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Fractional differential repetitive processes with Riemann–Liouville and Caputo derivatives

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Abstract In the paper, we study differential repetitive processes with fractional Riemann–Liouville and Caputo derivatives, in the context of the existence, uniqueness and continuous dependence of solutions on controls. Some applications to controllabilty of such processes are given as well.

Keywords Riemann–Liouville derivative · Caputo derivative · Differential repetitive process · Existence, uniqueness and continuous dependence of solutions on controls · Reachable set

1 Introduction

The theory of repetitive processes (multi-pass systems) is extensively studied for over thirty years (cf. Rogers et al. 2007 and references therein). The main idea of such processes consists, in general, in the repetition of the control system

$$\begin{cases} z^{(1)} = f(t, z, w, u), \\ w = g(t, z, u), \end{cases} \qquad t \in [a, b] \tag{1}$$

(here u is an input (control), z—a trajectory, w—an output and $z^{(1)}$ denotes the classical derivative of z) so that the output can be perfectly tracked as the operation repeats. In other words, we search for the sequence of controls with desired properties of the sequence of trajectories or outputs generated by these controls. One of the possible approaches relies on the describing of the recursive algorithm (learning law) of type

$$u_{k+1}(t) = \mathcal{L}(u_k(t), e_i(t)),\ \ k = 0, 1, \ldots, \tag{2}$$

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and investigating the properties of trajectories or outputs generated by these controls (here $\mathcal{L}$ is an operator and $e_i(t) = w_d(t) - w_i(t)$ where w_i is the output generated by input u_i, w_d is the desired output). System (1) with law (2) is called Iterative Learning Control (ILC) process. Repetitive processes (including discrete ones) are investigated mainly in the aspect of controllability and stability (Rogers et al. 2007; Idczak and Kamocki 2007; Idczak 2009; Srinivasan and Bonvin 2007; Galkowski et al. 2003; Cichy et al. 2013; Paszke and Bachelier 2013). They have numerous applications—in modeling of long-wall coal cutting, metal rolling and chemical batch processes, in programming of robotic manipulators and to investigation of the linear and nonlinear systems as well as systems with unknown structure information (Ahn et al. 2007; Arimoto et al. 1984; Lee and Chien 2008; Norrlof 2008; Sun and Wang 2008; Ye et al. 2009b).

Recent investigations show that the dynamics of many systems are described more accurately with the aid of fractional derivatives. They are used, among others things, to the modeling of capacitor (Westerlund and Ekstam 1994; Dzielinski et al. 2011, 2010), fluid flow through porous material (Langlands 2006), viscoelastic materials (Bagley and Torvik 1986). Fractional systems are investigated in many aspects: existence and uniqueness of solutions (Daftardar-Gejji and Babakhani 2004; Delbosco and Rodino 1996; Hayek et al. 1999; Idczak and Kamocki 2011; Kilbas et al. 2006), stability and positivity (Buslowicz 2008, 2011; Kaczorek 2011a, b), calculus of variations (Agrawal 2002; Baleanu and Muslih 2005; El-Nabulusi and Torres 2007; Idczak and Majewski 2012; Riewe 1996, 1997), controllability (Kaczorek 2011c) and optimal control (Agrawal 2004; Jelicic and Petrovacki 2009; Kamocki 2012; Tricaud and Chen 2010).

For over a decade, the ILC processes connected with the fractional systems of type

$$\begin{cases} z^{(\alpha)} = f(t, z, w, u) \\ w = g(t, z, u) \end{cases}$$

where $z^{(\alpha)}$ is the Riemann–Liouville or Caputo fractional derivative, and classical or fractional learning laws (i.e. containing the fractional derivatives) are studied (Chen and Moore 2001; Chen et al. 2012; Lazarevic 2004; Li et al. 2011a, b, c, 2012; Sabatier et al. 2007; Ye et al. 2009a). Such processes can be used to study repetitive models described with the aid of fractional derivatives. Using, for example, convergence of ILC algorithms (Li et al. 2011c, 2012) for systems of such a type, one can investigate fractional linear and nonlinear control systems, in particular, describe the controls generating outputs with desired properties or approximations of such outputs. As we read in Li et al. (2011c): "*In recent years, the application of ILC to the fractional-order system becomes a popular topic. The development of new fractional-order ILC algorithms, which belongs to a branch of fractional-order control* (Sabatier et al. 2007; Oustaloup 1994; Machado 1997; Podlubny 1999; Kilbas et al. 2006), *is urgently needed*".

In our paper, we consider the following fractional differential repetitive process (without learning law)

$$\begin{cases} (D^{\alpha}_{a+} z_{k+1})(t) = A_1 z_{k+1}(t) + A_2 w_k(t) + B u_{k+1}(t) \\ w_{k+1}(t) = C_1 z_{k+1}(t) + C_2 w_k(t) + D u_{k+1}(t) \end{cases} \tag{3}$$

where $k \in \mathbb{N} \cup \{0\}$, $t \in [a, b]$, $D^{\alpha}_{a+} z_{k+1}$ is a fractional derivative of order $\alpha \in (0, 1)$. We investigate the cases of Riemann–Liouville and Caputo derivatives. In the first case, by $D^{\alpha}_{a+} z_{k+1}$ we mean the fractional derivative in Riemann–Liouville sense and consider system (3) with initial conditions of the form

$$\begin{cases} (I_{a+}^{1-\alpha} z_k)(a) = c_k & \text{for } k \in \mathbb{N}, \\ w_0(t) = f(t) & \text{for } t \in \mathbb{R},\ a \le t \le b, \end{cases}$$

where $I_{a+}^{1-\alpha} z_k$ is an integral of order $1-\alpha$ of the function z_k. In the second case, $D_{a+}^{\alpha} z_{k+1}$ is replaced by ${}^C D_{a+}^{\alpha} z_{k+1}$—the fractional derivative in Caputo sense and boundary conditions take the form

$$\begin{cases} z_k(a) = c_k & \text{for } k \in \mathbb{N}, \\ w_0(t) = f(t) & \text{for } t \in \mathbb{R},\ a \le t \le b. \end{cases}$$

In both cases, we study existence, uniqueness and continuous dependence of solutions z_k on controls u_k. In the case of Caputo derivative, we study also a controllability property of (3), connected with the piecewise constant controls (taking the finite number of values and with a finite number of switching points). Such controls are very important from the practical point of view. More precisely, we show that the reachable set $\mathcal{A}_{M,P}$ for process (3), corresponding to the piecewise constant controls with the values in a fixed set M, is dense in the reachable set $\mathcal{A}_M$ corresponding to integrable controls taking their values in M. In other words, a point $(z_k(b))_{k\in\mathbb{N}} \in \prod_{k=1}^{\infty} \mathbb{R}^n$ that can be reached with the aid of an integrable control with values in M, can be approximated by points reachable with the aid of piecewise constant controls with values in M.

2 Preliminaries

In this section, we recall some definitions and basic facts concerning the fractional integrals and derivatives.

By $L^1 = L^1([a,b], \mathbb{R}^n)$ we shall denote the classical space of integrable functions $x : [a,b] \to \mathbb{R}^n$ and by Γ—the Euler gamma-function.

If $\alpha \in (0,1)$, $x \in L^1$, then the function

$$(I_{a+}^{\alpha} x)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}} d\tau,\ t \in [a,b] \text{ a.e.},$$

is called left-sided fractional integral of order α of the function x on the interval $[a,b]$, in the Riemann–Liouville sense. The function $D_{a+}^{\alpha} x(t) := \frac{d}{dx}(I_{a+}^{1-\alpha} x)(t)$ is called left-sided fractional derivative $D_{a+}^{\alpha} x$ of order α on the interval $[a,b]$, in the Riemann–Liouville sense, provided that $I_{a+}^{1-\alpha} x$ is absolutely continuous on $[a,b]$ (more precisely, has an absolutely continuous representative a.e. on $[a,b]$).

One can show that

$$(D_{a+}^{\alpha} x)(t) = 0,\ t \in [a,b] \text{ a.e.}$$

if and only if there exists a constant c such that

$$x(t) = \frac{c}{(t-a)^{1-\alpha}},\ t \in [a,b] \text{ a.e.}$$

By $AC_{a+}^{\alpha} = AC_{a+}^{\alpha}([a,b], R^n)$ we denote the set of all functions $x : [a,b] \to \mathbb{R}^n$ such that

$$x(t) = \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + (I_{a+}^{\alpha} \varphi)(t),\ t \in (a,b) \text{ a.e.,} \tag{4}$$

with $\varphi \in L^1$, $c \in \mathbb{R}^n$. One can show that x possesses the Riemann–Liouville derivative $D^{\alpha}_{a+}x$ if and only if $x \in AC^{\alpha}_{a+}$. In such a case $D^{\alpha}_{a+}x = \varphi$ a.e. on (a, b) and $(I^{1-\alpha}_{a+}x)(a) = c$ where φ and c are taken from (4). It is easy to see that AC^{α}_{a+} with the norm $\|x\|_{AC^{\alpha}_{a+}} = \left|(I^{1-\alpha}_{a+}x)(a)\right| + \left\|D^{\alpha}_{a+}x\right\|_{L^1}$ is complete.

More properties of fractional integrals and derivatives can be found for example in monographs (Kilbas et al. 2006; Samko et al. 1993).

3 Control system with Riemann–Liouville derivative

Let us consider the following Cauchy problem

$$\begin{cases} (D^{\alpha}_{a+}x)(t) = f(t, x(t)), & t \in [a, b] \text{ a.e.} \\ (I^{1-\alpha}_{a+}x)(a) = c, \end{cases} \tag{5}$$

where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$, $c \in \mathbb{R}^n$.

We have (cf. Idczak and Kamocki 2011)

Theorem 1 *If a function $f = f(t, x) : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in t, lipschitzian in x and the function $f(\cdot, 0)$ is integrable, then problem (5) has a unique solution $x \in AC^{\alpha}_{a+}$.*

Now, let us consider the following control system

$$\begin{cases} (D^{\alpha}_{a+}x)(t) = g(t, x(t), u(t)), & t \in [a, b] \text{ a.e.} \\ (I^{1-\alpha}_{a+}x)(a) = c, \end{cases} \tag{6}$$

where $g : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $u \in L^1([a, b], \mathbb{R}^m)$.

We have

Theorem 2 *If g is measurable in t, lipschitzian in x, continuous in u and there exist $c \geq 0$ and $r(\cdot) \in L^1$ such that*

$$|g(t, 0, u)| \leq c\,|u| + r(t)$$

for $t \in [a, b]$ a.e. and $u \in \mathbb{R}^m$, then, for any fixed control $u \in L^1$, system (6) has a unique solution $x \in AC^{\alpha}_{a+}$. Moreover, if $u_j \underset{j\to\infty}{\longrightarrow} u_0$ in L^1, then $x_j \underset{j\to\infty}{\longrightarrow} x_0$ in AC^{α}_{a+}. Consequently, $I^{1-\alpha}_{a+}z_j \underset{j\to\infty}{\rightrightarrows} I^{1-\alpha}_{a+}z_0$ uniformly on $[a, b]$ and $z_j \underset{j\to\infty}{\longrightarrow} z_0$ in L^1.

Proof Existence and uniqueness of a solution system (6) follows from Theorem 1. So, let $u_n \underset{n\to\infty}{\longrightarrow} u_0$ in L^1. It is easy to see that one can assume that $c = 0$. From the proof of Theorem 1 (cf. Idczak and Kamocki 2011) it follows that for any $j \in \mathbb{N} \cup \{0\}$ there exists a unique fixed point φ_j of the operator

$$\Phi_j : L^1 \ni \varphi \longmapsto g(t, (I^{\alpha}_{a+}\varphi)(t), u_j(t)) \in L^1$$

and the solution x_j corresponding to u_j is given by $x_j = I^{\alpha}_{a+}\varphi_j$. Let us observe that, for any fixed $k > 0$,

$$\begin{aligned} \left\|\varphi_j - \varphi_0\right\|_k &= \left\|\Phi_j(\varphi_j) - \Phi_0(\varphi_0)\right\|_k \leq \left\|\Phi_j(\varphi_j) - \Phi_j(\varphi_0)\right\|_k + \left\|\Phi_j(\varphi_0) - \Phi_0(\varphi_0)\right\|_k \\ &\leq Mk^{-\alpha}\left\|\varphi_j - \varphi_0\right\|_k + \left\|\Phi_j(\varphi_0) - \Phi_0(\varphi_0)\right\|_k, \end{aligned}$$

where $\|\varphi\|_k = \int_a^b \frac{\varphi(t)}{e^{kt}} dt$ for $\varphi \in L^1$ is the well known Bielecki norm in L^1, equivalent to the classical one. So, for sufficiently large $k > 0$ (such that $Mk^{-\alpha} \in (0, 1)$) we have

$$\left\|\varphi_j - \varphi_0\right\|_k \leq \frac{1}{1 - Mk^{-\alpha}} \left\|\Phi_j(\varphi_0) - \Phi_0(\varphi_0)\right\|_k \leq \frac{e^{-ka}}{1 - Mk^{-\alpha}} \left\|\Phi_j(\varphi_0) - \Phi_0(\varphi_0)\right\|$$
$$= \frac{e^{-ka}}{1 - Mk^{-\alpha}} \int_a^b \mid g(t, (I_{a+}^{\alpha}\varphi_0)(t), u_j(t)) - g(t, (I_{a+}^{\alpha}\varphi_0)(t), u_0(t)) \mid dt \underset{n\to\infty}{\longrightarrow} 0$$

(the last convergence follows from the continuity of the Nemytskii operator given by the Caratheodory function $h(t, u) = g(t, (I_{a+}^{\alpha}\varphi_0)(t), u))$. So, $x_j \underset{j\to\infty}{\longrightarrow} x_0$ in AC_{a+}^{α} because of

$$\left\|\varphi_j - \varphi_0\right\|_{L^1} \leq e^{kb} \left\|\varphi_j - \varphi_0\right\|_k$$

for $j \in \mathbb{N}$ and $\left\|x_j - x_0\right\|_{AC_{a+}^{\alpha}} = \left\|D_{a+}^{\alpha}x_j - D_{a+}^{\alpha}x_0\right\|_{L^1} = \left\|\varphi_j - \varphi_0\right\|_{L^1}$.

Consequently, since

$$\left(I_{a+}^{1-\alpha} z_j\right)(t) = I_{a+}^{1-\alpha}((I_{a+}^{\alpha} D_{a+}^{\alpha} z_j)(\cdot) + \frac{1}{\Gamma(\alpha)} \frac{c}{((\cdot) - a)^{1-\alpha}})(t) = \left(I_{a+}^{1} D_{a+}^{\alpha} z_j\right)(t) + c,$$

for $t \in [a, b]$ a.e., $j = 0, 1, \ldots$, we have (after identifying the sides of the above equality)

$$\left|(I_{a+}^{1-\alpha} z_j)(t) - (I_{a+}^{1-\alpha} z_0)(t)\right| = \left|(I_{a+}^{1} D_{a+}^{\alpha} z_j)(t) - (I_{a+}^{1} D_{a+}^{\alpha} z_0)(t)\right|$$
$$\leq \int_a^t \left|(D_{a+}^{\alpha} z_j)(s) - (D_{a+}^{\alpha} z_0)(s)\right| ds$$
$$\leq \left\|D_{a+}^{\alpha} z_j - D_{a+}^{\alpha} z_0\right\|_{L^1} = \left\|z_j - z_0\right\|_{AC_{a+}^{\alpha}}$$

for $t \in [a, b]$.

Moreover,

$$\left|z_j(t) - z_0(t)\right| \leq \left|(I_{a+}^{\alpha}(D_{a+}^{\alpha} z_j))(t) - (I_{a+}^{\alpha}(D_{a+}^{\alpha} z_0))(t)\right| = \left|(I_{a+}^{\alpha}(D_{a+}^{\alpha} z_j - D_{a+}^{\alpha} z_0))(t)\right|$$

for $t \in [a, b]$ a.e. So, using the boundedness of the operator $I_{a+}^{\alpha} : L^1 \to L^1$ (cf. Samko et al. 1993), we obtain

$$\left\|z_j - z_0\right\|_{L^1} \leq \left\|I_{a+}^{\alpha}(D_{a+}^{\alpha} z_j - D_{a+}^{\alpha} z_0)\right\|_{L^1} \leq K \left\|D^{\alpha} z_j - D^{\alpha} z_0\right\|_{L^1} = K \left\|z_j - z_0\right\|_{AC_{a+}^{\alpha}}$$

where $K > 0$. □

4 Repetitive processes with Riemann–Liouville derivative

Now, let us consider the fractional repetitive process of the form

$$\begin{cases} (D_{a+}^{\alpha} z_{k+1})(t) = A_1 z_{k+1}(t) + A_2 w_k(t) + B u_{k+1}(t) \\ w_{k+1}(t) = C_1 z_{k+1}(t) + C_2 w_k(t) + D u_{k+1}(t) \end{cases} \quad (7)$$

for $k \in \mathbb{N} \cup \{0\}$, $t \in \mathbb{R}$, $t \in [a, b]$ a.e., with initial conditions

$$\begin{cases} (I_{a+}^{1-\alpha} z_k)(a) = c_k & \text{for } k \in \mathbb{N}, \\ w_0(t) = f(t) & \text{for } t \in [a, b]. \end{cases} \quad (8)$$

Here $[a, b] \subset \mathbb{R}$ is a fixed interval, $z_k(t) \in \mathbb{R}^n$, $w_k(t) \in \mathbb{R}^m$, $u_k(t) \in \mathbb{R}^r$, A_1, A_2, B, C_1, C_2, D are matrices of appropriate dimensions. Constants $c_k \in \mathbb{R}^n$ and function $f : [a, b] \to \mathbb{R}^m$ are initial data, $D^{\alpha}_{a+} z_k$, $I^{1-\alpha}_{a+} z_k$ are derivative of order $\alpha \in (0, 1)$ and integral of order $1-\alpha$, in the Riemann–Liouville sense, of the function z_k, respectively. Functions $u_k(\cdot)$, $k \in \mathbb{N}$, are called controls on passes, $z_k(\cdot)$, $k \in \mathbb{N}$—trajectories on passes, $w_k(\cdot)$, $k \in \mathbb{N}$—outputs on passes.

When $\alpha = 1$ process (7)–(8) reduces to the classical repetitive process.

It is easy to see that process (7)–(8) can be written in the form of the following infinite system of equations

$$\begin{cases} D^{\alpha}_{a+} z_1 = A_1 z_1 + A_2 f + B u_1 \\ D^{\alpha}_{a+} z_2 = A_1 z_2 + A_2 C_1 z_1 + A_2 C_2 f + A_2 D u_1 + B u_2 \\ D^{\alpha}_{a+} z_3 = A_1 z_3 + A_2 C_1 z_2 + A_2 C_2 C_1 z_1 + A_2 C_2^2 f + A_2 C_2 D u_1 + A_2 D u_2 + B u_3 \\ \vdots \\ D^{\alpha}_{a+} z_k = A_1 z_k + \sum_{i=1}^{k-1} A_2 C_2^{k-1-i} C_1 z_i + A_2 C_2^{k-1} f + \sum_{j=1}^{k-1} A_2 C_2^{j-1} D u_{k-j} + B u_k \\ \vdots \end{cases} \tag{9}$$

with initial conditions

$$(I^{1-\alpha}_{a+} z_k)(a) = c_k \text{ for } k \in \mathbb{N}. \tag{10}$$

Our aim is to investigate existence, uniqueness and continuous dependence of solutions on controls for process (9)–(10). We consider this process in the spaces

$$\begin{aligned} \mathcal{AC}^{\alpha}_{a+} &= \mathcal{AC}^{\alpha}_{a+}\left([a, b], \prod_{k=1}^{\infty} \mathbb{R}^n\right) \\ &= \left\{ \mathfrak{z} = (z_k)_{k \in \mathbb{N}} : [a, b] \to \prod_{k=1}^{\infty} \mathbb{R}^n;\ z_k \in AC^{\alpha}_{a+},\ k \in \mathbb{N} \right\}, \end{aligned}$$

of trajectories and

$$\mathcal{L}^1 = \mathcal{L}^1\left([a, b], \prod_{k=1}^{\infty} \mathbb{R}^r\right) = \left\{ \mathfrak{u} = (u_k)_{k \in \mathbb{N}} : [a, b] \to \prod_{k=1}^{\infty} \mathbb{R}^r;\ u_k \in L^1,\ k \in \mathbb{N} \right\}$$

of controls.

The spaces $\mathcal{AC}^{\alpha}_{a+}$, $\mathcal{L}^1$ and $\prod_{k=1}^{\infty} \mathbb{R}^n$ are considered with product topologies. Let us recall that, for example, a sequence $(\mathfrak{u}^j)_{j \in \mathbb{N}}$ converges in $\mathcal{L}^1$ to an element $\mathfrak{u}^0$ with respect to the product topology if and only if

$$u_k^j \underset{j \to \infty}{\longrightarrow} u_k^0 \text{ in } L^p$$

for any $k \in \mathbb{N}$.

We have

Theorem 3 *For any control $\mathfrak{u} = (u_k)_{k \in \mathbb{N}} \in \mathcal{L}^1$ and initial data $c_k \in \mathbb{R}^n$, $k \in \mathbb{N}$, $f \in L^1$, there exists a unique solution $\mathfrak{z} = (z_k)_{k \in \mathbb{N}} \in \mathcal{AC}^{\alpha}_{a+}$ to process (9)–(10). Moreover, if a sequence $(\mathfrak{u}^j)_{j \in \mathbb{N}} = ((u_k^j)_{k \in \mathbb{N}})_{j \in \mathbb{N}}$ of controls converges in $\mathcal{L}^1$ to $\mathfrak{u}^0$, then $z_k^j \underset{j \to \infty}{\longrightarrow} z_k^0$ in*

AC^{α}_{a+} *for* $k \in \mathbb{N}$. *Consequently,* $I^{1-\alpha}_{a+} z^j_k \underset{j\to\infty}{\rightrightarrows} I^{1-\alpha}_{a+} z^0_k$ *uniformly on* $[a,b]$ *for* $k \in \mathbb{N}$ *and* $z^j_k \underset{j\to\infty}{\longrightarrow} z^0_k$ *in* L^1 *for* $k \in \mathbb{N}$.

Proof Let us fix $k \in \mathbb{N}$ and consider system

$$D^{\alpha}_{a+}\mathtt{z}(t) = \mathbb{A}_k \mathtt{z}(t) + \mathtt{F}_k f + \mathtt{B}_k \mathtt{u} \tag{1_k}$$

where $\mathtt{z} = (z_1, \ldots, z_k)$, $\mathtt{u} = (u_1, \ldots, u_k)$,

$$\mathbb{A}_k = \begin{bmatrix} A_1 & \ldots & 0 \\ \vdots & & \vdots \\ A_2 C_2^{k-2} C_1 & \ldots & A_1 \end{bmatrix}, \ \mathtt{F}_k = \begin{bmatrix} A_2 \\ \vdots \\ A_2 C_2^{k-1} \end{bmatrix}, \ \mathtt{B}_k = \begin{bmatrix} B & \ldots & 0 \\ \vdots & & \vdots \\ A_2 C_2^{k-2} D & \ldots & B \end{bmatrix}.$$

Existence and uniqueness of a solution to (1_k), satisfying initial condition

$$(I^{1-\alpha}_{a+}\mathtt{z})(a) = \mathtt{c} \tag{2_k}$$

where $\mathtt{c} = (c_1, \ldots, c_k)$, follows from the first part of Theorem 2. Convergences follows from the second part of this theorem. □

5 Control system with Caputo derivative

Let us denote

$$AC = \{x : [a,b] \to \mathbb{R}^n;\ x \text{ is absolutely continuous}\}.$$

By the left-sided Caputo fractional derivative ${}^C D^{\alpha}_{a+}x$ of order α of the function $x \in AC$ on the interval $[a,b]$ we mean the function

$$({}^C D^{\alpha}_{a+}x)(t) := D^{\alpha}_{a+}(x(\cdot) - x(a))(t),\ t \in [a,b] \text{ a.e.}$$

From (Samko et al. 1993, Lemma 2.1 and the next corollary) it follows that

$$({}^C D^{\alpha}_{a+}x)(t) = (D^{\alpha}_{a+}x)(t) - \frac{1}{\Gamma(1-\alpha)} \frac{x(a)}{(t-a)^{\alpha}} = I^{1-\alpha}_{a+}\left(\frac{d}{dt}x\right)(t). \tag{11}$$

It is easy to see that $({}^C D^{\alpha}_{a+}x)(\cdot) = 0$ if and only if $x(\cdot)$ is constant on $[a,b]$.

Now, let us consider the following Cauchy problem

$$\begin{cases} ({}^C D^{\alpha}_{a+}x)(t) = Ax(t) + v(t), & t \in [a,b] \text{ a.e.} \\ x(a) = c, \end{cases} \tag{12}$$

where $v : [a,b] \to \mathbb{R}^n$. By a solution to this problem we mean a function $x \in AC$.

Let us observe that if a function $x \in AC$ is a solution to (12), then (cf. Samko et al. 1993, Lemma 2.1)

$$\begin{cases} (D^{\alpha}_{a+}x)(t) = Ax(t) + \frac{1}{\Gamma(1-\alpha)}\frac{c}{(t-a)^{\alpha}} + v(t), & t \in [a,b] \text{ a.e.} \\ (I^{1-\alpha}_{a+}x)(a) = 0 \\ x(a) = c. \end{cases} \tag{13}$$

Conversely, if $x \in AC$ is a solution to (13), then $x(\cdot)$ is a solution to (12).

So, to obtain the existence of a unique solution to problem (12) it is sufficient to show that there exists a unique solution $x \in AC^{\alpha}_{a+}$ to problem

$$\begin{cases} (D^{\alpha}_{a+}x)(t) = Ax(t) + \frac{1}{\Gamma(1-\alpha)}\frac{c}{(t-a)^{\alpha}} + v(t),\ t \in [a,b] \text{ a.e.} \\ (I^{1-\alpha}_{a+}x)(a) = 0 \end{cases} \tag{14}$$

such that

$$\begin{cases} x \in AC \\ x(a) = c. \end{cases} \tag{15}$$

Indeed, we have

Theorem 4 *If $v \in I^{1-\alpha}_{a+}(L^1)$, then the unique solution $x \in AC^{\alpha}_{a+}$ to problem (14) satisfies (15) and, consequently, Cauchy problem (12) has a unique solution in AC.*

Proof Let us recall (cf. Idczak and Kamocki 2011) that the unique solution $x \in AC^{\alpha}_{a+}$ to problem (14) is given by

$$x(t) = (I^{\alpha}_{a+}\varphi_*)(t),\ t \in [a,b] \text{ a.e.},$$

where $\varphi_* \in L^1$ is a unique fixed point of the operator $\Phi : L^1 \to L^1$ given by

$$\Phi(\varphi)(t) = A\frac{1}{\Gamma(\alpha)}\int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}d\tau + \frac{1}{\Gamma(1-\alpha)}\frac{c}{(t-a)^{\alpha}} + v(t).$$

for $\varphi \in L^1$. In the same way as in Idczak and Kamocki (2011) one can show that

$$(I^{\alpha}_{a+}\varphi_*)(t) = A^m(I^{(m+1)\alpha}_{a+}\varphi_*)(t) + A^{m-1}(I^{(m-1)\alpha}_{a+}c)(t) + \cdots + A\left(I^{\alpha}_{a+}c\right)(t) + c$$
$$+A^{m-1}(I^{m\alpha}_{a+}v)(t) + \cdots + A\left(I^{2\alpha}_{a+}v\right)(t) + (I^{\alpha}_{a+}v)(t),\ t \in [a,b] \text{ a.e.}, \tag{16}$$

for any $m \in \mathbb{N}$.

All terms on the right hand side of the above equality, except the first one, are absolutely continuous (here, we use the fact that $v \in I^{1-\alpha}_{a+}(L^1)$) and all, except the first term and c, take the value 0 at the point $t = a$ (cf. Samko et al. 1993, Lemma 2.1). Of course, there exists $m \in \mathbb{N}$ such that $(m+1)\alpha \geq 1$ and $\beta := (m+1)\alpha - 1 \in (0,1)$. Consequently, one can write down the first term in the form

$$A^m\left(I^{(m+1)\alpha}_{a+}\varphi_*\right)(t) = A^m\left(I^1_{a+}(I^{\beta}\varphi_*)\right)(t),\ t \in [a,b] \text{ a.e.},$$

It means that it is an absolutely continuous function on $[a,b]$ and takes the value 0 at the point $t = a$.

So, the solution $x = I^{\alpha}_{a+}\varphi_*$ of problem (14) is absolutely continuous on $[a,b]$ and $x(a) = c$.

Moreover, if $x_1, x_2 \in AC$ are the solutions to (12), then, as we said, they are the solutions to (14) and, consequently, to $x_1 = x_2$. □

We also have the following theorem on the continuous dependence of solutions on controls.

Theorem 5 *If $v_j \in I^{1-\alpha}_{a+}(L^1)$, $j = 0, 1, \ldots$, and $v_j \underset{n\to\infty}{\longrightarrow} v_0$ in L^1, then $x_j \underset{n\to\infty}{\longrightarrow} x_0$ in AC^{α}_{a+} (here $x_j \in AC$ is a solution to (12), corresponding to v_j). Consequently, $I^{1-\alpha}_{a+}x_j \underset{j\to\infty}{\rightrightarrows} I^{1-\alpha}_{a+}x_0$ uniformly on $[a,b]$ and $x_j \underset{j\to\infty}{\longrightarrow} x_0$ in L^1.*

Proof Since x_j, $j \in \mathbb{N} \cup \{0\}$, are solutions to (13) corresponding to $\frac{1}{\Gamma(1-\alpha)} \frac{c}{(\cdot-a)^\alpha} + v(\cdot)$, the assertion follows from Theorem 2. □

Under a stronger assumption on the convergence of controls, the trajectories converge uniformly on $[a, b]$. Namely, we have

Theorem 6 *If $v_j \in I_{a+}^{1-\alpha}(L^1)$, $j \in \mathbb{N} \cup \{0\}$ are such that $v_j = I_{a+}^{1-\alpha} u_j$ and $u_j \underset{n\to\infty}{\longrightarrow} u_0$ in L^1, then $x_j(t) \underset{j\to\infty}{\rightrightarrows} x_0(t)$ on $[a, b]$.*

Proof Let $\varphi_j \in L^1$, $j \in \mathbb{N} \cup \{0\}$ be such that $x_j = I_{a+}^\alpha \varphi_j$. From the first part of Theorem 5 it follows that $\varphi_j \underset{n\to\infty}{\longrightarrow} \varphi_0$ in L^1. From (16) we obtain

$$
\begin{aligned}
x_j(t) - x_0(t) =& (I_{a+}^\alpha \varphi_j)(t) - (I_{a+}^\alpha \varphi_0)(t) \\
=& A^m \left(I_{a+}^{(m+1)\alpha}(\varphi_j - \varphi_0)\right)(t) + A^{m-1}\left(I_{a+}^{m\alpha}(v_j - v_0)\right)(t) \\
&+ \cdots + A\left(I_{a+}^{2\alpha}(v_j - v_0)\right)(t) + (I_{a+}^\alpha(v_j - v_0))(t) \\
=& A^m \left(I_{a+}^{1+\beta}(\varphi_j - \varphi_0)\right)(t) + A^{m-1}\left(I_{a+}^{m\alpha} I_{a+}^{1-\alpha}(u_j - u_0)\right)(t) \\
&+ \cdots + A\left(I_{a+}^{2\alpha} I_{a+}^{1-\alpha}(u_j - u_0)\right)(t) + (I_{a+}^\alpha I_{a+}^{1-\alpha}(u_j - u_0))(t) \\
=& A^m \left(I_{a+}^1 I_{a+}^\beta(\varphi_j - \varphi_0)\right)(t) + A^{m-1}\left(I_{a+}^1 I_{a+}^{(m-1)\alpha}(u_j - u_0)\right)(t) \\
&+ \cdots + A\left(I_{a+}^1 I_{a+}^\alpha(u_j - u_0)\right)(t) + \left(I_{a+}^1(u_j - u_0)\right)(t)
\end{aligned}
$$

for $t \in [a, b]$a.e., where $\beta \in (0, 1)$ is such as in the proof of Theorem 4. Since both left and right functions are absolutely continuous, the equality holds true at any point $t \in [a, b]$. The convergences $\varphi_j \underset{n\to\infty}{\longrightarrow} \varphi_0$ in L^1 and $u_j \underset{n\to\infty}{\longrightarrow} u_0$ in L^1 imply the convergences $I_{a+}^\beta(\varphi_j - \varphi_0) \underset{n\to\infty}{\longrightarrow} 0$, $I_{a+}^{(m-1)\alpha}(u_j - u_0) \underset{n\to\infty}{\longrightarrow} 0, \ldots, I_{a+}^\alpha(u_j - u_0) \underset{n\to\infty}{\longrightarrow} 0$ in L^1. So, all functions $I_{a+}^1 I_{a+}^\beta(\varphi_j - \varphi_0)$, $I_{a+}^1 I_{a+}^{(m-1)\alpha}(u_j - u_0), \ldots, I_{a+}^1 I_{a+}^\alpha(u_j - u_0)$, $I_{a+}^1(u_j - u_0)$ converge uniformly to zero function on $[a, b]$. □

The fact that the linear continuous mapping preserves the weak convergence we obtain

Corollary 1 *Under notations of the previous theorem, if $c = 0$ and $u_j \underset{n\to\infty}{\rightharpoonup} u_0$ weakly in L^1, then $x_j(t) \underset{j\to\infty}{\to} x_0(t)$ for any $t \in [a, b]$.*

6 Repetitive processes with Caputo derivative

Now, let us consider fractional differential repetitive process

$$
\begin{cases}
({}^C D_{a+}^\alpha z_{k+1})(t) = A_1 z_{k+1}(t) + A_2 w_k(t) + B v_{k+1}(t) \\
w_{k+1}(t) = C_1 z_{k+1}(t) + C_2 w_k(t) + D v_{k+1}(t)
\end{cases}
\tag{17}
$$

for $k \in \mathbb{N} \cup \{0\}$, $t \in \mathbb{R}$, $a \le t \le b$, with initial conditions

$$
\begin{cases}
z_k(a) = c_k & \text{for } k \in \mathbb{N} \\
w_0(t) = f(t) & \text{for } t \in [a, b].
\end{cases}
\tag{18}
$$

Denoting

$$\mathcal{AC} = \mathcal{AC}\left([a,b], \prod_{k=1}^{\infty}\mathbb{R}^n\right) = \left\{\mathfrak{z} = (z_k)_{k\in\mathbb{N}} : [a,b] \to \prod_{k=1}^{\infty}\mathbb{R}^n;\, z_k \in AC,\ k \in \mathbb{N}\right\},$$

in the same way as in the previous section, we obtain

Theorem 7 *For any* $\mathfrak{v} = (v_k)_{k\in\mathbb{N}} \in \mathcal{I}_{a+}^{1-\alpha}(\mathcal{L}^1) = \mathcal{I}_{a+}^{1-\alpha}(\mathcal{L}^1([a,b], \prod_{k=1}^{\infty}\mathbb{R}^r))$ *where*

$$\mathcal{I}_{a+}^{1-\alpha}(\mathcal{L}^1) = \left\{\mathfrak{v} = (v_k)_{k\in\mathbb{N}} : [a,b] \to \prod_{k=1}^{\infty}\mathbb{R}^r;\ v_k \in I_{a+}^{1-\alpha}(L^1),\ k \in \mathbb{N}\right\},$$

and initial data $c_k \in \mathbb{R}^n$, $k \in \mathbb{N}$, $f \in I_{a+}^{1-\alpha}(L^1)$, *there exists a unique solution* $\mathfrak{z} = (z_k)_{k\in\mathbb{N}} \in \mathcal{AC}$ *of problem (17)–(18). Moreover, if a sequence* $(\mathfrak{v}^j)_{j\in\mathbb{N}} = ((v_k^j)_{k\in\mathbb{N}})_{j\in\mathbb{N}} \subset \mathcal{I}_{a+}^{1-\alpha}(\mathcal{L}^1)$ *converges in* $\mathcal{L}^1$ *to* $\mathfrak{u}^0 \in \mathcal{I}_{a+}^{1-\alpha}(\mathcal{L}^1)$, *then* $z_k^j \underset{j\to\infty}{\longrightarrow} z_k^0$ *in* AC_{a+}^{α} *for* $k \in \mathbb{N}$. *Consequently,* $I_{a+}^{1-\alpha} z_k^j \underset{j\to\infty}{\rightrightarrows} I_{a+}^{1-\alpha} z_k^0$ *uniformly on* $[a,b]$ *and* $z_k^j \underset{j\to\infty}{\longrightarrow} z_k^0$ *in* L^1 *for* $k \in \mathbb{N}$.

Proof In the same way as in the proof of Theorem 3 we assert that problem (17)–(18) has a unique solution in $\mathcal{AC}$. Continuous dependence of solutions on controls follows from Theorem 5. □

From Corollary 1 we obtain

Theorem 8 *If* $f = 0$, $c_k = 0$ *for* $k \in \mathbb{N}$, $\mathfrak{v}^j = (v_k^j)_{k\in\mathbb{N}} \subset \mathcal{I}_{a+}^{1-\alpha}(\mathcal{L}^1)$, $j \in \mathbb{N}\cup\{0\}$ *are such that* $v_k^j = I_{a+}^{1-\alpha} u_k^j$ *and* $u_k^j \underset{j\to\infty}{\longrightarrow} u_k^0$ *weakly in* L^1 *for* $k \in \mathbb{N}$, *then* $z_k^j(t) \underset{j\to\infty}{\to} z_k^j(t)$ *for any* $t \in [a,b]$ *and* $k \in \mathbb{N}$.

7 Applications to controllability

Let us consider process (17) with homogenous initial conditions

$$\begin{cases} z_k(a) = 0 & \text{for } k \in \mathbb{N} \\ w_0(t) = 0 & \text{for } t \in [a,b]. \end{cases} \tag{19}$$

By $\mathcal{L}_M^1$ where $M \subset \mathbb{R}^r$ is a convex compact set, we denote the set

$$\left\{u = (u_k)_{k\in\mathbb{N}} : [a,b] \to \prod_{k=1}^{\infty}\mathbb{R}^r \in \mathcal{L}^1;\ u_k \text{ is measurable} \right.$$
$$\left. \text{and } u_k(t) \in M \text{ for } t \in [a,b], k \in \mathbb{N}\right\}$$

and by $\mathcal{I}_{a+}^{1-\alpha}(\mathcal{L}_M^1)$—the set

$$\left\{\mathfrak{v} = (v_k)_{k\in\mathbb{N}} : [a,b] \to \prod_{k=1}^{\infty}\mathbb{R}^r;\ v_k = I_{a+}^{1-\alpha} u_k \text{ where } u_k \text{ is measurable} \right.$$
$$\left. \text{and } u_k(t) \in M \text{ for } t \in [a,b] \text{ a.e.}, k \in \mathbb{N}\right\}.$$

In this section, it will be more convenient the functions u_k (not v_k) to call the controls.

By $\mathcal{A}_M$ we denote the reachable set for process (17)–(19), corresponding to the set of controls $\mathcal{L}^1_M$ i.e.

$$\mathcal{A}_M = \left\{ \mathfrak{z}(b) = (z_k(b))_{k\in\mathbb{N}} \in \prod_{k=1}^{\infty} \mathbb{R}^n;\ \text{there exists } \mathfrak{v} = (v_k)_{k\in\mathbb{N}} \subset \mathcal{I}^{1-\alpha}_{a+}(\mathcal{L}^1_M) \right.$$
$$\left. \text{such that } \mathfrak{z} = (z_k)_{k\in\mathbb{N}} \text{ is the solution to (17)–(19), corresponding to } \mathfrak{v} \right\}.$$

Analogously, by $\mathcal{I}^{1-\alpha}_{a+}(\mathcal{L}^1_{M,PC})$ we mean the set ([1])

$$\left\{ \mathfrak{v} = (v_k)_{k\in\mathbb{N}} : [a,b] \to \prod_{k=1}^{\infty} \mathbb{R}^r;\ v_k = I^{1-\alpha}_{a+} u_k \text{ where } u_k \text{ is piecewise constant} \right.$$
$$\left. \text{and } u_k(t) \in M \text{ for } t \in [a,b] \text{ a.e., } k \in \mathbb{N} \right\}$$

and by $\mathcal{A}_{M,PC}$ we mean the reachable set for process (17)–(19), corresponding to the set $\mathcal{L}_{M,PC}$ of piecewise constant controls , i.e.

$$\mathcal{A}_{M,PC} = \left\{ \mathfrak{z}(b) = (z_k(b))_{k\in\mathbb{N}} \in \prod_{k=1}^{\infty} \mathbb{R}^n;\ \text{there exists } \mathfrak{v} = (v_k)_{k\in\mathbb{N}} \subset \mathcal{I}^{1-\alpha}_{a+}(\mathcal{L}^1_{M,PC}) \right.$$
$$\text{such that } v_k = I^{1-\alpha}_{a+} u_k,\ u_k \in \mathcal{L}^1_{PC,M} \text{ and } \mathfrak{z} = (z_k)_{k\in\mathbb{N}}$$
$$\left. \text{is the solution to (17)–(19), corresponding to } \mathfrak{v} \right\}.$$

Using Theorem 8, in the same way as in Idczak and Kamocki (2007, Theorem 18) one can obtain

Theorem 9 *The set $\mathcal{A}_M$ is closed in $\prod_{k=1}^{\infty} \mathbb{R}^n$.*

Next, using theorem on the density of the set $L^1_{PC}([a,b],M)$ of piecewise constant functions on $[a,b]$ with values in M in the set $L^1([a,b],M)$ of integrable functions on $[a,b]$ with values in M (cf. Bacciotti 1981), in the same way as in Idczak and Kamocki (2007, Theorem 20) we obtain

Theorem 10 *The closure $\overline{\mathcal{A}_{M,PC}}$ of the set $\mathcal{A}_{M,PC}$ in the space $\prod_{k=1}^{\infty} \mathbb{R}^n$ coincides with the set $\mathcal{A}_M$.*

Example 1 Let us consider repetitive process (17)–(19). If $\mathfrak{z} = (z_k)_{k\in\mathbb{N}}$ is the trajectory of this system, corresponding to the control $\mathfrak{v} = (v_k)_{k\in\mathbb{N}}$ where $v_k(t) = I^{1-\alpha}_{a+}(u_k)$ with $u_k(t) = \sin kt$, then there exist piecewise constant functions $\widetilde{u}^j_k : [a,b] \to [-1,1]$ such that $\mathfrak{z}^j(b) = (z^j_k(b))_{k\in\mathbb{N}} \underset{j\to\infty}{\longrightarrow} \mathfrak{z}(b) = (z_k(b))_{k\in\mathbb{N}}$ in $\prod_{k=1}^{\infty} \mathbb{R}^n$, where $\mathfrak{z}^j$ is the trajectory of the process (17)–(19), corresponding to the control $\widetilde{\mathfrak{v}}^j = (\widetilde{v}^j_k)_{k\in\mathbb{N}}$ with $\widetilde{v}^j_k(t) = I^{1-\alpha}_{a+}(\widetilde{u}^j_k)$. Convergence $(z^j_k(b))_{k\in\mathbb{N}} \underset{j\to\infty}{\longrightarrow} (z_k(b))_{k\in\mathbb{N}}$ in $\prod_{k=1}^{\infty} \mathbb{R}^n$ means that $z^j_k(b) \underset{j\to\infty}{\longrightarrow} z_k(b)$ for any $k \in \mathbb{N}$.

[1] We say that a function $u : [a,b] \to \mathbb{R}^r$ is piecewise constant, if there exist a partition $a = a_0 < a_1 < \cdots < a_n = b$ of the interval $[a,b]$ such that u is constant on each subinterval $[a_{i-1}, a_i]$.

8 Conclusions

In the paper, we prove existence, uniqueness and continuous dependence of solutions on functional parameters (controls) for a fractional repetitive process. First, we consider the case of Riemann–Liouville derivatives with initial conditions (8) and next, basing ourselves on the obtained results for Riemann–Liouville derivatives, we investigate the case of Caputo derivatives with initial conditions (18). The obtained results have been applied to study some controllability property for fractional repetitive processes with Caputo derivative and homogenous initial conditions of type (18). This property is important from the practical point of view. Namely, it has been shown that the states that can be reached with the aid of integrable controls with values in a set M can be approximated with the aid of piecewise constant ones taking their values in M. An example illustrating this result is given. Future work will addres the issue of approximation of the reachable states by the points that are reachable with the aid of piecewise constant controls taking their values in the set of extreme points of the set M. Theorem of such a kind for repetitive processes of the first order has been proved in Idczak (2009).

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