Samreen et al. *Journal of Inequalities and Applications* 2014, **2014**:266
http://www.journalofinequalitiesandapplications.com/content/2014/1/266

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theorems for φ-contractions

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Abstract

This paper deals with the fixed point theorems for mappings satisfying a contractive condition involving a gauge function φ when the underlying set is endowed with a *b*-metric. Our results generalize/extend the main results of Proinov and thus we obtain as special cases some results of Mysovskih, Rheinboldt, Gel'man, and Huang. We also furnish an example to substantiate the validity of our results. Subsequently, an existence theorem for the solution of initial value problem has also been established.

MSC: 47H10; 54H25

Keywords: fixed points; *b*-metric space; gauge functions

1 Introduction and preliminaries

The Banach contraction principle has been extensively used to study the existence of solutions for the nonlinear Volterra integral equations and nonlinear integro-differential equations and to prove the convergence of algorithms in computational mathematics. These applications elicit the significance of fixed point theory. Therefore mathematicians have been propelled to contribute enormously in the field of fixed point theory by finding the fixed point(s) of self-mappings or nonself-mappings defined on several ambient spaces and satisfying a variety of conditions. Among these fixed point theorems only a few have practical importance, *i.e.*, they provide a constructive method for finding fixed point(s). This provides information on the convergence rate along with error estimates. The Banach contraction principle is one of such theorems wherein the proposed iterative scheme converges linearly. Commonly, the iterative procedures serve as constructive methods in fixed point theory. Furthermore, it is also of crucial importance to have prior and posterior estimates for such methods. In this context, Proinov [1] extended the Banach contraction principle with a higher order of convergence. He proposed an iterative scheme for a mapping satisfying a contractive condition which involves a gauge function of order $r \geq 1$ and obtained error estimates as well. His results include as special cases some results of Mysovskih [2], Rheinboldt [3], Gel'man [4], Huang [5], and others. In [6] the authors extended the results of Proinov to the case of multivalued mappings.

For the last few decades fixed point theory has rapidly been evolving, not only in metric structure but also in many different generalized spaces and the *b*-metric space is one of them. The notion of a *b*-metric space was initiated in some works of Bourbaki, Bakhtin, Czerwik, and Heinonen. Several papers appeared which deal with the fixed point theory for single valued and multivalued functions in a *b*-metric space [7–11] *etc.*

Inspired by the work of Proinov [1] in this paper we investigate whether the consequences of his results hold when the underlying structure is replaced with a *b*-metric space. We give an affirmative answer to this question. Our results generalize main results of Proinov [1] and thus subsume many results of authors [2–5]. We establish an example to substantiate the validity of our results. Consequently, in Section 3 we also obtain an existence theorem for the solution of an initial value problem.

In the following we recall some auxiliary notions and results in a b-metric space [7–9, 11] which are needed subsequently.

Definition 1.1 [7, 11] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric space if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(d1) $d(x,y) = 0$ if and only if $x = y$;
(d2) $d(x,y) = d(y,x)$;
(d3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

The pair (X,d) is called a b-metric space with the coefficient s.

The following example shows that the class of b-metric spaces is essentially larger than the class of metric spaces.

Example 1.2 [8, 11, 12] (1) Let $X := l_p(\mathbb{R})$ with $0 < p < 1$ where $l_p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Define $d : X \times X \to \mathbb{R}^+$ as

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p},$$

where $x = \{x_n\}$, $y = \{y_n\}$. Then (X,d) is a b-metric space with coefficient $s = 2^{1/p}$.

(2) Let $X := L_p[0,1]$ be the space of all real functions $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p \, dt < \infty$. Define $d : X \times X \to \mathbb{R}^+$ as

$$d(x,y) = \left(\int_0^1 \left| x(t) - y(t) \right|^p dt \right)^{1/p}.$$

Then (X,d) is a b-metric space with coefficient $s = 2^{1/p}$

A sequence $\{x_n\}$ in a b-metric space X is convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$; it is Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$. A b-metric space (X,d) is complete if every Cauchy sequence in X converges.

Let (X,d) be a b-metric space; then a convergent sequence has a unique limit; every convergent sequence is Cauchy; and in general the b-metric d is not a continuous functional [13].

Definition 1.3 Let (X,d) be a b-metric space and A be a nonempty subset of X then closure $\overline{A}$ of A is the set consisting of all points of A and its limit points. Moreover, A is closed if and only if $A = \overline{A}$.

In the following the b-metric version of Cantor's intersection theorem is given, which can easily be established running along the same lines as in the proof of its metric version.

Theorem 1.4 [14] *Let (X, d) be a complete b-metric space, then every nested sequence of closed balls has a nonempty intersection.*

Let $f : D \subset X \to X$ and there exist some $x \in D$ such that the set $\mathcal{O}(x) = \{x, fx, f^2x, \ldots\} \subset D$. The set $\mathcal{O}(x)$ is known as an orbit of $x \in D$. We recall that a function G from D into the set of real numbers is said to be f-orbitally lower semi-continuous at $t \in X$ if $\{x_n\} \subset \mathcal{O}(x)$ and $x_n \to t$ implies $G(t) \leq \liminf G(x_n)$ [15].

Throughout this paper let J always denote an interval in $\mathbb{R}^+$ containing 0 *i.e.*, an interval of the form $[0, R]$, $[0, R)$ or $[0, \infty)$ ($[0, 0] = \{0\}$ is a trivial interval). Let $P_n(t)$ denote a polynomial of the form $P_n(t) = 1 + t + \cdots + t^{n-1}$ and $P_0(t) = 0$. Let φ^n denote the nth iterate of a function $\varphi : J \to J$.

Definition 1.5 [1] Let $r \geq 1$; a function $\varphi : J \to J$ is said to be a gauge function of order r on J if it satisfies the following conditions:

(i) $\varphi(\lambda t) \leq \lambda^r \varphi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$,

(ii) $\varphi(t) < t$ for all $t \in J \setminus \{0\}$.

The condition (i) of Definition 1.5 elicits $\varphi(0) = 0$ and $\varphi(t)/t^r$ is nondecreasing on $J \setminus \{0\}$. A gauge function $\varphi : J \to J$ is said to be a Bianchini-Grandolfi gauge function if $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ for all $t \in J$ [16].

Subsequently, in this paper let (X, d) be a b-metric space (unless specified otherwise) with a coefficient $s \geq 1$. We assume that $f : D \subset X \to X$ be an operator and there exist some $x_0 \in D$ such that $\mathcal{O}(x_0) \subset D$. Let the operator f satisfy the following iterated contractive condition:

$$d(fx, f^2x) \leq \varphi(d(x, fx)) \quad \text{for all } x \in \mathcal{O}(x_0) \text{ such that } d(x, fx) \in J, \tag{1.1}$$

where φ is a gauge function of order $r \geq 1$ on an interval J. We establish two convergence theorems for iterative processes of the type

$$x_{n+1} = fx_n, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where f satisfies (1.1).

2 *b*-Bianchini-Grandolfi gauge functions

In [1] Proinov proved his main results by assuming Bianchini-Grandolfi gauge functions and the mapping f satisfying the contractive condition (3.5) when the underlying space is endowed with a metric (see Corollary 3.9). But in the setting of b-metric space for some technical reasons we have to restrict ourselves to the gauge functions satisfying $\sum_{n=0}^{\infty} s^n \varphi^n(t) < \infty$ for all $t \in J$ where s is the coefficient of b-metric space. Furthermore, taking into account such a crucial condition in order to calculate prior and posterior estimates we consider the gauge functions of the form

$$\varphi(t) = t\frac{\phi(t)}{s} \quad \text{for all } t \in J, \tag{2.1}$$

where $s \geq 1$ is the coefficient of b-metric d and ϕ is nonnegative nondecreasing function on J such that

$$0 \leq \phi(t) < 1 \quad \text{for all } t \in J. \tag{2.2}$$

Remark 2.1 One can always define a nonnegative nondecreasing function ϕ on J satisfying (2.1) and (2.2) as follows:

$$\phi(t) = \begin{cases} \frac{s\varphi(t)}{t}, & \text{if } t \in J \setminus \{0\}, \\ 0, & \text{if } t = 0, \end{cases} \tag{2.3}$$

where s is the coefficient of b-metric d.

For a fixed $s \geq 1$, let us consider the following simple examples of gauge functions of order r:

(i) $\varphi(t) = \frac{ct}{s}$, $0 < c < 1$ is a gauge function of order 1 on $J = [0, \infty)$;

(ii) $\varphi(t) = \frac{ct^r}{s}$ ($c > 0$, $r > 1$) is a gauge function of order r on $J = [0, h)$ where $h = (\frac{1}{c})^{\frac{1}{(r-1)}}$.

It is essential to mention here that to establish the fixed point theorem (see Theorem 3.7) we do not necessarily require the gauge functions φ satisfying (2.1), (2.2). But we consider the gauge function such that $\sum_{n=0}^{\infty} s^n \varphi^n(t) < \infty$ for all $t \in J$ where s is a coefficient of b-metric space.

Lemma 2.2 *Let φ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying* (2.1) *and* (2.2), *then*:

(1) $0 \leq \frac{\phi(t)}{s} < 1$ *for all* $t \in J$,

(2) $\phi(\mu t) \leq \mu^{r-1}\phi(t)$ *for all* $\mu \in (0,1)$ *and* $t \in J$.

Remark 2.3 When d is a simple metric, then $s = 1$. In such case every gauge function satisfying $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ is of the form $\varphi(t) = t\phi(t)$ where ϕ is nonnegative nondecreasing function on J (see [1]). Thus in such case the condition $0 \leq \phi(t) < 1$ for all $t \in J$ becomes superfluous and directly follows from Lemma 2.2.

The following lemma is fundamental to our main results.

Lemma 2.4 *Let φ be a gauge function of order $r \geq 1$ on J. If ϕ is a nonnegative and nondecreasing function on J satisfying* (2.1) *and* (2.2), *then for every $n \geq 0$ we have*:

(1) $\varphi^n(t) \leq t[\frac{\phi(t)}{s}]^{P_n(r)}$ *for all* $t \in J$,

(2) $\phi(\varphi^n(t)) \leq s[\frac{\phi(t)}{s}]^{r^n}$ *for all* $t \in J$.

Proof (1) Set $\mu = \frac{\phi(t)}{s}$ and let $t \in J$. Then from Lemma 2.2 we obtain $0 \leq \mu < 1$. For $\mu = 0$ the case is trivial. We shall prove (1) by using mathematical induction. For $n = 0, 1$ the property (1) is trivially satisfied as it reduces to an equality. Let it also hold for any integer $n \geq 1$, *i.e.*,

$$\varphi^n(t) \leq t\mu^{P_n(r)}.$$

Since φ is nondecreasing on J, we obtain (as $t\mu^{P_n(r)} \in J$ because $t \in J$ and $\mu < 1$)

$$\begin{aligned} \varphi^{n+1}(t) &\leq \varphi\left[t\mu^{P_n(r)}\right] \leq \mu^{rP_n(r)}\varphi(t) \leq \mu^{rP_n(r)} t \frac{\phi(t)}{s} \\ &= t\mu^{rP_n(r)+1} = t\mu^{P_{n+1}(r)}. \end{aligned}$$

(2) By making use of Lemma 2.2 and monotonicity of ϕ, (1) leads to the following:

$$\begin{aligned}\phi(\varphi^n(t)) &\le \phi\left(t\left[\frac{\phi(t)}{s}\right]^{P_n(r)}\right) \le \left[\frac{\phi(t)}{s}\right]^{(r-1)P_n(r)}\phi(t)\\ &= s\left[\frac{\phi(t)}{s}\right]^{1+(r-1)P_n(r)} = s\left[\frac{\phi(t)}{s}\right]^{r^n},\end{aligned}$$

which completes the proof. □

Definition 2.5 Let $q \ge 1$ be a fixed real number. A nondecreasing function $\varphi : J \to J$ is said to be a b-Bianchini-Grandolfi gauge function with a coefficient q on J if

$$\sigma(t) = \sum_{n=0}^{\infty} q^n \varphi^n(t) < \infty \quad \text{for all } t \in J. \tag{2.4}$$

We note that a b-Bianchini-Grandolfi gauge function also satisfies the following functional equation:

$$\sigma(t) = q\sigma(\varphi(t)) + t. \tag{2.5}$$

It is easy to see that every b-Bianchini-Grandolfi gauge function is also a Bianchini-Grandolfi [16] gauge function but the converse may not hold. A b-Bianchini-Grandolfi gauge function having coefficient $q_1 \ge 1$ is also a b-Bianchini-Grandolfi gauge function having coefficient $q_2 \ge 1$ for every $q_2 \le q_1$.

From now on, we always assume that the coefficient of the b-Bianchini-Grandolfi gauge function is at least as large as the coefficient of the b-metric space.

Lemma 2.6 *Every gauge function of order $r \ge 1$ defined by* (2.1) *and* (2.2) *is a b-Bianchini-Grandolfi gauge function with coefficient $s \ge 1$.*

Proof It is immediately follows from the first part of Lemma 2.4 and using the fact that $P_n(r) \ge n$ for $r \ge 1$ and $n \ge 0$. □

3 Fixed point theorems

For convenience we define a function $E : D \to \mathbb{R}^+$ by $E(x) = d(x, fx)$ and assume that there exist some $x_0 \in D$ such that $\mathcal{O}(x_0) \subset D$, so that the condition (1.1) can be put in the form

$$E(fx) \le \varphi(E(x)) \quad \text{for all } x \in \mathcal{O}(x_0) \text{ such that } E(x) \in J. \tag{3.1}$$

Lemma 3.1 *Suppose $x_0 \in X$ is such that $\mathcal{O}(x_0) \subset D$. Assume that $E(x_0) \in J$; then $E(x_n) \in J$ for all $n \ge 0$.*

Proof Note that $x_0, x_1, x_2, \dots, x_n$ are well defined and belong to D. From (3.1) we have

$$E(x_1) = d(x_1, x_2) \le \varphi(d(x_0, x_1)) = \varphi(E(x_0)) \in J \quad (\text{as } E(x_0) \in J).$$

Hence, $E(x_1) \in J$. Similarly, iterating successively we get $E(x_n) \in J$ for all $n \ge 0$. □

Definition 3.2 Suppose $x_0 \in D$ is such that $\mathcal{O}(x_0) \subset D$ and $E(x_0) \in J$. Then for every iterate $x_n \in D$, $n \geq 0$ we define the closed ball $\overline{B}(x_n, \rho_n)$ with center at x_n and radius $\rho_n = s\sigma(E(x_n))$, where $\sigma : J \to \mathbb{R}^+$ is defined by (2.4).

Lemma 3.3 *Suppose $x_0 \in D$ is such that $\mathcal{O}(x_0) \subset D$ and $E(x_0) \in J$. Assume that $\overline{B}(x_n, \rho_n) \subset D$ for some $n \geq 0$; then $x_{n+1} \in D$ and $\overline{B}(x_{n+1}, \rho_{n+1}) \subset \overline{B}(x_n, \rho_n)$.*

Proof Since $E(x_0) \in J$, Lemma 3.1 implies $E(x_n) \in J$ for all $n \geq 0$. The condition (2.5) implies $\sigma(t) \geq t$ for all $t \in J$. We have

$$d(x_n, x_{n+1}) \leq \sigma\big(d(x_n, x_{n+1})\big) \leq s\sigma\big(d(x_n, x_{n+1})\big) = \rho_n.$$

Thus $x_{n+1} \in \overline{B}(x_n, \rho_n) \subset D$. Now let $x \in \overline{B}(x_{n+1}, \rho_{n+1})$. As $E(x_n) \in J$ so that from (3.1) we have $E(x_{n+1}) \leq \varphi(E(x_n))$. By making use of (2.5) we get

$$\begin{aligned} d(x, x_n) &\leq s\big[d(x, x_{n+1}) + E(x_n)\big] \\ &\leq s\big[\rho_{n+1} + E(x_n)\big] = s\big[s\sigma\big(E(x_{n+1})\big) + E(x_n)\big] \\ &\leq s\big[s\sigma\big(\varphi\big(E(x_n)\big)\big) + E(x_n)\big] = s\sigma\big(E(x_n)\big) = \rho_n. \end{aligned}$$

Hence, $x \in \overline{B}(x_n, \rho_n)$. □

Definition 3.4 (Initial orbital point) We say that a point $x_0 \in D$ is an initial orbital point of f if $E(x_0) \in J$ and $\mathcal{O}(x_0) \subset D$.

The following lemma is obvious.

Lemma 3.5 *For every initial orbital point $x_0 \in D$ of f and every $n \geq 0$ we have*

$$E(x_{n+1}) \leq \varphi\big(E(x_n)\big) \quad \textit{and} \quad E(x_n) \leq \varphi^n\big(E(x_0)\big).$$

Furthermore, if φ is a gauge function of order $r \geq 1$ defined by (2.1) *and* (2.2), *then*

$$E(x_n) \leq E(x_0)\mu^{P_n(r)} \quad \textit{and} \quad \phi\big(E(x_n)\big) \leq s\mu^{r^n} = \phi(x_0)\mu^{r^n - 1},$$

where $\mu = \frac{\phi(E(x_0))}{s}$ and ϕ is nonnegative nondecreasing on J satisfying (2.1) *and* (2.2).

Proof By making use of Lemma 3.1 we obtain $E(x_{n+1}) \leq \varphi(E(x_n))$. Since φ is nondecreasing, it easily follows that $E(x_n) \leq \varphi^n(E(x_0))$. Now from Lemma 2.4(1) we have

$$E(x_n) \leq \varphi^n\big(E(x_0)\big) \leq E(x_0)\left[\frac{\phi(E(x_0))}{s}\right]^{P_n(r)} = E(x_0)\mu^{P_n(r)}.$$

By using Lemma 2.4(2) we obtain

$$\phi\big(E(x_n)\big) \leq \phi\big(\varphi^n\big(E(x_0)\big)\big) \leq s\left[\frac{\phi(E(x_0))}{s}\right]^{r^n} = s\mu^{r^n}.$$

□

The following lemma gives bounds for inclusion radii and throughout its proof we will make use of the following facts:

$$0 \le \phi(t) < 1, \qquad P_j(r) \ge j, \qquad 0 \le \mu^{r^n} < 1,$$

where $r \ge 1$, $\mu = \frac{\phi(E(x_0))}{s}$ and $j = 0, 1, 2, \ldots$.

Lemma 3.6 *Suppose $x_0 \in D$ is an initial orbital point of f and φ is a gauge function of order $r \ge 1$. Let ϕ be nonnegative and nondecreasing on J defined by* (2.1) *and* (2.2). *Then for radii $\rho_n = s\sigma(E(x_n))$, $n = 0, 1, 2, \ldots$, the following estimates hold*:

(1) $\rho_n \le sE(x_n)\sum_{j=0}^{\infty}[\phi(E(x_n))]^{P_j(r)} \le \frac{sE(x_n)}{1-\phi(E(x_n))}$,

(2) $\rho_n \le sE(x_n)\sum_{j=0}^{\infty}[\phi(E(x_0))\mu^{r^n-1}]^{P_j(r)} \le \frac{sE(x_n)}{1-\phi(E(x_0))\mu^{r^n-1}}$,

(3) $\rho_n \le sE(x_0)\mu^{P_n(r)}\sum_{j=0}^{\infty}[\phi(E(x_0))\mu^{r^n-1}]^{P_j(r)} \le sE(x_0)\frac{\mu^{P_n(r)}}{1-\phi(E(x_0))\mu^{r^n-1}}$,

(4) $\rho_{n+1} \le s\varphi(E(x_n))\sum_{j=0}^{\infty}[\phi(\varphi(E(x_n)))]^{P_j(r)} \le \frac{s\varphi(E(x_n))}{1-\phi(\varphi(E(x_n)))}$,

(5) $\rho_{n+1} \le s\varphi(E(x_n))\sum_{j=0}^{\infty}[\phi(E(x_0))\mu^{r^{n+1}-1}]^{P_j(r)} \le \frac{s\varphi(E(x_n))}{1-\phi(E(x_0))\mu^{r^{n+1}-1}}$,

where $\mu = \frac{\phi(E(x_0))}{s}$.

Proof (1) From definition of ρ_n we have

$$\begin{aligned}
\rho_n &= s\sigma\big(E(x_n)\big) = s\sum_{j=0}^{\infty} s^j\varphi^j\big(E(x_n)\big) \\
&\le s\sum_{j=0}^{\infty} s^j E(x_n)\left[\frac{\phi(E(x_n))}{s}\right]^{P_j(r)} \quad \text{(using Lemma 2.4)} \\
&= sE(x_n)\sum_{j=0}^{\infty} s^j\left[\frac{\phi(E(x_n))}{s}\right]^{P_j(r)} \\
&\le sE(x_n)\sum_{j=0}^{\infty}\big[\phi\big(E(x_n)\big)\big]^j = \frac{sE(x_n)}{1-\phi(E(x_n))}.
\end{aligned} \tag{3.2}$$

(2) From (3.2) we have

$$\begin{aligned}
\rho_n &\le sE(x_n)\sum_{j=0}^{\infty}\big[\phi\big(E(x_n)\big)\big]^{P_j(r)} \\
&\le sE(x_n)\sum_{j=0}^{\infty}\big[s\mu^{r^n}\big]^{P_j(r)} \quad \text{(using second part of Lemma 3.5)} \\
&= sE(x_n)\sum_{j=0}^{\infty}\big[\phi\big(E(x_0)\big)\mu^{r^n-1}\big]^{P_j(r)} \\
&\le sE(x_n)\sum_{j=0}^{\infty}\big[\phi\big(E(x_0)\big)\mu^{r^n-1}\big]^j \\
&= \frac{sE(x_n)}{1-\phi(E(x_0))\mu^{r^n-1}}.
\end{aligned}$$

(3) By making use of first part of Lemma 3.5 above we have

$$\begin{aligned}\rho_n &\le sE(x_n)\sum_{j=0}^{\infty}\left[\phi\left(E(x_0)\right)\mu^{r^n-1}\right]^{P_j(r)}\\ &\le sE(x_0)\mu^{P_n(r)}\sum_{j=0}^{\infty}\left[\phi\left(E(x_0)\right)\mu^{r^n-1}\right]^{j}\\ &\le \frac{sE(x_0)\mu^{P_n(r)}}{1-\phi(E(x_0))\mu^{r^n-1}}.\end{aligned}$$

(4) Now by making use of Lemma 2.4 we have

$$\begin{aligned}\rho_{n+1} &= s\sigma\left(E(x_{n+1})\right) = \sum_{j=0}^{\infty}s^j\varphi\left(E(x_{n+1})\right)\\ &\le sE(x_{n+1})\sum_{j=0}^{\infty}s^j\left[\frac{\phi(E(x_{n+1}))}{s}\right]^{P_j(r)}\\ &\le s\varphi\left(E(x_n)\right)\sum_{j=0}^{\infty}\left[\phi\left(\varphi\left(E(x_n)\right)\right)\right]^{P_j(r)}\\ &\quad\left(\text{as } E(x_{n+1})\le\varphi\left(E(x_n)\right) \text{ and } \phi \text{ is nondecreasing}\right)\\ &\le s\varphi\left(E(x_n)\right)\sum_{j=0}^{\infty}\left[\phi\left(\varphi\left(E(x_n)\right)\right)\right]^{j}\\ &= \frac{s\varphi(E(x_n))}{1-\phi(\varphi(E(x_n)))}.\end{aligned}$$

(5) From (4) we have

$$\begin{aligned}\rho_{n+1} &\le s\varphi\left(E(x_n)\right)\sum_{j=0}^{\infty}\left[\phi\left(E(x_{n+1})\right)\right]^{P_j(r)}\\ &\le s\varphi\left(E(x_n)\right)\sum_{j=0}^{\infty}\left[\phi\left(E(x_0)\right)\mu^{r^{n+1}-1}\right]^{P_j(r)} \quad \text{(using Lemma 3.5)}\\ &\le \frac{s\varphi(E(x_n))}{1-\phi(E(x_0))\mu^{r^{n+1}-1}}.\end{aligned}$$

□

Now we proceed to formulate the following fixed point theorems.

Theorem 3.7 *Let $f : D \subset X \to X$ be an operator on a complete b-metric space (X, d) such that the b-metric is continuous and f satisfies* (1.1) *with a b-Bianchini-Grandolfi gauge function φ of order $r \ge 1$ on an interval J with coefficient $s \ge 1$. Then starting from an initial orbital point x_0 of f the iterative sequence* (1.2) *remains in $\overline{B}(x_0, \rho_0)$ and converges to a point ξ which belongs to each of the closed balls $\overline{B}(x_n, \rho_n)$, $n = 0, 1, 2, \ldots$, where $\rho_n = s\sigma(d(x_n, x_{n+1}))$, σ defined in* (2.5) *and $s \ge 1$ is a coefficient of b-metric space. Furthermore, for each $n \ge 1$ we have*

$$d(x_{n+1}, x_n) \le \varphi\left(d(x_n, x_{n-1})\right).$$

If $\xi \in D$ and the function $E(x) = d(x,fx)$ on D is f-orbitally lower semi-continuous at ξ, then ξ is a fixed point of f.

Proof Since $x_0 \in D$ is an initial orbital point of f, from Lemma 3.3 we have

$$\overline{B}(x_{n+1}, \rho_{n+1}) \subset \overline{B}(x_n, \rho_n) \quad \text{for all } n \geq 0.$$

Thus $x_n \in \overline{B}(x_0, \rho_0)$ for all $n \geq 0$. According to the definition of ρ and using Lemma 3.5 we have

$$\begin{aligned}
\rho_n &= s\sigma\bigl(E(x_n)\bigr) \leq s\sigma\bigl(\varphi^n\bigl(E(x_0)\bigr)\bigr) \\
&= s\sum_{j=0}^{\infty} s^j \varphi^j\bigl(\varphi^n\bigl(E(x_0)\bigr)\bigr) \\
&= \frac{1}{s^{n-1}}\sum_{j=n}^{\infty} s^j \varphi^j\bigl(\varphi^n\bigl(E(x_0)\bigr)\bigr) \quad \text{for all } n \geq 0.
\end{aligned} \tag{3.3}$$

Since φ is a b-Bianchini-Grandolfi gauge function, from (3.3) we obtain

$$\rho_n \to 0 \quad \text{as } n \to \infty, \tag{3.4}$$

which implies that $\{\overline{B}(x_n, \rho_n)\}$ is a nested sequence of closed balls. By Cantor's theorem (for complete b-metric spaces), we deduce that there exists a unique point ξ such that $\xi \in \overline{B}(x_n, \rho_n)$ for all $n \geq 0$ and $x_n \to \xi$ or equivalently, $\lim_{n\to\infty} d(x_n, \xi) = 0$. From (d3) of Definition 1.1 we have

$$d(\xi, fx_n) \leq s\bigl[d(\xi, x_n) + d(x_n, fx_n)\bigr] = s\bigl[d(\xi, x_n) + d(x_n, x_{n+1})\bigr].$$

Letting $n \to \infty$ and since the b-metric is continuous, we obtain

$$\lim_{n\to\infty} d(\xi, fx_n) = 0.$$

If $\xi \in D$ and $E(x) = d(x,fx)$ is f-lower semi-continuous at ξ, then

$$d(\xi, f\xi) = E(\xi) \leq \lim_{n\to\infty} \inf E(x_n) = \lim_{n\to\infty} \inf d(x_n, x_{n+1}) = 0,$$

which infers $\xi = f\xi$. Furthermore, from Lemma 3.5 we obtain the following:

$$\begin{aligned}
d(x_n, x_{n+1}) = E(x_n) &\leq \varphi\bigl(E(x_{n-1})\bigr) \\
&= \varphi\bigl(d(x_{n-1}, x_n)\bigr).
\end{aligned}$$

□

Remark 3.8 Theorem 3.7 gives a generalization of [1, Theorem 4.1] and extends it to the case of b-metric spaces. It reduces to [1, Theorem 4.1] when $s = 1$. Hence Theorem 3.7 not only extends the result of Proinov [1] but in turn it also includes results of Bianchini and Grandolfi [16] and Hicks [17] as special cases.

Corollary 3.9 [1, Theorem 4.1] *Let (X,d) be a complete metric space and $f: D \subset X \to X$ be an operator satisfying*

$$d\big(fx,f^2x\big) \leq \varphi\big(d(x,fx)\big) \quad \textit{for all } x \in D \textit{ and } fx \in D \textit{ with } d(x,fx) \in J, \tag{3.5}$$

where φ is a Bianchini-Grandolfi gauge function on an interval J. Then starting from an initial orbital point x_0 of f the iterative sequence $\{x_n\}$ remains in $\overline{B}(x_0,\rho_0)$ and converges to a point ξ which belongs to each of the closed balls $\overline{B}(x_n,\rho_n)$, $n = 0,1,\ldots$ where $\rho_n = \sigma(d(x_n,fx_n))$ and $\sigma(t) = \sum_{n=0}^{\infty}\varphi^n(t)$. Moreover, if $\xi \in D$ and f is continuous at ξ, then ξ is a fixed point of f.

Proof It follows from Lemma 3.1 that $E(x_n) \in J$ for $n = 0,1,2,\ldots$. Thus from (3.5) we have

$$d\big(fx,f^2x\big) \leq \varphi\big(d(x,fx)\big) \quad \text{for all } x \in \mathcal{O}(x_0) \text{ with } d(x,fx) \in J. \tag{3.6}$$

Thus Theorem 3.7 yields $x_n \to \xi \in X$. Also since the iterative sequence $\{x_n\} \in \mathcal{O}(x_0)$ and the mapping f is continuous at point t, we have $fx_n \to f\xi$. Thus

$$E(\xi) = d(\xi,f\xi) \leq \lim_{n\to\infty} s\big[d(\xi,x_{n+1}) + d(x_{n+1},f\xi)\big] = 0 \leq \lim_{n\to\infty}\inf E(x_n).$$

This implies f-orbital lower semi-continuity of $E(x) = d(x,fx)$ at point ξ. Hence the conclusion follows from Theorem 3.7. □

Theorem 3.10 *Let $f: D \subset X \to X$ be an operator on a complete b-metric space (X,d) such that the b-metric is continuous and let f satisfy* (1.1) *with a b-Bianchini-Grandolfi gauge function φ of order $r \geq 1$ and a coefficient s on an interval J. Further, suppose that $x_0 \in D$ is an initial orbital point of f, then the following statements hold true.*

(1) *The iterative sequence* (1.2) *remains in $\overline{B}(x_0,\rho_0)$ and converges with rate of convergence at least $r \geq 1$ to a point ξ which belongs to each of the closed balls $\overline{B}(x_n,\rho_n)$, $n = 0,1,\ldots$, and*

$$\rho_n = sd(x_n,x_{n+1})\sum_{j=0}^{\infty}\big[\phi\big(d(x_n,x_{n+1})\big)\big] \leq \frac{sd(x_n,x_{n+1})}{1-\phi(d(x_n,x_{n+1}))}, \tag{3.7}$$

where ϕ is nonnegative and nondecreasing function on J satisfying (2.1) *and* (2.2).

(2) *For all $n \geq 0$ the following prior estimate holds*:

$$d(x_n,\xi) \leq \frac{E(x_0)}{s^{n-1}}\sum_{j=n}^{\infty}\phi\big(E(x_0)\big)^{P_j(r)} = d(x_0,fx_0)\frac{\phi(E(x_0))^{P_n(r)}}{s^{n-1}[1-\phi(E(x_0))^{r^n}]}. \tag{3.8}$$

(3) *For all $n \geq 1$ the following posterior estimate holds*:

$$\begin{aligned} d(x_n,\xi) &\leq s\varphi\big(d(x_n,x_{n-1})\big)\sum_{j=0}^{\infty}\big[\phi\big(\varphi\big(d(x_n,x_{n-1})\big)\big)\big]^{P_j(r)} \\ &\leq \frac{s\varphi(d(x_n,x_{n-1}))}{1-\phi[\varphi(d(x_n,x_{n-1}))]} \leq \frac{s\varphi(d(x_n,x_{n-1}))}{1-\phi(d(x_n,x_{n-1}))[\frac{\phi(d(x_n,x_{n-1}))}{s}]^{r-1}}. \end{aligned} \tag{3.9}$$

(4) *We have*

$$d(x_{n+1},x_n) \le \varphi\big(d(x_n,x_{n-1})\big) \le \mu^{P_n(r)} d(x_0,fx_0) \tag{3.10}$$

for all $n \ge 1$.

(5) *If* $\xi \in D$ *and the function* $G(x) = d(x,fx)$ *on* D *is* f*-orbitally lower semi-continuous at* ξ, *then* ξ *is a fixed point of* f.

Proof (1) From Theorem 3.7 it follows that the iterative sequence (1.2) converges to $t \in X$ and further $t \in \overline{B}(x_n, \rho_n)$ for all $n = 0,1,2,\ldots$. Moreover, estimate (1) of Lemma 3.6 implies

$$\rho_n \le sd(x_n,x_{n+1}) \sum_{j=0}^{\infty} \big[\phi\big(d(x_n,x_{n+1})\big)\big]^{P_j(r)} \le \frac{sd(x_n,x_{n+1})}{1-\phi(d(x_n,x_{n+1}))}.$$

(2) For $m > n$,

$$\begin{aligned} d(x_n,x_m) &\le sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + \cdots + s^{m-n-1} d(x_{m-2},x_{m-1}) + s^{m-n} d(x_{m-1},x_m) \\ &= \frac{1}{s^{n-1}} \sum_{j=n}^{m-1} s^j E(x_j) \\ &\le \frac{1}{s^{n-1}} \sum_{j=n}^{m-1} s^j \varphi^j\big(E(x_0)\big) \quad \big(\text{from Lemma 3.5}, E(x_n) \le \varphi^n\big(E(x_0)\big)\big) \\ &\le \frac{1}{s^{n-1}} \sum_{j=n}^{m-1} s^j E(x_0) \left[\frac{\phi(E(x_0))}{s}\right]^{P_j(r)} \quad (\text{using Lemma 2.4}) \\ &\le \frac{E(x_0)}{s^{n-1}} \sum_{j=n}^{m-1} \lambda^{P_j(r)}, \end{aligned}$$

where $\lambda = \phi(E(x_0))$. Keeping n fixed and letting $m \to \infty$ we get

$$d(x_n,\xi) \le \frac{E(x_0)}{s^{n-1}} \sum_{j=n}^{\infty} \lambda^{P_j(r)} = \frac{d(x_0,fx_0)}{s^{n-1}} \sum_{j=n}^{\infty} \lambda^{P_j(r)}. \tag{3.11}$$

Since

$$r^n + r^{n+1} \ge 2r^n, \qquad r^n + r^{n+1} + r^{n+2} \ge 3r^n, \qquad \ldots,$$

we have

$$\lambda^{r^n+r^{n+1}} \le \lambda^{2r^n}, \qquad \lambda^{r^n+r^{n+1}+r^{n+2}} \le \lambda^{3r^n}, \qquad \ldots.$$

Thus it implies

$$\begin{aligned} \sum_{j=n}^{\infty} \lambda^{P_j(r)} &= \lambda^{P_j(r)} + \lambda^{P_{j+1}(r)} + \cdots \\ &= \lambda^{P_n(r)}\big[1 + \lambda^{r^n} + \lambda^{r^n+r^{n+1}} + \lambda^{r^n+r^{n+1}+r^{n+2}} + \cdots\big] \end{aligned}$$

$$\begin{aligned} &\leq \lambda^{P_j(r)}\bigl[1+\lambda^{r^n}+\lambda^{2r^n}+\lambda^{3r^n}+\cdots\bigr] \\ &= \frac{\lambda^{P_n(r)}}{1-\lambda^{r^n}}. \end{aligned} \tag{3.12}$$

Hence from (3.11) we obtain

$$d(x_n,\xi) \leq \frac{E(x_0)}{s^{n-1}}\sum_{j=n}^{\infty}\phi\bigl(E(x_0)\bigr)^{P_j(r)} = d(x_0,fx_0)\frac{\phi(E(x_0))^{P_n(r)}}{s^{n-1}[1-\phi(E(x_0))^{r^n}]}.$$

(3) From (3.11) we have for $n \geq 0$,

$$d(x_n,\xi) \leq \frac{d(x_0,x_1)}{s^{n-1}}\sum_{j=n}^{\infty}\bigl[\phi\bigl(d(x_0,x_1)\bigr)\bigr]^{P_j(r)}.$$

Setting $n = 0$, $y_0 = x_0$, and $y_1 = x_1$ we have

$$d(y_0,\xi) \leq sd(y_0,y_1)\sum_{j=0}^{\infty}\bigl[\phi\bigl(d(y_0,y_1)\bigr)\bigr]^{P_j(r)}.$$

Setting again $y_0 = x_n$ and $y_1 = x_{n+1}$ gives

$$\begin{aligned} d(x_n,\xi) &\leq sd(x_n,x_{n+1})\sum_{j=0}^{\infty}\bigl[\phi\bigl(d(x_n,x_{n+1})\bigr)\bigr]^{P_j(r)} \\ &\leq s\varphi\bigl(d(x_n,x_{n-1})\bigr)\sum_{j=0}^{\infty}\bigl[\phi\bigl(\varphi\bigl(d(x_n,x_{n-1})\bigr)\bigr)\bigr]^{P_j(r)} \\ &\leq s\varphi\bigl(d(x_n,x_{n-1})\bigr)\sum_{j=0}^{\infty}\bigl[\phi\bigl(\varphi\bigl(d(x_n,x_{n-1})\bigr)\bigr)\bigr]^{j} \\ &= \frac{s\varphi(d(x_n,x_{n-1}))}{1-\phi(\varphi(d(x_n,x_{n-1})))}. \end{aligned} \tag{3.13}$$

From Lemma 2.4(2) we obtain

$$\phi\bigl(\varphi\bigl(d(x_n,x_{n-1})\bigr)\bigr) \leq s\biggl[\frac{\phi(d(x_n,x_{n-1}))}{s}\biggr]^r = \phi\bigl(d(x_n,x_{n-1})\bigr)\biggl[\frac{\phi(d(x_n,x_{n-1}))}{s}\biggr]^{r-1}, \tag{3.14}$$

which implies

$$\frac{1}{1-\phi(\varphi(d(x_n,x_{n-1})))} \leq \frac{1}{1-\phi(d(x_n,x_{n-1}))[\frac{\phi(d(x_n,x_{n-1}))}{s}]^{r-1}}. \tag{3.15}$$

Thus from (3.13) and (3.15) we deduce for $n \geq 1$,

$$\begin{aligned} d(x_n,\xi) &\leq \frac{s\varphi(d(x_n,x_{n-1}))}{1-\phi(\varphi(d(x_n,x_{n-1})))} \\ &\leq \frac{s\varphi(d(x_n,x_{n-1}))}{1-\phi(d(x_n,x_{n-1}))[\frac{\phi(d(x_n,x_{n-1}))}{s}]^{r-1}}. \end{aligned}$$

(4) We have

$$\begin{aligned} d(x_{n+1},x_n) &= E(x_n) \leq \varphi\big(E(x_{n-1})\big) \\ &= E(x_{n-1})\frac{\phi(E(x_{n-1}))}{s} \\ &\leq E(x_0)\mu^{P_{n-1}(r)}\mu^{r^{n-1}} \quad \text{(using Lemma 2.2)} \\ &= E(x_0)\mu^{P_{n-1}(r)+r^{n-1}} \\ &= E(x_0)\mu^{P_n(r)} = \mu^{P_n(r)}d(x_0,fx_0). \end{aligned}$$

(5) Its proof runs along the same lines as the proof of Theorem 3.7. □

Remark 3.11 For $s = 1$, Theorem 3.10 reduces to [1, Theorem 4.2]. It also generalizes (taking $s = 1$ and $\varphi(t) = \lambda t$, $0 < \lambda < 1$) results of Ortega and Rheinboldt [18, Section 12.3.2], Kornstaedt [19, Satz 4.1], Hicks and Rhoades [15], and Park [20, Theorem 2]. The first two conclusions of Theorem 3.10 are due to Gel'man [4, Theorem 3] (taking $s = 1$ and $\varphi(t) = ct^r$, $c \geq 0$, $r \geq 1$). It also yields some results of Hicks [17, Theorem 3].

Corollary 3.12 *Let $f : X \to X$ be an operator on a complete b-metric space (X, d) such that the b-metric is continuous. Further, assume that f satisfies*

$$d(fx,fy) \leq \varphi\big(d(x,y)\big) \quad \text{for all } x,y \in X \text{ with } d(x,y) \in J, \tag{3.16}$$

where φ is a b-Bianchini-Grandolfi gauge function of order $r \geq 1$ on an interval J and with coefficient $s \geq 1$. Assume that $x_0 \in X$ is such that $d(x_0,fx_0) \in J$. Then the following statements hold.

(i) *The iterative sequence* (1.2) *converges to a fixed point ξ of f.*
(ii) *The operator f has a unique fixed point in $S = \{x \in X : d(x,\xi) \in J\}$.*
(iii) *The estimates* (3.7)-(3.10) *are valid.*

Proof From (3.16) we have

$$d(fx,fy) \leq \varphi\big(d(x,y)\big) < d(x,y),$$

which gives the continuity of f in b-metric space (X, d). Thus conclusions (i) and (iii) follow immediately from Theorem 3.10. Let ξ' be another fixed point of f in S; then $d(\xi,\xi') \in J$. It follows from (3.16) that $d(\xi,\xi') \leq \varphi(d(\xi,\xi'))$, which yields $\xi' = \xi$. □

Remark 3.13 For $s = 1$ when the b-metric space under consideration is a simple metric space, the above corollary coincides with [1, Corollary 4.4]. Thus the conclusions of Corollary 3.12 are consequences of the results of Matkowski [21].

4 Application and illustrative example

The following example illuminates the degree of generality of our result.

Example 4.1 Let $X := \{x_1, x_2, x_3\}$. Define a function $d : X \times X \to \mathbb{R}^+$ as

$$d(x_1,x_2) = \frac{1}{k^2}, \qquad d(x_2,x_3) = \frac{1}{k-1}, \qquad d(x_1,x_3) = \frac{1}{k},$$

$$d(x_i,x_j) = d(x_j,x_i) \quad \text{and} \quad d(x_i,x_i) = 0 \quad \text{for all } i,j = 1,2,3,$$

where $k \geq 3$ is any positive integers. It is an easy exercise to see that d is a b-metric with coefficient $s \geq \frac{k^2}{k^2-1} > 1$. Define $f : X \to X$ as

$$fx_1 = x_1, \qquad fx_2 = x_1, \qquad fx_3 = x_2.$$

Setting $\varphi(t) = t^2$ on $J = [0, \frac{1}{k-1}]$ then φ is a b-Bianchini-Grandolfi gauge function with coefficient $\frac{k^2}{(k^2-1)}$ having order 2. Moreover, it is easily seen that all conditions of Theorem 3.7 are satisfied.

On the other hand, assume that x_1, x_2, x_3 are real numbers and the set $\{x_1, x_2, x_3\}$ is endowed with the Euclidean metric d_e. For each gauge function φ defined on some interval $[0, h)$ one can find $\frac{1}{n_0} \in [0, h)$ for some $n_0 \in \mathbb{N}$. In such a case, identifying $x_1 = \frac{1}{n_0}$, $x_2 = \frac{2}{n_0}$, $x_3 = \frac{3}{n_0}$, we assume f as defined above; then w.r.t. Euclidean metric d_e we have

$$d_e\left(f\frac{2}{n_0}, f\frac{3}{n_0}\right) = d_e\left(\frac{1}{n_0}, \frac{2}{n_0}\right) = \frac{1}{n_0} \leq \varphi\left(d_e\left(\frac{2}{n_0}, \frac{3}{n_0}\right)\right) = \varphi\left(\frac{1}{n_0}\right),$$

which contradicts the definition of φ. Hence, one cannot invoke the main results of Proinov [1, Theorems 4.1, 4.2, Corollary 4.4].

Theorem 4.2 *Consider the following initial value problem*:

$$x'(t) = f\bigl(t, x(t)\bigr), \qquad x(t_0) = x_0. \tag{4.1}$$

Assume that the following conditions hold:

(i) f *is continuous*;

(ii) f *satisfies the condition*

$$\bigl|f(t,x) - f(t,y)\bigr| \leq k\bigl|x(t) - y(t)\bigr|^r \quad \textit{for } (t,x), (t,y) \in R; \tag{4.2}$$

(iii) f *is bounded on* R, *i.e.*,

$$\bigl|f(t,x)\bigr| \leq \frac{k^r}{2}, \tag{4.3}$$

where $R = \{(t,x) : |t - t_0| \leq (\frac{1}{k})^{r-1}, |x - x_0| \leq \frac{k}{2}\}$, $r \geq 2$ *and* $0 < k < 1$.

Then the initial value problem (4.1) *has a unique solution on the interval* $I = [t_0 - (\frac{1}{k})^{r-1}, t_0 + (\frac{1}{k})^{r-1}]$.

Proof Let $C(I)$ be the space of all continuous real valued functions on I where $I = [t_0 - (\frac{1}{k})^{r-1}, t_0 + (\frac{1}{k})^{r-1}]$ with the usual supremum metric, *i.e.*,

$$d(x,y) = \max_{t \in I}\bigl|x(t) - y(t)\bigr|.$$

Integrating (4.1) gives

$$x(t) = x_0 + \int_{t_0}^{t} f\bigl(\tau, x(\tau)\bigr)\,d\tau. \tag{4.4}$$

Indeed finding the solution of initial value problem (4.1) is equivalent to finding the fixed point of the self-mapping $T : X \to X$ defined by

$$Tx(t) = x_0 + \int_{t_0}^{t} f\bigl(\tau, x(\tau)\bigr)\,d\tau, \tag{4.5}$$

where $X = \{x \in C(I) : |x(t) - x_0| \leq \frac{k}{2}; k > 0\}$; then X is a closed subspace of $C(I)$. We see that if $\tau \in I$, then $|\tau - t_0| \leq (\frac{1}{k})^{r-1}$ and $x \in X$ gives $|x(\tau) - x_0| \leq \frac{k}{2}$. Thus $(\tau, x(\tau)) \in R$, and since f is continuous on R, the integral in (4.5) exists and T is defined for each $x \in X$. To see this, observe that T maps X to itself. We use (4.5) to write

$$\begin{aligned}
\bigl|Tx(t) - x_0\bigr| &= \left|\int_{t_0}^{t} f\bigl(\tau, x(\tau)\bigr)\,d\tau\right| \\
&\leq \int_{t_0}^{t} \bigl|f\bigl(\tau, x(\tau)\bigr)\bigr|\,d\tau \\
&\leq \frac{k^r}{2}|t - t_0| \quad \text{(using (4.3))} \\
&\leq \frac{k^r}{2}\left(\frac{1}{k}\right)^{r-1} = \frac{k}{2}.
\end{aligned}$$

Now by using (4.2) we have

$$\begin{aligned}
\bigl|Tx(t) - Ty(t)\bigr| &\leq \int_{t_0}^{t} \bigl|f\bigl(\tau, x(\tau)\bigr) - f\bigl(\tau, y(\tau)\bigr)\bigr|\,d\tau \\
&\leq k\int_{t_0}^{t} \bigl|x(\tau) - y(\tau)\bigr|^r\,d\tau \\
&\leq k\Bigl(\max_{\tau \in I}\bigl|x(\tau) - y(\tau)\bigr|\Bigr)^r |t - t_0| \\
&\leq k\Bigl(\max_{\tau \in I}\bigl|x(\tau) - y(\tau)\bigr|\Bigr)^r \left(\frac{1}{k}\right)^{r-1} \\
&= \left(\frac{1}{k}\right)^{r-2}\Bigl(\max_{\tau \in I}\bigl|x(\tau) - y(\tau)\bigr|\Bigr)^r.
\end{aligned} \tag{4.6}$$

Thus (4.6) implies

$$d(Tx, Ty) \leq \left(\frac{1}{k}\right)^{r-2}\bigl(d(x, y)\bigr)^r. \tag{4.7}$$

We take $J = [0, k]$. Thus it suffices to take $\varphi(u) = (\frac{1}{k})^{r-2}u^r$ for $u \in [0, k]$, $k < 1$; then φ is a gauge function of order $r \geq 2$. Also, for $u \in J - \{0\}$ we have

$$\varphi(u) = \left(\frac{1}{k}\right)^{r-2}u^r = \left(\frac{1}{k}\right)^{r-2}u^2u^{r-2} \leq \left(\frac{1}{k}\right)^{r-2}u^2k^{r-2} = u^2 < u. \tag{4.8}$$

Thus from (4.7) we obtain $d(Tx, Ty) \leq \varphi(d(x, y))$ for all $x, y \in X$ and $d(x, y) \in J$. Further, for any $x \in X$ it is easily seen that $d(x, x_0) \leq \frac{k}{2}$, which yields $d(x, y) \leq k$ for $x, y \in X$. Therefore, all the conditions of Corollary 3.12 are satisfied. Hence the iterative sequence $x_n = Tx_{n-1}$;

$n = 1, 2, \ldots$ converges to the unique fixed point t of T at a rate of convergence $r \geq 2$. On the other hand, Picard's iterations converge to the solution linearly. □

5 Conclusion

In Section 3 we have established two convergence theorems in the setting of a b-metric such that the self-mapping satisfies a contraction condition involving a gauge function of order $r \geq 1$. The gauge function φ has to satisfy the condition $\sum_{n=0}^{\infty} s^n \varphi^n(t) < \infty$ where $s \geq 1$ is the coefficient of the underlying b-metric space. An example has been furnished to assess the degree of generality of our results. In Section 4 we established an existence theorem for the solution of an initial value problem, which not only gives the unique solution but also locates the domain for the solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Received: 19 February 2014 Accepted: 13 June 2014 Published: 22 July 2014

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doi:10.1186/1029-242X-2014-266
Cite this article as: Samreen et al.: **Fixed point theorems for φ-contractions.** *Journal of Inequalities and Applications* 2014 **2014**:266.