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Menger algebras of n-place interior operations

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Abstract. Algebraic properties of n-place interior operations on a fixed set are described. Conditions under which a Menger algebra of rank n can be represented by n-place interior operations are found.

1. Introduction

It is known [5] that on the topology on a set A one can talk in the language of open sets, the language of closed sets, the language of interior operations (also called opening operations), or the language of closure operations. Various types of closure operations on algebraic systems and their applications are well described (see for example [3]). So a natural question is about a similar characterization of interior operations having applications in topology and economics. Such operations were first studied from an algebraic point of view by Vagner [7]. Kulik observed in [4] that the superposition of two interior operations is not always an interior operation and found conditions under which the composition of two interior operations of a given set A is also an interior operation of this set. Moreover, he proved that a semigroup S is isomorphic to a semigroup of interior operations of some set if and only if S is idempotent and commutative.

Below, we introduce the concept of n-place interior operations and find conditions under which a Menger algebra of rank n can be isomorphically represented by n-place interior operations of some set.

2. Preliminaries

Let A be a nonempty set, $\mathfrak{P}(A)$ the family of all subsets of A, and $\mathcal{T}_n(\mathfrak{P}(A))$ the set of all n-place transformations of $\mathfrak{P}(A)$, i.e., maps $f\colon \prod^n \mathfrak{P}(A) \to \mathfrak{P}(A)$, where $\prod^n \mathfrak{P}(A)$ denotes the n-th Cartesian power of the set $\mathfrak{P}(A)$. For arbitrary $f, g_1, \ldots, g_n \in \mathcal{T}_n(\mathfrak{P}(A))$, we define the $(n+1)$-ary composition $f[g_1 \cdots g_n]$ by putting

$$f[g_1 \cdots g_n](X_1, \ldots, X_n) = f(g_1(X_1, \ldots, X_n), \ldots, g_n(X_1, \ldots, X_n))$$

for all $X_1, \ldots, X_n \in \mathfrak{P}(A)$.

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The $(n+1)$-ary operation $\mathcal{O}\colon (f, g_1, \ldots, g_n) \mapsto f[g_1 \cdots g_n]$ is called the *Menger superposition* of n-place functions (cf. [3, 6]). Then $(\mathcal{T}_n(\mathfrak{P}(A)), \mathcal{O})$ is a *Menger algebra* in the sense of [2] and [3], i.e., the operation $\mathcal{O}$ satisfies the so-called *superassociative law*:

$$f[g_1 \cdots g_n][h_1 \cdots h_n] = f[g_1[h_1 \cdots h_n] \cdots g_n[h_1 \cdots h_n]], \tag{2.1}$$

where $f, g_i, h_i \in \mathcal{T}_n(\mathfrak{P}(A))$, for $i = 1, \ldots, n$.

We say that n-place transformation $f \in \mathcal{T}_n(\mathfrak{P}(A))$ is

- *contractive* if for any $X_1, \ldots, X_n \in \mathfrak{P}(A)$, $f(X_1, \ldots, X_n) \subseteq X_1 \cap \cdots \cap X_n$;
- *idempotent* if $f[f \cdots f] = f$;
- *isotone* if for any $X_1, \ldots, X_n, Y_1, \ldots, Y_n \in \mathfrak{P}(A)$,
$$X_1 \subseteq Y_1 \wedge \cdots \wedge X_n \subseteq Y_n \Longrightarrow f(X_1, \ldots, X_n) \subseteq f(Y_1, \ldots, Y_n);$$
- $\cup$-*distributive* if for all $X, Y, H_1, \ldots, H_n \in \mathfrak{P}(A)$ and $i = 1, \ldots, n$,
$$f(H_1^{i-1}, X \cup Y, H_{i+1}^n) = f(H_1^{i-1}, X, H_{i+1}^n) \cup f(H_1^{i-1}, Y, H_{i+1}^n),$$
where H_s^r means $H_s, \ldots, H_r$ for $s \leqslant r$.
- $\cap$-*distributive* if for all $X, Y, H_1, \ldots, H_n \in \mathfrak{P}(A)$ and $i = 1, \ldots, n$,
$$f(H_1^{i-1}, X \cap Y, H_{i+1}^n) = f(H_1^{i-1}, X, H_{i+1}^n) \cap f(H_1^{i-1}, Y, H_{i+1}^n);$$
- *full* $\cup$-*distributive* if for any subsets $(X_k)_{k\in K}$ of A, all $H_1, \ldots, H_n \in \mathfrak{P}(A)$, and $i = 1, \ldots, n$,
$$f\big(H_1^{i-1}, \bigcup_{k\in K} X_k, H_{i+1}^n\big) = \bigcup_{k\in K} f\big(H_1^{i-1}, X_k, H_{i+1}^n\big);$$
- *full* $\cap$-*distributive* if for any subsets $(X_k)_{k\in K}$ of A, all $H_1, \ldots, H_n \in \mathfrak{P}(A)$, and $i = 1, \ldots, n$,
$$f\big(H_1^{i-1}, \bigcap_{k\in K} X_k, H_{i+1}^n\big) = \bigcap_{k\in K} f\big(H_1^{i-1}, X_k, H_{i+1}^n\big).$$

It is not difficult to see that the contractivity of an n-place transformation $f \in \mathcal{T}_n(\mathfrak{P}(A))$ is equivalent to the system of conditions

$$f(X_1, \ldots, X_n) \subseteq X_i, \text{ for } i = 1, \ldots, n.$$

The isotonicity is equivalent to the system of n implications

$$X \subseteq Y \Longrightarrow f(H_1^{i-1}, X, H_{i+1}^n) \subseteq f(H_1^{i-1}, Y, H_{i+1}^n), \text{ for } i = 1, \ldots, n,$$

where $X, Y, X_1, \ldots, X_n, H_1, \ldots, H_n \in \mathfrak{P}(A)$.

It is also easy to show that the Menger superposition of contractive (isotone) n-place transformations of $\mathfrak{P}(A)$ is again a contractive (isotone) n-place transformation of $\mathfrak{P}(A)$.

An n-place transformation of $\mathfrak{P}(A)$, which is contractive, idempotent, and isotone, is called an *n-place interior operation* or an *n-place interior operator* on the set A.

For $n = 1$, this definition coincides with the definition of interior operations proposed by Vagner (see [7]).

3. Properties of n-place interior operations

We start with the following characterization of n-place interior operations.

Theorem 3.1. *For an n-place transformation f of $\mathfrak{P}(A)$, the following conditions are equivalent:*

(a) *f is an n-place interior operation on A;*
(b) *for all $X_1,\ldots,X_n,Y_1,\ldots,Y_n\in\mathfrak{P}(A)$, we have*

$$f(X_1\cap Y_1,\ldots,X_n\cap Y_n)\subseteq f(f(X_1^n),\ldots,f(X_1^n))\cap f(Y_1^n)\cap Y_1\cap\cdots\cap Y_n. \quad (3.1)$$

Proof. (a) $\Longrightarrow$ (b): Suppose that f is an n-place interior operation on A. Then by the contractivity of f, for $X_i,Y_i\in\mathfrak{P}(A)$, for $i=1,\ldots,n$, we have

$$f(X_1\cap Y_1,\ldots,X_n\cap Y_n)\subseteq (X_1\cap Y_1)\cap\cdots\cap(X_n\cap Y_n)\subseteq Y_1\cap\cdots\cap Y_n. \quad (3.2)$$

As $X_i\cap Y_i\subseteq Y_i$, $i=1,\ldots,n$, the isotonity of f implies

$$f(X_1\cap Y_1,\ldots,X_n\cap Y_n)\subseteq f(Y_1,\ldots,Y_n)=f(Y_1^n). \quad (3.3)$$

Similarly, $X_i\cap Y_i\subseteq X_i$ for $i=1,\ldots,n$ implies

$$f(X_1\cap Y_1,\ldots,X_n\cap Y_n)\subseteq f(X_1,\ldots,X_n)=f(X_1^n).$$

Since $f(X_1^n)=f(f(X_1^n),\ldots,f(X_1^n))$, from the above we obtain

$$f(X_1\cap Y_1,\ldots,X_n\cap Y_n)\subseteq f(f(X_1^n),\ldots,f(X_1^n)),$$

which together with (3.2) and (3.3) gives (3.1). Thus, (a) implies (b).

(b) $\Longrightarrow$ (a): If an n-place transformation f satisfies (3.1), then setting $X_i=Y_i$ for $i=1,\ldots,n$ in (3.1), we obtain

$$f(X_1^n)\subseteq f(f(X_1^n),\ldots,f(X_1^n))\cap f(X_1^n)\cap X_1\cap\cdots\cap X_n. \quad (3.4)$$

So $f(X_1^n)\subseteq X_1\cap\cdots\cap X_n$, i.e., f is contractive. In addition, (3.4) implies $f(X_1^n)\subseteq f(f(X_1^n),\ldots,f(X_1^n))$. Since f is contractive, we have

$$f(f(X_1^n),\ldots,f(X_1^n))\subseteq f(X_1^n)\cap\cdots\cap f(X_1^n)=f(X_1^n),$$

which together with the previous inclusion proves that f is idempotent.

If $X_i\subseteq Y_i$, then obviously $X_i\cap Y_i=X_i$. Hence, by (3.1), for $X_i\subseteq Y_i$ for $i=1,\ldots,n$, we have

$$\begin{aligned}f(X_1^n)&=f(X_1\cap Y_1,\ldots,X_n\cap Y_n)\\&\subseteq (f(X_1^n),\ldots,f(X_1^n))\cap f(Y_1^n)\cap Y_1\cap\cdots\cap Y_n\subseteq f(Y_1^n),\end{aligned}$$

which means that f is isotone. Thus, f is an n-place interior operation. So (b) implies (a). □

Theorem 3.2. *For an n-place transformation f of $\mathfrak{P}(A)$, the following conditions are equivalent:*

(i) *f is contractive and full $\cup$-distributive;*
(ii) *f is contractive and $\cup$-distributive;*

(iii) *for all* $X_1, \dots, X_n \in \mathfrak{P}(A)$, *we have*

$$f(X_1^n) = f(A, \dots, A) \cap X_1 \cap \dots \cap X_n. \tag{3.5}$$

Proof. (i) $\Longrightarrow$ (ii): This is obvious.

(ii) $\Longrightarrow$ (iii): According to $\cup$-distributivity, for all subsets $X, H_1, \dots, H_n$ of A and $i = 1, \dots, n$ we have

$$f(H_1^{i-1}, X, H_{i+1}^n) \cup f(H_1^{i-1}, X', H_{i+1}^n) = f(H_1^{i-1}, A, H_{i+1}^n),$$

where $X' = A \setminus X$. Then clearly

$$\big(f(H_1^{i-1}, X, H_{i+1}^n) \cap X\big) \cup \big(f(H_1^{i-1}, X', H_{i+1}^n) \cap X\big) = f(H_1^{i-1}, A, H_{i+1}^n) \cap X. \tag{3.6}$$

Since f is contractive, we have $f(H_1^{i-1}, X, H_{i+1}^n) \subseteq X$, which implies that $f(H_1^{i-1}, X, H_{i+1}^n) \cap X = f(H_1^{i-1}, X, H_{i+1}^n)$. Similarly, $f(H_1^{i-1}, X', H_{i+1}^n) \subseteq X'$ and $X' \cap X = \varnothing$ imply $f(H_1^{i-1}, X', H_{i+1}^n) \cap X = \varnothing$. Thus, (3.6) has the form

$$f(H_1^{i-1}, X, H_{i+1}^n) = f(H_1^{i-1}, A, H_{i+1}^n) \cap X.$$

Using this identity, we obtain

$$\begin{aligned} &f(X_1, X_2, X_3, \dots, X_n) = f(A, X_2, X_3, \dots, X_n) \cap X_1 \\ &= f(A, A, X_3, \dots, X_n) \cap X_2 \cap X_1 = \cdots \\ &= f(A, A, A, \dots, A) \cap X_n \cap \dots \cap X_2 \cap X_1 = f(A, \dots, A) \cap X_1 \cap \dots \cap X_n. \end{aligned}$$

So (ii) implies (iii).

(iii) $\Longrightarrow$ (i): That (3.5) implies $f(X_1, \dots, X_n) \subseteq X_1 \cap \dots \cap X_n$ is not difficult to see. Thus, f is contractive. Moreover, in this case we also have

$$\begin{aligned} &f(A, \dots, A) \cap H_1 \cap \dots \cap H_{i-1} \cap \Big(\bigcup_{k \in K} X_k\Big) \cap H_{i+1} \cap \dots \cap H_n \\ &= \bigcup_{k \in K} f\big(H_1^{i-1}, X_k, H_{i+1}^n\big) \end{aligned}$$

by (3.5). So f is distributive with respect to the union. Thus (iii) implies (i), which completes the proof. $\square$

Corollary 3.3. *Every n-place transformation f on $\mathfrak{P}(A)$ satisfying* (3.5) *is an n-place interior operation on A.*

Proof. Any transformation satisfying (3.5) is clearly contractive. It is also idempotent, because

$$\begin{aligned} f(f(X_1^n), \dots, f(X_1^n)) &= f(A, \dots, A) \cap f(X_1^n) \\ &= f(A, \dots, A) \cap f(A, \dots, A) \cap X_1 \cap \dots \cap X_n \\ &= f(A, \dots, A) \cap X_1 \cap \dots \cap X_n = f(X_1^n) \end{aligned}$$

for all $X_1, \dots, X_n \in \mathfrak{P}(A)$.

For $X_1 \subseteq Y_1, \dots, X_n \subseteq Y_n$, we have $X_1 \cap \dots \cap X_n \subseteq Y_1 \cap \dots \cap Y_n$. Thus, $f(A, \dots, A) \cap X_1 \cap \dots \cap X_n \subseteq f(A, \dots, A) \cap Y_1 \cap \dots \cap Y_n$. So $f(X_1^n) \subseteq f(Y_1^n)$. Hence, f is isotone. $\square$

Corollary 3.4. *Every (full) $\cup$-distributive n-place interior operation is (full) $\cap$-distributive.*

Proof. Indeed, by Theorem 3.2, any $\cup$-distributive n-place interior operation f on the set A satisfies (3.5). Hence,

$$\begin{aligned} & f(H_1^{i-1}, X\cap Y, H_{i+1}^n) \\ & = f(A,\dots,A)\cap H_1\cap\cdots\cap H_{i-1}\cap(X\cap Y)\cap H_{i+1}\cap\cdots\cap H_n \\ & = \big(f(A,\dots,A)\cap H_1\cap\cdots\cap H_{i-1}\cap X\cap H_{i+1}\cap\cdots\cap H_n\big) \\ & \qquad\qquad \cap\big(f(A,\dots,A)\cap H_1\cap\cdots\cap H_{i-1}\cap Y\cap H_{i+1}\cap\cdots\cap H_n\big) \\ & = f(H_1^{i-1}, X, H_{i+1}^n)\cap f(H_1^{i-1}, Y, H_{i+1}^n) \end{aligned}$$

for $X, Y, H_1, \dots, H_n \in \mathfrak{P}(A)$ and $i = 1, \dots, n$. Thus, f is $\cap$-distributive.

Analogously, we can show that f is full $\cap$-distributive. □

4. Compositions of n-place interior operations

On the set $\mathcal{T}_n(\mathfrak{P}(A))$ of n-place transformations of the set A we introduce the binary relation $\preceq$ defined by

$$f \preceq g \iff (\forall\, X_1,\dots,X_n)\,\big(f(X_1^n)\subseteq g(X_1^n)\big).$$

It is easy to see that $\preceq$ is a partial order, i.e., it is reflexive, transitive, and antisymmetric.

Proposition 4.1. *The relation $\preceq$ has the following properties:*

(a) *If $f \in \mathcal{T}_n(\mathfrak{P}(A))$ is contractive, then $f[g_1\cdots g_n] \preceq g_i$ for $i = 1, \dots, n$ and all $g_1, \dots, g_n \in \mathcal{T}_n(\mathfrak{P}(A))$.*

(b) *If $f \in \mathcal{T}_n(\mathfrak{P}(A))$ is isotone, then*

$$g_1 \preceq h_1 \wedge \cdots \wedge g_n \preceq h_n \Longrightarrow f[g_1\cdots g_n] \preceq f[h_1\cdots h_n]$$

for all $g_1, \dots, g_n, h_1, \dots, h_n \in \mathcal{T}_n(\mathfrak{P}(A))$.

(c) *If $f \in \mathcal{T}_n(\mathfrak{P}(A))$ is isotone and $g \in \mathcal{T}_n(\mathfrak{P}(A))$ is contractive, then*

$$f[g\cdots g] \preceq f.$$

Proof. (a): If $f \in \mathcal{T}_n(\mathfrak{P}(A))$ is contractive, then for all $X_1, \dots, X_n \in \mathfrak{P}(A)$ and $i = 1, \dots, n$, we have $f[g_1\cdots g_n](X_1^n) = f(g_1(X_1^n), \dots, g_n(X_1^n)) \subseteq g_i(X_1^n)$. Hence, $f[g_1\cdots g_n] \preceq g_i$. So the first property is proved.

(b): If $f \in \mathcal{T}_n(\mathfrak{P}(A))$ is isotone and $g_i \preceq h_i$ for all $g_i, h_i \in \mathcal{T}_n(\mathfrak{P}(A))$ and all $i = 1, \dots, n$, then for all $X_1, \dots, X_n \in \mathfrak{P}(A)$, we have $g_i(X_1^n) \subseteq h_i(X_1^n)$ for all $i = 1, \dots, n$; consequently, $f(g_1(X_1^n), \dots, g_n(X_1^n)) \subseteq f(h_1(X_1^n), \dots, h_n(X_1^n))$. Therefore, $f[g_1\cdots g_n](X_1^n) \subseteq f[h_1\cdots h_n](X_1^n)$. So $f[g_1\cdots g_n] \preceq f[h_1\cdots h_n]$, which proves the second condition.

(c): This is a consequence of (a) and (b). □

Theorem 4.2. *The Menger superposition of given n-place interior operations $f, g_1, \ldots, g_n$ defined on the set A is an n-place interior operation on A if and only if for each $i = 1, \ldots, n$, we have*

$$g_i[f \cdots f][g_1 \cdots g_n] = f[g_1 \cdots g_n]. \tag{4.1}$$

Proof. ($\Longrightarrow$): Suppose that $f, g_1, \ldots, g_n$ and $f[g_1 \cdots g_n]$ are n-place interior operations on A. Since each g_i is contractive, according to (2.1) and Proposition 4.1, we obtain

$$g_i[f \cdots f][g_1 \cdots g_n] = g_i[f[g_1 \cdots g_n] \cdots f[g_1 \cdots g_n]] \preceq f[g_1 \cdots g_n].$$

On the other hand, using (2.1) and the fact that $f[g_1 \cdots g_n]$ is an idempotent n-place transformation, we get

$$\begin{aligned} f[g_1 \cdots g_n] &= f[g_1 \cdots g_n][f[g_1 \cdots g_n] \cdots f[g_1 \cdots g_n]] \\ &= f[g_1[f[g_1 \cdots g_n] \cdots f[g_1 \cdots g_n]] \cdots g_n[f[g_1 \cdots g_n] \cdots f[g_1 \cdots g_n]]] \\ &\preceq g_i[f[g_1 \cdots g_n] \cdots f[g_1 \cdots g_n]] = g_i[f \cdots f][g_1 \cdots g_n], \end{aligned}$$

which together with the previous inequality gives (4.1).

($\Longleftarrow$): As mentioned above, Menger superposition preserves contractivity and isotonicity. We show that $f[g_1 \cdots g_n]$ is idempotent. Indeed, according to (2.1) and (4.1), we have

$$\begin{aligned} &f[g_1 \cdots g_n][f[g_1 \cdots g_n] \cdots f[g_1 \cdots g_n]] \\ &= f[g_1[f[g_1 \cdots g_n] \cdots f[g_1 \cdots g_n]] \cdots g_n[f[g_1 \cdots g_n] \cdots f[g_1 \cdots g_n]]] \\ &= f[g_1[f \cdots f][g_1 \cdots g_n] \cdots g_n[f \cdots f][g_1 \cdots g_n]] \\ &= f[f[g_1 \cdots g_n] \cdots f[g_1 \cdots g_n]] = f[f \cdots f][g_1 \cdots g_n] = f[g_1 \cdots g_n]. \end{aligned}$$

Thus, $f[g_1 \cdots g_n]$ is an n-place interior operation. □

5. Algebras derived from their diagonal semigroups

Recall (cf. [2], [3]) that a *Menger algebra* (G, o) *of rank* n is a nonempty set G with an $(n+1)$-ary operation o: $(f, g_1, \ldots, g_n) \mapsto f[g_1 \cdots g_n]$ satisfying the identity (2.1). On such algebra we can define a binary operation $*$ by setting $x * y = x[y \cdots y]$ for any $x, y \in G$. It is easy to see that $(G, *)$ is a semigroup. It is called the *diagonal semigroup* of (G, o). In the case when

$$f[g_1 g_2 \cdots g_n] = f * g_1 * g_2 * \cdots * g_n,$$

we say that a Menger algebra (G, o) is *derived* from its diagonal semigroup $(G, *)$.

Proposition 5.1. *For any n-place interior operations f, g on the set A, the following three conditions are equivalent:*

(a) *$f * g$ is an n-place interior operation on A;*
(b) *$f * g = g * f * g$;*
(c) *$f * g = f * g * f$.*

Proof. (a) $\Longrightarrow$ (b): This implication follows from Theorem 4.2.

(b) $\Longrightarrow$ (c): By Proposition 4.1, we have $f[g \cdots g] \preceq f$, i.e., $f * g \preceq f$. This together with (b) shows $f * g = g * f * g \preceq g * f$. So $f * g \preceq g * f$, which in view of Proposition 4.1 (b), gives $f * f * g \preceq f * g * f$. Hence, $f * g = f * f * g \preceq f * g * f \preceq f * g$, and consequently, $f * g = f * g * f$.

(c) $\Longrightarrow$ (a): Since Menger superposition preserves contractivity and isotonicity, then $f * g$ is an isotone and contractive n-place transformation. We show that it is idempotent. In fact,

$$(f * g) * (f * g) = (f * g * f) * g = (f * g) * g = f * (g * g) = f * g.$$

Thus, $f * g$ is an n-place interior operation. □

6. Characterizations of algebras of n-place interior operations

An abstract characterization of Menger algebras of n-place interior operations is given in the following theorem.

Theorem 6.1. *A Menger algebra (G, o) of rank n is isomorphic to a Menger algebra of n-place interior operations on some set if and only if it satisfies the following three identities*

$$x[x \cdots x] = x, \tag{6.1}$$

$$x[y \cdots y] = y[x \cdots x], \tag{6.2}$$

$$x[y_1 \cdots y_n] = x[y_1 \cdots y_1] \cdots [y_n \cdots y_n]. \tag{6.3}$$

Proof. Necessity: Let $(\Phi, \mathcal{O})$, where $\Phi \subset \mathcal{T}_n(\mathfrak{P}(A))$, be a Menger algebra of n-place interior operations on the set A. Then obviously, $f[f \cdots f] = f$ for all $f \in \Phi$. Thus, the condition (6.1) is satisfied.

If $f, g \in \Phi$, then also $f[g \cdots g], g[f \cdots f] \in \Phi$. Therefore, $f * g$ and $g * f$ are n-place interior operations and by (b), (c) from Proposition 5.1, we have $f * g = g * f$. Thus, $f[g \cdots g] = g[f \cdots f]$. Hence, the condition (6.2) also is satisfied.

Further, using (2.1) and Theorem 4.2, for $f, g_1, \ldots, g_n \in \Phi$ and for each $i = 1, \ldots, n$, we obtain

$$g_i * f[g_1 \cdots g_n] = g_i[f[g_1 \cdots g_n] \cdots f[g_1 \cdots g_n]] = g_i[f \cdots f][g_1 \cdots g_n] = f[g_1 \cdots g_n].$$

Thus, $g_i * f[g_1 \cdots g_n] = f[g_1 \cdots g_n]$ for all $i = 1, \ldots, n$. Consequently,

$$\begin{aligned}(f * g_1 * \cdots * g_i) * f[g_1 \cdots g_n] &= (f * g_1 * \cdots * g_{i-1}) * (g_i * f[g_1 \cdots g_n]) \\ &= (f * g_1 * \cdots * g_{i-1}) * f[g_1 \cdots g_n] \\ &\ \ \vdots \\ &= f * f[g_1 \cdots g_n] = f[f[g_1 \cdots g_n] \cdots f[g_1 \cdots g_n]] \\ &= f[f \cdots f][g_1 \cdots g_n] = f[g_1 \cdots g_n].\end{aligned}$$

Hence,

$$(f * g_1 * \cdots * g_i) * f[g_1 \cdots g_n] = f[g_1 \cdots g_n] \tag{6.4}$$

for every $i = 1, \ldots, n$. Since $f[g_1 \cdots g_n] \preceq g_n$ and

$$(f * g_1 * \cdots * g_{n-1}) * f[g_1 \cdots g_n] \preceq (f * g_1 * \cdots * g_{n-1}) * g_n,$$

by Proposition 4.1, the equality (6.4) means that

$$f[g_1 \cdots g_n] \preceq f * g_1 * \cdots * g_n. \tag{6.5}$$

By Proposition 4.1, we also have $f * g_1 * \cdots * g_n \preceq f * g_1 * \cdots * g_i \preceq g_i$ for all $i = 1, \ldots, n$. Therefore, $f[(f * g_1 * \cdots * g_n) \cdots (f * g_1 * \cdots * g_n)] \preceq f[g_1 \cdots g_n]$, i.e., $f * (f * g_1 * \cdots * g_n) \preceq f[g_1 \cdots g_n]$, whence by $f * f = f$, we obtain

$$f * g_1 * \cdots * g_n \preceq f[g_1 \cdots g_n].$$

This, together with (6.5), gives $f[g_1 \cdots g_n] = f * g_1 * \cdots * g_n$. Thus, the condition (6.3) is satisfied too.

Sufficiency: Let (G, o) be a Menger algebra of rank n satisfying all the conditions of the theorem and let $(G, *)$ be its diagonal semigroup. Consider a binary relation ω defined on the set G as follows: $\omega = \{(x, y) \mid x * y = y\}$. Since the diagonal semigroup $(G, *)$ is semilattice, the relation ω is reflexive, transitive, and antisymmetric. So the relation ω is just the usual ordering on a semilattice (treated as a join semilattice), and $\omega\langle x\rangle = \{y \in G \mid (x, y) \in \omega\}$ is just the upset in this ordering.

We show that

$$\omega\langle x[y_1 \cdots y_n]\rangle = \omega\langle x\rangle \cap \omega\langle y_1\rangle \cap \cdots \cap \omega\langle y_n\rangle \tag{6.6}$$

for $x, y_1, \ldots, y_n \in G$.

Since the diagonal semigroup $(G, *)$ is semilattice, the equation $\omega\langle x * y\rangle = \omega\langle x\rangle \cap \omega\langle y\rangle$ holds in $(G, *)$.

Using this equation and (6.3) we obtain

$$\begin{aligned} \omega\langle x[y_1 \cdots y_n]\rangle &= \omega\langle x * y_1 * \cdots * y_n\rangle = \omega\langle x * (y_1 * \cdots * y_n)\rangle \\ &= \omega\langle x\rangle \cap \omega\langle y_1 * (y_2 * \cdots * y_n)\rangle = \omega\langle x\rangle \cap \omega\langle y_1\rangle \cap \omega\langle y_2 * \cdots * y_n\rangle \\ &= \cdots = \omega\langle x\rangle \cap \omega\langle y_1\rangle \cap \cdots \cap \omega\langle y_n\rangle, \end{aligned}$$

which proves (6.6).

Consider the set $\Phi = \{f_g \mid g \in G\} \subseteq \mathcal{T}_n(\mathfrak{P}(G))$ of all n-place operations f_g defined by

$$f_g(X_1^n) = \omega\langle g\rangle \cap X_1 \cap \cdots \cap X_n \tag{6.7}$$

and the map $P\colon g \mapsto f_g$.

Clearly, $f_g(G,\ldots,G)=\omega\langle g\rangle$. Thus, $f_g(X_1^n)=f_g(G,\ldots,G)\cap X_1\cap\cdots\cap X_n$. Hence, by Corollary 3.3, f_g is an n-place interior operation on the set G. Moreover, for all $g,g_1,\ldots,g_n$ and $X_1,\ldots,X_n\in\mathfrak{P}(G)$, we have

$$\begin{aligned}
&f_g[f_{g_1}\cdots f_{g_n}](X_1^n)=f_g(f_{g_1}(X_1^n),\ldots,f_{g_n}(X_1^n))\\
&=\omega\langle g\rangle\cap f_{g_1}(X_1^n)\cap\cdots\cap f_{g_n}(X_1^n)\\
&=\omega\langle g\rangle\cap\big(\omega\langle g_1\rangle\cap X_1\cap\cdots\cap X_n\big)\cap\cdots\cap\big(\omega\langle g_n\rangle\cap X_1\cap\cdots\cap X_n\big)\\
&=\big(\omega\langle g\rangle\cap\omega\langle g_1\rangle\cap\cdots\cap\omega\langle g_n\rangle\big)\cap X_1\cap\cdots\cap X_n\\
&\overset{(6.6)}{=}\omega\langle g[g_1\cdots g_n]\rangle\cap X_1\cap\cdots\cap X_n=f_{g[g_1\cdots g_n]}(X_1^n).
\end{aligned}$$

Thus, $P(g[g_1\cdots g_n])=P(g)[P(g_1)\cdots P(g_n)]$, i.e., P is a homomorphism of G onto Φ.

Suppose now that $P(g_1)=P(g_2)$, where $g_1,g_2\in G$. Then $f_{g_1}=f_{g_2}$, i.e., $f_{g_1}(X_1^n)=f_{g_2}(X_1^n)$ for all $X_1,\ldots,X_n\in\mathfrak{P}(G)$. Hence, $\omega\langle g_1\rangle\cap X_1\cap\cdots\cap X_n=\omega\langle g_2\rangle\cap X_1\cap\cdots\cap X_n$, which for $X_1=\cdots=X_n=G$, gives $\omega\langle g_1\rangle=\omega\langle g_2\rangle$. Since ω is an antisymmetric relation, from the above we conclude $g_1=g_2$. Thus, P is a bijection of G onto Φ. This means that a Menger algebra (G,o) of rank n is isomorphic to the constructed Menger algebra $(\Phi,\mathcal{O})$ of n-place interior operations. □

Corollary 6.2. *A Menger algebra (G,o) of rank n is isomorphic to a Menger algebra of n-place interior operations on some set if and only if it is derived from an idempotent commutative semigroup.*

The above corollary says that a Menger algebra isomorphic to a Menger algebra of n-place interior operations is derived from an idempotent commutative semigroup. Since a diagonal semigroup of a group-like Menger algebra is a group (see [1] or [3]), a group-like Menger algebra isomorphic to a Menger algebra of n-place interior operations has only one element. In view of Corollary 6.2, it is obvious that two Menger algebras of n-place interior operations are isomorphic if and only if their diagonal semigroups are isomorphic.

According to Theorem 3.2 and Corollary 3.4, each n-place interior operation defined by (6.7) is (full) $\cup$-distributive and (full) $\cap$-distributive. Therefore, we have the following corollary.

Corollary 6.3. *For a Menger algebra (G,o) of rank n the following conditions are equivalent:*

- (G,o) *is derived from an idempotent commutative semigroup.*
- (G,o) *is isomorphic to a Menger algebra of contractive (full) $\cup$-distributive n-place transformations on some set.*
- (G,o) *is isomorphic to a Menger algebra of (full) $\cup$-distributive n-place interior operations on some set.*
- (G,o) *is isomorphic to a Menger algebra of (full) $\cup$-distributive and $\cap$-distributive n-place interior operations on some set.*

7. Semigroups of interior operations

Menger algebras of rank $n = 1$ are (binary) semigroups. So, as a consequence of our results, we obtain some useful facts for semigroups.

Recall that Vagner (cf. [7]) defined *interior operations* on the set A as contractive, idempotent, and isotone transformations of $\mathfrak{P}(A)$. This definition coincides with our definition for $n = 1$. So, as a consequence of Theorem 3.1, we obtain the following corollary.

Corollary 7.1. *A transformation f of $\mathfrak{P}(A)$ is an interior operation on the set A if and only if*

$$f(X \cap Y) \subseteq f(f(X)) \cap f(Y) \cap Y$$

is valid for all $X, Y \in \mathfrak{P}(A)$.

From Theorem 3.2 and Corollary 3.3, we obtain

Corollary 7.2. *For a transformation f of $\mathfrak{P}(A)$, the following conditions are equivalent:*

- *f is contractive and full $\cup$-distributive;*
- *f is contractive and $\cup$-distributive;*
- *for every $X \in \mathfrak{P}(A)$, we have*

$$f(X) = f(A) \cap X. \tag{7.1}$$

Corollary 7.3. *Any transformation f of $\mathfrak{P}(A)$ satisfying* (7.1) *is an interior operation on A.*

As a consequence of Corollary 3.4, we obtain

Corollary 7.4. *Every* (*full*) *$\cup$-distributive interior operation is* (*full*) *$\cap$-distributive.*

Putting $n = 1$ in Theorem 4.2, we obtain one of the main results of the paper [4]:

Corollary 7.5. *The composition $f \circ g$ of two interior operations defined on the same set is an interior operation on this set if and only if $f \circ g = f \circ g \circ f$.*

The other main result of [4] is a consequence of our Corollary 6.3, which for $n = 1$ has the following form:

Corollary 7.6. *For a semigroup $(G, \cdot)$, the following statements are equivalent:*

- *$(G, \cdot)$ is an idempotent commutative semigroup;*
- *$(G, \cdot)$ is isomorphic to a semigroup of contractive and* (*full*) *$\cup$-distributive transformations on some set;*

- *$(G,\cdot)$ is isomorphic to a semigroup of (full) $\cup$-distributive interior operations on some set.*
- *$(G,\cdot)$ is isomorphic to a semigroup of (full) $\cup$-distributive and $\cap$-distributive interior operations on some set.*

From this, we obtain the following:

Corollary 7.7 (Podluzhnyak and Kulik [4]). *A semigroup $(G,\cdot)$ is isomorphic to some semigroup of interior operations on some set A if and only if it is idempotent and commutative.*

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