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Comments on Takahashi-Tanimoto's scalar solution

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Abstract: We study the identity-based solution of Witten's cubic bosonic open string field theory constructed by Takahashi and Tanimoto, which is claimed to describe the tachyon vacuum. We argue that the observables of the solution coincide with those of the tachyon vacuum using the method proposed by Kishimoto and Takahashi. We also discuss how to treat the kinetic term of the string field theory expanded around it.

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Contents

1 Introduction

Since the discovery of the tachyon vacuum solution by Schnabl [1], various kinds of analytic solutions of the equation of motion of the cubic bosonic open string field theory [2] have been constructed (for reviews, see [3–5]). It is now possible to construct a solution corresponding to any known open string background [6].

Most of the solutions found since [1] are so-called regular solutions which consist mainly of wedge states with non vanishing width with operator insertions. There exist some solutions which are not of this kind. An example is the solution

$$\Psi_{\rm TT} = \left[\int_{C_{\rm left}} \frac{d\xi}{2\pi i}\left(e^{h_a}-1\right) j_{\rm B}(\xi) - \int_{C_{\rm left}} \frac{d\xi}{2\pi i}\left(\partial h_a\right)^2 e^{h_a} c(\xi)\right]|I\rangle\ , \tag{1.1}$$

given by Takahashi and Tanimoto [7], which is called the scalar solution. Here $C_{\rm left}$ is a contour in the upper half plane depicted in figure 1, $j_{\rm B}$ is the BRST current

$$j_{\rm B}(\xi) = \left[cT + bc\partial c + \frac{3}{2}\partial^2 c\right](\xi)\ , \tag{1.2}$$

$|I\rangle$ is the identity string field and $h_a(\xi)$ is a function taken to be

$$h_a(\xi) = \ln\left(1+\frac{a}{2}\left(\xi+\frac{1}{\xi}\right)^2\right)\ , \tag{1.3}$$

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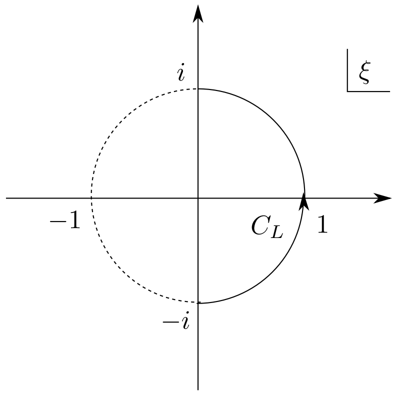

Figure 1. C_{left}.

for $a \geq -\frac{1}{2}$. Takahashi and Tanimoto claim that while the solution is a pure gauge solution for $a > -\frac{1}{2}$, it is a tachyon vacuum solution for $a = -\frac{1}{2}$.

The solution (1.1) is expressed as an identity state with local operator insertions. The solutions of such a form are called identity-based solutions. It is difficult to calculate observables like energy or Ellwood invariant of identity-based solutions. These quantities correspond to correlation functions of operators on a strip with vanishing width in the worldsheet theory and naive regularizations fail to yield definite values [8–10].

On the other hand, the identity-based solutions have some advantages. In general, the string field action expanded around a classical solution Ψ_{cl} can be given as

$$S'[\Psi] = -\frac{1}{g^2}\int\left[\frac{1}{2}\Psi Q'\Psi + \frac{1}{3}\Psi\Psi\Psi\right], \tag{1.4}$$

where

$$Q'A = QA + \Psi_{\text{cl}}A - (-1)^{|\Psi_{\text{cl}}||A|}A\Psi_{\text{cl}}\ .$$

In the case of regular solutions, Ψ_{cl} involves wedge states with finite width and it will be very difficult to study the string field theory action (1.4) with the kinetic operator Q'. However, if Ψ_{cl} is an identity-based solution, the Q' can be expressed by local operators on the worldsheet. For example, if Ψ_{cl} is the Takahashi-Tanimoto solution (1.1), the Q' becomes

$$\oint\frac{d\xi}{2\pi i}\left[e^{h_a}j_{\text{B}}(\xi) - (\partial h_a)^2 e^{h_a}c(\xi)\right]\ . \tag{1.5}$$

With Q' being an operator like this, we expect it is relatively easy to deal with the string field theory action (1.4).

Although the observables are not available, there are many evidences indicating that the Takahashi-Tanimoto solution (1.1) with $a = -\frac{1}{2}$ is a tachyon vacuum solution:

- There are no physical open string excitations around the background corresponding to $a = -\frac{1}{2}$. This fact has been shown by studying the BRST cohomology [11] or by constructing the homotopy operator [12].

- The open string amplitudes around the background can be shown to vanish [13].
- Solving the equation of motion in the background corresponding to $a = -\frac{1}{2}$ numerically, an unstable solution which is supposed to correspond to the perturbative vacuum can be found [14–16].

All these evidences imply that the solution corresponds to the tachyon vacuum. It should be interesting to explore the string field theory around such a background and see whether or not the closed string amplitudes can be reproduced from it. Since the solution is an identity-based solution, the string field theory expanded around the solution will have a tractable kinetic term.

In this paper, we would like to study the Takahashi-Tanimoto solution (1.1) with $a = -\frac{1}{2}$ and the string field theory expanded around it. What we will do first is to evaluate the observables of the solution in a rather indirect manner. In a recent paper [17], the authors consider the Erler-Schnabl solutions in the string field theory expanded around the identity-based marginal solutions found in [7, 18]. Since the Erler-Schnabl solutions will correspond to the tachyon vacuum, by calculating the observables of these solutions, they are able to evaluate the observables of the identity-based marginal solutions. We here apply this method to the scalar solution (1.1) with $a = -\frac{1}{2}$ and see what we can say about the observables of it. By doing so, we will get further evidences for the claim that the solution is a tachyon vacuum solution. In the latter half of the paper, we will discuss the string field theory expanded around the solution. We will show how we should treat the kinetic operator (1.5) in order for the solution to correspond to the tachyon vacuum.

The organization of this paper is as follows. In section 2, we evaluate the observables of the Takahashi-Tanimoto solution by calculating those of the Erler-Schnabl solution in the string field theory expanded around it. In section 3, we consider the string field theory around the Takahashi-Tanimoto solution and discuss how we should treat the kinetic operator. Section 4 is devoted to conclusions and discussions. In appendix A, we discuss the method proposed recently by Maccaferri [19] to construct regular solutions gauge equivalent to identity-based solutions. We explain what we can get by applying the method to the solution (1.1). In appendix B, we derive some identities concerning the operators U, U^{-1} which play important roles in the main text.

Note added. In the workshop "String field theory and related aspects VI, SFT2014" (28 July – 1 August 2014, SISSA Italy), where this work is presented [20], we have learned that Kishimoto, Masuda and Takahashi work on the same problem from a different point of view [21, 22]. Their results have some overlap with those in section 2.

While this paper was being typed, a paper [23] appeared on the arXiv, which also treat the same problem. There is some overlap with the contents of appendix A but the identity-based solution they deal with is different from ours.

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2 The Erler-Schnabl solution in the string field theory expanded around the Takahashi-Tanimoto solution

2.1 The Erler-Schnabl solution

The Erler-Schnabl solution [24]

$$\Psi_{\mathrm{ES}} = \frac{1}{1+K}\left(c + Q\left(Bc\right)\right) , \tag{2.1}$$

satisfies the equation of motion of the cubic string field theory. Here K, B, c are the string fields defined by

$$\begin{aligned} B &= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{dz}{2\pi i} b\left(z\right) |I\rangle \ , \\ c &= c\left(z\right)\big|_{z=\frac{1}{2}} |I\rangle \ , \\ K &= QB \\ &= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{dz}{2\pi i} T\left(z\right) |I\rangle \ , \end{aligned}$$

and the product of them is the star product. z is the sliver frame coordinate which is expressed by the upper half plane coordinate ξ in (1.1) as

$$z = \frac{2}{\pi} \arctan \xi \, .$$

K, B, c and Q satisfy the so-called KBc algebra [25, 26] and one can show that Ψ_{ES} is a solution by using the algebra. The Erler-Schnabl solution Ψ_{ES} describes the tachyon vacuum. This fact can be shown by calculating the observables or by showing that

$$A = B\frac{1}{1+K} \, ,$$

gives the homotopy operator for the background Ψ_{ES}, i.e. $QA + \Psi_{\mathrm{ES}}A + A\Psi_{\mathrm{ES}} = 1$ [27]. The existence of the homotopy operator implies that there exist no physical open string states around the background Ψ_{ES}.

As was pointed out in [28], it is straightforward to construct the Erler-Schnabl solution in the string field theory (1.4) expanded around an identity-based solution. Q' is a nilpotent operator and acts on string fields as a derivation. It is easy to see that

$$\Psi'_{\mathrm{ES}} = \frac{1}{1+K'}\left(c + Q'\left(Bc\right)\right) . \tag{2.2}$$

with

$$K' = Q'B \, ,$$

satisfies the equation of motion derived from the string field action (1.4), because the K', B, c and Q' satisfy the same algebra as the KBc and Q do. Moreover, the homotopy operator for the solution Ψ'_{ES} can be constructed as

$$A' = B\frac{1}{1+K'} \, .$$

Therefore one can argue that the solution $\Psi'_{\rm ES}$ describes the tachyon vacuum, provided $\frac{1}{1+K'}$ is a regular quantity.

Let us consider the Erler-Schnabl solution $\Psi'_{\rm ES}$ in the string field theory expanded around the Takahashi-Tanimoto solution given in (1.1) with $a=-\frac{1}{2}$. In this case, Q' is expressed by a contour integral

$$\oint \frac{dz}{2\pi i}\left[-\frac{\sin^2\pi z}{\cos^2\pi z}j_{\rm B}(z)+\frac{4\pi^2}{\cos^4\pi z}c(z)\right], \tag{2.3}$$

in the sliver frame and K' becomes

$$\begin{aligned} K' &= K+J\,, \\ J &\equiv \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{dz}{2\pi i}\left[-\frac{1}{\cos^2\pi z}T'(z)+\frac{4\pi^2}{\cos^4\pi z}\right]|I\rangle\,, \\ T'(z) &\equiv T^{\rm matter}(z)-b\partial c(z)\,. \end{aligned} \tag{2.4}$$

$\frac{1}{1+K'}$ can be expressed as

$$\frac{1}{1+K'}=\int_0^\infty dLe^{-L(1+K')}\,,$$

in the usual way and we need to define $e^{-LK'}$ to make sense of such quantities. In this section, we expand $e^{-LK'}$ as

$$\begin{aligned} e^{-LK'} &= e^{-L(K+J)} \\ &= \sum_{n=0}^\infty(-1)^n\lim_{\delta\to+0}\int_\delta^\infty dL_1\cdots\int_\delta^\infty dL_{n+1}\delta\left(\sum_{i=1}^{n+1}L_i-L\right)e^{-L_1K}Je^{-L_2K}J\cdots Je^{-L_{n+1}K}\,. \end{aligned} \tag{2.5}$$

and consider the right hand side as the definition of $e^{-LK'}$. From the point of view of the worldsheet theory, we define $e^{-LK'}$ perturbatively treating J as perturbation. The perturbation corresponds to adding

$$\int\frac{d^2z}{2\pi}\left[-\frac{1}{\cos^2\pi z}T'(z)+\frac{4\pi^2}{\cos^4\pi z}\right] \tag{2.6}$$

to the worldsheet action. Since it is a chiral quantity integrated over the bulk worldsheet, we do not encounter any ultraviolet divergences [17] and the expression is well-defined.[1] However, there is still a room for finite renormalizations. A prescription for such renormalization is fixed by introducing a cut-off δ.

Now let us consider the observables of the Erler-Schnabl solution $\Psi'_{\rm ES}$. The observables we consider are the action and the Ellwood invariant [29–31]. The action of $\Psi'_{\rm ES}$ in the string field theory (1.4) is equal to the difference of the energy between the background corresponding to $\Psi_{\rm TT}$ and that corresponding to $\Psi'_{\rm ES}$. The Ellwood invariant of $\Psi'_{\rm ES}$

[1]Notice that the normalization of J is fixed by the equation of motion and there is no reason to expect that the higher order terms in the expansion (2.5) are small in any sense. We will treat the operator K' without using such an expansion in section 3.

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becomes the difference of the 1-point function of a closed string vertex operator V between these backgrounds. Thus they can be expressed as

$$S\left[\Psi'_{\mathrm{ES}}\right] = E_{\mathrm{TT}} - E_{\Psi'_{\mathrm{ES}}}\,, \tag{2.7}$$

$$\mathrm{Tr}_V \Psi'_{\mathrm{ES}} = \langle Vc\rangle_{\Psi'_{\mathrm{ES}}} - \langle Vc\rangle_{\mathrm{TT}}\ , \tag{2.8}$$

Here the Ellwood invariant $\mathrm{Tr}_V\Phi$ is given as

$$\mathrm{Tr}_V\Phi = \langle I|\, V\,(i,-i)\,|\Phi\rangle\ , \tag{2.9}$$

where $V\,(i,-i) = c\bar{c}V^{\mathrm{m}}\,(i,-i)$ is a closed string vertex operator. E_{TT}, $E_{\Psi'_{\mathrm{ES}}}$, $\langle Vc\rangle_{\Psi'_{\mathrm{ES}}}$, $\langle Vc\rangle_{\mathrm{TT}}$ denote the energy and the one-point function of each background respectively. In the following, we will show $S\left[\Psi'_{\mathrm{ES}}\right] = \mathrm{Tr}_V\Psi'_{\mathrm{ES}} = 0$, which implies

$$E_{\Psi'_{\mathrm{ES}}} = E_{\mathrm{TT}}\,, \tag{2.10}$$

$$\langle Vc\rangle_{\Psi'_{\mathrm{ES}}} = \langle Vc\rangle_{\mathrm{TT}}\ . \tag{2.11}$$

Since we assume that Ψ'_{ES} corresponds to the tachyon vacuum, we can see that the observables $E_{\mathrm{TT}}, \langle Vc\rangle_{\mathrm{TT}}$ of the identity-based solution Ψ_{TT} coincide with those of the tachyon vacuum. Therefore showing $S\left[\Psi'_{\mathrm{ES}}\right] = \mathrm{Tr}_V\Psi'_{\mathrm{ES}} = 0$ gives evidences for the claim that the Takahashi-Tanimoto solution Ψ_{TT} describes the tachyon vacuum.

In this section, we use this indirect way proposed in [17] to calculate the observables $E_{\mathrm{TT}}, \langle Vc\rangle_{\mathrm{TT}}$ of the identity-based solution Ψ_{TT}. Recently there are somewhat more direct ways to calculate these quantities [28, 32][2] [19]. Especially Maccaferri [19] uses the so-called Zeze map [33] to construct regular solutions gauge equivalent to identity-based ones and calculate the observables of the regular ones. Moreover, the calculations eventually reduce to those of the $S\left[\Psi'_{\mathrm{ES}}\right], \mathrm{Tr}_V\Psi'_{\mathrm{ES}}$. In appendix A, we explain how we can apply Maccaferri's method to the Takahashi-Tanimoto solution (1.1) with $a = -\frac{1}{2}$.

2.2 Observables of Ψ'_{ES}

Now let us calculate the observables $S\left[\Psi'_{\mathrm{ES}}\right], \mathrm{Tr}_V\Psi'_{\mathrm{ES}}$ and show that both of them vanish.[3] From the expression (2.2), we obtain

$$\begin{aligned} S\left[\Psi'_{\mathrm{ES}}\right] &= -\frac{1}{6g^2}\mathrm{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K'}Q'c\right]\,, \\ \mathrm{Tr}_V\Psi'_{\mathrm{ES}} &= \mathrm{Tr}_V\left[\frac{1}{1+K'}c\right]\,. \end{aligned} \tag{2.12}$$

Therefore what we will prove are

$$\mathrm{Tr}_V\left[\frac{1}{1+K'}c\right] = 0\,, \tag{2.13}$$

$$\mathrm{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K'}Q'c\right] = 0\,. \tag{2.14}$$

[2] Kishimoto, Masuda and Takahashi [22] generalize the method of [28, 32] to the case of the scalar solutions.

[3] Kishimoto, Masuda and Takahashi [22] obtain the same results using a different method, considering more general solutions made from $K'Bc$.

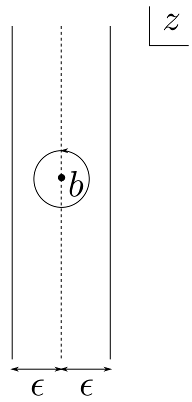

Figure 2. $e^{-\epsilon K}Q'be^{-\epsilon K}$.

These can be proved by using the following identities:

$$Q'\left(\frac{1}{\pi^2}b\right) = 1\,, \tag{2.15}$$

$$Q'c = 0\,. \tag{2.16}$$

Here

$$\frac{1}{\pi^2}b \equiv \frac{1}{\pi^2}b\left(z\right)\Big|_{z=\frac{1}{2}}\left|I\right\rangle = b\left(\xi\right)|_{\xi=1}\left|I\right\rangle\,,$$

and (2.15) suggests that the $\frac{1}{\pi^2}b$ works as a homotopy operator of the BRST charge Q'.

To be precise, one can show (2.15), (2.16) in the situation where we have some worldsheet around $\frac{1}{\pi^2}b, c$ without any local operator insertions. Namely we should consider

$$e^{-\epsilon K}Q'\left(\frac{1}{\pi^2}b\right)e^{-\epsilon K} = e^{-2\epsilon K}\,, \tag{2.17}$$

$$e^{-\epsilon K}Q'ce^{-\epsilon K} = 0\,, \tag{2.18}$$

in which we attach $e^{-\epsilon K}$'s to generate worldsheet as is depicted in figure 2.[4] With the worldsheet, one can express the action of Q' by the contour integral (2.3) and get

$$\begin{aligned}
&e^{-\epsilon K}Q'\left(\frac{1}{\pi^2}b\right)e^{-\epsilon K}\\
&\quad= e^{-\epsilon K}\left(\oint_0 \frac{dz}{2\pi i}\left[-\frac{\sin^2\pi z}{\cos^2\pi z}j_{\mathrm{B}}\left(z\right) + \frac{4\pi^2}{\cos^4\pi z}c\left(z\right)\right]\frac{1}{\pi^2}b\left(0\right)\right)e^{-\epsilon K}\\
&\quad= e^{-\epsilon K}\left(\oint_0 \frac{dz}{2\pi i}\left[-\frac{\sin^2\pi z}{\cos^2\pi z}\left(\frac{3}{2}\partial^2 c\left(z\right)\right) + \frac{4\pi^2}{\cos^4\pi z}c\left(z\right)\right]\frac{1}{\pi^2}b\left(0\right)\right)e^{-\epsilon K}\\
&\quad= e^{-2\epsilon K}\,.
\end{aligned} \tag{2.19}$$

Eq. (2.18) can be derived in the same way.

[4] In [19], such a prescription is used for the equation of motion. One can show that the Ψ_{TT} in (1.1) satisfies the equation of motion in the same way.

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Eq. (2.14) is an immediate consequence of (2.18). Eq. (2.13) can be derived from (2.17), (2.18) as follows. Inserting (2.15) into $\mathrm{Tr}_V\left[\frac{1}{1+K'}c\right]$, we get

$$\mathrm{Tr}_V\left[\frac{1}{1+K'}c\right]=\mathrm{Tr}_V\left[\frac{1}{\sqrt{1+K'}}Q'\left(\frac{1}{\pi^2}b\right)\frac{1}{\sqrt{1+K'}}c\right].$$

Here we use the definition

$$\frac{1}{\sqrt{1+K'}}=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^\infty dLL^{-\frac{1}{2}}e^{-L}e^{-LK'},$$

where $e^{-LK'}$ expressed as (2.5). With the cutoff δ, (2.15), (2.16) can be safely used because there are some worldsheets with no operator insertions around b, c. Thus we obtain

$$\begin{aligned}&\mathrm{Tr}_V\left[\frac{1}{1+K'}c\right]\\&=\mathrm{Tr}_V\left[\frac{1}{\sqrt{1+K'}}\frac{1}{\pi^2}b\frac{1}{\sqrt{1+K'}}Q'c\right]\\&=0\,.\end{aligned}$$

Before closing this section, one comment is in order. Using $Q'c=0$, one can see that from (2.2)

$$\Psi'_{\mathrm{ES}}=c\,.$$

Thus actually the Ψ'_{ES} itself is an identity-based solution,[5] although we do not have any trouble in calculating the right hand sides of (2.12). One can avoid this by replacing c by

$$c_y\equiv c\left(\frac{1}{2}+iy\right)|I\rangle\ \ (y\neq 0,\ \ y\in\mathbb{R})\,.$$

K', B, c_y satisfy the KBc algebra and one can construct the Erler-Schnabl solution

$$\Psi'_{\mathrm{ES},y}=\frac{1}{1+K'}\left(c_y+Q'\left(Bc_y\right)\right),$$

which is not identity-based, albeit it still includes an identity based piece. The observables to be calculated become

$$\begin{aligned}S\left[\Psi'_{\mathrm{ES},y}\right]&=-\frac{1}{6g^2}\mathrm{Tr}\left[\frac{1}{1+K'}c_y\frac{1}{1+K'}Q'c_y\right],\\ \mathrm{Tr}_V\Psi'_{\mathrm{ES},y}&=\mathrm{Tr}_V\left[\frac{1}{1+K'}c_y\right].\end{aligned}\tag{2.20}$$

One can show that these quantities are actually independent of y. Indeed, using the KBc identity

$$\{B,c_y\}=1\,,$$

[5]One may be able to calculate the observables for such a solution following [34] or [23].

the formulas given in [19] (eqs. (3.4), (3.10)–(3.19)) imply

$$\begin{aligned}
&\frac{1}{6}\mathrm{Tr}\left[\frac{1}{1+K'}c_y\frac{1}{1+K'}Q'c_y\right]\\
&\quad=\frac{1}{6}\mathrm{Tr}\left[\frac{1}{1+K}c_y\frac{1}{1+K}Qc_y\right]-\frac{1}{6}\mathrm{Tr}\left[\frac{1}{1+K}\Psi_{\mathrm{TT}}\frac{1}{1+K'}\Psi_{\mathrm{TT}}\frac{1}{1+K}\Psi_{\mathrm{TT}}\frac{1}{1+K'}\right],\\
&\mathrm{Tr}_V\left[\frac{1}{1+K'}c_y\right]\\
&\quad=\mathrm{Tr}_V\left[\frac{1}{1+K}c_y\right]-\mathrm{Tr}_V\left[\frac{1}{1+K}\Psi_{\mathrm{TT}}\frac{1}{1+K'}\right],
\end{aligned}$$

and the right hand sides are independent of y. Therefore evaluating them at $y=0$, we can see that the observables (2.20) all vanish.

3 String field theory expanded around the Takahashi-Tanimoto solution

The derivation in the previous section uses the perturbative definition (2.5) of $e^{-LK'}$. Since Ψ_{TT} is an identity-based solution, the kinetic term Q' is given by an integral of local operators on the worldsheet and we should be able to treat K' more directly. In this section, we will examine if we can derive the results in section 2 by doing so.

In the calculation of the observables in the previous section, the following relations were essential:

$$\begin{aligned}
e^{-\epsilon K}Q'\left(\frac{1}{\pi^2}b\right)e^{-\epsilon K}&=e^{-2\epsilon K},\\
e^{-\epsilon K}Q'ce^{-\epsilon K}&=0.
\end{aligned}$$

These relations hold for the perturbative definition of $e^{-LK'}$. In the treatment here, it will be more appropriate to consider

$$e^{-\epsilon K'}Q'\left(\frac{1}{\pi^2}b\right)e^{-\epsilon K'}=e^{-2\epsilon K'},\tag{3.1}$$

$$e^{-\epsilon K'}Q'ce^{-\epsilon K'}=0,\tag{3.2}$$

where the $e^{-\epsilon K'}$'s are expected to provide worldsheet with no operator insertions.

Actually, as we will see, the definition of $e^{-LK'}$ is very subtle and we need some regularization to define quantities involving it. There seem to be many ways to treat it, which should be related to the choice of the prescription of renormalization in the perturbative definition of $e^{-LK'}$ (2.5). Here we use the identities (3.1), (3.2) and their consequences (2.13), (2.14) as the guiding principle to find the definition of $e^{-LK'}$ so that the string field action (1.4) should describe the tachyon vacuum.

3.1 Similarity transformation

The K' given in (2.4) involves $T'(z)$ which is a twisted energy momentum tensor with central charge $c=24$. Therefore we need to take care of the conformal anomaly on the worldsheet to deal with the correlation functions on surfaces generated by $e^{-LK'}$ and the calculations will become cumbersome. Here we would like to use an alternative way of dealing with K' to do calculations.

As was pointed out by Kishimoto and Takahashi [11], the kinetic operator Q' of the string field theory expanded around the solution (1.1) with $a=-\frac{1}{2}$ can be expressed as

$$Q'=e^{-q}\left(-\frac{1}{4}Q_2+c_2\right)e^{q}\,, \tag{3.3}$$

where

$$q=-\oint\frac{d\xi}{2\pi i}\left(-bc\right)(\xi)\ln\left(1-\frac{1}{\xi^2}\right)^2\,, \tag{3.4}$$

$$Q_k=\oint\frac{d\xi}{2\pi i}\xi^k j_{\mathrm{B}}\left(\xi\right)\,, \tag{3.5}$$

$$c_k=\oint\frac{d\xi}{2\pi i}\xi^{k-2}c\left(\xi\right)\,. \tag{3.6}$$

Using the mode expansion of the ghost number current

$$-bc\left(\xi\right)=\sum_n j_n\xi^{-n-1}\,,$$

the q is expressed as

$$q=2\sum_{n=1}^{\infty}\frac{1}{n}j_{-2n}\,.$$

***bc*-shift operation.** Eq. (3.3) can be rewritten by using the bc-shift operation [11] defined for $k\in\mathbb{Z}$ as

$$\begin{aligned}
c_n &\rightarrow c_n^{(k)}=c_{n+k}\,,\\
b_n &\rightarrow b_n^{(k)}=b_{n-k}\,,\\
|0\rangle &\rightarrow |0\rangle^{(k)}=\begin{cases}b_{-k-1}b_{-k}\cdots b_{-2}\,|0\rangle & k>0\\ c_{k+2}c_{k+3}\cdots c_1\,|0\rangle & k<0\end{cases}\,,\\
\langle 0| &\rightarrow {}^{(k)}\langle 0|=\begin{cases}\langle 0|\,c_{-1}c_0\cdots c_{k-2} & k>0\\ \langle 0|\,b_2b_3\cdots b_{-k+1} & k<0\end{cases}\,,
\end{aligned}$$

and $\phi\rightarrow\phi^{(k)}=\phi$ if ϕ involves only matter fields. A state

$$|a\rangle=\phi_{-n_1}\cdots b_{-m_1}\cdots c_{-l_1}\cdots|0\rangle\,,$$

in the Fock space is mapped to

$$|a\rangle^{(k)}=\phi_{-n_1}^{(k)}\cdots b_{-m_1}^{(k)}\cdots c_{-l_1}^{(k)}\cdots|0\rangle^{(k)}\,,$$

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under this operation. $c_n^{(k)}, b_n^{(k)}, |0\rangle^{(k)}, {}^{(k)}\langle 0|$ satisfy

$$\begin{aligned} \left\{c_n^{(k)}, b_n^{(h)}\right\} &= \delta_{n+m,0}\,, \\ \left\{c_n^{(k)}, c_m^{(k)}\right\} = \left\{b_n^{(k)}, b_m^{(k)}\right\} &= 0\,, \\ b_n^{(k)}\,|0\rangle^{(k)} &= 0\ (n \geq -1)\ , \\ c_n^{(k)}\,|0\rangle^{(k)} &= 0\ (n \geq 2)\ , \\ {}^{(k)}\langle 0|\, b_n^{(k)} &= 0\ (n \leq 1)\ , \\ {}^{(k)}\langle 0|\, c_n^{(k)} &= 0\ (n \leq -2)\ , \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &{}^{(k)}\langle 0|\, c_{-1}^{(k)} c_0^{(k)} c_1^{(k)}\, |0\rangle^{(k)} \\ &= \langle 0|\, c_{-1} c_0 c_1\, |0\rangle \\ &= 1\,. \end{aligned} \tag{3.8}$$

Since we can evaluate all the correlation functions of the bc system using the relations (3.7), (3.8), we can see that for any states $\langle a|\,, |b\rangle$ in the Fock space,

$${}^{(k)}\langle a|b\rangle^{(k)} = \langle a|b\rangle\ .$$

Under the bc-shift operation, the BRST charge is transformed as

$$Q \to Q^{(k)} = Q_k - k^2 c_k\,.$$

Therefore (3.3) can be written as

$$Q' = -\frac{1}{4} e^{-q} Q^{(2)} e^{q}\,. \tag{3.9}$$

It is convenient to introduce operators $U_k\ \ (k \in \mathbb{Z})$ which are defined so that

$$\begin{aligned} U_k\,|a\rangle &= |a\rangle^{(k)}\ , \\ \langle a|\, U_k &= {}^{(-k)}\langle a|\ . \end{aligned}$$

U_k satisfies

$$\begin{aligned} U_k U_{-k}\,|a\rangle &= |a\rangle\ , \\ U_k \mathcal{O} U_{-k} &= \mathcal{O}^{(k)}\,, \end{aligned}$$

for any state $|a\rangle$ in the Fock space and any operator $\mathcal{O}$. It turns out that U_k can be expressed as

$$U_k = e^{-k\sigma_0}\,, \tag{3.10}$$

where σ_0 is the operator which appears in the bosonization formulas (B.5), (B.6). Indeed, $e^{-k\sigma_0}$ satisfies

$$
\begin{aligned}
e^{-k\sigma_0}c\left(\xi\right)e^{k\sigma_0} &= e^{-k\sigma_0}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}j_{-n}\xi^n\right]e^{\sigma_0}e^{j_0\ln\xi}\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}j_n\xi^{-n}\right]e^{k\sigma_0} = \xi^k c\left(\xi\right)\,,\\
e^{-k\sigma_0}b\left(\xi\right)e^{k\sigma_0} &= e^{-k\sigma_0}\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}j_{-n}\xi^n\right]e^{-\sigma_0}e^{-j_0\ln\xi}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}j_n\xi^{-n}\right]e^{k\sigma_0} = \xi^{-k}b\left(\xi\right)\,,\\
e^{-k\sigma_0}\left|0\right\rangle &= \begin{cases} b_{-k-1}b_{-k}\cdots b_{-2}\left|0\right\rangle = \left|0\right\rangle^{(k)} & k>0\\ c_{k+2}c_{k+3}\cdots c_1\left|0\right\rangle = \left|0\right\rangle^{(k)} & k<0\end{cases}\,,\\
\left\langle 0\right|e^{-k\sigma_0} &= \begin{cases} \left\langle 0\right|b_2b_3\cdots b_{k+1} ={}^{(-k)}\left\langle 0\right| & k>0\\ \left\langle 0\right|c_{-1}c_0\cdots c_{-k-2} ={}^{(-k)}\left\langle 0\right| & k<0\end{cases}\,.
\end{aligned}
$$

From (3.10) and

$$\left[j_0, e^{-k\sigma_0}\right] = -ke^{-k\sigma_0}\,,$$

we can see that U_k carries ghost number $-k$.

Eq. (3.9) can be written as

$$Q' = -\frac{1}{4}UQU^{-1}\,. \tag{3.11}$$

where

$$
\begin{aligned}
U &\equiv e^{-q}U_2\,,\\
U^{-1} &\equiv U_{-2}e^{q}\,.
\end{aligned}
\tag{3.12}
$$

Notice that U, U^{-1} are of ghost number $-2, 2$. U and U^{-1} are inverse to each other, when these operators act on the states in the Fock space. However, when we are dealing with the states outside of the Fock space, such a statement may become subtle, as is discussed in appendix B. Another thing to be noticed is that the BPZ conjugates of U, U^{-1} do not coincide with either U or U^{-1}.

Therefore, the Q' is related to the original kinetic operator Q by a similarity transformation (3.11), which implies that the solution Ψ_{TT} is formally in the pure gauge form. By the similarity transformation, K' is turned into an operator made from $T = \{Q, b\}$ and thus it is possible to evaluate quantities involving K' without dealing with the twisted energy momentum tensor T'.

3.2 U, U^{-1}

We need some identities satisfied by U, U^{-1} to perform calculations using the relation (3.11). From the definition (3.12) we obtain

$$\begin{aligned} Uc(\xi)\,U^{-1} &= \frac{\left(\xi^2-1\right)^2}{\xi^2}c(\xi) = -4e^{h_{-\frac{1}{2}}(\xi)}c(\xi)\,, && (3.13)\\ U^{-1}c(\xi)\,U &= \frac{\xi^2}{\left(\xi^2-1\right)^2}c(\xi) = -\frac{1}{4}e^{-h_{-\frac{1}{2}}(\xi)}c(\xi)\,, && (3.14)\\ Ub(\xi)\,U^{-1} &= \frac{\xi^2}{\left(\xi^2-1\right)^2}b(\xi) = -\frac{1}{4}e^{-h_{-\frac{1}{2}}(\xi)}b(\xi)\,, && (3.15)\\ U^{-1}b(\xi)\,U &= \frac{\left(\xi^2-1\right)^2}{\xi^2}b(\xi) = -4e^{h_{-\frac{1}{2}}(\xi)}c(\xi)\,. && (3.16)\end{aligned}$$

It is also possible to derive how U, U^{-1} act on the states $|0\rangle, |I\rangle, \langle 0|, \langle I|$:

$$\begin{aligned} U|0\rangle &= \frac{1}{16}\partial bb(1)\,\partial bb(-1)\,c_0c_1|0\rangle\,, && (3.17)\\ U^{-1}|0\rangle &= \frac{1}{16}\partial cc(1)\,\partial cc(-1)\,b_{-3}b_{-2}|0\rangle\,, && (3.18)\\ \langle 0|U &= \langle 0|\,b_2b_3\,, && (3.19)\\ \langle 0|U^{-1} &= \langle 0|\,c_{-1}c_0\,. && (3.20)\\ U|I\rangle &= \frac{1}{32}\partial bb(1)\,|I\rangle\,, && (3.21)\\ U^{-1}|I\rangle &= 2\partial cc(1)\,|I\rangle\,. && (3.22)\end{aligned}$$

Moreover, one can show that $\langle I|U$ and $\langle I|U^{-1}$ can be set to zero in the situations where no ghost operators are inserted at $\xi = \pm 1$. These properties are proved in appendix B.

Here let us comment on one thing concerning the operators U, U^{-1}, which will be relevant to the subsequent discussions. The pure gauge form (3.11) apparently contradicts the existence of the homotopy operator (2.15), as was pointed out in [12, 35]. Indeed, one can see from (3.11) that the representatives of the BRST cohomology of Q' are given by the states of the form [11]

$$\begin{aligned} UcV^{\mathrm{m}}(0)\,|0\rangle &: \mathrm{gh}\# = -1\,, && (3.23)\\ U\partial ccV^{\mathrm{m}}(0)\,|0\rangle &: \mathrm{gh}\# = 0\,. && (3.24)\end{aligned}$$

where V^{m} is a primary field made from the matter fields with weight 1. Therefore one can conclude that there exist no physical open string excitations because they correspond to the states with ghost number 1. On the other hand, the existence of the homotopy operator $b(1)$ implies that the states (3.23), (3.24) should be written in a BRST exact form

$$\begin{aligned} UcV^{\mathrm{m}}(0)\,|0\rangle &= Q'b(1)\,UcV^{\mathrm{m}}(0)\,|0\rangle\,,\\ U\partial ccV^{\mathrm{m}}(0)\,|0\rangle &= Q'b(1)\,U\partial ccV^{\mathrm{m}}(0)\,|0\rangle\,.\end{aligned}$$

Actually these do not hold. Indeed, using eqs. (3.14), (3.17), we obtain

$$U c V^{\mathrm{m}}(0)|0\rangle = \frac{1}{16}\partial b b(1)\,\partial b b(-1)\,c_{-1}c_0c_1V^{\mathrm{m}}(0)|0\rangle\ , \tag{3.25}$$

$$U\partial c c V^{\mathrm{m}}(0)|0\rangle = \frac{1}{16}\partial b b(1)\,\partial b b(-1)\,c_{-2}c_{-1}c_0c_1V^{\mathrm{m}}(0)|0\rangle\ , \tag{3.26}$$

and $b(1)\,UcV(0)|0\rangle = b(1)\,U\partial ccV(0)|0\rangle = 0$. The reason for this apparent contradiction is that the relation (2.15) holds only when there is some worldsheet around $b(1)$ without any local operator insertions, as we mentioned below eq. (2.18). Therefore, for the states (3.25), (3.26) which involve $\partial bb(1)$, $b(1)$ does not work as a homotopy operator of Q'.

3.3 Calculations of the observables

Now we would like to discuss how we can evaluate the observables (2.12) using the expression (3.11). In order to facilitate the calculation using eq. (3.11), we rewrite everything in terms of the first-quantized operators, rather than string fields. Here let us introduce $\mathcal{B}^+, \mathcal{L}'^+$ such that [1, 36]

$$\begin{aligned}\mathcal{B}^+ &= \oint\frac{d\xi}{2\pi i}\left(1+\xi^2\right)\left(\tan^{-1}\xi+\tan^{-1}\left(\frac{1}{\xi}\right)\right)b(\xi)\\ &= \frac{\pi}{2}\oint\frac{d\xi}{2\pi i}\left(1+\xi^2\right)\epsilon\left(\mathrm{Re}\xi\right)b(\xi)\ ,\\ \mathcal{L}'^+ &\equiv \left\{Q',\mathcal{B}^+\right\}\ .\end{aligned} \tag{3.27}$$

$\mathcal{L}'^+$ is the translation operator with respect to the sliver frame coordinate z for the left and right half of the string. Therefore, the action of $\mathcal{L}'^+$ on any state $|\phi\rangle$ can be expressed by the string field K' as

$$\mathcal{L}'^+|\phi\rangle = K' * |\phi\rangle + |\phi\rangle * K'\,. \tag{3.28}$$

$\mathcal{L}'^+$ can be used to express various quantities involving K' in our setup. For example, using (3.28) and (2.9), one can show that

$$\begin{aligned}&\langle I|\,e^{-\frac{L}{4}\mathcal{L}'^+}c(1)\,V(i,-i,)\,e^{-\frac{L}{4}\mathcal{L}'^+}|I\rangle\\ &\quad= \mathrm{Tr}_V\left[e^{-\frac{L}{4}K'}*c*e^{-\frac{L}{4}K'}*|I\rangle*e^{-\frac{L}{4}K'}*e^{-\frac{L}{4}K'}\right]\\ &\quad= \mathrm{Tr}_V\left[e^{-LK'}c\right]\ ,\end{aligned} \tag{3.29}$$

holds and the left hand side of eqs. (2.13) is expressed as

$$\mathrm{Tr}_V\left[\frac{1}{1+K'}c\right] = \int_0^\infty dLe^{-L}\langle I|\,e^{-\frac{L}{4}\mathcal{L}'^+}c(1)\,V(i,-i,)\,e^{-\frac{L}{4}\mathcal{L}'^+}|I\rangle\ . \tag{3.30}$$

In a similar way, one gets

$$\begin{aligned}&\langle I|\,e^{-\frac{L_1-L_2}{2}\mathcal{L}'^+}c(1)\,e^{-L_2\mathcal{L}'^+}Q'c(1)\,|I\rangle\\ &\quad= \mathrm{Tr}\left[e^{-\frac{L_1-L_2}{2}K'}*c*e^{-L_2K'}*Q'c*e^{-L_2K'}*e^{-\frac{L_1-L_2}{2}K'}\right]\\ &\quad= \mathrm{Tr}\left[e^{-L_1K'}ce^{-L_2K'}Q'c\right]\ .\end{aligned} \tag{3.31}$$

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We expect that $e^{-\frac{L_1-L_2}{2}\mathcal{L}'^+}$ is well-defined when $L_1 > L_2$ and this equation is valid only for $L_1 > L_2$. When $L_2 > L_1$, $\langle I| \, e^{-\frac{L_2-L_1}{2}\mathcal{L}'^+} c(-1) \, e^{-L_1\mathcal{L}'^+} Q'c(1) \, |I\rangle$ can be used to express $\mathrm{Tr}\left[e^{-L_1K'} c e^{-L_2K'} Q'c\right]$. Therefore the left hand side of (2.14) is expressed as

$$\begin{aligned}
\mathrm{Tr}\left[\frac{1}{1+K'} c \frac{1}{1+K'} Q'c\right] =& \int_0^\infty dL_2 e^{-L_2} \int_{L_2}^\infty dL_1 e^{-L_1} \\
& \times \langle I| \, e^{-\frac{L_1-L_2}{2}\mathcal{L}'^+} c(1) \, e^{-L_2\mathcal{L}'^+} Q'c(1) \, |I\rangle \\
& + \int_0^\infty dL_2 e^{-L_2} \int_0^{L_2} dL_1 e^{-L_1} \\
& \times \langle I| \, e^{-\frac{L_2-L_1}{2}\mathcal{L}'^+} c(-1) \, e^{-L_1\mathcal{L}'^+} Q'c(1) \, |I\rangle \, , \qquad (3.32)
\end{aligned}$$

Eqs. (3.1), (3.2) are also rewritten as

$$e^{-\epsilon\mathcal{L}'^+} Q'b(1) \, |I\rangle = e^{-\epsilon\mathcal{L}'^+} \, |I\rangle \, , \qquad (3.33)$$

$$e^{-\epsilon\mathcal{L}'^+} Q'c(1) \, |I\rangle = 0 \, . \qquad (3.34)$$

Let us check if one can prove (3.33), (3.34) by using the expression (3.11). Substituting (3.11) into the left hand side of (3.33), we get

$$-\frac{1}{4} e^{-\epsilon\mathcal{L}'^+} UQU^{-1} b(1) \, |I\rangle \, .$$

In order to avoid the singularity which appears in moving the operator U^{-1} to the right, we shift the position of b for regularization. Thus we consider

$$\begin{aligned}
& -\frac{1}{4} \lim_{\xi\to 1} e^{-\epsilon\mathcal{L}'^+} UQU^{-1} b(\xi) \, |I\rangle \\
& = -\frac{1}{4} \lim_{\xi\to 1} \left[\frac{\left(\xi^2-1\right)^2}{\xi^2} U e^{-\epsilon\tilde{\mathcal{L}}'^+} Qb(\xi) \, 2\partial cc(1) \, |I\rangle\right] . \qquad (3.35)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathcal{L}}'^+ &= U^{-1}\mathcal{L}'^+ U \\
&= \left\{Q, \frac{\pi}{2} \oint \frac{d\xi}{2\pi i} \left(1+\xi^2\right) \epsilon(\mathrm{Re}\xi) \, e^{h_{-\frac{1}{2}}(\xi)} b(\xi)\right\} \\
&= \frac{\pi}{2} \oint \frac{d\xi}{2\pi i} \left(1+\xi^2\right) \epsilon(\mathrm{Re}\xi) \, e^{h_{-\frac{1}{2}}(\xi)} T(\xi) \, , \qquad (3.36)
\end{aligned}$$

Instead of K' or $\mathcal{L}'^+$, $\tilde{\mathcal{L}}'^+$ is the fundamental translation operator to deal with in the subsequent calculation. Contrary to K', $\tilde{\mathcal{L}}'^+$ is made from $T(\xi)$ and we do not have to worry about the conformal anomaly. If the operator $e^{-\epsilon\tilde{\mathcal{L}}'^+}$ should generate worldsheet around $\{Q, b(\xi)\}$ in (3.35), we could express Q by a contour integral and proceed further.

The operator of the form (3.36) can be analyzed by the methods explained in [37]. Here it is convenient to go to the sliver frame and rewrite (3.36) as

$$\tilde{\mathcal{L}}'^+ = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{h\left(\frac{1}{2}+z\right)} T\left(\frac{1}{2}+z\right) + \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{h\left(-\frac{1}{2}+z\right)} T\left(-\frac{1}{2}+z\right) ,$$

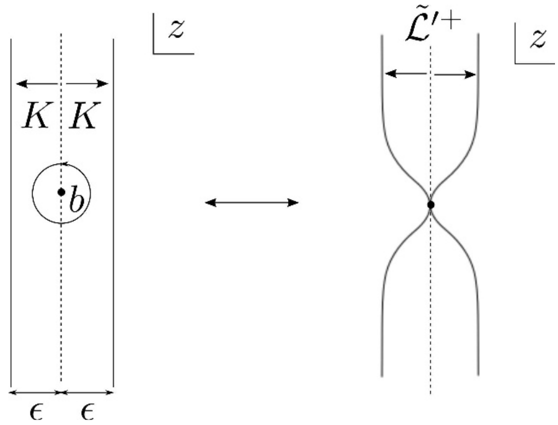

Figure 3. The worldsheet generated by $e^{-\epsilon\tilde{\mathcal{L}}'^{+}}$ in contrast to the one generated by $e^{-\epsilon K}$.

where

$$e^{h(z)} = -\frac{\cos^2 \pi z}{\sin^2 \pi z}\,.$$

We introduce a new coordinate w such that

$$\frac{\partial z}{\partial w} = e^{h(z)}\,,$$

which is integrated as

$$w\,(z) = z - \frac{1}{\pi}\frac{\sin \pi z}{\cos \pi z}\,. \tag{3.37}$$

Using these, $\tilde{\mathcal{L}}'^{+}$ is expressed as

$$\tilde{\mathcal{L}}'^{+} = \left[\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} + \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\right] \frac{dz}{2\pi i}\frac{\partial w}{\partial z} T\,(w)\,,$$

and $\tilde{\mathcal{L}}'^{+}$ generates translations with respect to the coordinate w. The map $w\,(z)$ (3.37) maps the region $0 < \mathrm{Im}z < \infty$ to $-\infty < \mathrm{Im}w < \infty$ and the region $-\infty < \mathrm{Im}z < 0$ to $-\infty < \mathrm{Im}w < \infty$ for $\mathrm{Re}z = \pm\frac{1}{2}$ and $z = \pm\frac{1}{2}$ are singular points. $z = \pm\frac{1}{2}$ are mapped to $w = \pm\infty$ and do not move under the translation generated by $\tilde{\mathcal{L}}'^{+}$. Therefore the operator $e^{-\epsilon\mathcal{L}'^{+}}$ acting on the identity state $|I\rangle$ generates the worldsheet of the form depicted in figure 3. Hence $e^{-\epsilon\mathcal{L}'^{+}}$ in (3.35) does not generate worldsheet around $Qb\,(\xi)$ and we cannot proceed from (3.35). The correlation functions which appear on the right hand sides of eqs. (3.30), (3.32) correspond to cylinders of the form $w \sim w + L$. Such a cylinder is mapped to two spheres whose coordinates are given by $e^{\frac{2\pi i}{L}w}$.

Regularization. The operator $e^{-\epsilon\tilde{\mathcal{L}}'^{+}}$ generates apparently singular surfaces, which should be defined as a limit of regular surfaces. There are problems in performing calculations on such singular surfaces. We are not able to prove the homotopy relation (3.34) on such surfaces because no worldsheet is generated around the point on the boundary. We would like to define the string field theory so that it describes the tachyon vacuum. Therefore what we need to do is to regularize the $\tilde{\mathcal{L}}'^{+}$, while preserving the relations (3.33), (3.34).

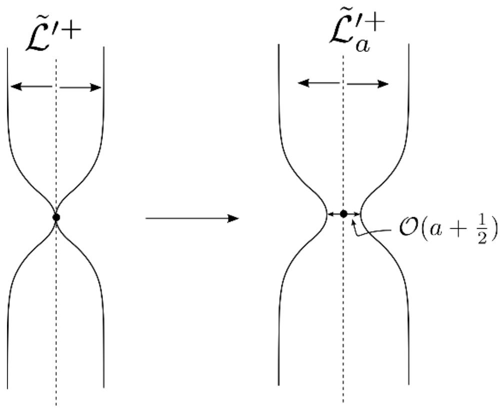

Figure 4. The surface generated by$e^{-L\tilde{\mathcal{L}}'^{+}_{a}}$ in contrast to the one generated by $e^{-L\tilde{\mathcal{L}}'^{+}}$.

The regularization we propose is to replace $\tilde{\mathcal{L}}'^{+}$ by

$$\tilde{\mathcal{L}}'^{+}_{a} \equiv \frac{\pi}{2}\oint \frac{d\xi}{2\pi i}\left(1+\xi^2\right)\epsilon\left(\mathrm{Re}\xi\right)e^{h_a(\xi)}T\left(\xi\right)\,. \tag{3.38}$$

$\left(a > -\frac{1}{2}\right)$ with $h_a\left(\xi\right)$ given in (1.3). We define $e^{-L\tilde{\mathcal{L}}'^{+}}$ as

$$\lim_{a\to-\frac{1}{2}} e^{-L\tilde{\mathcal{L}}'^{+}_{a}}\,. \tag{3.39}$$

For $a > -\frac{1}{2}$, the surface generated by $e^{-L\tilde{\mathcal{L}}'^{+}_{a}}$ is of the form depicted in figure 4 and we realize $e^{-L\tilde{\mathcal{L}}'^{+}}$ as a singular limit of $e^{-L\tilde{\mathcal{L}_{a}}'^{+}}$.

With such a regularization, the right hand side of (3.35) becomes

$$\begin{aligned}
&-\frac{1}{4}\lim_{a\to-\frac{1}{2}}\lim_{\xi\to1}\left[\frac{\left(\xi^2-1\right)^2}{\xi^2}Ue^{-\epsilon\tilde{\mathcal{L}}'^{+}_{a}}Qb\left(\xi\right)2\partial cc\left(1\right)\left|I\right\rangle\right]\\
&\quad=U\lim_{a\to-\frac{1}{2}}e^{-\epsilon\tilde{\mathcal{L}}'^{+}_{a}}2\partial cc\left(1\right)\left|I\right\rangle\,,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
U\lim_{a\to-\frac{1}{2}}e^{-\epsilon\tilde{\mathcal{L}}'^{+}_{a}}2\partial cc\left(1\right)\left|I\right\rangle &= U\lim_{a\to-\frac{1}{2}}e^{-\epsilon\tilde{\mathcal{L}}'^{+}_{a}}U^{-1}\left|I\right\rangle\\
&= e^{-\epsilon\mathcal{L}'^{+}}\left|I\right\rangle\,,
\end{aligned} \tag{3.40}$$

and we eventually get (3.33). Eq. (3.34) can be proved in the same way:

$$\begin{aligned}
&e^{-\epsilon\mathcal{L}'^{+}}Q'c\left(1\right)\left|I\right\rangle\\
&\quad=-\frac{1}{4}\lim_{\xi\to1}e^{-\epsilon\mathcal{L}'^{+}}UQU^{-1}c\left(\xi\right)\left|I\right\rangle\\
&\quad=-\frac{1}{4}\lim_{a\to-\frac{1}{2}}\lim_{\xi\to1}\left[\frac{\xi^2}{\left(\xi^2-1\right)^2}Ue^{-\epsilon\tilde{\mathcal{L}}'^{+}_{a}}Qc\left(\xi\right)2\partial cc\left(1\right)\left|I\right\rangle\right]\\
&\quad=0\,.
\end{aligned} \tag{3.41}$$

One can immediately show that the terms on the right hand side of (3.32) vanish by using (3.41). In order to show that the right hand side of (3.30) vanishes, we use (3.40) to get

$$\begin{aligned}
&\langle I| \, e^{-\frac{L}{4}\mathcal{L}'^{+}} c(1)\, V(i,-i,)\, e^{-\frac{L}{4}\mathcal{L}'^{+}} |I\rangle \\
&= \lim_{a\to -\frac{1}{2}} \lim_{\xi\to 1} \langle I| \, e^{-\frac{L}{4}\mathcal{L}'^{+}} c(\xi)\, V(i,-i,)\, U e^{-\frac{L}{4}\tilde{\mathcal{L}}'^{+}_{a}} 2\partial cc(1)\,|I\rangle \\
&= \lim_{a\to -\frac{1}{2}} \lim_{\xi\to 1} \langle I| \, U e^{-\frac{L}{4}\tilde{\mathcal{L}}'^{+}_{a}} \frac{\left(\xi^{2}-1\right)^{2}}{\xi^{2}} c(\xi)\, V(i,-i,)\, e^{-\frac{L}{4}\tilde{\mathcal{L}}'^{+}_{a}} 2\partial cc(1)\,|I\rangle \\
&= 0\,.
\end{aligned} \tag{3.42}$$

Here, with the regularization, $\langle I| \, U$ is away from the other operators $c(\xi)\,, \partial cc(1)$ and it can be set to zero. Thus we have shown how to regularize and define the operator $e^{-L\tilde{\mathcal{L}}'^{+}}$ so that we can derive (3.33), (3.34), (2.13), (2.14). These formulas imply that the string field theory describes the tachyon vacuum.

4 Conclusions and discussions

In this paper, we have evaluated the observables of the Takahashi-Tanimoto's scalar solution (1.1) with $a = -\frac{1}{2}$, by studying the Erler-Schnabl solution in the string field theory expanded around it. The results are consistent with the claim that the solution corresponds to the tachyon vacuum. In the calculations, the string field K' or its worldsheet operator counterpart plays crucial roles. In the latter half of this paper, we study the operator K' using the similarity transformation proposed by Kishimoto and Takahashi. We discuss how we should treat it in order to be consistent with the claim that the background is the tachyon vacuum.

The relation (3.11) will be useful to evaluate various other quantities in the string field theory expanded around Ψ_{TT}. Since the solution is supposed to describe the tachyon vacuum, we expect all the amplitudes involving open string states to vanish. On the other hand, we may be able to calculate closed string amplitudes using the string field theory [30, 38, 39]. In order to do such calculations, we should take Siegel gauge for example and construct the propagators. We will need some regularization like (3.39) to define the propagator. We leave it as a future problem.

The operator U, U^{-1} in (3.11) should be related to the boundary condition changing operators which play crucial roles in [6, 40]. Suppose that we formally[6] divide the operators U, U^{-1} into the left and right piece $U_L, U_R, \left(U^{-1}\right)_L, \left(U^{-1}\right)_R$ so that the operator U, U^{-1} acts on a string field A as

$$\begin{aligned}
UA &= U_L A U_R\,, \\
U^{-1}A &= \left(U^{-1}\right)_L A \left(U^{-1}\right)_R\,.
\end{aligned}$$

$U_L, U_R, \left(U^{-1}\right)_L, \left(U^{-1}\right)_R$ may be regarded as some kind of boundary condition changing operators and the identities given in subsection 3.2 imply the OPE's of them. It would

[6]Since U, U^{-1} involve operators like U_2, U_{-2}, we are not so sure if we could do such a decomposition.

be inspiring to study the Takahashi-Tanimoto background from the point of view of these operators.

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A Maccaferri's method

In a recent paper [19], Maccaferri considered a special case of Zeze map [33], which maps an identity-based solution to a regular solution. In the case of the Takahashi-Tanimoto solution (1.1) with $a = -\frac{1}{2}$, one obtains

$$\Psi_{\mathrm{TT}} \to \Psi_{\mathrm{reg.}} \equiv \left(1 + B\frac{1-F(K)}{K}\Psi_{\mathrm{TT}}\right)(Q+\Psi_{\mathrm{TT}})\left(1 + B\frac{1-F(K)}{K}\Psi_{\mathrm{TT}}\right)^{-1} . \quad \text{(A.1)}$$

The Zeze map (A.1) is a gauge transformation and we can get a regular solution gauge equivalent to Ψ_{TT} by choosing $F(K)$ appropriately. A convenient choice is $F(K) = \frac{1}{1+k}$ and we get

$$\Psi_{\mathrm{reg.}} = \frac{1}{1+K}\Psi_{\mathrm{TT}}\frac{1}{1+K'} - Q\left(\frac{1}{1+K}\Psi_{\mathrm{TT}}\frac{1}{1+K'}\right) , \quad \text{(A.2)}$$

which appears to be a regular solution. From the expression (A.2), it is straightforward to calculate the energy and the Ellwood invariant and one obtains [19]

$$\begin{aligned} S[\Psi_{\mathrm{reg.}}] &= -\frac{1}{6g^2}\mathrm{Tr}\left[\frac{1}{1+K}c\frac{1}{1+K}Qc\right] + \frac{1}{6g^2}\mathrm{Tr}\left[\frac{1}{1+K'}c\frac{1}{1+K'}Q'c\right] , \\ \mathrm{Tr}_V\Psi_{\mathrm{reg.}} &= \mathrm{Tr}_V\left[\frac{1}{1+K}c\right] - \mathrm{Tr}_V\left[\frac{1}{1+K'}c\right] . \end{aligned} \quad \text{(A.3)}$$

The right hand sides of eq. (A.3) can be written as

$$\begin{aligned} S[\Psi_{\mathrm{reg.}}] &= S[\Psi_{\mathrm{ES}}] - S\left[\Psi'_{\mathrm{ES}}\right] , \\ \mathrm{Tr}_V\Psi_{\mathrm{reg.}} &= \mathrm{Tr}_V\Psi_{\mathrm{ES}} - \mathrm{Tr}_V\Psi'_{\mathrm{ES}} , \end{aligned}$$

where $\Psi_{\mathrm{ES}}, \Psi'_{\mathrm{ES}}$ are the Erler-Schnabl solutions given in (2.1), (2.2). Thus the observables of $\Psi_{\mathrm{reg.}}$ are obtained from those of the Erler-Schnabl solution Ψ'_{ES}. Using $S[\Psi'_{\mathrm{ES}}] = \mathrm{Tr}_V\Psi'_{\mathrm{ES}} = 0$ derived in section 2, we can see that the observables of $\Psi_{\mathrm{reg.}}$ coincide with those of the tachyon vacuum solution Ψ_{ES}.

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Singularities. Actually, the calculation of the observables above suffers from singularities discussed by Maccaferri [19]. In calculating the action, one typically encounters quantities of the form

$$\langle c(z)\, c\partial c(0)\rangle_{C_L} = -\left(\frac{L}{\pi}\right)^2 \sin^2\frac{\pi z}{L}\,, \tag{A.4}$$

where $\langle\cdot\rangle_{C_L}$ denotes the correlation function on a semi-infinite cylinder with circumference L. Eq. (A.4) diverges in the limit $\mathrm{Im}z \to \pm\infty$ for small enough $L > 0$ or in the limit $L \to 0$ with $\mathrm{Im}z \neq 0$. Since the Takahashi-Tanimoto solution (1.1) involves an integral of the ghost c up to $\mathrm{Im}z = \pm\infty$, we have trouble in calculating the action.[7]

Therefore we need to find a good regularization to calculate the action.[8] In [19], a solution with

$$F(K) = F_\epsilon(K) = \frac{e^{-\epsilon K}}{1+(1-\epsilon)K}\,, \tag{A.5}$$

$(0 \leq \epsilon \leq 1)$ in (A.1) is considered as a regularization. Let Ψ_ϵ denote the $\Psi_{\mathrm{reg.}}$ with this choice of $F(K)$. It is easy to see that

$$\Psi_\epsilon = \frac{1}{1+K_\epsilon}\left(\Psi_{\mathrm{TT}} - \Psi_{\mathrm{TT}} B_\epsilon \frac{1}{1+K'_\epsilon}\Psi_{\mathrm{TT}}\right),$$

where

$$\begin{aligned} c_\epsilon &= c\frac{KB}{G_\epsilon(K)}c\,,\\ B_\epsilon &= B\frac{G_\epsilon(K)}{K}\,,\\ K_\epsilon &= QB_\epsilon = G_\epsilon(K)\,,\\ J_\epsilon &= \{B_\epsilon, \Psi_{\mathrm{TT}}\}\,,\\ K'_\epsilon &= K_\epsilon + J_\epsilon\,, \end{aligned} \tag{A.6}$$

and

$$\frac{1}{1+K_\epsilon} = \frac{1}{1+G_\epsilon(K)} = \frac{e^{-\epsilon K}}{1+(1-\epsilon)K}\,.$$

The Ψ_ϵ consists of wedge states of width not smaller than ϵ with operator insertions and we can avoid the above-mentioned divergences taking $\epsilon > \frac{1}{2}$.

$K_\epsilon, B_\epsilon, c_\epsilon$ in (A.6) satisfy the KBc algebra [34, 41, 42] and it is straightforward to show that the observables for the solution Ψ_ϵ coincide with the shift in those of the modified Erler-Schnabl solutions [19]

$$\Psi_{\mathrm{ES},\epsilon} = \frac{1}{1+K_\epsilon}\left(c_\epsilon + Q(B_\epsilon c_\epsilon)\right), \tag{A.7}$$

$$\Psi'_{\mathrm{ES},\epsilon} = \frac{1}{1+K'_\epsilon}\left(c_\epsilon + Q'(B_\epsilon c_\epsilon)\right), \tag{A.8}$$

[7] We do not encounter such divergences in the calculation of the Ellwood invariant or the overlap of $\Psi_{\mathrm{reg.}}$ with Fock space states.

[8] In [23], the author modifies the form of the solution (1.1) as was presented in [19] and avoids the singularity.

namely

$$
\begin{aligned}
S\left[\Psi_\epsilon\right] &= S\left[\Psi_{\mathrm{ES},\epsilon}\right] - S\left[\Psi'_{\mathrm{ES},\epsilon}\right] \\
&= -\frac{1}{6g^2}\mathrm{Tr}\left[\frac{1}{1+K_\epsilon}c_\epsilon\frac{1}{1+K_\epsilon}Qc_\epsilon\right] + \frac{1}{6g^2}\mathrm{Tr}\left[\frac{1}{1+K'_\epsilon}c_\epsilon\frac{1}{1+K'_\epsilon}Q'c_\epsilon\right]. \\
\mathrm{Tr}_V\Psi_\epsilon &= \mathrm{Tr}_V\Psi_{\mathrm{ES},\epsilon} - \mathrm{Tr}_V\Psi'_{\mathrm{ES},\epsilon} \\
&= \mathrm{Tr}_V\frac{1}{1+K_\epsilon}c_\epsilon - \mathrm{Tr}_V\frac{1}{1+K'_\epsilon}c_\epsilon\,,
\end{aligned}
\tag{A.9}
$$

Now we can use (A.9) to calculate the observables. As is pointed in [19], although Ψ_ϵ itself may involve singularities for small ϵ, $\Psi'_{\mathrm{ES},\epsilon}$ is regular for all $0 \le \epsilon \le 1$. Moreover one can show

$$
\begin{aligned}
\frac{\partial}{\partial\epsilon}\Psi_{\mathrm{ES},\epsilon} &= Q\Lambda + \Psi_{\mathrm{ES},\epsilon}\Lambda - \Lambda\Psi_{\mathrm{ES},\epsilon}\,, \\
\frac{\partial}{\partial\epsilon}\Psi'_{\mathrm{ES},\epsilon} &= Q'\Lambda' + \Psi'_{\mathrm{ES},\epsilon}\Lambda' - \Lambda'\Psi'_{\mathrm{ES},\epsilon}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda &= B_\epsilon\frac{1}{1+K_\epsilon}\frac{\partial}{\partial\epsilon}\Psi_{\mathrm{ES},\epsilon}\,, \\
\Lambda' &= B_\epsilon\frac{1}{1+K'_\epsilon}\frac{\partial}{\partial\epsilon}\Psi'_{\mathrm{ES},\epsilon}\,.
\end{aligned}
$$

Since the observables $\mathrm{Tr}_V\Psi_{\mathrm{ES},\epsilon}, \mathrm{Tr}_V\Psi'_{\mathrm{ES},\epsilon}, S\left[\Psi_{\mathrm{ES},\epsilon}\right], S\left[\Psi'_{\mathrm{ES},\epsilon}\right]$ are gauge invariant quantities, they are independent of ϵ provided the gauge parameters Λ, Λ' are regular string fields. Thus we can evaluate them choosing ϵ for which the calculation is easy. The most convenient choice is $\epsilon = 0$ and we get

$$
\mathrm{Tr}_V\Psi_\epsilon = \mathrm{Tr}_V\Psi_{\mathrm{ES}} - \mathrm{Tr}_V\Psi'_{\mathrm{ES}}\,, \tag{A.10}
$$

$$
S\left[\Psi_\epsilon\right] = S\left[\Psi_{\mathrm{ES}}\right] - S\left[\Psi'_{\mathrm{ES}}\right]. \tag{A.11}
$$

From (2.13), (2.14), we can see that the observables of Ψ_ϵ coincide with those of the tachyon vacuum solution Ψ_{ES}.

Thus, by using the Maccaferri's method, it is possible to construct regular solutions gauge equivalent to Ψ_{TT}, calculate the observables of them and show that they coincide with those of the tachyon vacuum. In a sense, this gives a more direct derivation of the observables of the identity-based solutions than the one given in section 2. On the other hand, since the gauge transformation (A.1) transforms an identity-based solution into a regular solution, the transformation itself might be somewhat singular. Therefore if the observables (A.3) can be identified with those of Ψ_{TT} may be debatable.

Before closing this appendix, one comment is in order. The string field theory expanded around the Takahashi-Tanimoto solution possesses a classical solution $-\Psi_{\mathrm{TT}}$ corresponding to the perturbative vacuum. Although the solution itself is an identity-based solution, one can construct a solution gauge equivalent to it

$$
-\frac{1}{1+K'}\Psi_{\mathrm{TT}}\frac{1}{1+K} + Q'\left(\frac{1}{1+K'}\Psi_{\mathrm{TT}}\frac{1}{1+K}\right),
$$

by Maccaferri's method. The observables can be calculated at least formally and they coincide with those of the perturbative vacuum.

B Properties of U, U^{-1}

In this appendix, we derive how the operators U, U^{-1} act on the states $|0\rangle, \langle 0|, |I\rangle, \langle I|$.

Let us first prove the following identities:

$$U\,|0\rangle\ =\ \frac{1}{16}\partial bb\,(1)\,\partial bb\,(-1)\,c_0c_1\,|0\rangle\ , \tag{B.1}$$

$$U^{-1}\,|0\rangle\ =\ \frac{1}{16}\partial cc\,(1)\,\partial cc\,(-1)\,b_{-3}b_{-2}\,|0\rangle\ , \tag{B.2}$$

$$\langle 0|\,U\ =\ \langle 0|\,b_2b_3\,, \tag{B.3}$$

$$\langle 0|\,U^{-1}\ =\ \langle 0|\,c_{-1}c_0\,. \tag{B.4}$$

Since $q = 2\sum_{n=1}^{\infty}\frac{1}{n}j_{-2n}$,

$$e^{\pm q}\,|0\rangle = \exp\left[\pm 2\sum_{n=1}^{\infty}\frac{1}{n}j_{-2n}\right]|0\rangle\ .$$

On the other hand, we have the bosonization formula

$$c\,(\xi)\ =\ \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}j_{-n}\xi^{n}\right]e^{\sigma_0}e^{j_0\ln\xi}\exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}j_{n}\xi^{-n}\right]\ , \tag{B.5}$$

$$b\,(\xi)\ =\ \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}j_{-n}\xi^{n}\right]e^{-\sigma_0}e^{-j_0\ln\xi}\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}j_{n}\xi^{-n}\right]\ , \tag{B.6}$$

where σ_0 is the canonical conjugate of j_0 satisfying

$$[j_0, \sigma_0] = 1\,.$$

Eqs. (B.5), (B.6) imply

$$\partial bb\,(1)\,\partial bb\,(-1)\,c_{-2}c_{-1}c_0c_1\,|0\rangle\ =\ 16e^{-q}\,|0\rangle\ ,$$

$$\partial cc\,(1)\,\partial cc\,(-1)\,b_{-5}b_{-4}b_{-3}b_{-2}\,|0\rangle\ =\ 16e^{q}\,|0\rangle\ .$$

From these, we get

$$\begin{aligned}
U\,|0\rangle\ &=\ e^{-q}U_2\,|0\rangle \\
&=\ e^{-q}\lim_{\varepsilon\to 0}\partial bb\,(\varepsilon)\,|0\rangle \\
&=\ \lim_{\varepsilon\to 0}\left(1-\frac{1}{\varepsilon^2}\right)^{-4}\partial bb\,(\varepsilon)\,\frac{1}{16}\partial bb\,(1)\,\partial bb\,(-1)\,c_{-2}c_{-1}c_0c_1\,|0\rangle \\
&=\ \frac{1}{16}\partial bb\,(1)\,\partial bb\,(-1)\,c_0c_1\,|0\rangle\ , \\
U^{-1}\,|0\rangle\ &=\ U_{-2}e^{q}\,|0\rangle \\
&=\ U_{-2}\frac{1}{16}\partial cc\,(1)\,\partial cc\,(-1)\,b_{-5}b_{-4}b_{-3}b_{-2}\,|0\rangle \\
&=\ \frac{1}{16}\partial cc\,(1)\,\partial cc\,(-1)\,b_{-3}b_{-2}\,|0\rangle\ .
\end{aligned}$$

Eqs. (B.3), (B.4) are obtained from

$$\langle 0| e^{\pm q} = \langle 0| \ .$$

Next, we examine how U, U^{-1} act on $|I\rangle$. We will show

$$U\,|I\rangle = \frac{1}{32}\partial bb\,(1)\,|I\rangle \ , \tag{B.7}$$

$$U^{-1}\,|I\rangle = 2\partial cc\,(1)\,|I\rangle \ . \tag{B.8}$$

These are shown by using the defining relation [8, 36, 43] of $\langle I|$

$$\langle I|\,\phi\,(0)\,|0\rangle = \langle f\circ\phi\,(0)\rangle_{\mathrm{UHP}} \ , \tag{B.9}$$

where

$$f\,(\xi) = \frac{2\xi}{1-\xi^2} \,,$$

and $\langle\cdot\rangle_{\mathrm{UHP}}$ denotes the correlation function on the upper half plane. In order to derive (B.8), for example, what we should do is to calculate

$$\langle 0|\,\phi\,(0)\,U\,|I\rangle \ ,$$

and show that it is equal to $\langle 0|\,\phi\,(0)\,2\partial cc(1)|I\rangle$ for any $\phi\,(0)$. Since U only changes the ghost part of $\langle I|$, we only have to deal with the case where $\phi\,(0)$ is made from ghost operators. Therefore what we should calculate are the quantities of the form

$$\langle 0|\prod_i c\,(\xi_i)\prod_j b\,(\xi'_j)\,U\,|I\rangle \ . \tag{B.10}$$

Using eqs. (B.4), (3.13), (3.15), we obtain

$$\begin{aligned}
&\langle 0|\prod_i c\,(\xi_i)\prod_j b\,(\xi'_j)\,U^{-1}\,|I\rangle \\
&= \langle 0|\,c_{-1}c_0\prod_i\left(\frac{\left(\xi_i^2-1\right)^2}{\xi_i^2}c\,(\xi_i)\right)\prod_j\left(\frac{\xi_j'^2}{\left(\xi_j'^2-1\right)^2}b\,(\xi'_j)\right)|I\rangle \\
&= \langle I|\prod_i\left(\frac{\left(\xi_i^2-1\right)^2}{\xi_i^2}I\circ c\,(\xi_i)\right)\prod_j\left(\frac{\xi_j'^2}{\left(\xi_j'^2-1\right)^2}I\circ b\,(\xi'_j)\right)c_0c_1\,|0\rangle \\
&= \langle 0|\prod_i\left(\left(\frac{2}{f\,(\xi_i)}\right)^2 f\circ I\circ c\,(\xi_i)\right)\prod_j\left(\left(\frac{f\left(\xi'_j\right)}{2}\right)^2 f\circ I\circ b\,(\xi'_j)\right)\frac{1}{2}c_0c_1\,|0\rangle \\
&= 2\,\langle 0|\,U_{-2}\prod_i f\circ I\circ c\,(\xi_i)\prod_j f\circ I\circ b\,(\xi'_j)\,|0\rangle \\
&= 2\,\langle 0|\,c_{-1}c_0\prod_i f\circ I\circ c\,(\xi_i)\prod_j f\circ I\circ b\,(\xi'_j)\,|0\rangle \\
&= 2\,\langle 0|\,f\circ I\circ(\partial cc)\,(1)\prod_i f\circ I\circ c\,(\xi_i)\prod_j f\circ I\circ b\,(\xi'_j)\,|0\rangle \\
&= \langle 0|\prod_i c\,(\xi_i)\prod_j b\,(\xi'_j)\,2\partial cc\,(1)\,|I\rangle \ ,
\end{aligned} \tag{B.11}$$

where

$$I:\ \xi\to-\frac{1}{\xi}\,,$$

is the inversion map. Eq. (B.11) implies $U^{-1}\left|I\right\rangle=2\partial cc\left(1\right)\left|I\right\rangle$. Eq. (B.7) can be shown in the same way.

Although the state $\left|I\right\rangle$ is not included in the Fock space, the operators U,U^{-1} are inverse to one another, when they are acting on it. Indeed,

$$\begin{aligned}U\left(U^{-1}|I\rangle\right)&=2U\partial cc\left(1\right)\left|I\right\rangle\\&=2U\lim_{\xi\to1}\partial cc\left(\xi\right)\left|I\right\rangle\\&=2U\lim_{\xi\to1}\left(\frac{\left(\xi^2-1\right)^2}{\xi^2}\right)^2\partial cc\left(\xi\right)\frac{1}{32}\partial bb\left(1\right)\left|I\right\rangle\,,\\&=\left|I\right\rangle\,,\end{aligned}\tag{B.12}$$

and we can also get $U^{-1}\left(U|I\rangle\right)=|I\rangle$ in the same way.

Now let us consider the action of U,U^{-1} on $\langle I|$. In order to get $\langle I|U$, we need to calculate

$$\langle I|U\prod_i c\left(\xi_i\right)\prod_j b\left(\xi_j'\right)|0\rangle\ .\tag{B.13}$$

Using (3.14), (3.16), (B.1), it is straightforward to get

$$\begin{aligned}&U\prod_i c\left(\xi_i\right)\prod_j b\left(\xi_j'\right)|0\rangle\\&=\frac{1}{16}\prod_i\left(\frac{\left(\xi_i^2-1\right)^2}{\xi_i^2}c\left(\xi_i\right)\right)\prod_j\left(\frac{\xi_j'^2}{\left(\xi_j'^2-1\right)^2}b\left(\xi_j'\right)\right)\partial bb\left(1\right)\partial bb\left(-1\right)c_0c_1\left|0\right\rangle\ .\end{aligned}$$

Now using (B.9), we obtain

$$\begin{aligned}&\langle I|\,U\prod_i c\left(\xi_i\right)\prod_j b\left(\xi_j'\right)|0\rangle\\&\quad=\langle 0|\,f\circ\left(\partial bb\right)\left(1\right)f\circ\left(\partial bb\right)\left(-1\right)\prod_i\left(\frac{\left(\xi_i^2-1\right)^2}{\xi_i^2}f\circ c\left(\xi_i\right)\right)\\&\qquad\times\prod_j\left(\frac{\xi_j'^2}{\left(\xi_j'^2-1\right)^2}f\circ b\left(\xi_j'\right)\right)f\circ\left(\partial cc\right)\left(0\right)|0\rangle\ .\end{aligned}\tag{B.14}$$

Since

$$\begin{aligned}f\circ\left(\partial bb\right)\left(\pm1\right)&=\lim_{\varepsilon\to0}\left(\frac{\partial f}{\partial\xi}\right)^5\partial bb\left(\frac{2\xi}{1-\xi^2}\right)\Big|_{\xi=\pm1+\varepsilon}\\&=\lim_{\varepsilon\to0}\varepsilon^{-10}\partial bb\left(-\frac{1}{\varepsilon}\right)\\&\sim b_2b_3\,,\end{aligned}$$

acting on $\langle 0|$,

$$\langle I|\, U \prod_i c\,(\xi_i) \prod_j b\,(\xi'_j)\, |0\rangle = 0\,, \tag{B.15}$$

provided none of ξ_i, ξ'_j coincides with ± 1. We can also derive, for example,

$$\langle I|\, U \partial cc(\pm 1) \prod_i c\,(\xi_i) \prod_j b\,(\xi'_j)\, |0\rangle = 32\, \langle I| \prod_i c\,(\xi_i) \prod_j b\,(\xi'_j)\, |0\rangle\ , \tag{B.16}$$

if none of ξ_i, ξ'_j coincides with ± 1. Therefore we can set $\langle I|U$ to zero in the case where there are no ghost operator insertions at $\xi = \pm 1$. One can show that $\langle I|U^{-1}$ can be set to zero in such situations, in the same way. However $\langle I|U\partial cc(\pm 1)$ and $\langle I|U^{-1}\partial bb(\pm 1)$ are not zero identically. We do not know how to express $\langle I|U$ and $\langle I|U^{-1}$ with such properties in a closed form.

The vanishing of $\langle I|U$, $\langle I|U^{-1}$ in some situations does not mean that the operators U, U^{-1} are not invertible. For example, if one considers correlation function of the form

$$(\langle I|\, U)\, U^{-1} \prod_i c\,(\xi_i) \prod_j b\,(\xi'_j)\, |0\rangle\ , \tag{B.17}$$

with $\xi_i \neq \pm 1, \xi'_j \neq \pm 1$, one can see from eq. (B.2) that the operator U^{-1} induces insertions of $\partial cc(\pm 1)$:

$$\begin{aligned}
&(\langle I|\, U)\, U^{-1} \prod_i c\,(\xi_i) \prod_j b\,(\xi'_j)\, |0\rangle \\
&= \langle I|\, U \frac{1}{16} \partial cc(1) \partial cc(-1) \prod_i \left(\frac{\xi_i^2}{(\xi_i^2 - 1)^2} c(\xi_i) \right) \prod_j \left(\frac{(\xi_j'^2 - 1)^2}{\xi_j'^2} b(\xi'_j) \right) b_{-3} b_{-2} |0\rangle
\end{aligned} \tag{B.18}$$

Hence we cannot set $\langle I|\, U$ to zero but rather we obtain

$$(\langle I|\, U)\, U^{-1} \prod_i c\,(\xi_i) \prod_j b\,(\xi'_j)\, |0\rangle = \langle I| \prod_i c\,(\xi_i) \prod_j b\,(\xi'_j)\, |0\rangle\ . \tag{B.19}$$

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