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2 + 1-dimensional traversable wormholes supported by positive energy

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Abstract We revisit the shapes of the throats of wormholes, including thin-shell wormholes (TSWs) in 2 + 1 dimensions. In particular, in the case of TSWs this is done in a flat 2 + 1-dimensional bulk spacetime by using the standard method of cut-and-paste. Upon departing from a pure time-dependent circular shape i.e., $r = a(t)$ for the throat, we employ a θ-dependent closed loop of the form $r = R(t, \theta)$, and in terms of $R(t, \theta)$ we find the surface energy density σ on the throat. For the specific convex shapes we find that the total energy which supports the wormhole is positive and finite. In addition, we analyze the general wormhole's throat. By considering the specific equation of $r = R(\theta)$ instead of $r = r_0 = $ const., and upon certain choices of functions for $R(\theta)$, we find the total energy of the wormhole to be positive.

1 Introduction

In the theory of wormholes the prime important issue concerns the energy which turns out to be negative (i.e. exotic matter) to resist against gravitational collapse. This and stability related matters informed by Morris and Thorne [1] were restructured later on by Hochberg and Visser [2]. The 2 + 1-dimensional version of the wormholes was considered first in [3,4] and more recent work can be found in [5–7]. The nonexistence of negative energy in classical physics/Einstein's general relativity persisted as a serious handicap. Given this fact, and without reference to quantum theory in which negative energy has room at the smallest scales to resolve the problem of microscopic wormholes, how can one tackle the large scale wormholes? In this study, first we restrict ourselves to thin-shell wormholes (TSWs) which are tailored by the cut-and-paste technique of spacetimes [8–12]. In our view the theorems proved in [2] for general wormholes should be taken cautiously and mostly relaxed when

the subject matter is TSWs [13]. One important point that we emphasize/exploit is that the throat need not have a circular topology. It may depend on the angular variable as well, for example. This is the case that we naturally confront in static, non-spherical spacetimes. One such example is the Zipoy–Voorhees (ZV)-geometry, which deviates from spherical symmetry by a deformation/oblateness parameter [14–16]. We employed this to show that the overall/total energy can be made positive although locally, depending on the angular location it may take negative values [17]. We construct the simplest possible TSW in 2 + 1 dimensions whose bulk is made of flat Minkowski spacetime. Such a wormhole was constructed first by Visser [8], for the spherical throat case in 3 + 1 dimensions. In our case of 2 + 1 dimensions the only non-zero curvature is at the throat which consists of a ring, apt for the proper junction conditions. For the shape of the throat we assume an arbitrary angular dependence in order to attain ultimately a positive total energy (i.e. normal matter). In other words, the throat surface is chosen as $F = r - R(t, \theta) = 0$, for an appropriate function $R(t, \theta)$. For $R(t, \theta) = a(t)$, we recover the circular topology considered to date. Note that t is the coordinate time measured by an external observer. As a possible choice we employ $R(0, \theta) = R_0(\theta) = \frac{1}{\sqrt{\left|\cos \frac{5}{2}\theta\right| + 1}}$ which represents a starfish shape. The crucial point about the path of the throat is that it must be convex rather than concave in order to attain anything but exotic matter. A circle, which is concave yields the undesired negative energy. We present various alternatives for the starfish geometrical shapes to justify our argument. The limiting case of almost zero periodic dependence on the angle brings us to that of total energy zero (=a vacuum) on the throat, which amounts to making a TSW from a vacuum.

With this much information as regards 2 + 1-dimensional TSWs we extend our argument to the 2 + 1-dimensional general traversable wormhole which is considered to be a brane in 3+1-dimensional flat spacetime. We provide explicit examples to show that 2 + 1-dimensional wormholes can be

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fueled by a total positive energy and the null energy condition is satisfied.

The organization of the paper is as follows. In Sect. 2 we study TSWs with general throat shapes. $2+1$-dimensional wormholes induced from $3+1$-dimensional flat spacetime is considered in Sect. 3. The paper ends with our conclusions in Sect. 4.

2 Thin-shell wormholes with general throat shape

In this section we consider a model of TSW in $2+1$-dimensional flat spacetime. Hence, the bulk metric is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2. \tag{1}$$

Following [9–12] we introduce $M^{\pm} = \{r > R(t, \theta)\}$ as two incomplete manifolds from the original bulk and then we paste them on an identical hypersurface with equation

$$F(t, r, \theta) = r - R(t, \theta) = 0 \tag{2}$$

to make upon them a complete manifold known as the TSW. Let us note that in [9–12] and subsequent papers the proper time τ is used instead of the coordinate time t, which we consider here. As one can see, the results found in terms of t can be easily transformed to terms of τ. For instance, in the case we consider here by setting $\left(\frac{\partial R}{\partial t}\right)^2 = \left(\frac{\partial R}{\partial \tau}\right)^2 / \left(1 + \left(\frac{\partial R}{\partial \tau}\right)^2\right)$ the results will be expressed in terms of τ. This relation also shows that in the static equilibrium case $\frac{\partial R}{\partial t} = \frac{\partial R}{\partial \tau} = 0$ and therefore the physical properties such as the energy density and pressures are the same.

The throat is located on the shell $r = R(t, \theta)$ and therefore $R(t, \theta)$ is a general function of θ and t but not arbitrary. As, $r = R(t, \theta)$ is going to be the throat which connects two different spacetimes, in $2 + 1$ dimensions it must be a closed loop. We choose $x^{\alpha} = (t, r, \theta)$ for the bulk and $\xi^i = (t, \theta)$ for the hypersurface. Therefore while the bulk metric is given by

$$g_{\mu\nu} = \text{diag}\left(-1, 1, r^2\right), \tag{3}$$

the induced metric on the shell h_{ij} is obtained by using

$$h_{ij} = \frac{\partial x^{\alpha}}{\partial \xi^i} \frac{\partial x^{\beta}}{\partial \xi^j} g_{\alpha\beta}. \tag{4}$$

One finds

$$ds_{\Sigma}^2 = -\left(1 - \dot{R}^2\right) dt^2 + \left(R^2 + R'^2\right) d\theta^2 + 2\dot{R}R' dt d\theta, \tag{5}$$

in which a prime and a dot stand for derivative with respect to θ and t, respectively. Next, we find the extrinsic curvature

tensor defined as

$$K_{ij}^{\pm} = -n_{\gamma}^{\pm} \left(\frac{\partial^2 x^{\gamma}}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^{\gamma} \frac{\partial x^{\alpha}}{\partial \xi^i} \frac{\partial x^{\beta}}{\partial \xi^j}\right), \tag{6}$$

in which

$$n_{\gamma}^{\pm} = \pm \frac{1}{\sqrt{\Delta}} \frac{\partial F(t, r, \theta)}{\partial x^{\gamma}} \tag{7}$$

with

$$\Delta = \frac{\partial F(t, r, \theta)}{\partial x^{\alpha}} \frac{\partial F(t, r, \theta)}{\partial x^{\beta}} g^{\alpha\beta}. \tag{8}$$

Using (2), we find

$$\Delta = 1 + \left(\frac{R'}{R}\right)^2 - \dot{R}^2. \tag{9}$$

The exact form of the normal vector is found to be

$$n_t^{\pm} = \pm \frac{1}{\sqrt{\Delta}} \left(-\dot{R}\right), \tag{10}$$

$$n_r^{\pm} = \pm \frac{1}{\sqrt{\Delta}}, \tag{11}$$

and

$$n_{\theta}^{\pm} = \pm \frac{1}{\sqrt{\Delta}} \left(-R'\right), \tag{12}$$

such that $|\vec{n}| = 1$. The bulk's line element (1) admits $\Gamma_{r\theta}^{\theta} = \Gamma_{\theta r}^{\theta} = \frac{1}{r}$, $\Gamma_{\theta\theta}^r = -r$, while the remaining Christoffel symbols are zero. Therefore, the extrinsic curvature tensor elements become

$$K_{t\theta}^{\pm} = -n_r^{\pm} \dot{R}' - n_{\theta}^{\pm} \left(\frac{\dot{R}}{R}\right), \tag{13}$$

$$K_{tt}^{\pm} = -n_r^{\pm} \ddot{R}, \tag{14}$$

$$K_{\theta\theta}^{\pm} = -n_r^{\pm} \left(R'' - R\right) - 2n_{\theta}^{\pm} \frac{R'}{R}. \tag{15}$$

The Israel junction conditions [18–22] read

$$k_i^j - k\delta_i^j = -8\pi S_i^j, \tag{16}$$

in which

$$S_i^j = \begin{pmatrix} -\sigma & q_1 \\ q_2 & p \end{pmatrix} \tag{17}$$

is the energy-momentum tensor on the thin shell, $k_i^j = K_i^{j+} - K_i^{j-}$ and $k = \text{trace}\left(k_i^j\right)$. Note that q_1 and q_2 are appropriate pressure terms. Combining the results found above we get

$$k_{tt} = -\frac{2\ddot{R}}{\sqrt{\Delta}}, \tag{18}$$

$$k_{\theta\theta} = \frac{-2}{\sqrt{\Delta}} \left(R'' - R - \frac{2R'^2}{R}\right), \tag{19}$$

and

$$k_{t\theta} = \frac{-2}{\sqrt{\Delta}} \left(\dot{R}' - \frac{R'\dot{R}}{R} \right). \tag{20}$$

Furthermore, one finds

$$k^t_t = h^{tt} k_{tt} + h^{t\theta} k_{t\theta}, \tag{21}$$

$$k^\theta_\theta = h^{\theta\theta} k_{\theta\theta} + h^{\theta t} k_{\theta t}, \tag{22}$$

$$k^\theta_t = h^{\theta t} k_{tt} + h^{\theta\theta} k_{t\theta}, \tag{23}$$

and

$$k^t_\theta = h^{tt} k_{\theta t} + h^{t\theta} k_{\theta\theta}. \tag{24}$$

We recall that

$$h_{ij} = \begin{pmatrix} -B & H \\ H & A \end{pmatrix}, \tag{25}$$

which implies

$$h^{ij} = \begin{pmatrix} \frac{-A}{AB+H^2} & \frac{H}{AB+H^2} \\ \frac{H}{AB+H^2} & \frac{B}{AB+H^2} \end{pmatrix}, \tag{26}$$

in which $A = (R'^2 + R^2)$, $B = 1 - \dot{R}^2$ and $H = \dot{R}R'$. Considering (26) in (21–25) we find

$$k^t_t = \frac{-2\left[(1 - \dot{R}^2)\left(\dot{R}' - \frac{R'\dot{R}}{R} \right) - (R'^2 + R^2)\ddot{R} \right]}{\left[(R'^2 + R^2)(1 - \dot{R}^2) + \dot{R}^2 R'^2 \right] \sqrt{1 + \left(\frac{R'}{R} \right)^2 - \dot{R}^2}}, \tag{27}$$

$$k^\theta_\theta = \frac{-2\left[(1 - \dot{R}^2)\left(R'' - R - \frac{2R'^2}{R} \right) + \dot{R}R'\left(\dot{R}' - \frac{R'\dot{R}}{R} \right) \right]}{\left[(R'^2 + R^2)(1 - \dot{R}^2) + \dot{R}^2 R'^2 \right] \sqrt{1 + \left(\frac{R'}{R} \right)^2 - \dot{R}^2}}, \tag{28}$$

$$k^\theta_t = \frac{-2\left[\dot{R}R'\ddot{R} + (1 - \dot{R}^2)\left(\dot{R}' - \frac{R'\dot{R}}{R} \right) \right]}{\left[(R'^2 + R^2)(1 - \dot{R}^2) + \dot{R}^2 R'^2 \right] \sqrt{1 + \left(\frac{R'}{R} \right)^2 - \dot{R}^2}}, \tag{29}$$

and

$$k^t_\theta = \frac{-2\left[\dot{R}R'\left(R'' - R - \frac{2R'^2}{R} \right) - (R'^2 + R^2)\left(\dot{R}' - \frac{R'\dot{R}}{R} \right) \right]}{\left[(R'^2 + R^2)(1 - \dot{R}^2) + \dot{R}^2 R'^2 \right] \sqrt{1 + \left(\frac{R'}{R} \right)^2 - \dot{R}^2}}. \tag{30}$$

Therefore the Israel junction conditions imply

$$\sigma = -\frac{1}{8\pi} k^\theta_\theta \tag{31}$$

and

$$p = \frac{1}{8\pi} k^t_t. \tag{32}$$

In static equilibrium, one may set $R = R_0(\theta)$ and $\dot{R}_0 = \ddot{R}_0 = 0$, which consequently yield

$$\sigma_0 = \frac{1}{4\pi} \frac{\left(R''_0 - R_0 - \frac{2R'^2_0}{R_0} \right)}{(R'^2_0 + R^2_0)\sqrt{1 + \left(\frac{R'_0}{R_0} \right)^2}}, \tag{33}$$

and

$$p_0 = q_{10} = q_{20} = 0. \tag{34}$$

This is not surprising since the bulk spacetime is flat. Therefore in the static equilibrium, the only non-zero component of the energy-momentum tensor on the throat is the energy density σ_0. We note that the total matter supporting the wormhole is given by

$$\Omega = \int_0^{2\pi} \int_0^\infty \sqrt{-g}\,\sigma\,\delta(r - R)\,\mathrm{d}r\,\mathrm{d}\theta. \tag{35}$$

Let us add that a physically acceptable energy-momentum tensor must at least satisfy the weak energy conditions, which implies (i) $\sigma_0 \geq 0$ and (ii) $\sigma_0 + p_0 \geq 0$ and the total energy to be positive and finite. Therefore for the energy-momentum tensor which we have found on the surface with only energy density non-zero, our task reduces naturally to finding the specific case with $\sigma_0 \geq 0$ and $0 \leq \Omega < \infty$.

In the sequel we consider the various possibilities of the shape of the throat including the circular one. The first case to be checked is the circular throat i.e., $R_0 = $ constant. This leads to $\sigma_0 = \frac{-1}{4\pi}\frac{1}{R_0}$ and clearly violates the null energy condition which states that $\sigma_0 + p_0 \geq 0$.

For a specific function of $R_0(\theta)$, there are four different possibilities: (i) $\sigma_0 < 0$ on entire domain of $\theta \in [0, 2\pi]$, (ii) $\sigma_0 \leq 0$ or $\sigma_0 \geq 0$ but the total energy $\Omega < 0$, (iii) $\sigma_0 \leq 0$ or $\sigma_0 \geq 0$ with the total energy $\Omega > 0$ and (iv) $\sigma \geq 0$. Here

$$\Omega = \int_0^{2\pi} R_0 \sigma_0 \mathrm{d}\theta \tag{36}$$

is the total energy on the throat. In what follows we present illustrative examples for all cases. We note that the specific cases given below can easily be replaced by other functions but we must keep in mind that although $R_0(\theta)$ is a general function, $r = R_0$ must present a closed path in $2 + 1$ dimensions.

2.1 $\sigma_0 < 0$, $\Omega < 0$

In the first example, the throat is deformed from a perfect circle to an oval shape given by

$$R_0(\theta) = \frac{1}{0.5\cos^2\theta + 1}. \tag{37}$$

The shape of the throat and σ_0 are shown in Fig. 1a, b, respectively. As we observe in Fig. 1b, the energy density is negative
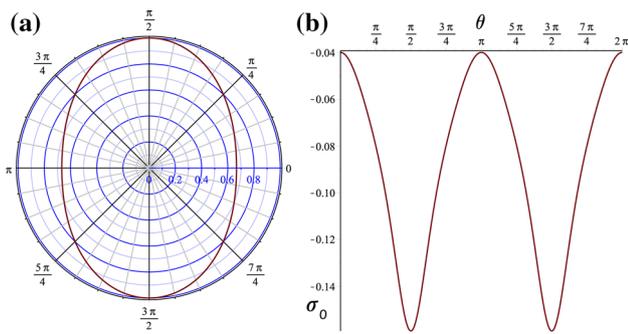
Fig. 1 The geometry of the throat for $R_0(\theta) = \frac{1}{0.5\cos^2\theta + 1}$ with its energy density distribution σ_0. We see that the signature of the curvature is positive everywhere and as a result the matter is exotic everywhere
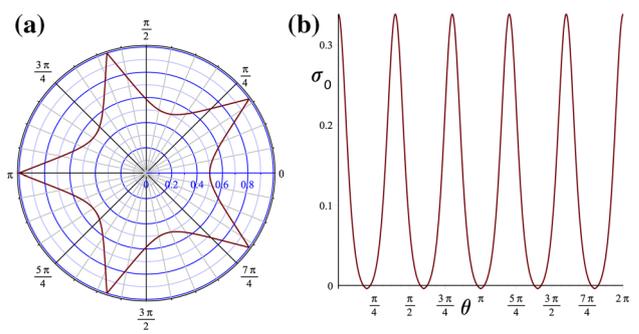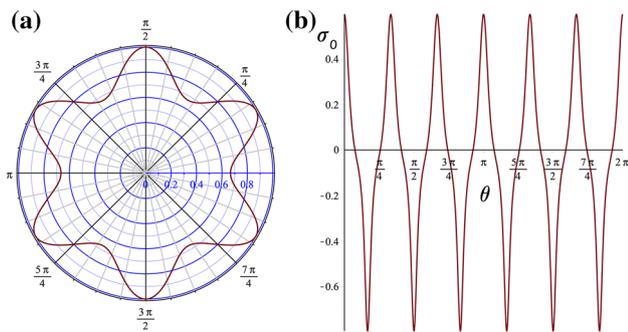

Fig. 2 The geometry of the throat when $R_0(\theta) = \frac{1}{0.5\cos^2(3\theta)+1}$ and its energy density distribution σ_0. This figure shows that σ_0 is positive when the curvature is negative and vice versa. The total energy which supports the wormhole, however, is negative

everywhere for $\theta \in [0, 2\pi]$. This is also seen from the shape of the throat whose curvature is positive on $\theta \in [0, 2\pi]$. Although the total exotic matter for the throat of the form of a circle of radius one is -0.5, in the case of (37) the total exotic matter is -0.48111 in geometrical units. This shows that a small deformation causes the total exotic matter to be less.

2.2 $\sigma_0 \lesssim 0$, $\Omega < 0$

As our second case we consider

$$R_0(\theta) = \frac{1}{0.5\cos^2(3\theta) + 1}, \tag{38}$$

which admits a throat of the shape shown in Fig. 2a. The corresponding energy density σ_0 is shown in Fig. 2b. As one can see the energy density is positive wherever the curvature of the throat is negative and vise versa. The overall energy is negative: $\Omega = -0.39339$ unit.

Fig. 3 The geometry of the throat when $R_0(\theta) = \frac{1}{\left|\cos\left(\frac{5}{2}\theta\right)\right| + 1}$ and σ_0 in terms of θ. Note that σ_0 is overwhelmingly positive and so is the total energy

2.3 $\sigma_0 \lesssim 0$, $\Omega > 0$

For

$$R_0(\theta) = \frac{1}{\left|\cos\left(\frac{5}{2}\theta\right)\right| + 1} \tag{39}$$

the shape of the throat looks like a starfish as is displayed in Fig. 3a. The behavior of the energy density σ_0 is depicted in Fig. 3b. As is clear σ_0 is positive everywhere except at the neighborhood of the corners of the throat where the curvature is positive. The total energy, however, is positive i.e., $\Omega = 0.38888$ unit.

2.4 $\sigma_0 > 0$, $\Omega > 0$

For this case let us consider

$$R_0(\theta) = \frac{1}{\sqrt{\left|\cos\left(\frac{5}{2}\theta\right)\right| + 1}}, \tag{40}$$

which is shown in Fig. 4a. In this case σ_0 is positive everywhere as is shown in Fig. 4b and the total energy is positive i.e. $\Omega = 0.40561$ unit. We should admit, however, that although $R_0(\theta)$ is well defined everywhere in the range of θ i.e., $\theta \in [0, 2\pi]$ at five spike points in Fig. 4a its derivative does not exist. This feature of $R_0(\theta)$ causes the energy density to be discontinuous at the same values of θ. This means that σ_0 is not defined at those points although its limit exists and is positive and finite. Figure 4c shows that σ_0 is positive around those critical points.

2.5 Parametric ansatz for $R_0(\theta)$

To complete our analysis we look at the case given in Sect. 2.2 and generalize the form of $R_0(\theta)$ as given by

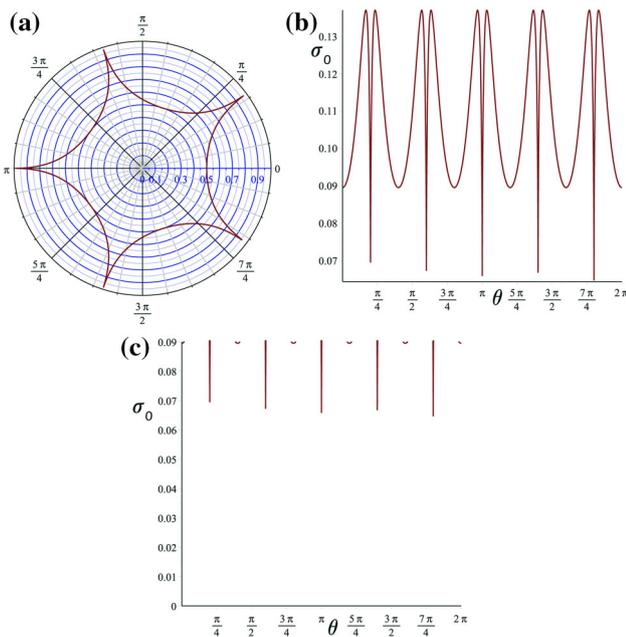$$R_0(\theta) = \frac{1}{\epsilon\cos^2(n\theta) + 1}, \tag{41}$$

(a)

(b)

(c)

Fig. 4 The geometry of the throat when $R_0(\theta) = \dfrac{1}{\sqrt{\left|\cos\left(\frac{5}{2}\theta\right)\right|+1}}$ and σ_0 in terms of θ. In this figure σ_0 is positive everywhere as the curvature is negative, and the total energy is positive
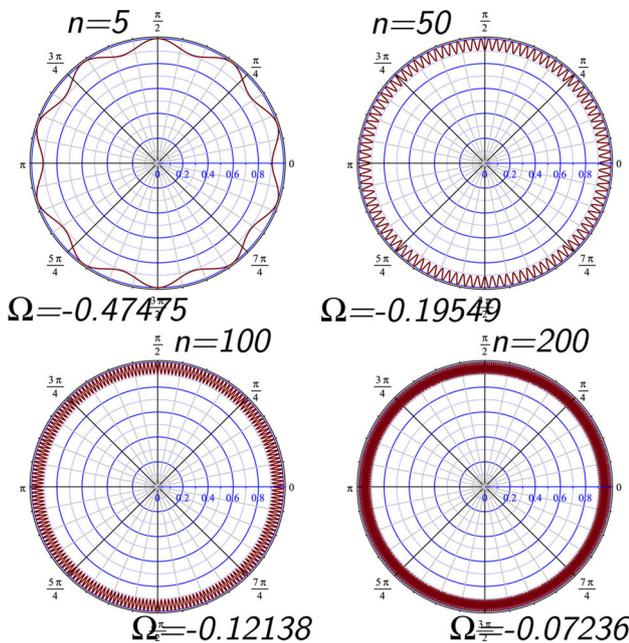

Fig. 5 The geometry of the throat when $R_0(\theta) = \dfrac{1}{\epsilon \cos^2(n\theta)+1}$ in terms of θ for $\epsilon = 0.1$ and $n = 5, 50, 100,$ and 200. The total exotic matter Ω for each case is also given. We observe that larger n yields smaller amount of exotic matter such that $\lim_{n\to\infty} \Omega = 0$

in which $\epsilon \in \mathbb{R}^+$ and $n = 2, 3, 4, \ldots$. In Fig. 5 we plot $R_0(\theta)$ in terms of different n and $\epsilon = 0.1$. The total exotic matter for each case is also calculated. We observe that increasing n decreases the magnitude of the exotic matter. When n

goes to infinity (i.e. an infinite oscillation) the total energy goes to zero, while at each point the energy density shows a positive or negative fluctuation. Of course the assumption of infinite frequency takes us away from the domain of classical physics, probably to the quantum domain. In the latter, particle creation from a vacuum is well known. Here, instead of particles we have the formation of wormholes. The ansatz (41) shows how one can go to the vacuum case $\Omega \to 0$ with $n \to \infty$. Yet we wish to abide by the classical domain with $n \ll \infty$.

3 2 + 1-dimensional wormhole induced by 3 + 1-dimensional flat spacetime

We consider the $3 + 1$-dimensional Minkowski spacetime in the cylindrical coordinates

$$ds^2 = -dt^2 + dr^2 + dz^2 + r^2 d\theta^2 \tag{42}$$

with the substitution $z = \xi(r, \theta)$. This gives the line element

$$ds^2 = -dt^2 + \left(1 + \xi_r(r, \theta)^2\right) dr^2 + \left(r^2 + \xi_\theta(r, \theta)^2\right) d\theta^2 + 2\xi_r(r, \theta)\xi_\theta(r, \theta) dr d\theta, \tag{43}$$

in which $\xi_r(r, \theta) = \frac{\partial \xi(r,\theta)}{\partial r}$ and $\xi_\theta(r, \theta) = \frac{\partial \xi(r,\theta)}{\partial \theta}$ where $\xi(r, \theta)$ is a function of r and θ. Using this line element, the Einstein tensor is obtained with only one non-zero component, i.e.,

$$G_t^t = -\frac{r^3 \xi_r \xi_{rr} + \xi_{rr} \xi_{\theta\theta} r^2 - \xi_\theta^2 + 2\xi_{r\theta}\xi_\theta r - \xi_{r\theta}^2 r}{\left(r^2 + \xi_\theta^2 + \xi_r^2 r^2\right)^2}. \tag{44}$$

Einstein's equation ($8\pi G = 1 = c$) reads

$$G_\mu^\nu = T_\mu^\nu, \tag{45}$$

in which T_μ^ν is the energy-momentum tensor. The latter implies that the only non-zero component of the energy-momentum tensor is the $T_t^t = -\rho$ component and therefore

$$\rho = \frac{r^3 \xi_r \xi_{rr} + \xi_{rr} \xi_{\theta\theta} r^2 - \xi_\theta^2 + 2\xi_{r\theta}\xi_\theta r - \xi_{r\theta}^2 r}{\left(r^2 + \xi_\theta^2 + \xi_r^2 r^2\right)^2}. \tag{46}$$

The total energy which supports the wormhole is obtained by

$$\Omega = \int_0^{2\pi} \int_0^\infty \rho \sqrt{-g}\, dr\, d\theta. \tag{47}$$

To complete this section we recall the case of a physical energy-momentum tensor and the weak energy condition which we have discussed before. Here, the situation is almost the same i.e., the only non-zero component of the energy-momentum tensor is the energy density ρ. Therefore in order to say that our wormhole is physical, which means it is

supported by normal (non-exotic) matter, the conditions are $\rho \geq 0$ and $0 \leq \Omega < \infty$.

3.1 Flare-out conditions

To have a wormhole we observe that $z = \xi(r, \theta)$ must be chosen aptly for a general wormhole structure. For instance, one may consider in the first attempt $\xi(r, \theta) = \xi(r)$ and thus we find

$$ds^2 = -dt^2 + \left(1 + \xi'(r)^2\right)dr^2 + r^2 d\theta^2, \tag{48}$$

so that the form of the energy density becomes

$$\rho = \frac{\xi'\xi''}{r\left(1 + \xi'^2\right)^2}. \tag{49}$$

The expression (48) is comparable with Morris–Thorne's static wormhole,

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{1}{1 - \frac{b(r)}{r}}dr^2 + r^2 d\theta^2, \tag{50}$$

in which $\Phi(r)$ and $b(r)$ are the red-shift and shape functions, respectively. In the specific case (48), one finds $\Phi(r) = 0$ and $b(r) = r\left(\frac{\xi'(r)^2}{1 + \xi'(r)^2}\right)$. The well-known flare-out condition introduced by Morris and Thorne implies that if $r = r_0$ is the location of the throat, (i) $b(r_0) = r_0$ and (ii) for $r > r_0$, $b'(r) < \frac{b(r)}{r}$. In terms of the new setting, (i) implies that at the throat $\xi' = \pm\infty$ and (ii) states that $\xi'\xi'' < 0$ for $r > r_0$. In addition to these conditions at the throat we have $z = \xi(r_0) = 0$.

Next, we introduce the location of the throat at $z = 0$ and $r = R_0(\theta)$ in which $R_0(\theta)$ is a periodic function of θ. These mean that $z = \xi(R_0, \theta) = 0$. Now, for a general function for $z = \xi(r, \theta)$ we impose the same conditions as the Morris–Thorne wormholes i.e., $\xi_r\xi_{rr} < 0$ for $r > R_0(\theta)$ and at the location of the throat (where $z = 0$ and $r = R_0(\theta)$) $\xi_r(R_0, \theta) = \pm\infty$.

3.2 An illustrative example

Here we present an explicit example. Let us consider

$$\xi = \pm 2r_0\sqrt{\left(\frac{r}{R_0(\theta)} - 1\right)} \tag{51}$$

in which the first condition i.e., $\xi = 0$ at the location of the throat $r = R_0(\theta)$ is fulfilled. Next, the expression

$$\xi_r\xi_{rr} = -\frac{1}{2}\frac{r_0^2}{\left(\frac{r}{R_0} - 1\right)^2 R_0^3} < 0 \tag{52}$$

imposes $R_0(\theta) > 0$ on the entire domain of θ i.e. $\theta \in [0, 2\pi]$. We note also that $R_0(\theta)$ must be a periodic function

of θ to make $r = r_0 R_0(\theta)$ a closed loop, which is going to be our throat. The forms of

$$\xi_r = \frac{\pm r_0}{R_0\sqrt{\frac{r}{R_0} - 1}} \tag{53}$$

and

$$\xi_{rr} = -\frac{\pm r_0}{2R^2\left(\frac{r}{R_0} - 1\right)^{3/2}} \tag{54}$$

suggest that at the throat $\xi_r \to \pm\infty$ and also $\xi_{rr} \to \mp\infty$ as expected. The form of the energy density in terms of R, however, becomes

$$\rho = \frac{R_0^3 r_0^2 \left(R_0'' R_0 - R_0^2 - 2R_0'^2\right)}{2r\left(r_0^2 R_0'^2 + R_0^2\left(r_0^2 + rR_0 - r_0 R_0^2\right)\right)^2}. \tag{55}$$

The latter implies that any periodic function of $R_0(\theta)$ which satisfies $R_0'' R_0 - R_0^2 - 2R_0'^2 > 0$ can represent a traversable wormhole with positive energy. In the case of $R_0 = r_0$ or $\xi = \pm 2r_0\sqrt{\frac{r}{r_0} - 1}$, the wormhole is shown in Fig. 6 whose energy density is given by

$$\rho = -\frac{r_0}{2r^3}. \tag{56}$$

In Figs. 6, 7, and 8 we plot the wormholes with $R_0(\theta)$ given in (37), (38), (39), (40), and (41), respectively. Also in Fig. 9, the energy density ρ corresponds to the individual cases of (a), (b), (c), and (d) for (37), (38), (39), and (40), respectively, which are given in terms of r and θ with $r_0 = 1$.
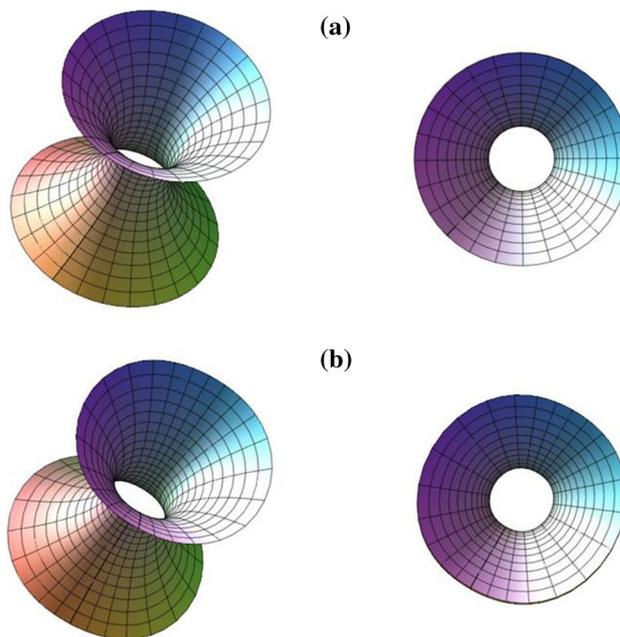
(a)

(b)

Fig. 6 The geometry of the throat when $R_0(\theta) = 1$ for **(a)** and $R_0(\theta) = \frac{1}{0.5\cos^2\theta + 1}$ for **(b)** with $r_0 = 1$
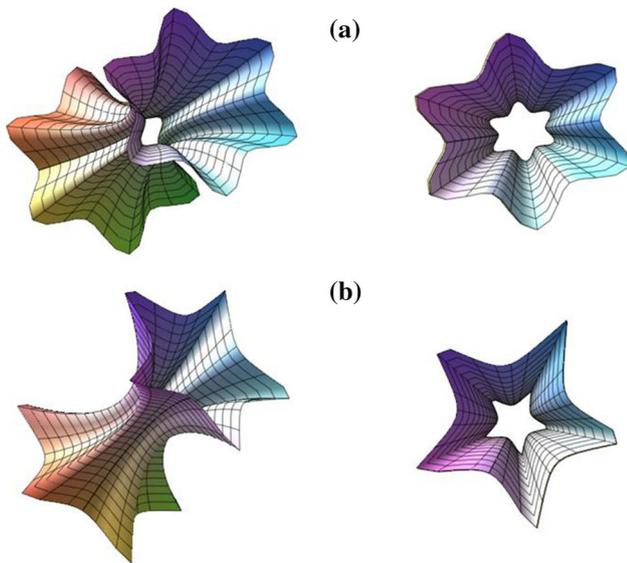
Fig. 7 The geometry of the throat when $R_0(\theta) = \frac{1}{0.5\cos^2(3\theta)+1}$ for (**a**) and $R_0(\theta) = \frac{1}{\left|\cos\left(\frac{5}{2}\theta\right)\right|+1}$ for (**b**) with $r_0 = 1$
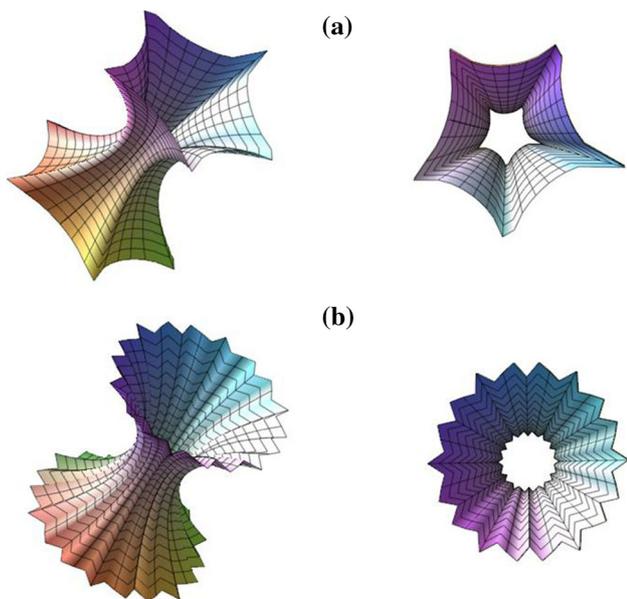

Fig. 8 The geometry of the throat when $R_0(\theta) = \frac{1}{\sqrt{\left|\cos\left(\frac{5}{2}\theta\right)\right|+1}}$ for (**a**) and $R_0(\theta) = \frac{1}{\epsilon\cos^2(n\theta)+1}$ for (**b**) with $r_0 = 1$, $\epsilon = 0.1$, and $n = 30$

We see, for instance, that in Fig. 9d the energy is positive everywhere.

Regarding Fig. 9d we must admit again that for the same reason as in the TSW, $R_0(\theta)$ does not admit derivative at the spike points, therefore the energy density ρ is not defined at the same points. On the other hand its limits exist at these critical points and are both finite and positive. This can be seen from (55) whose numerator and denominator become
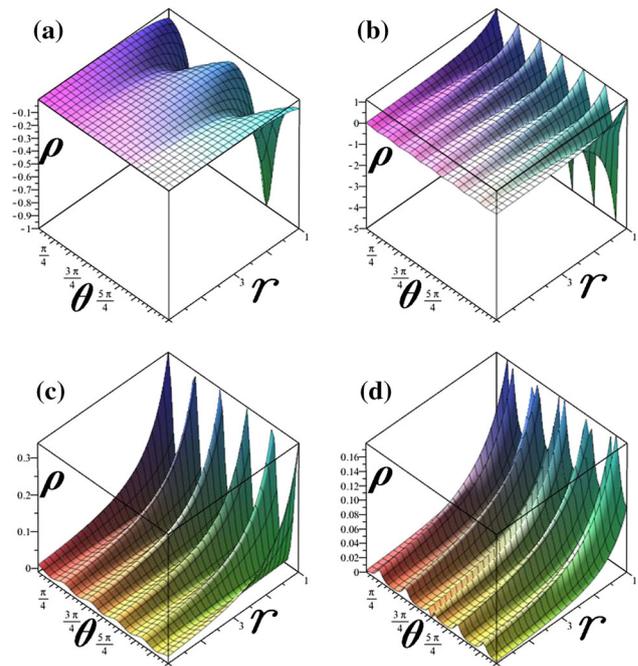

Fig. 9 The energy densities of the wormholes given in (37) for (**a**), (38) for (**b**), (39) for (**c**) and (40) for (**d**). As we observe, in the cases (**b**) and (**c**) the energy density gets a positive value for some interval while for (**d**) $\rho > 0$ everywhere. In (**a**) the energy density is negative everywhere

infinity near the critical points, while the ratio, which is the limit of ρ, remains finite. This is similar to the behavior of a function like $\frac{\sin(x)}{x}$ in the neighborhood of $x = 0$.

4 Conclusion

First of all let us admit that traversable wormholes in $2 + 1$ dimensions were considered much earlier, namely in the 1990s [3,4], and became fashionable recently [5–7]. Our principal aim in this study is to establish a traversable wormhole with normal (i.e. non-exotic) matter in $2+1$ dimensions. For the TSWs the strategy is to assume a closed angular path of the form $r = R(t, \theta)$, where for $R(t, \theta) = a(t)$, we recover the circular throat topology. Since this leads to exotic matter it is not attractive by our assessment. Let us note that throughout our study we use the coordinate time instead of the proper time. Our analysis shows that any concave-shaped $R_0(\theta) = R(0, \theta)$ around the origin undergoes the same fate of exotic matter. However, a convex-shaped $R_0(\theta)$ seems promising in obtaining a normal matter. This is shown by explicit ansatzes whose plots suggest starfish-shaped closed curves for the throat of a $2 + 1$-dimensional wormhole. Locally, for specific angular range it may yield negative energy, but in total the energy accumulates on the positive side. This result supports our previous finding that for the

non-spherical spacetimes, i.e. the ZV-metrics, the throats can be non-spherical and in turn one may obtain a TSW in Einstein's theory with a positive total energy [17]. The conclusion drawn herein for $2 + 1$ dimensions therefore can be generalized to higher dimensions. A similar construction method has been employed for the general wormholes. We have treated the $2 + 1$-dimensional wormhole as a brane in $3 + 1$-dimensional Minkowski space and show the possibility of physical traversable wormholes. It turns out, however, that the rabbit emerges from only very special hats, not from all hats. Extension of our work to $3 + 1$-dimensional wormholes is under construction.

References

1. M.S. Morris, K.S. Thorne, Am. J. Phys. **56**, 395 (1988)
2. D. Hochberg, M. Visser, Phys. Rev. D **56**, 4745 (1997)
3. M.S. Delgaty, R.B. Mann, Int. J. Mod. Phys. D **4**, 231 (1995)
4. G.P. Perry, R.B. Mann, Gen. Relativ. Gravit. **24**, 305 (1992)
5. F. Rahaman, A. Banerjee, I. Radinschi, Int. J. Theor. Phys. **51**, 1680 (2012)
6. A. Banerjee, Int. J. Theor. Phys. **52**, 2943 (2013)
7. C. Bejarano, E.F. Eiroa, C. Simeone, Eur. Phys. J. C **74**, 3015 (2014)
8. M. Visser, Phys. Rev. D **39**, 3182 (1989)
9. M. Visser, Nucl. Phys. B **328**, 203 (1989)
10. P.R. Brady, J. Louko, E. Poisson, Phys. Rev. D **44**, 1891 (1991)
11. E. Poisson, M. Visser, Phys. Rev. D **52**, 7318 (1995)
12. M. Visser, *Lorentzian Wormholes from Einstein to Hawking* (American Institute of Physics, New York, 1995)
13. S.H. Mazharimousavi, M. Halilsoy, Phys. Rev. D **90**, 087501 (2014)
14. H. Weyl, Ann. Physik **54**, 117 (1917)
15. D.M. Zipoy, J. Math. Phys. (N.Y.) **7**, 1137 (1966)
16. B.H. Voorhees, Phys. Rev. D **2**, 2119 (1970)
17. S.H. Mazharimousavi, M. Halilsoy, Eur. Phys. J. C **74**, 3067 (2014)
18. W. Israel, Nuovo Cimento **44B**, 1 (1966)
19. V. de la Cruz, W. Israel, Nuovo Cimento **51A**, 774 (1967)
20. J.E. Chase, Nuovo Cimento **67B**, 136 (1970)
21. S.K. Blau, E.I. Guendelman, A.H. Guth, Phys. Rev. D **35**, 1747 (1987)
22. R. Balbinot, E. Poisson, Phys. Rev. D **41**, 395 (1990)