A note on 3D $\mathcal{N} = 2$ dualities: real mass flow and partition function

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ABSTRACT: We study two well-known classes of dualities in three dimensional $\mathcal{N} = 2$ supersymmetric field theories. In the first class there are non trivial interactions involving monopole operators while in the second class the dual gauge theories have Chern-Simons terms in the action. An RG flows connecting the first dual pair to the second one has been studied in the past and tested on the partition function on the squashed three sphere. Recently an opposite RG flow connecting the second dual pair to the first one has been studied in the case of unitary gauge groups. In this paper we study this flow on the partition function on the squashed three sphere. We verify that the equality between the partition functions of the original dual models is preserved in the IR, where the other dual pair is reached. We generalize the analysis to the case of symplectic and of orthogonal groups.

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Contents

1 Introduction

At low energies asymptotically free quantum field theories can flow to strong coupling, where a perturbative analysis is impossible. A complete non-perturbative analysis of these models is complicate and some effective description is necessary. One possibility consists of finding a completely different theory that in the IR describes the same degrees of freedom and correlators of the original one. In such a case the two models are dual in the IR. Anyway it is not a trivial task to find the dual model of a strongly coupled quantum field theory.

An useful laboratory in this search of dual models is supersymmetry. Indeed many examples of dualities have been known for a long time in four dimensions, like the Montonen-Olive duality in the maximal supersymmetric case [1], the $\mathcal{N} = 2$ Seiberg-Witten duality [2] and the $\mathcal{N} = 1$ Seiberg duality of SQCD [3]. These four dimensional dualities map the strongly coupled *electric* regime of one theory to the weakly coupled *magnetic* regime of the dual theory. Many extensions of these dualities have been studied in four dimensions.

The search of analogous dualities in other dimensions is a natural problem. In the last years the three dimensional case attracted a great interest. For example a rich class of dualities exists when $\mathcal{N} = 2$. This case shares the same number of supercharges, four, as the $\mathcal{N} = 1$ four dimensional case, and one may expect some analogy with Seiberg duality.[1]

Indeed it was shown in [5] that a *similar* duality exists in three dimensions. This duality is usually called Aharony duality. This duality is only *similar* to Seiberg duality because in the three dimensional case some differences arise in the analysis of the moduli space and they affect the structure of the dual field content and superpotential. More precisely, the monopole operators that parameterize a branch of the moduli space of one theory couple with the monopole operators of the dual theory.

There is a second class of dualities in the three dimensional case that looks close to Seiberg duality. In three dimensions one can write a topological Chern-Simons (CS) action. This action modifies the vacuum structure of a theory and gives rise to the Giveon-Kutasov duality [6]. This duality maps two CS gauge theories and looks similar to the four dimensional Seiberg duality in terms of the field content even if it involves CS theories. More recently other dualities that mix the Aharony and Giveon-Kutasov cases have been found in [7]. These cases involve a chiral like matter content where the number of fundamentals and anti-fundamentals does not coincide.

Even if Aharony and Giveon-Kutasov dualities look different, at the level of the action and of the field content, they are strongly related. The reason is that in three dimensions one can assign a real mass to the matter fields charged under some global symmetries. When a gauge charged chiral fermion is massive it can be integrated out. This process shifts the effective CS level by a semi-integer number. This real mass flow connects theories without CS terms to theories with CS terms, and one can in principle flow from one pair of dual theories to another thanks to this mechanism. While the flow from the Aharony pair to the Giveon-Kutasov is simple to understand the opposite one is more mysterious, because one has to understand the origin of the superpotential interaction between the monopoles of the electric and of the magnetic phase. This flow has been recently found in [8] by observing that if one assigns some large mass term to some of the electric fields of the Giveon-Kutasov duality it does not simply reflects in a set of masses for some dual fields. Indeed in the large mass regime there are points in the moduli space that become singular, and one has to map the vacua correctly. Indeed at these points some massless fields acquire a mass and some massive fields become massless. The flow investigated in [8] was restricted to the case of CS level k equals to -1. It was then studied in some more general case in [9].

In this paper we generalise the construction of [8] to generic level k. Then we study the partition function on the squashed three sphere with the real mass terms. We start from the equivalence of the partition functions of the Giveon-Kutasov dual theories. By adding the real masses on both sides of this duality and by taking the large mass limit we arrive at the expected relation for the Aharony dual pair.

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[1]In three dimensions there is another kind of duality, called mirror symmetry, that we will not discuss in this paper [4].

An important aspect of the the computation is related to the structure of the partition function. It is a matrix integral over the real scalar component of the vector multiplet, reduced to the Cartan subgroup. In some phase this scalar may take a vev proportional to the large mass that we introduced. Naively one may think that this shift cannot affect the integral, but it is only true for a finite mass. In the large mass regime this vev affects the dominant contribution to the partition function, such that the expected result is recovered. We finally study the analogous flow in the case of symplectic and orthogonal groups, showing that similar results hold.

The paper is organised as follows. In section 2 we review some useful aspect of three dimensional $\mathcal{N} = 2$ gauge theories that are relevant in our analysis. In section 3 we discuss the Giveon-Kutasov and the Aharony dualities and we study the real mass flow connecting them. In section 4 we review the matrix integral describing the partition function on the squashed three sphere and the role of the masses in the partition function. In section 5 we study the RG flow from the Giveon-Kutasov to the Aharony duality from the perspective of the partition function. In section 6 and 7 we study the symplectic and the orthogonal cases respectively. In section 8 we conclude.

2 Review material

In this section we review some basic aspects of $\mathcal{N} = 2$ three dimensional gauge theories. We refer the reader to [10] for a more complete review.

Three dimensional $\mathcal{N} = 2$ supersymmetry has four supercharges Q_α and $\widetilde{Q}_\alpha$ with $\alpha = 1, 2$. Their non vanishing anticommutator is

$$\{Q_\alpha, \widetilde{Q}_\beta\} = \sigma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z \tag{2.1}$$

where Z is the central charge, corresponding to the reduced momentum along P_3.

As in four dimensions there are a vector multiplet and a chiral (and antichiral multiplet). The vector multiplet V is composed by a gauge boson A_μ, the gaugini λ_α and $\widetilde{\lambda}_\alpha$ the D-term D and a real scalar σ. When this real scalar gets an expectation value it generically breaks the gauge symmetry to $U(1)^r$ (r being the rank of the gauge group) and one is at a generic point of the Coulomb branch. In the abelian case the photon can be dualized into a scalar $F_{\mu\nu,i} = \epsilon_{\mu\nu\lambda}\partial^\lambda\phi_i$ ($i = 1, \ldots, r$) and one can construct a supermultiplet $\Phi_i = \sigma_i + i\phi_i$. This supermultiplet parameterizes the classical Coulomb branch and is associated to a monopole operator $Y_i \simeq e^{\Phi_i}$. The Coulomb branch is usually lifted by quantum correction and only some direction may remain flat. There is also an Higgs branch, parameterized by the charged chiral multiplets.

In three dimensions there are usually more global symmetries than in the four dimensional case. Indeed the anomalous symmetries of the four dimensional case become non anomalous in three dimensions. In addition we have the usual $U(1)_R$ charge that rotates the supercharges, as in four dimensions. Another new symmetry is the topological $U(1)_J$ symmetry, that is generated by the current $J^i_\mu = \epsilon_{\mu\nu\lambda}F^{\nu\lambda,i}$ and shifts the dual photon. There are r $U(1)_J$ currents, but at quantum level, where the Coulomb branch is lifted, just one combination may be left.

Another multiplet that one can construct in three dimensions is the linear multiplet. A linear multiplet, say Σ is defined by $\epsilon^{\alpha\beta}D_\alpha D_\beta \Sigma = \epsilon^{\alpha\beta}\overline{D}_\alpha \overline{D}_\beta \Sigma = 0$ and its lowest component is a real scalar. One can define a linear multiplet for every globally conserved current.

The linear multiplet is useful to understand the relation between the central charge and the real masses. In three dimensions we can indeed turn on a real mass m for a chiral field X, if this last is charged under a global symmetry.

$$\int d^4\theta X^\dagger e^{m\theta\bar\theta} X \qquad (2.2)$$

There is a linear multiplet that contains the global current under which the chiral field X is charged. The central charge is associated to a background superfield and it is the scalar component of this linear multiplet, in this case $Z = m$. In general $Z = \sum q_i m_i$ where m_i are background linear multiplets.

There is another contribution to the central charge coming from the real FI parameter. If one turns on a background vector multiplet V_b for the topological $U(1)_J$ one has can add a term $\int d^4\theta V_b\Sigma$ that integrating by parts corresponds to $\int d^4\theta V\Sigma_b$. The background linear multiplet Σ_b is a FI for the gauge multiplet V and contributes to the central charge.

Lastly in three dimensions there is also a topological CS action $k^{ij}\int d^4\theta \Sigma_i V_j$, that is gauge invariant (under large gauge transformations) if the CS level k^{ij} is an integer. It is important to observe that chiral fermions with a real mass and CS levels are strongly connected. A massive fermion ψ has in general real mass $m_\psi = m + \sum_i q_\psi^i \sigma_i$. By integrating it out we have at one loop a shift in the CS level $(k_{\text{eff}})^{ij} = k^{ij} + \frac{1}{2}\sum q_\psi^i q_\psi^i sgn(m)$. By gauge invariance k_{eff} has to be integer. It implies that if $\sum q_\psi^i q_\psi^j$ is odd and k^{ij} is not vanishing then parity is broken. This has to be compensated by $k^{ij} \in \mathbb{Z} + \frac{1}{2}$. This phenomenon is named parity anomaly [11, 12].

3 Giveon-Kutasov and Aharony duality

In this section we present the models that we investigate in the rest of the paper. They are three dimensional $\mathcal{N} = 2$ supersymmetric gauge theories with $U(N_c)$ gauge groups and N_f matter fields in the fundamental and in the antifundamental representation of the gauge group. These theories are called vector like because the number of fundamentals and of antifundamentals is the same. We leave possible generalisation to dualities between theories with a chiral field content [7] for future investigations.

First we discuss the Giveon-Kutasov duality [6]. The electric theory consists of a $U(N_c)$ gauge theory with a CS action at level k. There are N_f fields Q in the fundamental and N_f fields $\widetilde{Q}$ in the antifundamental of the gauge group. In absence of superpotential there is a $SU(N_f) \times SU(N_f)$ flavor symmetry acting on these quarks. Moreover we have an $U(1)_A$ global symmetry under which both the fields have the same charge $+1$ and an $U(1)_R$ symmetry.

The dual theory is a $U(N_f - N_c + |k|)$ gauge theory with a CS action at level $-k$. There are N_f fields q in the fundamental and N_f fields $\widetilde{q}$ in the antifundamental of the gauge group. The N_f^2 electric mesons $M = Q\widetilde{Q}$ are elementary singlets in the dual description

and couple to the quarks through the superpotential

$$W = Mq\widetilde{q} \tag{3.1}$$

The charges of the fields under the global symmetries are

	U(N_c)	U$(\widetilde{N}_c)$	SU(N_f)	SU(N_f)	U$(1)_A$	U$(1)_R$
Q	N_c	1	$\overline{N}_f$	1	1	Δ
$\tilde{Q}$	$\overline{N}_c$	1	1	N_f	1	Δ
q	1	$\overline{\widetilde{N}_c}$	N_f	1	-1	$1-\Delta$
$\tilde{q}$	1	$\widetilde{N}_c$	1	$\overline{N}_f$	-1	$1-\Delta$
M	1	1	$\overline{N}_f$	N_f	2	2Δ

$$\tag{3.2}$$

In this paper we are interested in connecting these two models with another pair of dual models by an RG flow. This second pair of dual models, the Aharony duality, was found in [5]. On the electric side we consider a U$(N_c)_0$ gauge theory, where the subscript indicates that there is not CS action, and N_f fundamentals and antifundamentals Q and $\tilde{Q}$ respectively. The dual model has gauge group U$(N_f - N_c)_0$, the CS level is vanishing as well, and there are N_f fundamentals and antifundamentals q and $\tilde{q}$ respectively. As in the case of the Giveon-Kutasov duality the electric mesons are elementary degrees of freedom in the dual description and couple to the quarks through the superpotential (3.1).

The Coulomb branch is not completely lifted by the quantum corrections. There are still combinations of monopole operators that remains flat, and they correspond to the ones with flux $(\pm 1, 0, \ldots, 0)$ in the Cartan of the gauge group. We will refer to these monopole operators in the electric theory as $X_\pm$. They have charge ± 1 under the topological U(1) that shifts the dual photon. In the magnetic theory they are singlets that interact with the dual monopole operators $x_\pm$ by a superpotential

$$\Delta W = x_+ X_- + x_- X_+ \tag{3.3}$$

The charges of the fields under the global symmetries are

	U(N_c)	U$(\widetilde{N}_c)$	SU(N_f)	SU(N_f)	U$(1)_A$	U$(1)_R$	U$(1)_J$
Q	N_c	1	$\overline{N}_f$	1	1	Δ	0
$\tilde{Q}$	$\overline{N}_c$	1	1	N_f	1	Δ	0
q	1	$\overline{\widetilde{N}_c}$	N_f	1	-1	$1-\Delta$	0
$\tilde{q}$	1	$\widetilde{N}_c$	1	$\overline{N}_f$	-1	$1-\Delta$	0
M	1	1	$\overline{N}_f$	N_f	2	2Δ	0
X_+	1	1	1	1	$-N_f$	$N_f(1-\Delta)-N_c+1$	1
X_-	1	1	1	1	$-N_f$	$N_f(1-\Delta)-N_c+1$	-1
x_+	1	1	1	1	N_f	$N_f(\Delta-1)+N_c+1$	1
x_-	1	1	1	1	N_f	$N_f(\Delta-1)+N_c+1$	-1

$$\tag{3.4}$$

3.1 Flowing from Aharony to Giveon-Kutasov duality

In this section we review the RG flow connecting the Aharony dual pair to the Giveon-Kutasov dual pair studied in [15]. The CS terms are generated by assigning real masses to some of the fermions and by integrating them out.

We consider, on the electric side, a $U(N_c)_0$ gauge theory with $N_f + k$ fundamentals and antifundamentals. We turn on positive real masses for these matter fields such that there are N_f light and k heavy fields. By integrating the heavy massive fermions out we shift the CS level from 0 to k. We are left with a $U(N_c)_k$ model with N_f fundamentals and antifundamentals.

The dual theory has an $U(N_f + k - N_c)_0$ gauge group with N_f light and k heavy flavors. Moreover this theory has new elementary singlets, consisting of mesons and monopoles of the electric theory. The monopoles and some components of the meson acquire a large mass term, in accordance to the global symmetries. By integrating the heavy fields out we are left with a $U(N_f - N_c + k)_{-k}$ gauge theory, with N_f light flavors and N_f^2 light singlets. It corresponds to the expected Giveon-Kutasov dual phase.

Observe that we could also have inverted the sign of the real masses. It would have changed the electric level from k to $-k$ and the dual CS level in the opposite way. In any case the rank of the dual is $(N_f - N_c + k)$, for positive k. In general, the dual of $U(N_c)_k$ is $U(N_f - N_c + |k|)_{-k}$ for any choice of sign of k.

3.2 Flowing from Giveon-Kutasov to Aharony duality

In this section we discuss the RG flow connecting the Giveon-Kutasov dual pair to the Aharony dual pair. This flow has been recently studied in [8]. The analysis of [8] is restricted to the electric $U(N_c)_{-1}$ gauge theory with N_f flavors, dual to $U(N_f - N_c + 1)_1$ with N_f flavors and N_f^2 singlets. The analysis has been extended to the cases with $k = 2$ and $k = 4$ in [9]. Here we generalise part of the analysis of [8, 9] to general k.

In the case of $k = -1$ two different flows are possible. In one case one has in the IR the usual Aharony dual pair. This has been named the $1 - 4$ duality in [8] and we will keep the same name in this paper. In the second case, named $2 - 3$, one flows to a slightly different pair of dual theories. The matter content is the same as in the usual case, but one of the interaction between the electric and magnetic monopoles in the dual phase is lifted and it appears in the electric superpotential. The electric theory has $W = x_- X_+$, a monopole of the magnetic theory is a singlet in the electric phase. The dual superpotential is $W = Mq\tilde{q} + x_+ X_-$. This duality can be obtained from Aharony duality by adding a superpotential mass term for the electric monopole (or antimonopole) in the dual theory. If $|k| > 1$ more complicate structures are possible [9]. We will not discuss these possibilities here.

First we study the flow from the Giveon-Kutasov dual pair at level $-k$ ($k > 0$) to the $1-4$ Aharony dual pair. We consider an $U(N_c)_{-k}$ gauge theory with N_f light fundamentals and k heavy ones, with real masses. For simplicity we take all the heavy masses with the same value $m > 0$. After integrating out the heavy matter we obtain an $U(N_c)_0$ gauge

theory with N_f massless flavors. The level shifts because we integrate out $2k$ chiral fermions with the same positive real mass. This is the electric version of the Aharony duality.

The dual theory is a $\mathrm{U}(N_f - N_c + 2k)_k$ gauge theory with N_f light and k heavy flavors, with mass $-m$. There are $(N_f + k)^2$ singlets M, with N_f^2 light components. This theory has superpotential $W = Mq\tilde{q}$. In presence of the large real mass one has to shift the vacuum parameterized by σ_i. The shifted location on the Coulomb branch is interesting because some new light degrees of freedom arise here. The magnetic Giveon-Kutasov phase flows to the magnetic Aharony phase if one chooses $\sigma_1 = \ldots, \sigma_{N_f - N_c} = 0$ and $\sigma_{N_f - N_c + 1}, \ldots, \sigma_{N_f - N_c + k} = -\sigma_{N_f - N_c + k + 1}, \ldots, -\sigma_{N_f - N_c + 2k} = m$. All the other matter fields are at the origin.

This non trivial vacuum is crucial in the analysis, because it corresponds to a point in the moduli space where the gauge symmetry is broken and some of the real masses for the quarks become light. The gauge group $\mathrm{U}(N_f - N_c + 2k)_k$ is broken to $\mathrm{U}(N_f - N_c) \times \mathrm{U}(k)^2$. After integrating out the heavy quarks the $\mathrm{U}(N_f - N_c)$ factor has N_f light flavors and vanishing CS level. The two $\mathrm{U}(k)$ factors are more involved. Indeed in this case the light N_f quarks become heavy because their mass is shifted by the vev of σ. The mass of the extra k heavy fields is shifted in the two sectors by an amount of $\pm m$. In one case we are left with k light fundamentals and k antifundamentals with mass $-2m$, in the other case the situation is the opposite, there are k light antifundamentals and k heavy fundamentals, with mass $-2m$. In both cases an effective CS level $\frac{k}{2}$ is obtained. Summarizing the $\mathrm{U}(N_f - N_c)_0$ sector has N_f flavors and N_f^2 singlets, one $\mathrm{U}(k)_{k/2}$ sector hase k charged chirals and the other $\mathrm{U}(k)_{k/2}$ has k charged antichirals.

The last two sectors are the key ingredients to obtain the monopole interactions in the Aharony duality. We consider one of these sectors but the discussion applies in the same way to the second one. A $\mathrm{U}(k)_{\frac{k}{2}}$ gauge theory with one chiral superfield with charge $+1$ is dual to a single chiral superfield X_+ [7].[2] This operator *must* couple with the magnetic monopoles x_- associated to $\mathrm{U}(1) \subset \mathrm{U}(N_f - N_c)$. The reason is that the original $\mathrm{U}(N_f - N_c + 2k)$ theory has a topological symmetry and under this symmetry the operators X_+ and x_- have opposite charges. The coupling is through a superpotential $W = x_- X_+$ [8]. In the second case we can repeat the same analysis and we eventually obtain $W = x_+ X_-$. Finally we get a $\mathrm{U}(N_f - N_C)_0$ with superpotential

$$W = Mq\tilde{q} + x_- X_+ + x_+ X_- \tag{3.5}$$

as predicted by Aharony duality. Observe that here we are restricting to the case of level $-k$ in the electric theory with positive k. The case of $k < 0$ can be studied by inverting the sign of the real mass.

As discussed in [8] there is a second possibility. It consists of studying the electric theory in the non trivial vacuum $\sigma_{N_c - k}, \ldots, \sigma_k = m$, and all the other components vanishing. The magnetic vacuum that preserves the duality is $\sigma_{N_f - N_c + k + 1}, \ldots, \sigma_{N_f - N_c + 2k} = -m$,

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[2]One can understand this duality starting from Aharony duality with an $\mathrm{U}(k)_0$ gauge group and k flavors. By adding a large positive (negative) real mass to the k chiral or to the k antichiral fields one generates a positive (negative) CS level. On the other hand, in the dual theory, that is composed just by singlets, many fields become massive and one remains with just one singlet.

and all the other components are vanishing. In the large m limit the electric theory is a $U(N_c - k)_0 \times U(k)_{-k/2}$ gauge theory with N_f light fundamentals and antifundamentals, charged under $U(N_c - k)_0$, and k light fundamentals, charged under $U(k)_{-k/2}$. The magnetic theory is a $U(N_f - N_c + k)_0 \times U(k)_{k/2}$ gauge theory with N_f light fundamentals and antifundamentals, charged under $U(N_c - k)_0$, and light k fundamentals, charged under $U(k)_{k/2}$. The two extra $U(k)$ sectors can be studied as before and we obtain the electric superpotential $W = x_- X_+$ and the magnetic one $W = M q \tilde{q} + x_+ X_-$. We refer to this duality as the $2 - 3$ duality.

4 The squashed three sphere partition function

In this section we review some aspect concerning the partition function on a squashed three sphere, S_b^3, preserving a $U(1)^2$ isometry of the original $SO(4)$ of the round case.

Partition functions computed on curved backgrounds that preserve some supercharges are powerful objects, because they give one loop exact results in supersymmetry. Localization is the most general technique to perform these calculations and it was first used in [16] for the partition function on S^4 of $\mathcal{N} = 2$ four dimensional theories. The case of the three dimensional sphere was first studied in [17] for $\mathcal{N} > 2$. The extension to $\mathcal{N} = 2$ was done in [13, 18] for the round sphere and in [19] for the case we are interested in.

The possibility of computing the partition function of a quantum field theory exactly is useful in checking the dualities. Indeed many dualities proposed in three dimensions have been checked by matching their partition functions on S_b^3.

Moreover one can add mass contributions to the partition function and study properties of the RG flows. Indeed in this paper we are interested in checking the flow by considering the partition functions of the Giveon-Kutasov dual pair and by checking that, at the end of the flow, the two partition functions still match and describe the correct Aharony dual pair.

The general structure of the partition function on the squashed sphere for a gauge group of rank G and charged matter is

$$Z = \frac{1}{|W|} \int \prod_{i=1}^{G} \frac{d\sigma_i}{\sqrt{-\omega_1 \omega_2}} e^{\frac{ik\pi\sigma_i^2}{\omega_1 \omega_2} + \frac{\pi i \lambda \sigma_i}{\omega_1 \omega_2}} \frac{\prod_I \Gamma_h \left(\omega \Delta_I + \rho_I(\sigma) + \tilde{\rho}_I(\mu) \right)}{\prod_{\alpha \in R_+} \Gamma_h \left(\alpha(\sigma) \right) \Gamma_h \left(-\alpha(\sigma) \right)} \quad (4.1)$$

The integral is performed over the Cartan subgroup of the gauge group. It is parameterized by the diagonal entries of the real scalar σ in the gauge group. The exponential receives contributions from the classical action, from the CS term at level k and from the real FI parameter λ. $|W|$ represents the sum over the Weyl degeneracies.

The Gamma Γ_h functions are obtained by computing the one loop superdeterminants of the vector and matter multiplets. They are usually divergent expressions that require a regularization. The function Γ_h is referred in the literature as hyperbolic Gamma function and it can be written as

$$\Gamma_h(z; \omega_1, \omega_2) \equiv \Gamma_h(z) \equiv \prod_{n,m=1}^{\infty} \frac{(n+1)\omega_1 + (m+1)\omega_2 - z}{n\omega_1 + m\omega_2 + z} \quad (4.2)$$

The contribution of the vector multiplet corresponds to the denominator of (4.1) and it is parameterized by the positive roots of the algebra.[3] The contribution of the matter multiplet is the last term in the numerator of (4.1). Each term corresponds to the contribution of the I-chiral multiplet with R charge Δ. The I-field is in the representation r of the gauge group G with weight $\rho_I(\sigma)$ and in the representation $\widetilde{r}$ of the flavour group F, with weight $\rho_I(\mu)$. Sometimes in the rest of the paper we will use the shortcut $\Gamma_h(x)\Gamma_h(-x) = \Gamma_h(\pm x)$ to simplify the expressions.

In the rest of this section we discuss the relation between the masses and the partition function, because it is crucial in the analysis of the RG flow. As observed, the partition function has an explicit dependence on the Cartan subgroup of the flavor symmetry. This is related to appearance of the central charge in the supersymmetry algebra. The central charge is $Z = \sum q_i m_i$ where the sum is performed over the real masses of the matter fields and q_i are the charges of these fields under the flavor symmetries.

Observe that these global non-R currents usually mix with the $U(1)_R$ current. This mixing is associated to the assignation of a non zero imaginary part to the real masses. The exact R-charge is obtained by finding the combination that minimizes the absolute value of the partition function [13, 14].

Observe that the monopole operators are charged also under the global topological $U(1)_J$ that shifts the dual photon. It follows that this charge mixes with the R-charge for these operators, and we can take this mixing into account in the partition function by assigning an imaginary part to the parameter λ.

One can consider another type of mass term. It consists of a complex mass in the superpotential. Suppose we have a massive combination

$$W = mQ\tilde{Q} \tag{4.3}$$

where Q and $\widetilde{Q}$ are in the and anti-fundamental of the gauge group G and their R charges are $\Delta_Q = \Delta = 2 - \Delta_{\widetilde{Q}}$. Thanks to the relation $\Gamma_h(z)\Gamma_h(2\omega - z) = 1$ the contribution of Q and $\widetilde{Q}$ to the partition function is 1.

The relation of the partition function with the real masses is more interesting. Indeed integrating out a real mass term modifies the partition function. Let us consider a field with R-charge Δ_I charged under the gauge group with weight $\rho_I(\sigma)$ and real mass $\rho(\mu_I) + m$. In the large m limit Γ_h reduces to

$$\lim_{m \to \pm\infty} \Gamma_h\left(\omega\Delta_I + \rho_I(\sigma) + \widetilde{\rho}_I(\mu) + m\right) =$$
$$= \zeta^{-sgn(m)} e^{\frac{i\pi}{2\omega_1\omega_2} sgn(m)(\omega(\Delta_I - 2) + \rho_I(\sigma) + \widetilde{\rho}_I(\mu) + m)^2} \tag{4.4}$$

We will often use this relation in this paper. Indeed as discussed in section 3.2 the flow from the Giveon-Kutasov to the Aharony duality is performed by assigning large real masses to some of the fields.

By using the relation (4.4) we have to decouple the effect of the massive fields on both sides of the duality. This is necessary to match the partition functions of the new dual

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[3]Actually in the one loop determinant of the vector multiplet there is another term that cancels against the Vandermonde determinant in the measure.

pair, where the heavy fields disappear. However the large mass dependence is not only in the matter fields, because extra mass dependencies come from the vacuum structure. Indeed in some cases the scalar σ in the vector multiplet takes an expectation value. In these cases one has to shift σ by an amount of m, where m is the large mass. This shift affects the integral in the large m limit, modifying the dominant contribution. We refer the reader to [20–22] for more discussions on this point.

Moreover, there are extra contributions of the real masses to the partition function, related to CS contact terms associated to the global symmetries [7, 14, 23]. These contributions are necessary in the flows that we are considering for the decoupling of the heavy masses on the two sides of the duality.

We conclude this section by reviewing the relations between the two sides of the Aharony and Giveon-Kutasov dualities. These relations have discovered in [21] and studied in the physical literature in [7, 15, 20, 22, 24]. First we fix the gauge group $G = \mathrm{U}(N_c)$ (we will review the $O(N_c)$ and $\mathrm{SP}(2N_c)$ case later). Then we consider $N_f^{(1)}$ quarks Q and $N_f^{(2)}$ antiquarks $\tilde{Q}$. We consider a real mass μ_a for the fundamentals and ν_b for the antifundamentals, where $a = 1, \ldots, N_f^{(1)}$ and $b = 1, \ldots, N_f^{(b)}$. From now on we absorb the R charge inside the real masses as well. This is done by turning on an imaginary deformation (this can be done because the $\mathrm{U}(1)_R$ mixes with the abelian global symmetries). In presence of a CS action, at level k and of a real FI parameter λ the partition function for such a model is

$$I_{\mathrm{U}(N_c)_k}^{\left(N_f^{(1)}, N_f^{(2)}\right)}(\mu; \nu; \lambda) = \int \prod_{i=1}^{N_c} d\sigma_i e^{\frac{i\pi(k\sigma_i^2 + \lambda\sigma_i)}{\omega_1\omega_2}} \frac{\prod_{a=1}^{N_f^{(1)}} \Gamma_h\left(\sigma_i + \mu_a\right) \prod_{b=1}^{N_f^{(2)}} \Gamma_h\left(-\sigma_i + \nu_b\right)}{\prod_{1 \leq i < j \leq N_c} \Gamma_h^{-1}\left(\pm(\sigma_i - \sigma_j)\right)} \quad (4.5)$$

Now we consider the case of Aharony duality. The electric side is obtained by turning off the CS level and by fixing $N_f^{(1)} = N_f^{(2)}$. The parameters μ_a and ν_b are

$$\mu_a = m_A + m_a + \omega\Delta \qquad \nu_b = m_A + \widetilde{m}_b + \omega\Delta \qquad (4.6)$$

with the balancing condition $\sum_{a=1}^{N_f} \mu_a = \sum_{b=1}^{N_f} \nu_b = m_A$. The equivalence between the partition functions is encoded in the relation

$$I_{\mathrm{U}(N_c)_0}^{(N_f, N_f)}(\mu; \nu; \lambda) = I_{\mathrm{U}(N_f - N_c)_0}^{(N_f, N_f)}(\omega - \mu; \omega - \nu; -\lambda)$$

$$\times \Gamma_h\left((N_f - N_c + 1)\omega - N_f m_A \pm \frac{\lambda}{2}\right) \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \nu_b) \quad (4.7)$$

The relation between the two phases of the Giveon-Kutasov duality is given by (here we fix $k > 0$)

$$I_{\mathrm{U}(N_c)_k}^{(N_f, N_f)}(\mu; \nu; \lambda) = I_{\mathrm{U}(N_f - N_c + k)_{-k}}^{(N_f, N_f)}(\omega - \mu; \omega - \nu; -\lambda) \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \nu_b) \zeta^{-k^2 - 2} e^{\frac{i\pi}{2\omega_1\omega_2}\phi} \quad (4.8)$$

where $\zeta = e^{\frac{i\pi}{24\omega_1\omega_2}}$ and the exponent ϕ is

$$\phi = k(\sum_{a=1}^{N_f}(\mu_a^2 + \nu_a^2) + k(k - 2(N_f - N_c + k))\omega^2 + \frac{1}{2}\lambda^2 - 2k\omega\sum_{a=1}^{N_f}(\mu_a + \nu_a)$$

$$+ \lambda\sum_{a=1}^{N_f}(\mu_a - \nu_a) + \frac{1}{2}(2(N_f - N_c + k)\omega - \sum_{a=1}^{N_f}(\mu_a + \nu_a))^2) \qquad (4.9)$$

observe that here we always referred to $k > 0$. The case $k < 0$ is obtained from equation (5.5.7) of [21].

5 Following the flow on the partition function

In this section we study the flow discussed in section 3.2. We first write the partition function of the dual Giveon-Kutasov pair, by turning on the contribution of the real masses as well. Then we integrate out the heavy states by using formula (4.4). The decoupling of the large mass is triggered not only by the massive fields but also by the vacuum structure. Indeed, as already discussed, the shift in the vev of sigma is not just a variable redefinition in the large m limit. If the vacuum is not at the origin an heavy quark can be effectively light and the integral receives the dominant contribution from such a vacuum.

5.1 The 1-4 case

In this first case the vacuum of the electric theory is $\langle\sigma_1\rangle = \cdots = \langle\sigma_{N_c}\rangle = 0$ in the electric case. The gauge group is $\mathrm{U}(N_c)_{-k}$ with $k > 0$ and N_f light and k heavy flavors. The mass structure is

$$\begin{cases} \mu_a = m_a + m_A & a = 1, \ldots, N_f \\ \mu_a = m + m_A & a = N_f + 1, \ldots, N_f + k \\ \nu_b = \widetilde{m}_b + m_A & b = 1, \ldots, N_f \\ \nu_b = m + m_A & b = N_f + 1, \ldots, N_f + k \end{cases} \qquad (5.1)$$

In the large m limit the partition function is

$$Z_e = \lim_{m\to\infty} \zeta^{-2kN_c} e^{\frac{i\pi k}{\omega_1\omega_2}N_c(m+m_A-\omega)^2} I_{\mathrm{U}(N_c)_0}^{(N_f,N_f)}(\mu; \nu; \lambda) \qquad (5.2)$$

where we absorbed the R charge inside the real masses.

The dual theory has gauge group $\mathrm{U}(N_f - N_c + 2k)_k$ with N_f light and k heavy flavors. The mass structure is

$$\begin{cases} \mu_a = \omega - m_a - m_A & a = 1, \ldots, N_f \\ \mu_a = \omega - m - m_A & a = N_f + 1, \ldots, N_f + k \\ \nu_b = \omega - \widetilde{m}_b - m_A & b = 1, \ldots, N_f \\ \nu_b = \omega - m - m_A & b = N_f + 1, \ldots, N_f + k \end{cases} \qquad (5.3)$$

The mass structure of the extra singlets M can be read from the global charges of Q and $\widetilde{Q}$.

The vacuum is given as in section 3.2. In the large m limit this changes the behaviour of the partition function and it is not a simple shift of the variables. This can be understood

by writing the explicit form of the partition function at large m. In this first example we separate each contribution coming from the vector multiplet, the charged and the uncharged matter and the classical action.

We start with the vector multiplet. The non trivial expectation value of σ leaves three sectors, $U(N_f - N_c) \times U(k)^2$, and in addition some mass dependent terms. These mass dependent terms come from the limit

$$\lim_{m \to \infty} \prod_{i=1}^{N_f - N_c} \prod_{j=1}^{k} \Gamma_h(\pm(\sigma_i - \widetilde{\sigma}_j + m))\Gamma_h(\pm(\sigma_i - \hat{\sigma}_j - m)) \times \prod_{i=1}^{k} \prod_{j=1}^{k} \Gamma_h(\pm(\widetilde{\sigma}_i - \hat{\sigma}_j - 2m))$$

$$= \lim_{m \to \infty} e^{\frac{4i\pi km\omega\left(N_f - N_c + k\right)}{\omega_1 \omega_2}} \prod_{i=1}^{k} e^{-\frac{2i\pi\omega(N_f - N_c + k)(\widetilde{\sigma}_i - \hat{\sigma}_i)}{\omega_1 \omega_2}} \tag{5.4}$$

The first term depends on the mass while the second term is a shift in the FI parameters of the unbroken $U(k)$ sectors.

In the charged matter sector there are three light contributions. There are N_f fundamentals and antifundamentals in the unbroken $U(N_f - N_c)$, k chiral fields in one of the unbroken $U(k)$ and k antichiral fields in the other unbroken $U(k)$. The other charged fields are heavy and one has to integrate them out. At large m their contribution is

$$\lim_{m \to \infty} \prod_{i=1}^{N_f - N_c} \Gamma_h^k(\pm\sigma_i + \omega - m_A - m) \prod_{i=1}^{k} \Gamma_h^k(\widetilde{\sigma}_i + \omega - m_A - 2m)\Gamma_h^k(-\hat{\sigma}_i + \omega - m_A - 2m)$$

$$\prod_{i=1}^{k} \prod_{a=1}^{N_f} \Gamma_h(\widetilde{\sigma}_i - m + \omega - \mu_a)\Gamma_h(-\widetilde{\sigma}_i + m + \omega - \nu_a)\Gamma_h(\hat{\sigma}_i + m + \omega - \mu_a)\Gamma_h(-\hat{\sigma}_i - m + \omega - \nu_a)$$

$$= \zeta^{2k(N_f - N_c + k)} \lim_{m \to \infty} e^{\frac{i\pi k}{\omega_1 \omega_2} + m^2(N_f - N_c + 4k) - 2mm_A(N_c - 3N_f - 2k) + m_A^2(N_f - N_c + k)}$$

$$\times \prod_{i=1}^{k} e^{\frac{i\pi\left(m_A\left(2N_f + k\right) + 2km\right)}{\omega_1 \omega_2}(\widetilde{\sigma}_i - \hat{\sigma}_i)} e^{-\frac{i\pi k}{2\omega_1 + \omega_2}(\widetilde{\sigma}_i^2 + \hat{\sigma}_i^2)} \prod_{i=1}^{N_f - N_c} e^{-\frac{i\pi k}{2\omega_1 + \omega_2}\sigma_i^2} \tag{5.5}$$

where we imposed the constraint $\sum_{a=1}^{N_f}(\mu_a + \nu_a) = 2m_A$. This constraint, called balancy condition, has to be imposed on the masses in the case of Aharony duality.

The $(N_f + k)^2$ uncharged singlets split in three sectors. One contains light fields, i.e. the N_f^2 mesons of Aharony duality, the other two sectors are heavy. At large m this heavy sector contributes as

$$\lim_{m \to \infty} \prod_{a=1}^{N_f} \Gamma_h^k(\mu_a + m_A + m)\Gamma_h^k(\nu_a + m_A + m) \prod_{a,b=1}^{k} \Gamma_h(2m_A + 2m) \tag{5.6}$$

$$= \lim_{m \to \infty} \zeta^{-k^2 - 2kN_f} e^{\frac{i\pi k\left(\sum_{a=1}^{N_f}\left(\mu_a^2 + \nu_a^2\right) + 2N_f(m_A + m - \omega)(3m_A + m - \omega) + 4km_A(m_A + 2m - \omega) + k(\omega - 2m)^2\right)}{2\omega_1 \omega_2}}$$

The contributions from the classical action are shifted accordingly to the vacuum structure. The last contribution is the exponent (4.9). By imposing the constraint $\sum_{a=1}^{N_f}(\mu_a + \nu_a) =$

$2m_A$ it is

$$\zeta^{-k^2-2}e^{\frac{i\pi\left(-2k\left(\sum_{a=1}^{N_f}(\mu_a^2+\nu_a^2)+2k(m_A+m)^2\right)+4\left(m_A(N_f+k)-\omega(N_f-N_c+2k)N_c+km\right)^2\right)}{4\omega_1\omega_2}}$$

$$\times e^{\frac{i\pi\left(8k\omega\left(m_A(N_f+k)+km\right)-2k\omega^2\left(-2N_c+2N_f+3k\right)+\lambda^2\right)}{4\omega_1\omega_2}} \tag{5.7}$$

Finally at large m the partition function of the dual magnetic theory is

$$Z_m = \lim_{m\to\infty} e^{\frac{i\pi\left(2\omega m_A\left(N_f(N_c-2k)-N_f^2+k^2\right)+m_A^2\left(kN_c+2kN_f+N_f^2-2k^2\right)+\omega^2\left(N_c^2+N_f\left(N_f+2k\right)-N_c\left(2N_f+k\right)\right)\right)}{\omega_1\omega_2}}$$

$$e^{\frac{i\pi km N_c\left(2m_A+m-2\omega\right)}{\omega_1\omega_2}+\frac{i\pi\lambda^2}{4\omega_1\omega_2}}\ \zeta^{-2kN_c-2}\ \prod_{a,b=1}^{N_f}\Gamma_h(\mu_a+\nu_b)\ I_{\mathrm{U}(N_f-N_c)_0}^{(N_f,N_f)}(\omega-\mu;\omega-\nu;-\lambda)$$

$$I_{\mathrm{U}(k)_{\frac{k}{2}}}^{(0,k)}(0;\omega-m_A;\lambda_1)\ I_{\mathrm{U}(k)_{\frac{k}{2}}}^{(k,0)}(\omega-m_A;0;\lambda_2) \tag{5.8}$$

with

$$\lambda_1 = 2\omega(N_c-N_f)-m_A(k-2N_f)-\lambda,\quad \lambda_2 = -2\omega(N_c-N_f)+m_A(k-2N_f)-\lambda \tag{5.9}$$

The last line of (5.8) represents two chiral $\mathrm{U}(k)_{\frac{k}{2}}$ sectors, the first with k fundamentals and the second with k antifundamentals. In the field theory interpretation each of these sectors is dual to a single chiral superfield. These integrals represent chiral CS gauge theories with level $\frac{k}{2}$, for integer k. They have been studied in [7] and in this case one can compute the integral. We find

$$I_{\mathrm{U}(k)_{\frac{k}{2}}}^{(0,k)}(0;\omega-m_A;\lambda_1) = \zeta\Gamma_h\left(\frac{km_A}{2}-k\omega-\frac{\lambda_1}{2}+\omega\right)e^{\frac{i\pi\left(2km_A(2k\omega+3\lambda_1)+\lambda_1(\lambda_1-4k\omega)-3k^2m_A^2\right)}{8\omega_1\omega_2}}$$

$$I_{\mathrm{U}(k)_{\frac{k}{2}}}^{(k,0)}(\omega-m_A;0;\lambda_2) = \zeta\Gamma_h\left(\frac{km_A}{2}-k\omega+\frac{\lambda_2}{2}+\omega\right)e^{\frac{i\pi\left(2km_A(2k\omega-3\lambda_2)+\lambda_2(\lambda_2+4k\omega)-3k^2m_A^2\right)}{8\omega_1\omega_2}}$$

$$\tag{5.10}$$

By substituting the expression in (5.9) to λ_1 and λ_2 we obtain the expected monopole contribution with N_c colours and N_f flavors

$$\Gamma_h\left(\omega(N_f-N_c+1)-N_f m_A \pm \frac{\lambda}{2}\right) \tag{5.11}$$

By putting everything together we reproduce the formula (4.7).

5.2 The 2-3 case

In this section we consider the second possibility that we discussed in section 3.1. In this case there is a different vacuum structure in both the electric and magnetic case. In the electric theory some of the σ_i acquire a large expectation value proportional to the real mass m. The duality with the massive Giveon-Kutasov magnetic theory is preserved by modifying the vacuum of the magnetic theory appropriately, as discussed in section 3.2.

This different choice of vacua does not modify the structure of the $U(N_c)_0$ and $U(N_f - N_c)_0$ unbroken parts of the two Aharony dual gauge theories but it changes the superpotential interactions.

We start by considering an electric CS gauge theory with $U(N_c + k)_{-k}$ and with the same masses as (5.1). We fix $k > 0$ and study the limit of large $m \to \infty$, with the vacuum given by $\sigma_i = 0$ for $i = 1, \ldots, N_c$ and $\sigma_i = -m$ $i = N_c + 1, \ldots, N_c + k$.

At large m the partition function of the electric theory becomes

$$
Z_e = \lim_{m \to \infty} \zeta^{-k^2 - 2kN_c} e^{-\frac{i\pi km\left(2m_A\left(-N_c - N_f + k\right) - mN_c + 2\omega N_f + \lambda\right)}{\omega_1 \omega_2} - \frac{i\pi k\left(k\omega^2 - 2N_c(\omega - m_A)^2 + km_A(m_A - 2\omega)\right)}{2\omega_1 \omega_2}}
$$
$$
\times I_{U(k)_{-\frac{k}{2}}}^{(0,k)}(0; m_A; \lambda_1) I_{U(N_f - N_c)_0}^{(N_f, N_f)}(\mu; \nu; \lambda) \tag{5.12}
$$

with

$$
\lambda_1 = \lambda + 2\omega\left(N_f - N_c\right) + \omega - m_A(k + 2N_f) \tag{5.13}
$$

The gauge group of the dual phase is $U(N_f - N_c + k)_k$ and the masses of the charged matter fields are given in (5.3). There are also N_f^2 light singlets, while the other components are heavy. The vacuum is $\sigma_i = 0$ for $i = 1, \ldots, N_f - N_c$ and $\sigma_i = m$ for $i = N_f + 1, \ldots, N_f + k$. At large m the dual partition function becomes

$$
Z_m = \lim_{m \to \infty} \zeta^{-k^2 - 2kN_c - 2} e^{-\frac{i\pi km\left(2m_A\left(-N_c - N_f + k\right) - mN_c + 2\omega N_f + \lambda\right)}{\omega_1 \omega_2}}
$$
$$
\times e^{\frac{i\pi\left(8\omega m_A\left(N_f\left(N_c - N_f - 2k\right) + k^2\right) + m_A^2\left(4kN_c + 4N_f\left(N_f + 2k\right) - 10k^2\right) + 4\omega^2\left(N_c^2 - N_c\left(2N_f + k\right) + N_f\left(N_f + 2k\right)\right) + \lambda^2\right)}{4\omega_1 \omega_2}}
$$
$$
\times \prod_{a,b} \Gamma_h(\mu_a + \nu_b) I_{U(N_f - N_c)_0}^{(N_f, N_f)}(\omega - \mu; \omega - \nu; -\lambda) I_{U(k)_{\frac{k}{2}}}^{(0,k)}(0; \omega - m_A; \lambda_2) \tag{5.14}
$$

with

$$
\lambda_2 = -\lambda + 2\omega\left(N_f - N_c\right) - m_A(k + 2N_f) \tag{5.15}
$$

By computing the two integrals of the two chiral sectors one obtains the correct contributions from the monopoles in the electric and magnetic sectors, and the correct result is found.

At this point of the discussion we can discuss the difference between the $1 - 4$ case and in the $2 - 3$ case. In the $1 - 4$ case the electric gauge group, in the flow from the Giveon-Kutasov to the Aharony duality, remains unbroken, the saddle is in the vacuum $\langle \sigma_i \rangle = 0$. In the $2 - 3$ case the electric gauge group in the flow from the Giveon-Kutasov to the Aharony duality is broken in a product of two subgroups, one represents the gauge group associated to the Aharony electric theory while the second gauge group is a sector with *chiral* like matter,[4] with CS level $\frac{k}{2}$. This sector generates the superpotential interaction involving the monopoles, that in the $2 - 3$ case arise also in the electric case.

Notice that we made two different choices for the electric Giveon-Kutasov gauge group in the $1 - 4$ and in the $2 - 3$ case. Indeed in the first case the electric rank is N_c while in

---

[4]By *chiral* we mean that there are fundamental but not antifundamentals in the large m limit.

the second case it is $N_c + k$. In both cases the electric models flow to the Aharony case with rank N_c. The magnetic theories are modified accordingly.

Anyway there is an alternative way to perform this calculation, basically the study of the flow in the $1-4$ case can be enough to obtain the flow $2-3$ case.[5] One can start from the $1-4$ case and add, in the magnetic theory, a massive interaction, coupling one of the electric monopoles (singlets) to a new singlet. The quantum numbers of this singlets are determined by the superpotential interaction itself. By integrating out the heavy fields and by dualizing, one obtains the expected $2-3$ duality. At the level of the partition function one has to add the contribution of this new singlet. By using the relation

$$\Gamma_h(z)\Gamma_h(2\omega - z) = 1 \tag{5.16}$$

one can show that, due to the presence of the mass term involving the old and the new singlets, the contribution of these singlets to the partition function is equal to the unity. For consistency the same contribution from the new singlet has to be added on the other side of the equality between the two partition functions of the $1-4$ case. This has the same quantum numbers of the magnetic monopoles acting as a singlet in the electric theory. The expected relation for the $2-3$ case is then recovered without any other complications.

However at the level of the partition the integration of the monopole operators by the formula (5.16) does not seem to affect the structure of the saddle point, while in the explicit calculation for the $2-3$ case we had to choose a different saddle in the flow from the theory without CS to the theory with non zero level. The reason is that the monopole are charged under the topological symmetry, that is associated to the FI term in the theory. The integration of one of the electric monopoles in the magnetic theory changes the flow. The effetive FI of the electric and of the magnetic theory are modified in the large mass limit and it affects the saddle of the partition function as we explicitely checked in the calculation.

6 The symplectic group

In this section we extend our results to the case of the symplectic gauge group $\mathrm{SP}(2N_c)$. First we discuss the Aharony and Giveon-Kutasov dualities for symplectic groups then we review some useful result for the partition function. Then we will discuss the flow from the Giveon-Kutasov dual pair to the Aharony dual pair. Finally we test the result on the partition function as in the unitarity case.

6.1 Aharony and Giveon-Kutasov dualities

First we discuss the case of Aharony duality. The electric theory is a $\mathrm{SP}(2N_c)_0$ gauge theory with $2N_f$ flavors Q. Note that the group is real here and the field content is chiral like.

The magnetic theory is a $\mathrm{SP}(2(N_f - N_c - 1))$ gauge theory with N_f flavors q and a singlet in the $2N_f(N_f - 1)$ representation of the $\mathrm{SU}(2N_f)$ flavor symmetry. In addition the monopole operator Y of the electric theory is a singlet in the dual theory and couples to the magnetic monopole y. The superpotential is

$$W = Mqq + Yy \tag{6.1}$$

The symplectic version of Giveon-Kutasov duality was first discussed in [15]. In this case the electric theory has $SP(2N_c)_{2k}$ gauge theory with N_f light flavors. The factor of 2 in front of the CS level is related to the normalization of the generators of the Lie algebra [15]. The dual theory is a $SP(2(N_f - N_c - 1 + |k|)$ gauge theory with N_f flavors and the meson in the $N_f(2N_f - 1)$ representation, with superpotential

$$W = Mqq \tag{6.2}$$

6.2 Partition functions

As in the unitary case also in the symplectic case the identities relating these two dualities have been first derived in [21]. Here we explicitly show these relations.

First we give the general expression for the partition function of an $SP(2N_c)_{2k}$ gauge theory with $2N_f$ fundamental

$$I^{2N_f}_{SP(2N_c)_{2k}}(\mu) = \frac{1}{2^{N_c} N_c!} \int \prod_{i=1}^{N_c} \frac{d\sigma_i e^{\frac{2i\pi k}{\omega_1 \omega_2}\sigma_i^2}}{\sqrt{-\omega_1\omega_2}} \prod_{a=1}^{2N_f} \Gamma_h(\pm\sigma_i + \mu_a)\Gamma_h^{-1}(\pm 2\sigma_i) \prod_{1\leq i<j\leq N_c} \Gamma_h^{-1}(\pm\sigma_i \pm \sigma_j) \tag{6.3}$$

When the CS term is vanishing we have the Aharony dual pair. In this case the equivalence between the electric and magnetic partition functions is encoded in the relation

$$I^{2N_f}_{SP(2N_c)_0}(\mu) = I^{2N_f}_{SP(2(N_f-N_c-1))_0} \Gamma_h(2(N_f - N_c)\omega - 2N_f m_A) \prod_{1\leq a<b\leq 2N_f} \Gamma_h(\mu_a + \mu_b) \tag{6.4}$$

with the balancy condition $\sum_{a=1}^{2N_f} \mu_a = 2N_f m_A$. The second relation can be derived from (6.4) by integrating out some massive quarks. The duality between the electric $SP(2N_c)_{2k}$ and the magnetic $SP(2(N_f - N_c + |k| - 1)_{-2k}$ CS gauge theories is summarised in the relation

$$I^{2N_f}_{SP(2N_c)_{2k}}(\mu) = I^{2N_f}_{SP(2(N_f-N_c-1))_{-2k}} \prod_{1\leq a<b\leq 2N_f} \Gamma_h(\mu_a + \mu_b)\zeta^{sign(k)(|k|-1)(2|k|-1)} \tag{6.5}$$

$$e^{\frac{i\pi}{2\omega_1\omega_2}\left(-4N_c k\omega^2 - sign(k)\left((2N_f(1-\Delta)-2N_c+1)\omega - \sum_{a=1}^{2N_f}\mu_a\right)^2 + 2k\sum_{a=1}^{N_f}(\mu_a+\omega(1-\Delta))^2 + k(2|k|-1)\omega^2\right)}$$

6.3 Flowing from Giveon-Kutasov duality to Aharony duality

We consider a $SP(2N_c)_{-2k}$ gauge theory with $2N_f$ light and $2k$ heavy flavors. The masses are

$$\begin{cases} \mu_a = m_A + m_a & a = 1, \ldots, 2N_f \\ \mu_a = m_A + m & a = 2N_f + 1, \ldots, 2(N_f + k) \end{cases} \tag{6.6}$$

with $m \to \infty$. This theory flows to a $SP(2N_c)_0$ gauge theory with $2N_f$ light fundamentals. The dual theory has $SP(2(N_f - N_c - 1 + 2k))_{2k}$ gauge group and $2(N_f + k)$ fundamentals and $2(N_f + k)^2 - (N_f + k)$ singlets. The masses can be read from the global symmetries as usual. The duality is preserved if the vacuum is chosen as

$$\begin{cases} \sigma_i = 0 & i = 1, \ldots, N_f - N_c - 1 \\ \sigma_i = m & i = N_f - N_c, \ldots, N_f - N_c + 2k - 1 \end{cases} \tag{6.7}$$

This breaks the dual gauge group into $\text{SP}(2(N_f - N_c - 1)_0 \times \text{U}(2k)_k$. By integrating out the massive matter we have, in the unbroken symplectic sector, $2N_f$ light quarks while in the unitary sector we are left with k light fundamentals. This chiral theory can be studied as in the case of the unitary groups and it gives raise to the term yY required by the Aharony duality. In the rest of this section we will test the validity of this RG flow on the partition function.

6.4 Following the flow on the partition function

In this section we study the flow just explained on the partition function. At large m the partition function of the electric theory is

$$Z_E = \lim_{m \to \infty} \zeta^{-4N_c k} e^{\frac{2i\pi k N_c (m_A + m - \omega)^2}{\omega_1 \omega_2}} I^{2N_f}_{\text{SP}(2N_c)_0}(\mu) \tag{6.8}$$

The dual partition function in the large m limit is[6]

$$Z_M = \lim_{m \to \infty} \zeta^{-4N_c k - 1} e^{\frac{i\pi}{2\omega_1 \omega_2}\phi} \prod_{1 \le a < b \le 2N_f} \Gamma_h(\mu_a + \mu_b) I^{2N_f}_{\text{SP}(2(N_f - N_c - 1))_0}(\omega - \mu) I^{(0,2k)}_{\text{U}(2k)_k}(0; \omega - m_A; \lambda) \tag{6.9}$$

where the exponent ϕ is

$$\begin{aligned}
\phi = {} & 8k\omega m_A N_c + 8kmm_A N_c - 16k^2 \omega m_A + 4k^2 m_A^2 + 8k\omega m_A + 4km^2 N_c \\
& -8km\omega N_c - 12k\omega^2 N_c + 4\omega^2 N_c + 12k^2 \omega^2 - 8k\omega^2 + \omega^2 \\
& +8\omega m_A N_c N_f + 4km_A^2 N_c - 32k\omega m_A N_f + 16km_A^2 N_f - 8\omega m_A N_f^2 \\
& +4\omega m_A N_f + 4m_A^2 N_f^2 - 8\omega^2 N_c N_f + 4\omega^2 N_c^2 + 16k\omega^2 N_f + 4\omega^2 N_f^2 - 4\omega^2 N_f
\end{aligned} \tag{6.10}$$

and the complex FI term in the $\text{U}(2k)_k$ sector is

$$\lambda = 4(N_f - N_c + k)\omega - 2\omega - 2m_A(k + 2N_f) \tag{6.11}$$

This integral can be computed explicitly ant it gives

$$Z_{\text{U}(2k)_k} = e^{-\frac{i\pi\left(4km_A(4k\omega - 3\lambda) + \lambda(8k\omega + \lambda) - 12k^2 m_A^2\right)}{8\omega_1 \omega_2}} \Gamma_h\left(\omega + km_A - 2k\omega + \frac{\lambda}{2}\right) \tag{6.12}$$

After substituting (6.11) into (6.12) the argument of Γ_h is $2\omega(N_f - N_c) - 2m_A N_f$ that represents the correct contribution of the electric monopole appearing as an elementary field in the magnetic phase of Aharony duality. Finally by eliminating the m dependence on both sides we recover formula (6.4) as expected.

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[6]As in the unitary case we impose the balancing condition on the light fields $\sum_{a=1}^{2N_f} \mu_a = 2N_f m_A$.

7 The orthogonal case

In this section we repeat the analysis for the orthogonal groups. Giveon-Kutasov and Aharony dualities have been studied in [7, 26–28].

First we analyse the Giveon-Kutasov theory. We consider a $O(N_c)_{-k}$ gauge theory with N_f light and k heavy flavors. The real masses are

$$\begin{cases} \mu_a = m_A + m_a & a = 1, \ldots, N_f \\ \mu_a = m_A + m & a = N_f + 1, \ldots, N_f + k \end{cases} \tag{7.1}$$

with large positive m. The dual theory is a $O(N_f - N_c + 2 + 2k)_k$ gauge theory with N_f light and k heavy flavor and $\frac{(N_f+k)(N_f+k+1)}{2}$ singlets. Their mass structure is obtained from the global charges as usual. The vacuum of this dual theory is

$$\begin{cases} \sigma_i = 0 & i = 1, \ldots, N_f - N_c + 2 \\ \sigma_i = -m & i = N_f - N_c + 3, \ldots, N_f - N_c + 2 + 2k \end{cases} \tag{7.2}$$

This flow generates the Aharony dual pair for the orthogonal group.

As in the unitary and symplectic cases we study this flow on the partition function. However, in the orthogonal case, some subtleties arise because the rank of the gauge group can be even or odd, and this modifies some terms in the partition function. The partition function for a model with an $O(N_c)$ gauge group (with $N_c = 2N$ or $N_c = 2N + 1$) with N_f flavors is given by the following two relations

$$I_{O(2N)_k}^{n_f}(\mu) = \frac{1}{2^{N-1} N!} \int \prod_{i=1}^{N} \frac{d\sigma_i e^{\frac{i\pi k}{\omega_1 \omega_2} \sigma_i^2}}{\sqrt{-\omega_1 \omega_2}} \prod_{a=1}^{N_f} \Gamma_h(\pm \sigma_i + \mu_a) \prod_{1 \le i < j \le n} \Gamma_h^{-1}(\pm \sigma_i \pm \sigma_j) \tag{7.3}$$

$$I_{O(2N+1)_k}^{n_f}(\mu) = \frac{\prod_{a=1}^{N_f} \Gamma_h(\mu_a)}{2^N N!} \int \prod_{i=1}^{N} \frac{d\sigma_i e^{\frac{i\pi k}{\omega_1 \omega_2} \sigma_i^2}}{\sqrt{-\omega_1 \omega_2}} \prod_{a=1}^{N_f} \Gamma_h(\pm \sigma_i + \mu_a) \Gamma_h^{-1}(\pm \sigma_i) \prod_{1 \le i < j \le N} \Gamma_h^{-1}(\pm \sigma_i \pm \sigma_j)$$

Even if the two expressions look different it has been shown in [7] that one can write a single formula to match the cases of the Aharony and Giveon-Kutasov duality. For the Aharony duality one has

$$I_{O(N_c)_0}^{N_f} = I_{O(N_f - N_c + 2)_0}^{N_f} \Gamma_h \left(\omega(N_f - N_c + 2) - \sum_{a=1}^{N_f} \mu_a \right) \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \mu_b) \tag{7.4}$$

with the constraint $\sum_{a=1}^{N_f} \mu_a = N_f m_A$. In any case one has to distinguish in the explicit calculations if the gauge group has rank even or odd. The relation (7.4) allows to derive the analogous equality for the Giveon-Kutasov duality. This was done in [7] and one has

$$I_{O(N_c)_k}^{N_f} = I_{O(N_f - N_c + 2 + 2|k|)_{-k}}^{N_f} \zeta^{\frac{sgn(k)(|k|+1)(|k|+2)}{2}} \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \mu_b) \tag{7.5}$$

$$\times e^{\frac{i\pi}{2\omega_1 \omega_2} \left(-N_c k \omega^2 - sgn(k) \left((N_f - N_c + 2 + 2|k|)\omega - \sum_{a=1}^{N_f} \mu_a \right)^2 + k \sum_{a=1}^{N_f} (\mu_a - \omega)^2 + \frac{k}{2}(|k|+1) \right)}$$

In the rest of this section we study the flow explained above and show that it connects (7.5) to (7.4).

As anticipated the explicit calculation requires the knowledge of the parity of the rank of the gauge group in the electric and in the magnetic theory. In general there are eight possibilities, depending on the parity of N_c, N_f and k. By explicit computations it is possible to observe that they reduce to the same formula. In general starting from the $O(N_c)_k$ theory with $N_f + k$ flavors, N_f light and k heavy as usual one arrives to an $O(N_c)_0$ model with N_f light flavors. In the large m limit we have

$$Z_E = \lim_{m\to\infty} \zeta^{-2kN_c} I_{O(N_c)_0}^{N_f}(\mu) e^{\frac{i\pi k N_c (m_A + m - \omega)^2}{\omega_1 \omega_2}} \tag{7.6}$$

In the dual magnetic case, we assign the proper masses to the fields according to the global symmetries. Then we shift σ as in (7.2). After fixing $\sum_{a=1}^{N_f} \mu_a = N_f m_A$ in the large m limit we have

$$Z_M = \lim_{m\to\infty} \zeta^{-2kN_c-1} \lim_{m\to\infty} e^{\frac{i\pi k}{\omega_1 \omega_2}\phi} \prod_{a,b=1}^{n_f} \Gamma_h(\mu_a + \mu_b) I_{O(N_f - N_c + 2)_0}^{N_f}(\omega - \mu) I_{U(k)_{\frac{k}{2}}}^{(k,0)}(m_A; 0, \lambda) \tag{7.7}$$

where the exponent ϕ is

$$\begin{aligned} \phi = {} & m_A^2 \left(kN_c + 4kN_f + N_f^2 + k^2\right) + N_c \left(-2\omega^2 N_f + k\left(m^2 - 2m\omega - 3\omega^2\right) - 2\omega^2\right) \\ & + \omega^2((4k+2)N_f + N_f^2 + 3k^2 + 4k + 1) - 2m_A(\omega((4k+1)N_f + N_f^2 + 2k(k+1)) \\ & + \omega^2 N_c^2 - N_c(\omega N_f + k(m+\omega))) \end{aligned} \tag{7.8}$$

and the complex FI parameter is

$$\lambda = 2\omega \left(N_f - N_c + k + 1\right) - m_A \left(2N_f + k\right) \tag{7.9}$$

The integral $I_{U(k)_{\frac{k}{2}}}^{(k,0)}(m_A; 0, \lambda)$ can be computed with the usual technique of [7]. It gives

$$I_{U(k)_{k/2}}^{(k,0)}(m_A; 0, \lambda) = e^{-\frac{i\pi\left(-3k^2 m_A^2 + 2km_A(2k\omega - 3\lambda) + \lambda(4k\omega + \lambda)\right)}{8\omega_1 \omega_2}} \Gamma_h\left(\frac{km_A}{2} - k\omega + \frac{\lambda}{2} + \omega\right) \tag{7.10}$$

After substituting λ all the exponents cancel when one equates (7.6) to (7.7). The hyperbolic Γ function in (7.10) becomes

$$\Gamma_h(\omega\left(N_f - N_c + 2\right) - m_A N_f) \tag{7.11}$$

and as expected it coincides with the expected contribution of the electric monopole.

8 Conclusions

In this paper we studied RG flow connecting two sets of dual pairs in three dimensions. We showed that the matching between the two partition functions in the UV pair is preserved by the flow once the IR pair is reached. The interesting fact is that one can reconstruct from the partition function the contribution of the monopole sector in the

magnetic Aharony phase. We have also shown that the analysis holds in the symplectic and in the orthogonal cases.

Some generalization of our work are possible. First one may consider the flows studied in [9]. Indeed it may happen in these cases that some of the extra sectors, that we used to reconstruct the monopole contributions, decouple. Studying these flows with the partition function should be useful for an understanding of this decoupling. Another possibility consists of flowing from the Giveon-Kutasov case to the dual pairs with a chiral field content studied in [7]. One may also study real mass flows from UV duals with adjoint matter [29, 30] or more complicate representations and gauge groups [31, 32] and obtain IR dualities, and check the behaviour of these flows on the partition function.

Another interesting aspect that we did not address in this paper is the role of accidental symmetries. Indeed already at the level of the Aharony dual there are critical values of the levels and ranks at which some singlet become free [33]. This is expected also for theories with a representations different than the fundamental [24, 25, 34]. It would be interesting to understand if and how accidental symmetries may be an obstruction for these flow.

Acknowledgments

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