2}\mathrm{s}(l)\bigr\}\cdot D,\\
0 &= \bigl(\kappa^3 - \kappa + \sqrt{2}\bigr)\mathrm{e}(-l)\cdot A - \bigl(\kappa^3 - \kappa - \sqrt{2}\bigr)\mathrm{e}(l)\cdot B\\
&\quad + \bigl\{\sqrt{2}\mathrm{c}(l) - \bigl(\kappa^3 + \kappa\bigr)\mathrm{s}(l)\bigr\}\cdot C + \bigl\{-\bigl(\kappa^3 + \kappa\bigr)\mathrm{c}(l) - \sqrt{2}\mathrm{s}(l)\bigr\}\cdot D,
\end{aligned}$$

which are equivalent collectively to

$$\mathbf{Q}\cdot(A \quad B \quad C \quad D)^T = \mathbf{O}, \qquad (12)$$

where **O** is the 4 × 1 zero matrix and **Q** is the following 4 × 4 matrix:

$$\mathbf{Q} = \begin{pmatrix} (\kappa^2+\sqrt{2}\kappa+1)\mathrm{e}(l) & -(\kappa^2-\sqrt{2}\kappa+1)\mathrm{e}(-l) \\ (\kappa^2-\sqrt{2}\kappa+1)\mathrm{e}(-l) & -(\kappa^2+\sqrt{2}\kappa+1)\mathrm{e}(l) \\ (\kappa^3-\kappa-\sqrt{2})\mathrm{e}(l) & -(\kappa^3-\kappa+\sqrt{2})\mathrm{e}(-l) \\ (\kappa^3-\kappa+\sqrt{2})\mathrm{e}(-l) & -(\kappa^3-\kappa-\sqrt{2})\mathrm{e}(l) \end{pmatrix.$$
$$\begin{matrix} -\sqrt{2}\kappa\,\mathrm{c}(l)+(\kappa^2-1)\mathrm{s}(l) & -(\kappa^2-1)\mathrm{c}(l)-\sqrt{2}\kappa\,\mathrm{s}(l) \\ \sqrt{2}\kappa\,\mathrm{c}(l)-(\kappa^2-1)\mathrm{s}(l) & -(\kappa^2-1)\mathrm{c}(l)-\sqrt{2}\kappa\,\mathrm{s}(l) \\ -\sqrt{2}\mathrm{c}(l)+(\kappa^3+\kappa)\mathrm{s}(l) & -(\kappa^3+\kappa)\mathrm{c}(l)-\sqrt{2}\mathrm{s}(l) \\ \sqrt{2}\mathrm{c}(l)-(\kappa^3+\kappa)\mathrm{s}(l) & -(\kappa^3+\kappa)\mathrm{c}(l)-\sqrt{2}\mathrm{s}(l) \end{pmatrix}.$$

By Proposition 2, the assumption that u is a nonzero eigenfunction of $\mathcal{K}_{l,\alpha,k}$ is equivalent to the existence of nontrivial $(A\ B\ C\ D)$ satisfying (12), which again is equivalent to $\det\mathbf{Q} = 0$. Thus λ is an eigenvalue of $\mathcal{K}_{l,\alpha,k}$, if and only if $\det\mathbf{Q} = 0$.

A long and tedious computation, which can be facilitated by utilizing Computer Algebra Systems, produces the following determinant of **Q**:

$$\begin{aligned} \det\mathbf{Q} = 4e^{L\kappa}\Big[&-2e^{-L\kappa}\left(\kappa^4+1\right)^2 \\ &+\left\{\left(\kappa^4-4\kappa^2+1\right)\cos(L\kappa)+2\sqrt{2}\kappa\left(\kappa^2-1\right)\sin(L\kappa)\right\} \\ &\cdot\left\{e^{-2L\kappa}\left(\kappa^4-2\sqrt{2}\kappa^3+4\kappa^2-2\sqrt{2}\kappa+1\right)\right. \\ &\left.+\left(\kappa^4+2\sqrt{2}\kappa^3+4\kappa^2+2\sqrt{2}\kappa+1\right)\right\}\Big], \end{aligned} \tag{13}$$

where $L = 2l\alpha$ is the *intrinsic* length of the beam. For checking the validity of (13), we provide a Mathematica notebook file. See Additional files 1 and 2.

3.2 Simplification of det Q

Since $(\kappa^4-4\kappa^2+1)^2 + \{2\sqrt{2}\kappa(\kappa^2-1)\}^2 = (\kappa^4+1)^2$, we have

$$\begin{aligned} &\left(\kappa^4-4\kappa^2+1\right)\cos(L\kappa)+2\sqrt{2}\kappa\left(\kappa^2-1\right)\sin(L\kappa) \\ &= \left(\kappa^4+1\right)\left\{\frac{\kappa^4-4\kappa^2+1}{\kappa^4+1}\cos(L\kappa)+\frac{2\sqrt{2}\kappa(\kappa^2-1)}{\kappa^4+1}\sin(L\kappa)\right\} \\ &= \left(\kappa^4+1\right)\left\{\cos\hat{h}(\kappa)\cos(L\kappa)+\sin\hat{h}(\kappa)\sin(L\kappa)\right\} \\ &= \left(\kappa^4+1\right)\cos\left(L\kappa-\hat{h}(\kappa)\right) \end{aligned} \tag{14}$$

for some function $\hat{h}(\kappa)$ of κ. Specifically, we define $\hat{h}$ by

$$\hat{h}(\kappa) := \begin{cases} \arctan\{\frac{2\sqrt{2}\kappa(\kappa^2-1)}{\kappa^4-4\kappa^2+1}\} & \text{if } 0 \le \kappa < \frac{\sqrt{3}-1}{\sqrt{2}}, \\ -\frac{\pi}{2} & \text{if } \kappa = \frac{\sqrt{3}-1}{\sqrt{2}}, \\ -\pi + \arctan\{\frac{2\sqrt{2}\kappa(\kappa^2-1)}{\kappa^4-4\kappa^2+1}\} & \text{if } \frac{\sqrt{3}-1}{\sqrt{2}} < \kappa < \frac{\sqrt{3}+1}{\sqrt{2}}, \\ -\frac{3\pi}{2} & \text{if } \kappa = \frac{\sqrt{3}+1}{\sqrt{2}}, \\ -2\pi + \arctan\{\frac{2\sqrt{2}\kappa(\kappa^2-1)}{\kappa^4-4\kappa^2+1}\} & \text{if } \kappa > \frac{\sqrt{3}+1}{\sqrt{2}}, \end{cases} \tag{15}$$

where the branch of arctan is taken such that arctan(0) = 0. Note that

$$\begin{aligned}\kappa^4 - 4\kappa^2 + 1 &= \{\kappa^2 - (2-\sqrt{3})\}\{\kappa^2 - (2+\sqrt{3})\} \\ &= \left(\kappa + \frac{\sqrt{3}-1}{\sqrt{2}}\right)\left(\kappa - \frac{\sqrt{3}-1}{\sqrt{2}}\right)\left(\kappa + \frac{\sqrt{3}+1}{\sqrt{2}}\right)\left(\kappa - \frac{\sqrt{3}+1}{\sqrt{2}}\right),\end{aligned}$$

and hence

$$\frac{2\sqrt{2}\kappa(\kappa^2-1)}{\kappa^4-4\kappa^2+1} = \frac{2\sqrt{2}(\kappa+1)}{(\kappa+\frac{\sqrt{3}-1}{\sqrt{2}})(\kappa+\frac{\sqrt{3}+1}{\sqrt{2}})} \cdot \frac{\kappa(\kappa-1)}{(\kappa-\frac{\sqrt{3}-1}{\sqrt{2}})(\kappa-\frac{\sqrt{3}+1}{\sqrt{2}})}.$$

So it is easy to see that $\hat{h}$ thus defined is continuous. See Figure 1 for the graph of $\hat{h}(\kappa)$.

Note that

$$\begin{aligned}\hat{h}'(\kappa) &= \frac{1}{1+(\frac{2\sqrt{2}\kappa(\kappa^2-1)}{\kappa^4-4\kappa^2+1})^2} \cdot \left(\frac{2\sqrt{2}\kappa(\kappa^2-1)}{\kappa^4-4\kappa^2+1}\right)' \\ &= -\frac{(\kappa^4-4\kappa^2+1)^2}{(\kappa^4+1)^2} \cdot \frac{2\sqrt{2}(\kappa^4+1)(\kappa^2+1)}{(\kappa^4-4\kappa^2+1)^2} \\ &= -\frac{2\sqrt{2}(\kappa^2+1)}{\kappa^4+1} < 0. \end{aligned} \tag{16}$$

This shows that $\hat{h}$ is in fact real-analytic and strictly decreasing. We also have $\hat{h}(0) = 0$ and $\lim_{\kappa\to\infty} \hat{h}(\kappa) = -2\pi$ from (15).

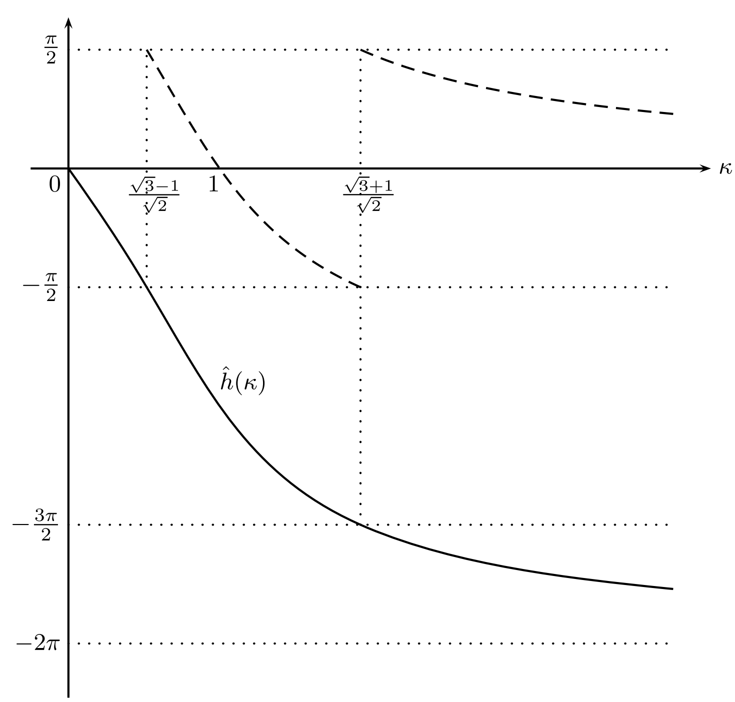

Figure 1 Graph of $\hat{h}(\kappa)$. The solid curve represents $\hat{h}(\kappa)$ which decreases on $[0,\infty)$ approaching -2π. The dashed curves represent the function $\arctan\{\frac{2\sqrt{2}\kappa(\kappa^2-1)}{\kappa^4-4\kappa^2+1}\}$.

Define

$$h(\kappa) := L\kappa - \hat{h}(\kappa). \tag{17}$$

Then (14) becomes

$$(\kappa^4 - 4\kappa^2 + 1)\cos(L\kappa) + 2\sqrt{2}\kappa(\kappa^2 - 1)\sin(L\kappa) = (\kappa^4 + 1)\cos h(\kappa). \tag{18}$$

By (16) and (17), we have

$$h'(\kappa) = L + \frac{2\sqrt{2}(\kappa^2 + 1)}{(\kappa^4 + 1)} > 0. \tag{19}$$

The properties of the function $h(\kappa)$, which we will need later, are summarized in Lemma 1.

Lemma 1

(a) *$h(\kappa)$ is real-analytic, and is strictly increasing with $h(0) = 0$, $\lim_{\kappa\to\infty} h(\kappa) = \infty$.*

(b) *$h'(\kappa)$ is strictly increasing on $[0, \sqrt{\sqrt{2}-1}]$ from $h'(0) = L + 2\sqrt{2}$ to $h'(\sqrt{\sqrt{2}-1}) = L + 2 + \sqrt{2}$, and strictly decreasing on $[\sqrt{\sqrt{2}-1}, \infty)$ approaching $\lim_{\kappa\to\infty} h'(\kappa) = L$. In particular, $L < h'(\kappa) \leq L + 2 + \sqrt{2}$ for every $\kappa \geq 0$, and hence $\lim_{\kappa\to\infty} h(\kappa)/\kappa = L$ implying $h(\kappa) \sim \kappa$.*

Proof (a) follows immediately from (15), (17), (19). Since

$$\begin{aligned} h''(\kappa) &= \left\{\frac{2\sqrt{2}(\kappa^2+1)}{(\kappa^4+1)}\right\}' = -\frac{4\sqrt{2}\kappa(\kappa^4 + 2\kappa^2 - 1)}{(\kappa^4+1)^2} \\ &= -\frac{4\sqrt{2}(\kappa^2 + (\sqrt{2}+1))(\kappa + \sqrt{\sqrt{2}-1})}{(\kappa^4+1)^2} \cdot \kappa(\kappa - \sqrt{\sqrt{2}-1}), \end{aligned}$$

h' is strictly increasing on $[0, \sqrt{\sqrt{2}-1}]$ from $h'(0) = L + 2\sqrt{2}$ to $h'(\sqrt{\sqrt{2}-1}) = L + 2 + \sqrt{2}$, and is strictly decreasing on $[\sqrt{\sqrt{2}-1}, \infty)$ to $\lim_{\kappa\to\infty} h'(\kappa) = L$. Hence, (b) follows. □

Using (18), the determinant of $\mathbf{Q}$ in (13) can be rewritten as

$$\begin{aligned} \det \mathbf{Q} &= 4e^{L\kappa}\bigl[-2e^{-L\kappa}(\kappa^4+1)^2 + (\kappa^4+1)\cos h(\kappa) \\ &\quad \cdot \bigl\{e^{-2L\kappa}(\kappa^4 - 2\sqrt{2}\kappa^3 + 4\kappa^2 - 2\sqrt{2}\kappa + 1) \\ &\quad + (\kappa^4 + 2\sqrt{2}\kappa^3 + 4\kappa^2 + 2\sqrt{2}\kappa + 1)\bigr\}\bigr] \\ &= 4(\kappa^4+1)e^{L\kappa}\bigl[-2(\kappa^4+1)\cdot e^{-L\kappa} + (\kappa^2 - \sqrt{2}\kappa + 1)^2\cos h(\kappa)\cdot(e^{-L\kappa})^2 \\ &\quad + (\kappa^2 + \sqrt{2}\kappa + 1)^2\cos h(\kappa)\bigr], \end{aligned} \tag{20}$$

since $(\kappa^2 \pm \sqrt{2}\kappa + 1)^2 = \kappa^4 \pm 2\sqrt{2}\kappa^3 + 4\kappa^2 \pm 2\sqrt{2}\kappa + 1$. It follows from (20) that the equation $\det \mathbf{Q} = 0$, regarding it as a quadratic equation in $e^{-L\kappa}$, is equivalent to

$$\begin{aligned} e^{-L\kappa} &= \frac{1}{(\kappa^2 - \sqrt{2}\kappa + 1)^2 \cdot \cos h(\kappa)} \\ &\quad \cdot \Bigl[(\kappa^4+1) \pm \sqrt{(\kappa^4+1)^2 - (\kappa^2 + \sqrt{2}\kappa + 1)^2(\kappa^2 - \sqrt{2}\kappa + 1)^2\cos^2 h(\kappa)}\Bigr], \end{aligned}$$

which, using the identity

$$\left(\kappa^2+\sqrt{2}\kappa+1\right)\left(\kappa^2-\sqrt{2}\kappa+1\right)=\kappa^4+1, \tag{21}$$

is again equivalent to

$$\frac{\kappa^2-\sqrt{2}\kappa+1}{\kappa^2+\sqrt{2}\kappa+1}=e^{L\kappa}\cdot\frac{1\pm\sin h(\kappa)}{\cos h(\kappa)}. \tag{22}$$

Note from (20) that $\det\mathbf{Q}\neq 0$, when $\cos(h(\kappa))=0$.

Define

$$p(\kappa):=\frac{\kappa^2-\sqrt{2}\kappa+1}{\kappa^2+\sqrt{2}\kappa+1} \tag{23}$$

and

$$\begin{aligned}\varphi_+(\kappa)&:=e^{L\kappa}\cdot\frac{1+\sin h(\kappa)}{\cos h(\kappa)},\\ \varphi_-(\kappa)&:=e^{L\kappa}\cdot\frac{1-\sin h(\kappa)}{\cos h(\kappa)}.\end{aligned} \tag{24}$$

We also use the notation

$$\varphi_\pm(\kappa):=e^{L\kappa}\cdot\frac{1\pm\sin h(\kappa)}{\cos h(\kappa)}.$$

Then (22), and hence the characteristic equation $\det\mathbf{Q}=0$ for $\kappa>0$, is finally reduced to the following equivalent form:

$$p(\kappa)=\varphi_\pm(\kappa)\quad\text{for }\kappa>0, \tag{25}$$

which means $p(\kappa)=\varphi_+(\kappa)$ or $p(\kappa)=\varphi_-(\kappa)$ for $\kappa>0$.

3.3 Properties of the functions $p(\kappa)$ and $\varphi_\pm(\kappa)$

Note from (23) that

$$\begin{aligned}p'(\kappa)&=\frac{(2\kappa-\sqrt{2})(\kappa^2+\sqrt{2}\kappa+1)-(2\kappa+\sqrt{2})(\kappa^2-\sqrt{2}\kappa+1)}{(\kappa^2+\sqrt{2}\kappa+1)^2}\\ &=\frac{2\sqrt{2}(\kappa^2-1)}{(\kappa^2+\sqrt{2}\kappa+1)^2}=\frac{2\sqrt{2}(\kappa+1)}{(\kappa^2+\sqrt{2}\kappa+1)^2}\cdot(\kappa-1).\end{aligned} \tag{26}$$

The following lemma on the property of the function $p(\kappa)$ immediately follows from (23) and (26). See Figure 2 for the graph of $p(\kappa)$.

Lemma 2 *$p(\kappa)$ is strictly decreasing on $[0,1]$ from $p(0)=1$ to $p(1)=3-2\sqrt{2}$, and is strictly increasing on $[1,\infty)$ approaching $\lim_{\kappa\to\infty}p(\kappa)=1$. In particular, we have $0<3-2\sqrt{2}<p(\kappa)<1$ for every $\kappa>0$.*

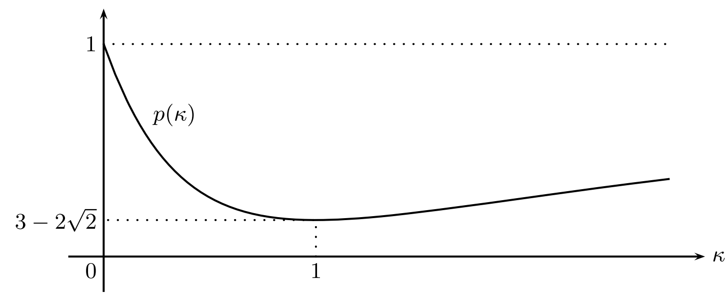

Figure 2 Graph of $p(\kappa)$. $p(\kappa)$ decreases on $[0,1]$ from $p(0)=1$ to $p(1)=3-2\sqrt{2}\approx 0.17157$, and increases on $[1,\infty)$ approaching 1.

By Lemma 1(a), the inverse h^{-1} of the function h is well defined from $[0,\infty)$ onto $[0,\infty)$, and is also strictly increasing. From the definition (24) of $\varphi_\pm$, we have

$$\begin{aligned}\varphi_\pm\bigl(h^{-1}(2\pi n)\bigr) &= e^{L\cdot h^{-1}(2\pi n)}\cdot\frac{1\pm\sin(2\pi n)}{\cos(2\pi n)} = \exp\bigl(L\cdot h^{-1}(2\pi n)\bigr)>1,\\ \varphi_\pm\bigl(h^{-1}(2\pi n+\pi)\bigr) &= e^{L\cdot h^{-1}(2\pi n+\pi)}\cdot\frac{1\pm\sin(2\pi n+\pi)}{\cos(2\pi n+\pi)}\\ &= -\exp\bigl(L\cdot h^{-1}(2\pi n+\pi)\bigr)\end{aligned}\tag{27}$$

and

$$\begin{aligned}&\lim_{\kappa\to h^{-1}(2\pi n+\pi/2)-}\varphi_+(\kappa)=\infty, &&\lim_{\kappa\to h^{-1}(2\pi n+\pi/2)+}\varphi_+(\kappa)=-\infty,\\ &\lim_{\kappa\to h^{-1}(2\pi n-\pi/2)-}\varphi_-(\kappa)=-\infty, &&\lim_{\kappa\to h^{-1}(2\pi n-\pi/2)+}\varphi_-(\kappa)=\infty\end{aligned}$$

for every $n=0,\pm1,\pm2,\ldots$ Note that

$$\begin{aligned}\varphi_\pm(\kappa) &= e^{L\kappa}\frac{1\pm\sin h(\kappa)}{\cos h(\kappa)} = e^{L\kappa}\frac{(1\pm\sin h(\kappa))\cos h(\kappa)}{\cos^2 h(\kappa)}\\ &= e^{L\kappa}\frac{(1\pm\sin h(\kappa))\cos h(\kappa)}{1-\sin^2 h(\kappa)} = e^{L\kappa}\frac{\cos h(\kappa)}{1\mp\sin h(\kappa)}.\end{aligned}$$

So φ_+ (respectively, φ_-) has removable singularities at $h^{-1}(2\pi n-\pi/2)$ (respectively, $h^{-1}(2\pi n+\pi/2)$) for $n=0,\pm1,\pm2,\ldots$ We regard these singularities all to be removed in the definition of $\varphi_\pm$, so that

$$\varphi_\pm\left(h^{-1}\left(2\pi n\mp\frac{\pi}{2}\right)\right):=0\tag{28}$$

for $n=0,\pm1,\pm2,\ldots$ Thus φ_+ and φ_- are continuous, respectively, on the intervals $(h^{-1}(2\pi n+\pi/2),h^{-1}(2\pi(n+1)+\pi/2))$ and $(h^{-1}(2\pi n-\pi/2),h^{-1}(2\pi(n+1)-\pi/2))$ for every $n=0,\pm1,\pm2,\ldots$ In fact, φ_+ and φ_- are real-analytic in these respective intervals, since $h(\kappa)$ is real-analytic by Lemma 1(a). Since

$$\frac{d}{dt}\left(\frac{1\pm\sin t}{\cos t}\right)=\frac{\pm\cos t\cdot\cos t-(1\pm\sin t)\cdot(-\sin t)}{\cos^2 t}=\pm\frac{1\pm\sin t}{\cos^2 t},\tag{29}$$

we have

$$\begin{aligned}\varphi'_{\pm}(\kappa) &= \frac{d}{d\kappa}\left(e^{L\kappa}\frac{1 \pm \sin h(\kappa)}{\cos h(\kappa)}\right) \\ &= e^{L\kappa}\left\{L \cdot \frac{1 \pm \sin h(\kappa)}{\cos h(\kappa)} \pm \frac{1 \pm \sin h(\kappa)}{\cos^2 h(\kappa)} \cdot h'(\kappa)\right\},\end{aligned} \tag{30}$$

hence, by (19),

$$\begin{aligned}&\varphi'_{\pm}(\kappa) \\ &\quad = e^{L\kappa}\left\{\frac{L(1 \pm \sin h(\kappa))}{\cos h(\kappa)} \pm \frac{1 \pm \sin h(\kappa)}{\cos^2 h(\kappa)} \cdot \left(L + \frac{2\sqrt{2}(\kappa^2+1)}{\kappa^4+1}\right)\right\} \\ &\quad = \pm\frac{e^{L\kappa}(1 \pm \sin h(\kappa))}{(\kappa^4+1)\cos^2 h(\kappa)}\left\{L\left(\kappa^4+1\right)\left(1 \pm \cos h(\kappa)\right) + 2\sqrt{2}\left(\kappa^2+1\right)\right\} \\ &\quad = \pm\frac{e^{L\kappa}}{(\kappa^4+1)(1 \mp \sin h(\kappa))}\left\{L\left(\kappa^4+1\right)\left(1 \pm \cos h(\kappa)\right) + 2\sqrt{2}\left(\kappa^2+1\right)\right\}.\end{aligned} \tag{31}$$

Here we used the fact that

$$\frac{1 \pm \sin t}{\cos^2 t} = \frac{1 \pm \sin t}{(1+\sin t)(1-\sin t)} = \frac{1}{1 \mp \sin t}.$$

Since $1 \pm \sin t$ and $1 \pm \cos t$ are positive except at discrete points, (31) shows that φ_+ is strictly increasing and φ_- is strictly decreasing on the intervals where they are defined.

We summarize properties of $\varphi_\pm$ in Lemma 3. See Figure 3 for the graphs of $\varphi_\pm$.

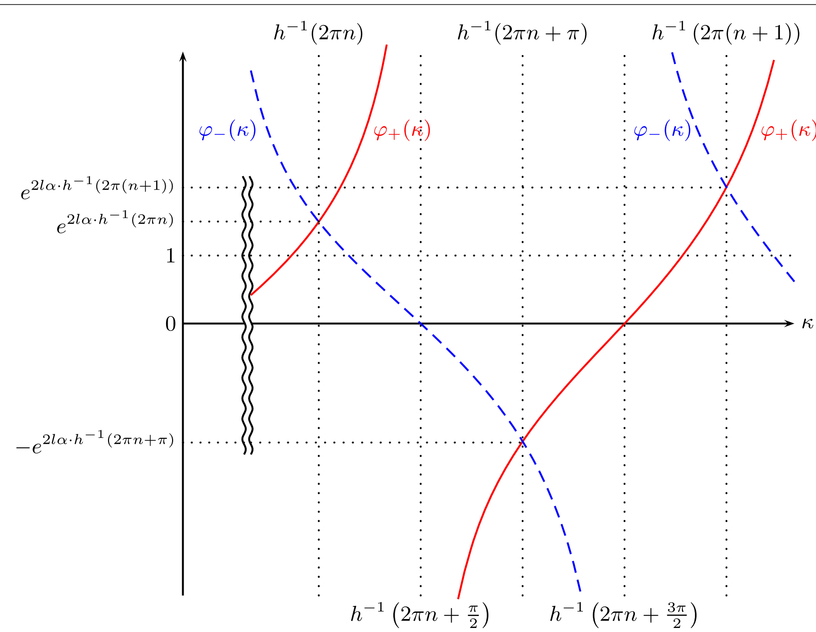

Figure 3 Graphs of $\varphi_+(\kappa)$ and $\varphi_-(\kappa)$. Solid red lines (——) represent $\varphi_+(\kappa)$, and dashed blue lines (- - -) represent $\varphi_-(\kappa)$. φ_+ increases on $(h^{-1}(2\pi n+\pi/2), h^{-1}(2\pi(n+1)+\pi/2))$ from $-\infty$ to ∞, and φ_- decreases on $(h^{-1}(2\pi n-\pi/2), h^{-1}(2\pi(n+1)-\pi/2))$ from ∞ to $-\infty$. $\varphi_\pm(h^{-1}(2\pi n)) = \exp\{L\cdot h^{-1}(2\pi n)\}$, $\varphi_\pm(h^{-1}(2\pi n+\pi)) = -\exp\{L\cdot h^{-1}(2\pi n+\pi)\}$, $\varphi_\pm(h^{-1}(2\pi n \mp \pi/2)) = 0$.

Lemma 3

(a) *For every* $n = 0, \pm 1, \pm 2, \ldots$, $\varphi_+(\kappa)$ *is strictly increasing on the interval* $(h^{-1}(2\pi n + \pi/2), h^{-1}(2\pi(n+1) + \pi/2))$ *from* $-\infty$ *to* ∞, *and* $\varphi_-(\kappa)$ *is strictly decreasing on the interval* $(h^{-1}(2\pi n - \pi/2), h^{-1}(2\pi(n+1) - \pi/2))$ *from* ∞ *to* $-\infty$. $\varphi_\pm(\kappa)$, *where defined, are real-analytic.*

(b) *Suppose* $\kappa > 0$. *If* $0 < \varphi_+(\kappa) < 1$, *then* $h^{-1}(2\pi n - \pi/2) < \kappa < h^{-1}(2\pi n)$ *for* $n = 1, 2, 3, \ldots$ *If* $0 < \varphi_-(\kappa) < 1$, *then* $h^{-1}(2\pi n) < \kappa < h^{-1}(2\pi n + \pi/2)$ *for* $n = 0, 1, 2, \ldots$

The next result on the relationship between p and $\varphi_\pm$, will play a crucial role in analyzing the characteristic equation (25). Note that, by Lemma 2, (25) would hold only when $0 < \varphi_\pm(\kappa) < 1$.

Lemma 4

(a) $\varphi_+'(\kappa) > p'(\kappa)$ *for every* $\kappa > 0$ *such that* $p(\kappa) \le \varphi_+(\kappa) < 1$.

(b) $\varphi_-'(\kappa) < p'(\kappa)$ *for every* $\kappa > 0$ *such that* $p(\kappa) \le \varphi_-(\kappa) < 1$.

Proof By (30), we have

$$\begin{aligned}\varphi_\pm'(\kappa) &= e^{L\kappa}\frac{1 \pm \sin h(\kappa)}{\cos h(\kappa)}\bigl\{L \pm h'(\kappa)\sec h(\kappa)\bigr\} \\ &= \varphi_\pm(\kappa)\bigl\{L \pm h'(\kappa)\sec h(\kappa)\bigr\}.\end{aligned} \tag{32}$$

Suppose $\kappa > 0$. Since $p(\kappa) > 0$ by Lemma 2, both of the conditions $p(\kappa) \le \varphi_+(\kappa) < 1$ and $p(\kappa) \le \varphi_-(\kappa) < 1$ imply $0 < \cos h(\kappa) < 1$, and hence $\sec h(\kappa) > 1$ by Lemma 3(b). (See also Figure 3.) Note also that $h'(\kappa) > L > 0$ by Lemma 1(b).

Suppose $p(\kappa) \le \varphi_+(\kappa) < 1$. Then $\varphi_+(\kappa) > 0$, $\sec h(\kappa) > 1$. Hence from (32), we have

$$\varphi_+'(\kappa) > \varphi_+(\kappa)\bigl\{L + h'(\kappa)\cdot 1\bigr\} = \varphi_+(\kappa)\bigl\{h'(\kappa) - L\bigr\} \ge p(\kappa)\bigl\{h'(\kappa) - L\bigr\},$$

where we used the assumption $\varphi_+(\kappa) \ge p(\kappa)$ for the last inequality. So (a) will follow if we show $p(\kappa)\{h'(\kappa) - L\} > p'(\kappa)$, which, by (19), (23), (26), is equivalent to

$$\frac{\kappa^2 - \sqrt{2}\kappa + 1}{\kappa^2 + \sqrt{2}\kappa + 1}\cdot\frac{2\sqrt{2}(\kappa^2+1)}{\kappa^4+1} > \frac{2\sqrt{2}(\kappa^2-1)}{(\kappa^2+\sqrt{2}\kappa+1)^2}. \tag{33}$$

Using (21), (33) is reduced to $\kappa^2 + 1 > \kappa^2 - 1$, which is true. Thus (33) is true, and this show (a).

Suppose $p(\kappa) \le \varphi_-(\kappa) < 1$. Then $\varphi_-(\kappa) > 0$, $\sec h(\kappa) > 1$. From (32), we have

$$\varphi_-'(\kappa) < \varphi_-(\kappa)\bigl\{L - h'(\kappa)\cdot 1\bigr\} = -\varphi_-(\kappa)\bigl\{h'(\kappa) - L\bigr\} \le -p(\kappa)\bigl\{h'(\kappa) - L\bigr\},$$

where we used the assumption $\varphi_-(\kappa) \ge p(\kappa)$ for the last inequality. So (b) will follow if we show $-p(\kappa)\{h'(\kappa) - L\} < p'(\kappa)$, which, by (19), (23), (26), is equivalent to

$$\frac{\kappa^2 - \sqrt{2}\kappa + 1}{\kappa^2 + \sqrt{2}\kappa + 1}\cdot\frac{2\sqrt{2}(\kappa^2+1)}{\kappa^4+1} > -\frac{2\sqrt{2}(\kappa^2-1)}{(\kappa^2+\sqrt{2}\kappa+1)^2}. \tag{34}$$

Using (21) again, (34) is reduced to $\kappa^2 + 1 > -\kappa^2 + 1$, which is true since $\kappa > 0$. Thus (34) is true, and this show (b). □

4 The eigenstructure of $\mathcal{K}_{l,\alpha,k}$: proof of Theorem 1

We now analyze the eigenstructure of the operator $\mathcal{K}_{l,\alpha,k}$ by proving Theorem 1. It is precisely the solution structure of the equation $\det \mathbf{Q} = 0$ in λ, which is equivalent to that of (25) in λ. Remember that we only need to consider the case when $0 < \lambda < 1/k$, which is equivalent to $\kappa > 0$ by (8).

By Lemma 2, (25) has a solution only when $0 < \varphi_+(\kappa) < 1$ or $0 < \varphi_-(\kappa) < 1$. By (27), (28), and Lemma 3(a), the set of $\kappa > 0$ satisfying $0 < \varphi_+(\kappa) < 1$ is contained in the union of the intervals

$$A_n^+ := \left(h^{-1}\left(2\pi n - \frac{\pi}{2}\right), h^{-1}(2\pi n)\right), \quad n = 1, 2, 3, \ldots$$

Similarly, the set of $\kappa > 0$ satisfying $0 < \varphi_-(\kappa) < 1$ is contained in the union of the intervals

$$A_n^- := \left(h^{-1}(2\pi n), h^{-1}\left(2\pi n + \frac{\pi}{2}\right)\right), \quad n = 0, 1, 2, \ldots$$

In fact, by the intermediate value theorem, there exists at least one κ in each A_n^+, for $n = 1, 2, 3, \ldots$, satisfying $p(\kappa) = \varphi_+(\kappa)$, since

$$\begin{aligned} &p\left(h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right) > 0 = \varphi_+\left(h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right), \\ &p\left(h^{-1}(2\pi n)\right) < 1 < \varphi_+\left(h^{-1}(2\pi n)\right) \end{aligned} \tag{35}$$

for $n = 1, 2, 3, \ldots$, by Lemma 2 and (27), (28). Similarly, there exists at least one κ in each A_n^-, for $n = 1, 2, 3, \ldots$, satisfying $p(\kappa) = \varphi_-(\kappa)$, since

$$\begin{aligned} &p\left(h^{-1}(2\pi n)\right) < 1 < \varphi_-\left(h^{-1}(2\pi n)\right), \\ &p\left(h^{-1}\left(2\pi n + \frac{\pi}{2}\right)\right) > 0 = \varphi_-\left(h^{-1}\left(2\pi n + \frac{\pi}{2}\right)\right) \end{aligned} \tag{36}$$

for $n = 1, 2, 3, \ldots$ Note that we cannot apply the intermediate value theorem to A_0^-, since $p(0) = 1 = \varphi_-(0)$. In fact, it will be shown in Lemma 5 that A_0^- contains no κ satisfying $p(\kappa) = \varphi_-(\kappa)$.

Since the functions $p(\kappa)$ and $\varphi_\pm(\kappa)$ are real-analytic (and different), the set of κ satisfying (25) is discrete. Thus we can take the smallest β_n in A_n^+ satisfying $p(\kappa) = \varphi_+(\kappa)$, and the largest γ_n in A_n^- satisfying $p(\kappa) = \varphi_-(\kappa)$ for $n = 1, 2, 3, \ldots$ Then we have

$$h^{-1}\left(2n\pi - \frac{\pi}{2}\right) < \beta_n < h^{-1}(2n\pi) < \gamma_n < h^{-1}\left(2n\pi + \frac{\pi}{2}\right), \quad n = 1, 2, 3, \ldots \tag{37}$$

Lemma 5 *The set of κ satisfying the characteristic equation* (25) *is*

$$\{\beta_n \mid n = 1, 2, 3, \ldots\} \cup \{\gamma_n \mid n = 1, 2, 3, \ldots\}.$$

Proof It is sufficient to show that there is no κ in A_0^- satisfying $p(\kappa) = \varphi_-(\kappa)$, and there is at most one κ in A_n^+ (respectively, A_n^-) satisfying $p(\kappa) = \varphi_+(\kappa)$ (respectively, $p(\kappa) = \varphi_-(\kappa)$) for $n = 1, 2, 3, \ldots$

Let $n = 1, 2, 3, \dots$ Note that, by (35) and the definition of β_n, we have $p(\kappa) > \varphi_+(\kappa)$ for every $\kappa \in (h^{-1}(2\pi n - \pi/2), \beta_n)$. Suppose there exists another κ in A_n^+ satisfying $p(\kappa) = \varphi_+(\kappa)$, which we denote $\tilde{\beta}_n$. By the definition of β_n, we have $\beta_n < \tilde{\beta}_n$. We can assume $\tilde{\beta}_n$ is chosen such that there is no κ between β_n and $\tilde{\beta}_n$ satisfying $p(\kappa) = \varphi_+(\kappa)$, since the set of solutions of (25) is discrete. So we have either $p(\kappa) > \varphi_+(\kappa)$ for every $\kappa \in (\beta_n, \tilde{\beta}_n)$, or $p(\kappa) < \varphi_+(\kappa)$ for every $\kappa \in (\beta_n, \tilde{\beta}_n)$. Suppose the former. Then the graphs of $p(\kappa)$ and $\varphi_+(\kappa)$ should be tangent to each other at $\kappa = \beta_n$, which implies $p'(\beta_n) = \varphi_+'(\beta_n)$. Since $p(\beta_n) = \varphi_+(\beta_n)$, this contradicts Lemma 4(a), and it follows that $p(\kappa) < \varphi_+(\kappa)$ for every $\kappa \in (\beta_n, \tilde{\beta}_n)$. Then by Lemma 4(a) again, we have $p'(\kappa) < \varphi_+'(\kappa)$ for every $\kappa \in (\beta_n, \tilde{\beta}_n)$. Applying the mean value theorem to the function $p(\kappa) - \varphi_+(\kappa)$ on $[\beta_n, \tilde{\beta}_n]$, we have

$$0 = \{p(\tilde{\beta}_n) - \varphi_+(\tilde{\beta}_n)\} - \{p(\beta_n) - \varphi_+(\beta_n)\} = \{p'(\tilde{\kappa}) - \varphi_+{}'(\tilde{\kappa})\} \cdot (\tilde{\beta}_n - \beta_n)$$

for some $\tilde{\kappa} \in (\beta_n, \tilde{\beta}_n)$. Then we have $p'(\tilde{\kappa}) = \varphi_+'(\tilde{\kappa})$, which is a contradiction. Thus we conclude that there is no κ in A_n^+ other than β_n, which satisfies $p(\kappa) = \varphi_+(\kappa)$.

Let $n = 1, 2, 3, \dots$ Note that, by (36) and the definition of γ_n, we have $p(\kappa) > \varphi_-(\kappa)$ for every $\kappa \in (\gamma_n, h^{-1}(2\pi n + \pi/2))$. Suppose there exists another κ in A_n^- satisfying $p(\kappa) = \varphi_-(\kappa)$, which we denote $\tilde{\gamma}_n$. By the definition of γ_n, we have $\tilde{\gamma}_n < \gamma_n$. We can assume $\tilde{\gamma}_n$ is chosen such that there is no κ between $\tilde{\gamma}_n$ and γ_n satisfying $p(\kappa) = \varphi_-(\kappa)$, since the set of solutions of (25) is discrete. So we have either $p(\kappa) > \varphi_-(\kappa)$ for every $\kappa \in (\tilde{\gamma}_n, \gamma_n)$, or $p(\kappa) < \varphi_-(\kappa)$ for every $\kappa \in (\tilde{\gamma}_n, \gamma_n)$. Suppose the former. Then the graphs of $p(\kappa)$ and $\varphi_-(\kappa)$ should be tangent to each other at $\kappa = \gamma_n$, which implies $p'(\gamma_n) = \varphi_-'(\gamma_n)$. Since $p(\gamma_n) = \varphi_-(\gamma_n)$, this contradicts Lemma 4(b), and it follows that $p(\kappa) < \varphi_-(\kappa)$ for every $\kappa \in (\tilde{\gamma}_n, \gamma_n)$. Then by Lemma 4(b) again, we have $p'(\kappa) > \varphi_-'(\kappa)$ for every $\kappa \in (\tilde{\gamma}_n, \gamma_n)$. Applying the mean value theorem to the function $p(\kappa) - \varphi_-(\kappa)$ on $[\tilde{\gamma}_n, \gamma_n]$, we have

$$0 = \{p(\gamma_n) - \varphi_-(\gamma_n)\} - \{p(\tilde{\gamma}_n) - \varphi_-(\tilde{\gamma}_n)\} = \{p'(\tilde{\kappa}) - \varphi_-'(\tilde{\kappa})\} \cdot (\gamma_n - \tilde{\gamma}_n)$$

for some $\tilde{\kappa} \in (\tilde{\gamma}_n, \gamma_n)$. Then we have $p'(\tilde{\kappa}) = \varphi_-'(\tilde{\kappa})$, which is a contradiction. Thus we conclude that there is no κ in A_n^- other than γ_n, which satisfies $p(\kappa) = \varphi_-(\kappa)$.

Suppose there exists κ in A_0^- satisfying $p(\kappa) = \varphi_-(\kappa)$. Since the set of solutions of (25) is discrete, we can take γ_0 to be the largest among such κ. Then we have $p(\kappa) > \varphi_-(\kappa)$ for every $\kappa \in (\gamma_0, h^{-1}(\pi/2))$, since $p(h^{-1}(\pi/2)) > 0 = \varphi_-(h^{-1}(\pi/2))$ by Lemma 2 and (28). Let $\tilde{\gamma}_0$ be the largest in $[0, \gamma_0)$ satisfying $p(\kappa) = \varphi_-(\kappa)$. Note that $\tilde{\gamma}_0$ exists, since $p(0) = \varphi_-(0) = 1$. Replacing $\tilde{\gamma}_n$, γ_n by $\tilde{\gamma}_0$, γ_0, respectively, and applying the same argument in the above paragraph again, results in a contradiction. Thus we conclude that there is no κ in A_0^- satisfying $p(\kappa) = \varphi_-(\kappa)$, and the proof is complete. □

Note that the inverse function h^{-1} of h is strictly increasing from $[0, \infty)$ onto $[0, \infty)$ by Lemma 1(a). Putting $t = h(\kappa)$, (17) can be written as

$$L \cdot h^{-1}(t) = t + \hat{h}\big(h^{-1}(t)\big) \quad \text{for } t \geq 0. \tag{38}$$

Lemma 6

(a) $1/(L + 2 + \sqrt{2}) \leq (h^{-1})'(t) < 1/L$ *for* $t \geq 0$.

(b) $h^{-1}(t) \sim t$ *and* $h^{-1}(t) - (t - 2\pi)/L \sim t^{-1}$.

Proof (a) follows immediately from Lemma 1(b), since $(h^{-1})'(t) = 1/\{h'(h^{-1}(t))\} = 1/h'(\kappa)$, where we put $t = h(\kappa)$.

By (38), we have

$$\begin{aligned}
&\lim_{t\to\infty} t\left(h^{-1}(t) - \frac{t-2\pi}{L}\right)\\
&\quad= \lim_{t\to\infty} t\left\{\frac{t+\hat{h}(h^{-1}(t))}{L} - \frac{t-2\pi}{L}\right\}\\
&\quad= \frac{1}{L}\lim_{t\to\infty} t\left\{\hat{h}\left(h^{-1}(t)\right) + 2\pi\right\} = \frac{1}{L}\lim_{\kappa\to\infty} h(\kappa)\left\{\hat{h}(\kappa) + 2\pi\right\}\\
&\quad= \frac{1}{L}\lim_{\kappa\to\infty}\frac{h(\kappa)}{\kappa}\cdot\lim_{\kappa\to\infty}\kappa\left\{\tilde{h}(\kappa) + 2\pi\right\} = \frac{1}{L}\cdot L\cdot\lim_{\kappa\to\infty}\frac{\hat{h}(\kappa)+2\pi}{\frac{1}{\kappa}},
\end{aligned}$$

where the last equality comes from Lemma 1(b). Since $\lim_{\kappa\to\infty}\hat{h}(\kappa) = -2\pi$, we can use l?Hôspital?s rule to get

$$\lim_{t\to\infty} t\left(h^{-1}(t) - \frac{t-2\pi}{L}\right) = \lim_{\kappa\to\infty}\frac{\hat{h}'(\kappa)}{-\frac{1}{\kappa^2}} = \lim_{\kappa\to\infty}\frac{2\sqrt{2}\kappa^2(\kappa^2+1)}{\kappa^4+1} = 2\sqrt{2} \tag{39}$$

by (16). This shows $|h^{-1}(t) - (t-2\pi)/L| \sim t^{-1}$, which also implies $h^{-1}(t) \sim t$. □

Note that, for $0 < t < \pi/2$, we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1-\cos t}{\sin t}\right) &= \frac{\sin t\cdot\sin t - (1-\cos t)\cdot\cos t}{\sin^2 t} = \frac{1-\cos t}{\sin^2 t} > 0,\\
\frac{d^2}{dt^2}\left(\frac{1-\cos t}{\sin t}\right) &= \frac{\sin t\cdot\sin^2 t - (1-\cos t)\cdot 2\sin t\cos t}{\sin^4 t}\\
&= \frac{1+\cos^2 t - 2\cos t}{\sin^3 t} = \frac{(1-\cos t)^2}{\sin^3 t} > 0.
\end{aligned}$$

This implies that the function $(1-\cos t)/\sin t$ is increasing and convex on $(0,\pi/2)$, and hence $t/2 < (1-\cos t)/\sin t < 2t/\pi$ for $0 < t < \pi/2$, since $\lim_{t\to 0}\{(1-\cos t)/\sin t\} = 0$, $(1-\cos(\pi/2))/\sin(\pi/2) = 1$, and $\lim_{t\to 0}\{(1-\cos t)/\sin t\}' = \lim_{t\to 0}\{(1-\cos t)/\sin^2 t\} = 1/2$. It follows that

$$\frac{t}{2} < \frac{1+\sin(2\pi n - \frac{\pi}{2} + t)}{\cos(2\pi n - \frac{\pi}{2} + t)} = \frac{1-\sin(2\pi n + \frac{\pi}{2} - t)}{\cos(2\pi n + \frac{\pi}{2} - t)} < \frac{2t}{\pi} \quad \text{for } 0 < t < \frac{\pi}{2}, \tag{40}$$

since

$$\frac{1+\sin(2\pi n - \frac{\pi}{2} + t)}{\cos(2\pi n - \frac{\pi}{2} + t)} = \frac{1-\sin(\frac{\pi}{2} - t)}{\cos(\frac{\pi}{2} - t)} = \frac{1-\cos t}{\sin t}.$$

Note that $0 < p(\kappa) < 1$ for $\kappa > 0$ by Lemma 2. For each $n = 1, 2, 3, \ldots$, we can take $0 < \epsilon_n^+ < \delta_n^+ < \pi/2$ such that

$$\varphi_+\left(h^{-1}\left(2\pi n - \frac{\pi}{2} + \epsilon_n^+\right)\right) = p\left(h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right), \tag{41}$$

$$\varphi_+\left(h^{-1}\left(2\pi n - \frac{\pi}{2} + \delta_n^+\right)\right) = 1, \tag{42}$$

since φ_+ is strictly increasing on A_n^+ from $\varphi_+(h^{-1}(2\pi n - \pi/2)) = 0$ to $\varphi_+(h^{-1}(2\pi n)) > 1$ by (27), (28), Lemma 3(a). Similarly, we can take $0 < \epsilon_n^- < \delta_n^- < \pi/2$ for each $n = 1, 2, 3, \ldots$, such that

$$\varphi_-\left(h^{-1}\left(2\pi n + \frac{\pi}{2} - \delta_n^-\right)\right) = 1, \tag{43}$$

$$\varphi_-\left(h^{-1}\left(2\pi n + \frac{\pi}{2} - \epsilon_n^-\right)\right) = p\left(h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right), \tag{44}$$

since φ_- is strictly decreasing on A_n^- from $\varphi_+(h^{-1}(2\pi n)) > 1$ to $\varphi_+(h^{-1}(2\pi n + \pi/2)) = 0$ by (27), (28), Lemma 3(a).

Suppose n is sufficiently large, so that $h^{-1}(2\pi n - \pi/2) > 1$. This is possible, since h^{-1} is one-to-one and onto from $[0, \infty)$ to $[0, \infty)$ by Lemma 1(a). Then, since p is strictly increasing on $(1, \infty)$ by Lemma 2, we have

$$p\left(h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right) < p\left(h^{-1}\left(2\pi n - \frac{\pi}{2} + \epsilon_n^+\right)\right) < p\left(h^{-1}\left(2\pi n + \frac{\pi}{2} - \epsilon_n^-\right)\right),$$

and hence by (41), (42), (43), (44),

$$\varphi_+\left(h^{-1}\left(2\pi n - \frac{\pi}{2} + \epsilon_n^+\right)\right) < p\left(h^{-1}\left(2\pi n - \frac{\pi}{2} + \epsilon_n^+\right)\right),$$

$$\varphi_+\left(h^{-1}\left(2\pi n - \frac{\pi}{2} + \delta_n^+\right)\right) > p\left(h^{-1}\left(2\pi n - \frac{\pi}{2} + \delta_n^+\right)\right),$$

$$\varphi_-\left(h^{-1}\left(2\pi n + \frac{\pi}{2} - \delta_n^-\right)\right) > p\left(h^{-1}\left(2\pi n + \frac{\pi}{2} - \delta_n^-\right)\right),$$

$$\varphi_-\left(h^{-1}\left(2\pi n + \frac{\pi}{2} - \epsilon_n^-\right)\right) < p\left(h^{-1}\left(2\pi n + \frac{\pi}{2} - \epsilon_n^-\right)\right).$$

It follows from the intermediate value theorem that, for sufficiently large n,

$$h^{-1}\left(2\pi n - \frac{\pi}{2}\right) < h^{-1}\left(2\pi n - \frac{\pi}{2} + \epsilon_n^+\right) < \beta_n < h^{-1}\left(2\pi n - \frac{\pi}{2} + \delta_n^+\right), \tag{45}$$

$$h^{-1}\left(2\pi n + \frac{\pi}{2} - \delta_n^-\right) < \gamma_n < h^{-1}\left(2\pi n + \frac{\pi}{2} - \epsilon_n^-\right) < h^{-1}\left(2\pi n + \frac{\pi}{2}\right), \tag{46}$$

since β_n (respectively, γ_n) is the only κ in A_n^+ (respectively, A_n^-) satisfying $p(\kappa) = \varphi_+(\kappa)$ (respectively, $p(\kappa) = \varphi_-(\kappa)$).

Lemma 7 *$\beta_n \sim \gamma_n \sim n$, and $\beta_n - h^{-1}(2\pi n - \pi/2) \sim h^{-1}(2\pi n + \pi/2) - \gamma_n \sim e^{-2\pi n}$, $\beta_n - (2\pi(n-1) - \pi/2)/L \sim \gamma_n - (2\pi(n-1) + \pi/2)/L \sim n^{-1}$.*

Proof Suppose n is sufficiently large so that (45), (46) hold. The fact $\beta_n \sim \gamma_n \sim n$ immediately follows from (45), (46), since $h^{-1}(t) \sim t$ by Lemma 6(b). By (45), (46), we have

$$\beta_n - h^{-1}\left(2\pi n - \frac{\pi}{2}\right) > h^{-1}\left(2\pi n - \frac{\pi}{2} + \epsilon_n^+\right) - h^{-1}\left(2\pi n - \frac{\pi}{2}\right), \tag{47}$$

$$\beta_n - h^{-1}\left(2\pi n - \frac{\pi}{2}\right) < h^{-1}\left(2\pi n - \frac{\pi}{2} + \delta_n^+\right) - h^{-1}\left(2\pi n - \frac{\pi}{2}\right), \tag{48}$$

$$h^{-1}\left(2\pi n+\frac{\pi}{2}\right)-\gamma_n>h^{-1}\left(2\pi n+\frac{\pi}{2}\right)-h^{-1}\left(2\pi n+\frac{\pi}{2}-\epsilon_n^-\right), \tag{49}$$

$$h^{-1}\left(2\pi n+\frac{\pi}{2}\right)-\gamma_n<h^{-1}\left(2\pi n+\frac{\pi}{2}\right)-h^{-1}\left(2\pi n+\frac{\pi}{2}-\delta_n^-\right). \tag{50}$$

By applying the mean value theorem to h^{-1}, we have

$$\begin{aligned}
&h^{-1}\left(2\pi n-\frac{\pi}{2}+\epsilon_n^+\right)-h^{-1}\left(2\pi n-\frac{\pi}{2}\right)=\left(h^{-1}\right)'\left(2\pi n-\frac{\pi}{2}+\tilde{\epsilon}_n^+\right)\cdot\epsilon_n^+,\\
&h^{-1}\left(2\pi n-\frac{\pi}{2}+\delta_n^+\right)-h^{-1}\left(2\pi n-\frac{\pi}{2}\right)=\left(h^{-1}\right)'\left(2\pi n-\frac{\pi}{2}+\tilde{\delta}_n^+\right)\cdot\delta_n^+,\\
&h^{-1}\left(2\pi n+\frac{\pi}{2}\right)-h^{-1}\left(2\pi n+\frac{\pi}{2}-\epsilon_n^-\right)=\left(h^{-1}\right)'\left(2\pi n+\frac{\pi}{2}-\tilde{\epsilon}_n^-\right)\cdot\epsilon_n^-,\\
&h^{-1}\left(2\pi n+\frac{\pi}{2}\right)-h^{-1}\left(2\pi n+\frac{\pi}{2}-\delta_n^-\right)=\left(h^{-1}\right)'\left(2\pi n+\frac{\pi}{2}-\tilde{\delta}_n^-\right)\cdot\delta_n^-
\end{aligned}$$

for some $0\le\tilde{\epsilon}_n^+\le\epsilon_n^+$, $0\le\tilde{\delta}_n^+\le\delta_n^+$, $0\le\tilde{\epsilon}_n^-\le\epsilon_n^-$, $0\le\tilde{\delta}_n^-\le\delta_n^-$. So by Lemma 6(a), we have

$$\begin{aligned}
&h^{-1}\left(2\pi n-\frac{\pi}{2}+\epsilon_n^+\right)-h^{-1}\left(2\pi n-\frac{\pi}{2}\right)\ge\frac{\epsilon_n^+}{L+2+\sqrt{2}},\\
&h^{-1}\left(2\pi n-\frac{\pi}{2}+\delta_n^+\right)-h^{-1}\left(2\pi n-\frac{\pi}{2}\right)<\frac{\delta_n^+}{L},\\
&h^{-1}\left(2\pi n+\frac{\pi}{2}\right)-h^{-1}\left(2\pi n+\frac{\pi}{2}-\epsilon_n^-\right)\ge\frac{\epsilon_n^-}{L+2+\sqrt{2}},\\
&h^{-1}\left(2\pi n+\frac{\pi}{2}\right)-h^{-1}\left(2\pi n+\frac{\pi}{2}-\delta_n^-\right)<\frac{\delta_n^-}{L},
\end{aligned}$$

and hence by (47), (48), (49), (50),

$$\frac{\epsilon_n^+}{L+2+\sqrt{2}}<\beta_n-h^{-1}\left(2\pi n-\frac{\pi}{2}\right)<\frac{\delta_n^+}{L}, \tag{51}$$

$$\frac{\epsilon_n^-}{L+2+\sqrt{2}}<h^{-1}\left(2\pi n+\frac{\pi}{2}\right)-\gamma_n<\frac{\delta_n^-}{L}. \tag{52}$$

Using (40), (41), (42), (43), (44), and the definition (24) of $\varphi_\pm$, we have

$$\begin{aligned}
&p\left(h^{-1}\left(2\pi n-\frac{\pi}{2}\right)\right)\\
&\quad=\varphi_+\left(h^{-1}\left(2\pi n-\frac{\pi}{2}+\epsilon_n^+\right)\right)\\
&\quad=\exp\left\{L\cdot h^{-1}\left(2\pi n-\frac{\pi}{2}+\epsilon_n^+\right)\right\}\cdot\frac{1+\sin(2\pi n-\frac{\pi}{2}+\epsilon_n^+)}{\cos(2\pi n-\frac{\pi}{2}+\epsilon_n^+)}\\
&\quad<\exp\left\{L\cdot h^{-1}(2\pi n)\right\}\cdot\frac{2}{\pi}\epsilon_n^+,\\
&p\left(h^{-1}\left(2\pi n-\frac{\pi}{2}\right)\right)\\
&\quad=\varphi_-\left(h^{-1}\left(2\pi n+\frac{\pi}{2}-\epsilon_n^-\right)\right)
\end{aligned}$$

$$= \exp\left\{L \cdot h^{-1}\left(2\pi n + \frac{\pi}{2} - \epsilon_n^-\right)\right\} \cdot \frac{1 - \sin(2\pi n + \frac{\pi}{2} - \epsilon_n^-)}{\cos(2\pi n + \frac{\pi}{2} - \epsilon_n^-)}$$
$$< \exp\left\{L \cdot h^{-1}\left(2\pi n + \frac{\pi}{2}\right)\right\} \cdot \frac{2}{\pi}\epsilon_n^-$$

and

$$1 = \varphi_+\left(h^{-1}\left(2\pi n - \frac{\pi}{2} + \delta_n^+\right)\right)$$
$$= \exp\left\{L \cdot h^{-1}\left(2\pi n - \frac{\pi}{2} + \delta_n^+\right)\right\} \cdot \frac{1 + \sin(2\pi n - \frac{\pi}{2} + \delta_n^+)}{\cos(2\pi n - \frac{\pi}{2} + \delta_n^+)}$$
$$> \exp\left\{L \cdot h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right\} \cdot \frac{1}{2}\delta_n^+,$$
$$1 = \varphi_-\left(h^{-1}\left(2\pi n + \frac{\pi}{2} - \delta_n^-\right)\right)$$
$$= \exp\left\{L \cdot h^{-1}\left(2\pi n + \frac{\pi}{2} - \delta_n^-\right)\right\} \cdot \frac{1 - \sin(2\pi n + \frac{\pi}{2} - \delta_n^-)}{\cos(2\pi n + \frac{\pi}{2} - \delta_n^-)}$$
$$> \exp\left\{L \cdot h^{-1}(2\pi n)\right\} \cdot \frac{1}{2}\delta_n^-,$$

and hence

$$\epsilon_n^+ > \frac{\pi}{2} \cdot p\left(h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right) \exp\left\{-L \cdot h^{-1}(2\pi n)\right\}, \tag{53}$$
$$\epsilon_n^- > \frac{\pi}{2} \cdot p\left(h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right) \exp\left\{-L \cdot h^{-1}\left(2\pi n + \frac{\pi}{2}\right)\right\}, \tag{54}$$
$$\delta_n^+ < 2\exp\left\{-L \cdot h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right\}, \tag{55}$$
$$\delta_n^- < 2\exp\left\{-L \cdot h^{-1}(2\pi n)\right\}. \tag{56}$$

Note that, for any constant c, we have $\lim_{n\to\infty} p(h^{-1}(2\pi n + c)) = 1$ by Lemma 2 and

$$\lim_{n\to\infty}\left[e^{2\pi n} \cdot \exp\left\{-L \cdot h^{-1}(2\pi n + c)\right\}\right]$$
$$= \lim_{n\to\infty} \exp\left\{2\pi n - L \cdot h^{-1}(2\pi n + c)\right\}$$
$$= \lim_{t\to\infty} \exp\left\{t - c - L \cdot h^{-1}(t)\right\} = \lim_{t\to\infty} \exp\left\{t - 2\pi + 2\pi - c - L \cdot h^{-1}(t)\right\}$$
$$= \lim_{t\to\infty} \exp\left[L \cdot \left\{\frac{t - 2\pi}{L} - h^{-1}(t)\right\} + (2\pi - c)\right] = e^{2\pi - c}$$

by Lemma 6(b). So by combining (51), (52), and (53), (54), (55), (56), we have

$$\frac{\pi e^{2\pi}}{2(L + 2 + \sqrt{2})} \le \lim_{n\to\infty}\left[e^{2\pi n} \cdot \left\{\beta_n - h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right\}\right] \le \frac{2e^{2\pi + \frac{\pi}{2}}}{L}, \tag{57}$$
$$\frac{\pi e^{2\pi - \frac{\pi}{2}}}{2(L + 2 + \sqrt{2})} \le \lim_{n\to\infty}\left[e^{2\pi n} \cdot \left\{h^{-1}\left(2\pi n + \frac{\pi}{2}\right) - \gamma_n\right\}\right] \le \frac{2e^{2\pi}}{L}, \tag{58}$$

which shows $\beta_n - h^{-1}(2\pi n - \pi/2) \sim h^{-1}(2\pi n + \pi/2) - \gamma_n \sim e^{-2\pi n}$.

By (57), (58), we have

$$\begin{aligned}
0 &\le \lim_{n\to\infty} n\left\{\beta_n - h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right\} \\
&= \lim_{n\to\infty} ne^{-2\pi n} \cdot \lim_{n\to\infty}\left[e^{2\pi n}\cdot\left\{\beta_n - h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right\}\right] \\
&\le \frac{2e^{2\pi+\frac{\pi}{2}}}{L}\cdot \lim_{n\to\infty} ne^{-2\pi n} = 0, \\
0 &\le \lim_{n\to\infty} n\left\{h^{-1}\left(2\pi n + \frac{\pi}{2}\right) - \gamma_n\right\} \\
&= \lim_{n\to\infty} ne^{-2\pi n} \cdot \lim_{n\to\infty}\left[e^{2\pi n}\cdot\left\{h^{-1}\left(2\pi n + \frac{\pi}{2}\right) - \gamma_n\right\}\right] \\
&\le \frac{2e^{2\pi}}{L}\cdot \lim_{n\to\infty} ne^{-2\pi n} = 0,
\end{aligned}$$

and hence

$$\lim_{n\to\infty} n\left\{\beta_n - h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right\} = \lim_{n\to\infty} n\left\{h^{-1}\left(2\pi n + \frac{\pi}{2}\right) - \gamma_n\right\} = 0.$$

So by (39), we have

$$\begin{aligned}
&\lim_{n\to\infty} n\left\{\beta_n - \frac{1}{L}\left(2\pi(n-1) - \frac{\pi}{2}\right)\right\} \\
&\quad = \lim_{n\to\infty} n\left\{\beta_n - h^{-1}\left(2\pi n - \frac{\pi}{2}\right)\right\} \\
&\qquad + \lim_{n\to\infty} n\left\{h^{-1}\left(2\pi n - \frac{\pi}{2}\right) - \frac{1}{L}\left(2\pi n - \frac{\pi}{2}\right) + \frac{2\pi}{L}\right\} \\
&\quad = \lim_{t\to\infty} \frac{t+\frac{\pi}{2}}{2\pi}\left(h^{-1}(t) - \frac{t-2\pi}{L}\right) \\
&\quad = \lim_{t\to\infty} \frac{t+\frac{\pi}{2}}{2\pi t}\cdot \lim_{t\to\infty} t\left(h^{-1}(t) - \frac{t-2\pi}{L}\right) = \frac{1}{2\pi}\cdot 2\sqrt{2} = \frac{\sqrt{2}}{\pi}, \\
&\lim_{n\to\infty} n\left\{\gamma_n - \frac{1}{L}\left(2\pi(n-1) + \frac{\pi}{2}\right)\right\} \\
&\quad = \lim_{n\to\infty} n\left\{\gamma_n - h^{-1}\left(2\pi n + \frac{\pi}{2}\right)\right\} \\
&\qquad + \lim_{n\to\infty} n\left\{h^{-1}\left(2\pi n + \frac{\pi}{2}\right) - \frac{1}{L}\left(2\pi n + \frac{\pi}{2}\right) + \frac{2\pi}{L}\right\} \\
&\quad = \lim_{t\to\infty} \frac{t-\frac{\pi}{2}}{2\pi}\left(h^{-1}(t) - \frac{t-2\pi}{L}\right) \\
&\quad = \lim_{t\to\infty} \frac{t-\frac{\pi}{2}}{2\pi t}\cdot \lim_{t\to\infty} t\left(h^{-1}(t) - \frac{t-2\pi}{L}\right) = \frac{1}{2\pi}\cdot 2\sqrt{2} = \frac{\sqrt{2}}{\pi},
\end{aligned}$$

which shows $\beta_n - (2\pi(n-1) - \pi/2)/L \sim \gamma_n - (2\pi(n-1) + \pi/2)/L \sim n^{-1}$, and the proof is complete. □

Lemma 8 *Suppose positive sequences* $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$, $\{c_n\}_{n=1}^\infty$ *satisfy* $a_n \sim b_n \sim n$ *and* $a_n - b_n \sim c_n$. *Then* $1/(1+b_n^4) - 1/(1+a_n^4) \sim n^{-5}c_n$.

Proof Let $f(x) = 1/(1+x^4)$. By the mean value theorem, we have

$$\begin{aligned}\frac{1}{1+b_n^4} - \frac{1}{1+a_n^4} &= f(b_n) - f(a_n) = f'(\xi_n)\cdot(b_n - a_n)\\ &= \frac{4\xi_n^3}{(1+\xi_n^4)^2}\cdot(a_n - b_n)\end{aligned}$$

for some $b_n \le \xi_n \le a_n$ for $n = 1,2,3,\ldots$ Note that $\xi_n \sim a_n \sim b_n \sim n$. So we have

$$n^5 c_n^{-1}\cdot\left(\frac{1}{1+b_n^4} - \frac{1}{1+a_n^4}\right) = \frac{4(\frac{\xi_n}{n})^3}{\{\frac{1}{n^4} + (\frac{\xi_n}{n})^4\}^2}\cdot\frac{a_n - b_n}{c_n},$$

which is bounded below and above by some positive constants for every sufficiently large n, since $\xi_n \sim n$ and $a_n - b_n \sim c_n$. This implies $1/(1+b_n^4) - 1/(1+a_n^4) \sim n^{-5}c_n$. □

Proof of Theorem 1 By Proposition 3, $\mathcal{K}_{l,\alpha,k}$ has no eigenvalues outside the interval $(0,1/k)$. By (8) and Lemma 5, the eigenvalues in $(0,1/k)$ are μ_n/k, ν_n/k, $n = 1,2,3,\ldots$, where we put

$$\mu_n := \frac{1}{1+\beta_n^4}, \qquad \nu_n := \frac{1}{1+\gamma_n^4} \tag{59}$$

for $n = 1,2,3,\ldots$ Note that L is the only parameter involved with the characteristic equation (25). So its solutions β_n, γ_n, and hence μ_n, ν_n, depend only on L for $n = 1,2,3,\ldots$ The bounds on μ_n, ν_n in (a) follow from (37) and (59), and thus we showed (a).

Since $\beta_n \sim \gamma_n \sim n$ by Lemma 7, it follows easily from (59) that $\mu_n \sim \nu_n \sim n^{-4}$. Note that $h^{-1}(2\pi n - \pi/2) \sim h^{-1}(2\pi n + \pi/2) \sim n$ by Lemma 6(b). So by Lemma 8 and (59), we have

$$\begin{aligned}
&\frac{1}{1+\{h^{-1}(2\pi n - \frac{\pi}{2})\}^4} - \mu_n = \frac{1}{1+\{h^{-1}(2\pi n - \frac{\pi}{2})\}^4} - \frac{1}{1+\beta_n^4} \sim n^{-5}e^{-2\pi n},\\
&\nu_n - \frac{1}{1+\{h^{-1}(2\pi n + \frac{\pi}{2})\}^4} = \frac{1}{1+\gamma_n^4} - \frac{1}{1+\{h^{-1}(2\pi n + \frac{\pi}{2})\}^4} \sim n^{-5}e^{-2\pi n},\\
&\frac{1}{1+\frac{1}{L^4}(2\pi(n-1) - \frac{\pi}{2})^4} - \mu_n = \frac{1}{1+\frac{1}{L^4}(2\pi(n-1) - \frac{\pi}{2})^4} - \frac{1}{1+\beta_n^4} \sim n^{-6},\\
&\frac{1}{1+\frac{1}{L^4}(2\pi(n-1) + \frac{\pi}{2})^4} - \nu_n = \frac{1}{1+\frac{1}{L^4}(2\pi(n-1) + \frac{\pi}{2})^4} - \frac{1}{1+\gamma_n^4} \sim n^{-6},
\end{aligned}$$

since $\beta_n - h^{-1}(2\pi n - \pi/2) \sim h^{-1}(2\pi n + \pi/2) - \gamma_n \sim e^{-2\pi n}$ and $\beta_n - (2\pi(n-1) - \pi/2)/L \sim \gamma_n - (2\pi(n-1) + \pi/2)/L \sim n^{-1}$ by Lemma 7. This shows (b), and the proof is complete. □

5 Behavior of the eigenvalues with respect to the beam length: proof of Theorem 2

In this section, we prove Theorem 2 by investigating the behavior of the eigenvalues of $\mathcal{K}_{l,\alpha,k}$ obtained in Theorem 1, as the intrinsic length L of the given beam changes.

Lemma 9 β_n *and* γ_n *are strictly decreasing with respect to* L *for* $n = 1,2,3,\ldots$

Proof Since β_n and γ_n are solutions of the equations $\varphi_+(\kappa) - p(\kappa) = 0$ and $\varphi_-(\kappa) - p(\kappa) = 0$, respectively, we have $\varphi_+(\beta_n) - p(\beta_n) = 0$, and $\varphi_-(\gamma_n) - p(\gamma_n) = 0$. Differentiation of these equations with respect to L gives

$$\begin{aligned}
0 &= \frac{d}{dL}\varphi_+(\beta_n) - \frac{d}{dL}p(\beta_n) \\
&= \left\{ \frac{\partial\varphi_+}{\partial\kappa}(\beta_n) \cdot \frac{d\beta_n}{dL} + \frac{\partial\varphi_+}{\partial L}(\beta_n) \right\} - \frac{dp}{d\kappa}(\beta_n) \cdot \frac{d\beta_n}{dL} \\
&= \left\{ \varphi_+'(\beta_n) - p'(\beta_n) \right\} \cdot \frac{d\beta_n}{dL} + \frac{\partial\varphi_+}{\partial L}(\beta_n), \\
0 &= \frac{d}{dL}\varphi_-(\gamma_n) - \frac{d}{dL}p(\gamma_n) \\
&= \left\{ \frac{\partial\varphi_-}{\partial\kappa}(\gamma_n) \cdot \frac{d\gamma_n}{dL} + \frac{\partial\varphi_-}{\partial L}(\gamma_n) \right\} - \frac{dp}{d\kappa}(\gamma_n) \cdot \frac{d\gamma_n}{dL} \\
&= \left\{ \varphi_-'(\gamma_n) - p'(\gamma_n) \right\} \cdot \frac{d\gamma_n}{dL} + \frac{\partial\varphi_-}{\partial L}(\gamma_n),
\end{aligned}$$

and hence

$$\frac{d\beta_n}{dL} = -\frac{\partial\varphi_+}{\partial L}(\beta_n) \cdot \frac{1}{\varphi_+{}'(\beta_n) - p'(\beta_n)}, \tag{60}$$

$$\frac{d\gamma_n}{dL} = -\frac{\partial\varphi_-}{\partial L}(\gamma_n) \cdot \frac{1}{\varphi_-{}'(\gamma_n) - p'(\gamma_n)}. \tag{61}$$

By differentiating (24) with respect to L, we have

$$\begin{aligned}
\frac{\partial\varphi_\pm}{\partial L}(\kappa) &= \frac{\partial}{\partial L}\left\{ e^{L\kappa} \cdot \frac{1 \pm \sin(L\kappa - \hat{h}(\kappa))}{\cos(L\kappa - \hat{h}(\kappa))} \right\} \\
&= e^{L\kappa}\left\{ \kappa \cdot \frac{1 \pm \sin(L\kappa - \hat{h}(\kappa))}{\cos(L\kappa - \hat{h}(\kappa))} \pm \frac{1 \pm \sin(L\kappa - \hat{h}(\kappa))}{\cos^2(L\kappa - \hat{h}(\kappa))} \cdot \kappa \right\} \\
&= \pm\frac{\kappa e^{L\kappa}\{1 \pm \sin(L\kappa - \hat{h}(\kappa))\}\{1 \pm \cos(L\kappa - \hat{h}(\kappa))\}}{\cos^2(L\kappa - \hat{h}(\kappa))},
\end{aligned}$$

where we used (29) for the second equality. So we have $(\partial\varphi_+/\partial L)(\beta_n) > 0$ and $(\partial\varphi_-/\partial L)(\gamma_n) < 0$. Since $p(\beta_n) = \varphi_+(\beta_n)$ and $p(\gamma_n) = \varphi_-(\gamma_n)$, we have $\varphi_+'(\beta_n) - p'(\beta_n) > 0$ and $\varphi_-'(\gamma_n) - p'(\gamma_n) < 0$ by Lemma 4. Thus, by (60) and (61), we have $d\beta_n/dL < 0$ and $d\gamma_n/dL < 0$, which completes the proof. □

Lemma 10 *For any fixed* $t > 0$, $h^{-1}(t)$ *is strictly decreasing with respect to L, and* $\lim_{L\to\infty} h^{-1}(t) = 0$,

$$\lim_{L\to 0} h^{-1}(t) = \begin{cases} \hat{h}^{-1}(-t) & \text{if } 0 < t < 2\pi, \\ \infty & \text{if } t \geq 2\pi. \end{cases}$$

Proof Fix $t > 0$. Differentiating both sides of (38) with respect to L, we have

$$h^{-1}(t) + L \cdot \frac{d}{dL}h^{-1}(t) = \hat{h}'\big(h^{-1}(t)\big) \cdot \frac{d}{dL}h^{-1}(t).$$

Hence, by putting $\kappa = h^{-1}(t) > 0$, we have

$$\frac{d}{dL}h^{-1}(t) = -\frac{h^{-1}(t)}{L - \hat{h}'(h^{-1}(t))} = -\frac{\kappa}{L - \hat{h}'(\kappa)} = -\frac{\kappa}{h'(\kappa)} < 0$$

by (17) and Lemma 1(b). This shows that $h^{-1}(t)$ is strictly decreasing with respect to L.

From (38), we have

$$\lim_{L\to\infty} h^{-1}(t) = t \cdot \lim_{L\to\infty}\frac{1}{L} + \lim_{L\to\infty}\frac{\hat{h}(h^{-1}(t))}{L} = \lim_{L\to\infty}\frac{\hat{h}(\kappa)}{L} = 0,$$

since $-2\pi < \hat{h}(\kappa) < 0$ for every $\kappa > 0$.

Since $h^{-1}(t)$ is strictly decreasing with respect to L, either $\lim_{L\to 0} h^{-1}(t) = \infty$, or $\lim_{L\to 0} h^{-1}(t) = c$ for some constant $c > 0$. Suppose the latter. Taking the limits as $L \to 0$ on both sides of (38), we have

$$0 = c \cdot \lim_{L\to 0} L = \lim_{L\to 0}\{L \cdot h^{-1}(t)\} = \lim_{L\to 0}\{t + \hat{h}(h^{-1}(t))\} = t + \lim_{L\to 0}\hat{h}(h^{-1}(t)) = t + \hat{h}(c). \quad (62)$$

But this is impossible for $t \ge 2\pi$, since $\hat{h}(c) > -2\pi$ for every $c > 0$. Thus $\lim_{L\to 0} h^{-1}(t) = \infty$ for $t \ge 2\pi$.

Let $0 < t < 2\pi$, and suppose $\lim_{L\to 0} h^{-1}(t) = \infty$. From (38), we have $t = L \cdot h^{-1}(t) - \hat{h}(h^{-1}(t))$, and hence

$$\begin{aligned} 2\pi > t &= \lim_{L\to 0}\{L \cdot h^{-1}(t)\} - \lim_{L\to 0}\hat{h}(h^{-1}(t)) = \lim_{L\to 0}\{L \cdot h^{-1}(t)\} - \lim_{\kappa\to\infty}\hat{h}(\kappa) \\ &= \lim_{L\to 0}\{L \cdot h^{-1}(t)\} - (-2\pi) \ge 2\pi, \end{aligned}$$

since $\lim_{\kappa\to\infty} \hat{h}(\kappa) = -2\pi$ by (15). This is a contradiction, and we conclude that $\lim_{L\to 0} h^{-1}(t) = c$ for some $c > 0$ when $0 < t < 2\pi$. The value of c can be obtained from (62) so that $\lim_{L\to 0} h^{-1}(t) = \hat{h}^{-1}(-t)$. □

Note that $h^{-1}(3\pi/2) < \beta_1 < h^{-1}(2\pi)$ by (37). In proving the following result, this fact makes the case $\lim_{L\to 0}\beta_1$ subtler than the others. For this case, we need to utilize additionally the fact that it is a solution of the equation $p(\kappa) = \varphi_+(\kappa)$. Note that $\lim_{L\to 0}\beta_1 \to \infty$ is equivalent to $\lim_{L\to 0} h(\beta_1) = 2\pi$.

Lemma 11 $\lim_{L\to 0}\beta_n = \lim_{L\to 0}\gamma_n = \infty$ *and* $\lim_{L\to\infty}\beta_n = \lim_{L\to\infty}\gamma_n = 0$ *for* $n = 1, 2, 3, \ldots$

Proof By (37) and Lemma 10, we have

$$\begin{aligned} &\lim_{L\to 0}\beta_n \ge \lim_{L\to 0} h^{-1}\left(2\pi n - \frac{\pi}{2}\right) = \infty, \quad n = 2, 3, 4, \ldots, \\ &\lim_{L\to 0}\gamma_n \ge \lim_{L\to 0} h^{-1}(2\pi n) = \infty, \quad n = 1, 2, 3, \ldots, \\ &0 \le \lim_{L\to\infty}\beta_n \le \lim_{L\to\infty} h^{-1}(2\pi n) = 0, \quad n = 1, 2, 3, \ldots, \\ &0 \le \lim_{L\to\infty}\gamma_n \le \lim_{L\to\infty} h^{-1}\left(2\pi n + \frac{\pi}{2}\right) = 0, \quad n = 1, 2, 3, \ldots, \end{aligned}$$

which shows $\lim_{L\to 0}\beta_n = \infty$ for $n = 2,3,4,\dots$, and $\lim_{L\to 0}\gamma_n = \infty$, $\lim_{L\to\infty}\beta_n = 0$, $\lim_{L\to\infty}\gamma_n = 0$ for $n = 1,2,3,\dots$

It remains to show $\lim_{L\to 0}\beta_1 = \infty$. Note that we cannot directly use Lemma 10, as we did above for the others, because $\beta_1 < h^{-1}(2\pi)$. Since β_1 is strictly decreasing with respect to L by Lemma 10, either $\lim_{L\to 0}\beta_1 = \infty$ or $\lim_{L\to 0}\beta_1 = \overline{\beta}_1$ for some $\overline{\beta}_1 < \infty$. Suppose the latter. Then, since $h^{-1}(3\pi/2) < \beta_1$, we have

$$\frac{\sqrt{3}+1}{\sqrt{2}} = \hat{h}^{-1}\left(-\frac{3\pi}{2}\right) = \lim_{L\to 0} h^{-1}\left(\frac{3\pi}{2}\right) \le \lim_{L\to 0}\beta_1 = \overline{\beta}_1 < \infty \tag{63}$$

by Lemma 10 and (15). Since β_1 satisfies the equation $p(\beta_1) = \varphi_+(\beta_1)$, we have

$$p(\beta_1) = e^{L\beta_1}\frac{1+\sin(L\beta_1 - \hat{h}(\beta_1))}{\cos(L\beta_1 - \hat{h}(\beta_1))},$$

and hence

$$p(\beta_1)\cos\bigl(L\beta_1 - \hat{h}(\beta_1)\bigr) - e^{L\beta_1}\bigl\{1+\sin\bigl(L\beta_1 - \hat{h}(\beta_1)\bigr)\bigr\} = 0.$$

Taking the limits of the both sides as $L \to 0$, we have

$$\begin{aligned} 0 &= \lim_{L\to 0}\Bigl[p(\beta_1)\cos\bigl(L\beta_1 - \hat{h}(\beta_1)\bigr) - e^{L\beta_1}\bigl\{1+\sin\bigl(L\beta_1 - \hat{h}(\beta_1)\bigr)\bigr\}\Bigr] \\ &= p(\overline{\beta}_1)\cos\bigl(-\hat{h}(\overline{\beta}_1)\bigr) - \bigl\{1+\sin\bigl(-\hat{h}(\overline{\beta}_1)\bigr)\bigr\} = p(\overline{\beta}_1)\cos\hat{h}(\overline{\beta}_1) + \sin\hat{h}(\overline{\beta}_1) - 1. \end{aligned} \tag{64}$$

Note that

$$\begin{aligned} &\frac{d}{d\kappa}\bigl\{p(\kappa)\cos\hat{h}(\kappa) + \sin\hat{h}(\kappa) - 1\bigr\} \\ &\quad = p'(\kappa)\cos\hat{h}(\kappa) - p(\kappa)\sin\hat{h}(\kappa)\cdot\hat{h}'(\kappa) + \cos\hat{h}(\kappa)\cdot\hat{h}'(\kappa) \\ &\quad = \bigl\{p'(\kappa) + \hat{h}'(\kappa)\bigr\}\cos\hat{h}(\kappa) - p(\kappa)\hat{h}'(\kappa)\sin\hat{h}(\kappa). \end{aligned} \tag{65}$$

For every $\kappa > 0$, we have $p(\kappa) > 0$ by Lemma 2, $\hat{h}'(\kappa) < 0$ by (16), and

$$\begin{aligned} p'(\kappa) + \hat{h}'(\kappa) &= \frac{2\sqrt{2}(\kappa^2-1)}{(\kappa^2+\sqrt{2}\kappa+1)^2} - \frac{2\sqrt{2}(\kappa^2+1)}{\kappa^4+1} \\ &= \frac{2\sqrt{2}\{(\kappa^2-1)(\kappa^4+1) - (\kappa^2+1)(\kappa^2+\sqrt{2}\kappa+1)^2\}}{(\kappa^2+\sqrt{2}\kappa+1)^2(\kappa^4+1)} \\ &= -\frac{2\sqrt{2}(2\sqrt{2}\kappa^5 + 6\kappa^4 + 4\sqrt{2}\kappa^3 + 4\kappa^2 + 2\sqrt{2}\kappa + 2)}{(\kappa^2+\sqrt{2}\kappa+1)^2(\kappa^4+1)} < 0 \end{aligned}$$

by (16) and (26). Suppose $\kappa > (\sqrt{3}+1)/\sqrt{2}$. Then $-2\pi < \hat{h}(\kappa) < -3\pi/2$ by (15), and hence $\cos\hat{h}(\kappa) > 0$ and $\sin\hat{h}(\kappa) < 0$. From these facts, we conclude that (65) is always negative for $\kappa > (\sqrt{3}+1)/\sqrt{2}$, and hence $p(\kappa)\cos\hat{h}(\kappa) + \sin\hat{h}(\kappa) - 1$ is strictly decreasing for $\kappa \ge (\sqrt{3}+1)/\sqrt{2}$. It follows that $p(\kappa)\cos\hat{h}(\kappa) + \sin\hat{h}(\kappa) - 1 < 0$ for $\kappa \ge (\sqrt{3}+1)/\sqrt{2}$, since

$$\begin{aligned} &p\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\cos\left\{\hat{h}\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\right\} + \sin\left\{\hat{h}\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\right\} - 1 \\ &\quad = p\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\cos\left(-\frac{3\pi}{2}\right) + \sin\left(-\frac{3\pi}{2}\right) - 1 = -2 < 0 \end{aligned}$$

by (15). This is a contradiction to (63) and (64), and thus we conclude that $\lim_{L\to 0}\beta_1 = \infty$. □

Proof of Theorem 2 The proof follows immediately from (59) and Lemmas 9, 11. □

6 Numerical computation of the eigenvalues

We use Newton?s method for our numerical computation. We first compute approximate values of β_n and γ_n. To compute β_n (respectively, γ_n), we have to solve the equation $p(\kappa) = \varphi_+(\kappa)$ (respectively, $p(\kappa) = \varphi_-(\kappa)$). By Lemma 5, β_n (respectively, γ_n) is the unique solution in the interval $A_n^+ = (h^{-1}(2\pi n - \pi/2), h^{-1}(2\pi))$ (respectively, $A_n^- = (h^{-1}(2\pi n), h^{-1}(2\pi + \pi/2))$). As an initial guess for β_n (respectively, γ_n), we use $h^{-1}(2\pi n - \pi/4)$ (respectively, $h^{-1}(2\pi n + \pi/4)$), an approximate value of which is obtained by solving (again by Newton?s method) the equation $h(\kappa) = 2\pi n - \pi/4$ (respectively, $h(\kappa) = 2\pi n + \pi/4$). Note that h is one-to-one and onto, and so $h(\kappa) = c$ has a unique global solution for any $c > 0$.

For example, to compute β_1 when $L = 1$, we first solve the equation $h(\kappa) = 2\pi - \pi/4$ when $L = 1$, which is $\kappa - \hat{h}(\kappa) = 7\pi/4$, to get

$$h^{-1}(2\pi - \pi/4) \approx 1.419670987525799.$$

With this value as an initial guess, we use Newton?s method to the equation$p(\kappa) = \varphi_+(\kappa)$ when $L = 1$, which is

$$\frac{\kappa^2 - \sqrt{2}\kappa + 1}{\kappa^2 + \sqrt{2}\kappa + 1} = e^{\kappa}\frac{1 + \sin(\kappa - \hat{h}(\kappa))}{\cos(\kappa - \hat{h}(\kappa))},$$

to get $\beta_1 \approx 1.191421197714390$. We mention that, in view of the approximation in Theorem 1(b), it is more advantageous to use $h^{-1}(2\pi n \mp \pi/2)$ as initial guesses for large n. We list the result of our computation of a few initial β_n and γ_n when $L = 1$ in Table 2. To illustrate the bounds in (37) and the approximations in Lemma 7, we also list there corresponding values of $h^{-1}(2\pi)$, $h^{-1}(2\pi \pm \pi/2)$, and $(2\pi(n-1) \pm \pi/2)/L$ when $L = 1$.

The computation of μ_n (respectively, ν_n) can be done by using the relations (59) and the result of computation of β_n (respectively, γ_n) above. For example, we compute μ_1 when $L = 1$ as

$$\mu_1 \approx 1/\bigl(1 + 1.191421197714390^4\bigr) \approx 0.331681981441542.$$

Using (8), we could also apply Newton?s method directly to the equations

$$p\left(\sqrt[4]{\frac{1}{\lambda} - 1}\right) = \varphi_{\pm}\left(\sqrt[4]{\frac{1}{\lambda} - 1}\right)$$

with the initial guesses $1/\{1 + (h^{-1}(2\pi n \mp \pi/2))^4\}$, but we mention that this method can be quite sensitive to initial guesses. We list the result of our computation of a few initial μ_n and ν_n when $L = 1$ in Table 3. There, we also list corresponding values of $1/\{1 + (h^{-1}(2\pi))^4\}$, $1/\{1 + (h^{-1}(2\pi \pm \pi/2))^4\}$, and $1/\{1 + (2\pi(n-1) \pm \pi/2)^4/L^4\}$ when $L = 1$ to illustrate the bounds and the approximations in Theorem 1.

Table 2 Numerical values of β_n and γ_n when $L = 1$

n	Name	Value	$(2\pi(n-1) \mp \pi/2)/L$
1	$h^{-1}(2\pi - \pi/2)$	1.158670738392296	
	β_1	1.191421197714390	–1.570796326794896
	$h^{-1}(2\pi)$	1.750980760482237	
	γ_1	2.637856739191656	1.570796326794896
	$h^{-1}(2\pi + \pi/2)$	2.673553841718542	
2	$h^{-1}(4\pi - \pi/2)$	5.256787217675680	
	β_2	5.262300407849289	4.712388980384689
	$h^{-1}(4\pi)$	6.707921416840514	
	γ_2	8.200207778135508	7.853981633974483
	$h^{-1}(4\pi + \pi/2)$	8.200581481509233	
3	$h^{-1}(6\pi - \pi/2)$	11.247700835446595	
	β_3	11.247720678493973	10.995574287564276
	$h^{-1}(6\pi)$	12.787998043974640	
	γ_3	14.334797074430887	14.137166941154069
	$h^{-1}(6\pi + \pi/2)$	14.334798038235459	
4	$h^{-1}(8\pi - \pi/2)$	17.441107108879219	
	β_4	17.441107153760840	17.278759594743862
	$h^{-1}(8\pi)$	18.998568977749238	
	γ_4	20.558043111829927	20.420352248333656
	$h^{-1}(8\pi + \pi/2)$	20.558043113872500	
5	$h^{-1}(10\pi - \pi/2)$	23.681452204590053	
	β_5	23.681452204681734	23.561944901923449
	$h^{-1}(10\pi)$	25.244839588317457	
	γ_5	26.809088990153228	26.703537555513242
	$h^{-1}(10\pi + \pi/2)$	26.809088990157306	

The last column lists values of the approximations $(2\pi(n-1) - \pi/2)/L$ to β_n and $(2\pi(n-1) + \pi/2)/L$ to γ_n.

Table 3 Numerical values of μ_n and ν_n when $L = 1$

n	Name	Value	$1/\{1 + (2\pi(n-1) \mp \pi/2)^4/L^4\}$
1	$1/\{1 + (h^{-1}(2\pi - \pi/2))^4\}$	0.356842821387149	
	μ_1	0.331681981441542	0.141082164173265
	$1/\{1 + (h^{-1}(2\pi))^4\}$	0.096154317825982	
	ν_1	0.020235634105536	0.141082164173265
	$1/\{1 + (h^{-1}(2\pi + \pi/2))^4\}$	0.019196682744858	
2	$1/\{1 + (h^{-1}(4\pi - \pi/2))^4\}$	0.001307826261601	
	μ_2	0.001302361278230	0.002023744499666
	$1/\{1 + (h^{-1}(4\pi))^4\}$	0.000493666532259	
	ν_2	0.000221108040807	0.000262740095219
	$1/\{1 + (h^{-1}(4\pi + \pi/2))^4\}$	0.000221067748587	
3	$1/\{1 + (h^{-1}(6\pi - \pi/2))^4\}$	0.000062476665124	
	μ_3	0.000062476224272	0.000068406697161
	$1/\{1 + (h^{-1}(6\pi))^4\}$	0.000037391554101	
	ν_3	0.000023682280310	0.000025034538029
	$1/\{1 + (h^{-1}(6\pi + \pi/2))^4\}$	0.000023682273941	
4	$1/\{1 + (h^{-1}(8\pi - \pi/2))^4\}$	0.000010806849662	
	μ_4	0.000010806849551	0.000011218760557
	$1/\{1 + (h^{-1}(8\pi))^4\}$	0.000007675613651	
	ν_4	0.000005598484481	0.000005751016121
	$1/\{1 + (h^{-1}(8\pi + \pi/2))^4\}$	0.000005598484479	
5	$1/\{1 + (h^{-1}(10\pi - \pi/2))^4\}$	0.000003179547340	
	μ_5	0.000003179547340	0.000003244546827
	$1/\{1 + (h^{-1}(10\pi))^4\}$	0.000002462115765	
	ν_5	0.000001935846573	0.000001966635852
	$1/\{1 + (h^{-1}(10\pi + \pi/2))^4\}$	0.000001935846573	

The last column lists values of the approximations $1/\{1 + (2\pi(n-1) - \pi/2)^4/L^4\}$ to μ_n and $1/\{1 + (2\pi(n-1) + \pi/2)^4/L^4\}$ to ν_n.

Finally, Table 1 in Section 1 lists the result of our computation of μ_1, ν_1, μ_2, ν_2 for various L, which illustrates Theorem 2. Especially, the μ_1 part in Table 1 lists the L^2-norm of the operator $\mathcal{K}_{l,\alpha,k}$ for various L.

Additional material

Additional file 1: This Mathematica notebook file is for checking the validity of (13) in Section 3.1. Open it with Mathematica, and execute (shift + enter) the series of commands there.
Additional file 2: This pdf file is just a printed version of the file choi.nb, as it looks after it is opened with Mathematica and all the commands therein are executed.

Competing interests

The author declares to have no competing interests.

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