Wang et al. *Fixed Point Theory and Applications* 2014, 2014:80
http://www.fixedpointtheoryandapplications.com/content/2014/1/80

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

General iterative algorithms for mixed equilibrium problems, variational inequalities and fixed point problems

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Abstract
In this paper, we introduce and analyze a general iterative algorithm for finding a common solution of a mixed equilibrium problem, a general system of variational inequalities and a fixed point problem of infinitely many nonexpansive mappings in a real Hilbert space. Under some mild conditions, we derive the strong convergence of the sequence generated by the proposed algorithm to a common solution, which also solves some optimization problem. The result presented in this paper improves and extends some corresponding ones in the earlier and recent literature.
MSC: 49J30; 47H10; 47H15

Keywords: mixed equilibrium problem; nonexpansive mapping; variational inequality; fixed point; strongly positive bounded linear operator; inverse strongly monotone mapping

1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty, closed and convex subset of H, and let $T: C \to C$ be a nonlinear mapping. Throughout this paper, we use $F(T)$ to denote the fixed point set of T. A mapping $T: C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

Let $F: C \times C \to R$ be a real-valued bifunction and $\varphi: C \to R$ be a real-valued function, where R is the set of real numbers. The so-called mixed equilibrium problem (MEP) is to find $x \in C$ such that

$$F(x,y) + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C, \tag{1.2}$$

which was considered and studied in [1, 2]. The set of solutions of MEP (1.2) is denoted by $\mathrm{MEP}(F,\varphi)$. In particular, whenever $\varphi \equiv 0$, MEP (1.2) reduces to the equilibrium problem (EP) of finding $x \in C$ such that

$$F(x,y) \ge 0, \quad \forall y \in C,$$

which was considered and studied in [3–7]. The set of solutions of the EP is denoted by $\mathrm{EP}(F)$. Given a mapping $A: C \to H$, let $F(x,y) = \langle Ax, y - x\rangle$ for all $x, y \in C$. Then $x \in \mathrm{EP}(F)$

if and only if $\langle Ax, y-x\rangle \geq 0$ for all $y \in C$. Numerous problems in physics, optimization and economics reduce to finding a solution of the EP.

Throughout this paper, assume that $F : C \times C \to R$ is a bifunction satisfying conditions (A1)-(A4) and that $\varphi : C \to R$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, *i.e.*, for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) $F(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) C is a bounded set.

The mappings $\{T_n\}_{n=1}^{\infty}$ are said to be an infinite family of nonexpansive self-mappings on C if

$$\|T_n x - T_n y\| \leq \|x - y\|, \quad \forall x, y \in C, n \geq 1, \tag{1.3}$$

and denoted by $F(T_n)$ is the fixed point set of T_n, *i.e.*, $F(T_n) := \{x \in C : T_n x = x\}$. Finding an optimal point in the intersection $\bigcap_{n=1}^{\infty} F(T_n)$ of fixed point sets of mappings T_n, $n \geq 1$, is a matter of interest in various branches of sciences.

Recently, many authors considered some iterative methods for finding a common element of the set of solutions of MEP (1.2) and the set of fixed points of nonexpansive mappings; see, *e.g.*, [2, 8, 9] and the references therein.

A mapping $A : C \to H$ is said to be

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C.$$

In such a case, A is said to be η-strongly monotone;

(iii) inverse-strongly monotone if there exists a constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \zeta\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

In such a case, A is said to be ζ-inverse-strongly monotone.

Let $A : C \to H$ be a nonlinear mapping. The classical variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

We use $\mathrm{VI}(C, A)$ to denote the set of solutions to VIP (1.4). One can easily see that VIP (1.4) is equivalent to a fixed point problem. That is, $u \in C$ is a solution of VIP (1.4) if and only if u is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant. Variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems. It is now well known that the variational inequalities are equivalent to the fixed point problems, the origin of which can be traced back to Lions and Stampacchia [10]. Not only the existence and uniqueness of solutions are important topics in the study of VIP (1.4), but also how to actually find a solution of VIP (1.4) is important. Up to now, there have been many iterative algorithms in the literature for finding approximate solutions of VIP (1.4) and its extended versions; see *e.g.*, [3, 11–14].

Recently, Plubtieng and Punpaeng [15] and Ceng *et al.* [16, 17] considered some iterative methods for VIP (1.4) and its extended versions and got some strong convergence theorems. As a generalization of VIP (1.4), the general system of variational inequalities (GSVI) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.5}$$

which μ_1 and μ_2 are two positive constants. GSVI (1.5) is considered and studied in [17], and the solution set of GSVI (1.5) is denoted by $\mathrm{GSVI}(C, B_1, B_2)$. In particular, whenever $B_1 = B_2 = A$ and $x^* = y^*$, GSVI (1.5) reduces to VIP (1.4). Ceng *et al.* [17] transformed GSVI (1.5) into a fixed point problem in the following way.

Lemma 1.1 (see [17]) *For given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of GSVI* (1.5) *if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$Gx = P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2) x, \quad \forall x \in C,$$

where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$ and P_C is the projection of H onto C.

In particular, if the mapping $B_i : C \to H$ is ζ_i-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\zeta_i)$ for $i = 1, 2$. We denote by Γ the fixed point set of the mapping G.

On the other hand, Moudafi [1] introduced the viscosity approximation method for nonexpansive mappings (see also [18] for further developments in both Hilbert spaces and Banach spaces).

A mapping $f : C \to C$ is called α-contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

Let f be a contraction on C. Starting with an arbitrary initial $x_1 \in C$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.6}$$

where T is a nonexpansive mapping of C into itself and $\{\alpha_n\}$ is a sequence in $(0,1)$. It is proved in [1, 18] that under appropriate conditions imposed on $\{\alpha_n\}$ the sequence $\{x_n\}$ generated by (1.6) strongly converges to the unique solution $x^* \in F(T)$ to the VIP

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T).$$

A linear bounded operator A is said to be $\overline{\gamma}$-strongly positive on H if there exists a constant $\overline{\gamma} \in (0,1)$ such that

$$\langle Ax, x \rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.7}$$

Recently, Marino and Xu [19] introduced the following general iterative process:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall n \geq 0, \tag{1.8}$$

where A is a strongly positive bounded linear operator on H. They proved that under appropriate conditions imposed on $\{\alpha_n\}$ the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution $x^* \in F(T)$ to the VIP

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T), \tag{1.9}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for all $x \in H$).

In 2007, Takahashi and Takahashi [5] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the EP and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Let $S : C \to H$ be a nonexpansive mapping. Starting with arbitrary initial $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, & \forall n \geq 1. \end{cases} \tag{1.10}$$

They proved that under appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in F(S) \cap \mathrm{EP}(F)$, where $x^* = P_{F(S)\cap \mathrm{EP}(F)} f(x^*)$.

Subsequently, Plubtieng and Punpaeng [15] introduced a general iterative process for finding a common element of the set of solutions of the EP and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Let $S : H \to H$ be a nonexpansive mapping. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)Su_n, & \forall n \geq 1. \end{cases} \tag{1.11}$$

They proved that under appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequence $\{x_n\}$ generated by (1.11) converges strongly to the unique solution $x^* \in F(S) \cap \mathrm{EP}(F)$ to the VIP

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad \forall x \in F(S) \cap \mathrm{EP}(F), \tag{1.12}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)\cap \mathrm{EP}(F)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for all $x \in H$).

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a mapping W_n of C into itself as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\ \ldots, \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\ \ldots, \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I. \end{cases} \tag{1.13}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Recently, Yao *et al.* [6] proved the following strong convergence result.

Theorem 1.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction satisfying conditions* (A1)-(A4). *Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{r_n\} \subset (0,\infty)$. Suppose that the following conditions are satisfied*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.

Let f be a contraction on H, and let $x_0 \in H$ be given arbitrarily. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated iteratively by

$$\begin{cases} F(y_n, x) + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0, & \forall x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \end{cases}$$

converge strongly to $x^* \in \Omega$, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{1}{2}\|x\|^2 - h(x),$$

where h is a potential function for f.

Very recently, Chen [20] proved the following strong convergence theorem.

Theorem 1.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that the common fixed point set $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $f : C \to H$ be an α-contraction, and let $A : H \to H$ be a strongly positive bounded linear operator with a constant $\overline{\gamma} > 0$. Let γ be a constant such that $0 < \gamma\alpha < \overline{\gamma}$. For an arbitrary initial point $x_0 \in C$, one defines a sequence $\{x_n\}$ iteratively*

$$x_{n+1} = P_C\big[\alpha_n \gamma f(x_n) + (1 - \alpha_n A) W_n x_n\big], \quad \forall n \geq 0, \tag{1.14}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$. Assume that the sequence $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ generated by (1.14) *converges in norm to the unique solution $x^* \in \Omega$, which solves the VIP*

$$\big\langle (A - \gamma f)x^*, x^* - x\big\rangle \leq 0, \quad \forall x \in \Omega. \tag{1.15}$$

More recently, Rattanaseeha [7] introduced an iterative algorithm:

$$\begin{cases} x_1 \in H \quad \text{arbitrarily given,} \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n A) W_n x_n], \quad \forall n \geq 1, \end{cases} \tag{1.16}$$

and proved the following strong convergence theorem.

Theorem 1.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)-(A4). *Let f be an α-contraction on H with $\alpha \in (0,1)$, and let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F) \neq \emptyset$. Let $A : H \to H$ be a $\overline{\gamma}$-strongly positive bounded linear operator with $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\lambda_1, \lambda_2, \ldots$ be a sequence of real numbers such that $0 < \lambda_n \leq b < 1$, $n = 1, 2, \ldots$. Let W_n be the W-mapping of C into itself generated by* (1.13). *Let W be defined by $Wx = \lim_{n\to\infty} W_n x$, $\forall x \in C$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by* (1.16), *where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{r_n\}$ is a sequence in $(0,\infty)$ such that the following conditions hold*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\lim_{n\to\infty} r_n = r > 0$.

Then both $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $x^* \in \Omega$, *which is the unique solution to the VIP*

$$\langle (A - \gamma f)x^*, x^* - x\rangle \le 0, \quad \forall x \in \Omega.$$

Equivalently, $x^* = P_\Omega(I - A + \gamma f)x^*$.

Let $F : C \times C \to R$ be a real-valued bifunction, $\varphi : C \to R$ be a real-valued function, A be a strongly positive bounded linear operator on H, and f be an l-Lipschitz continuous mapping on H. Motivated by the above facts, in this paper we propose and analyze a general iterative algorithm:

$$\begin{cases} x_1 \in H \quad \text{arbitrarily given}, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)W_n G u_n], \quad \forall n \ge 1, \end{cases} \tag{1.17}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$, $G = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, and W_n is the W-mapping defined by (1.13), for finding a common solution of MEP (1.2), GSVI (1.5) and the fixed point problem of an infinite family of nonexpansive self-mappings $\{T_n\}_{n=1}^\infty$ on C. It is proven that under mild conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.17) converge strongly to $x^* \in \Omega := \bigcap_{n=1}^\infty F(T_n) \cap \mathrm{MEP}(F, \varphi) \cap \Gamma$, which is the unique solution of the VIP

$$\langle (A - \gamma f)x^*, x^* - x\rangle \le 0, \quad \forall x \in \Omega, \tag{1.18}$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x \in \Omega} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for all $x \in H$). Whenever $B_1 \equiv B_2 \equiv 0$ and $\varphi \equiv 0$, our problem of finding $x^* \in \bigcap_{n=1}^\infty F(T_n) \cap \mathrm{MEP}(F, \varphi) \cap \Gamma$ reduces to the problem of finding $x^* \in \bigcap_{n=1}^\infty F(T_n) \cap \mathrm{EP}(F)$ in Theorem 1.3. The results presented in this paper improve and extend the corresponding theorems in [7].

2 Preliminaries

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, and let C be a closed convex subset of H. We indicate weak convergence and strong convergence by using the notations $\to$ and $\rightharpoonup$, respectively. A mapping $f : C \to H$ is called l-Lipschitz continuous if there exists a constant $l \ge 0$ such that

$$\|f(x) - f(y)\| \le l\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $l = 1$ then f is called a nonexpansive mapping; if $l \in [0,1)$ then f is called a contraction. Recall that a mapping $T : H \to H$ is said to be a firmly nonexpansive mapping if

$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in H.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1 *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0, \forall y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \forall y \in H$.

Consequently, P_C is a firmly nonexpansive mapping of H onto C and hence nonexpansive and monotone.

Let $A : C \to H$ be an α-inverse-strongly monotone mapping. Then it is obvious that A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \bigl\|(I - \lambda A)x - (I - \lambda A)y\bigr\|^2 &= \bigl\|x - y - \lambda(Ax - Ay)\bigr\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle Ax - Ay, x - y\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2. \end{aligned} \tag{2.1}$$

So, whenever $\lambda \leq 2\alpha$, $I - \lambda A$ is a nonexpansive mapping.

Given a positive number $r > 0$. Let $T_r^{(F,\varphi)} : H \to C$ be the solution set of the auxiliary mixed equilibrium problem, that is, for each $x \in H$,

$$T_r^{(F,\varphi)} x := \biggl\{ y \in C : F(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0, \forall z \in C \biggr\}.$$

Proposition 2.2 (see [2, 8]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction satisfying conditions* (A1)-(A4), *and let $\varphi : C \to R$ be a lower semicontinuous and convex function with restriction* (B1) *or* (B2). *Then the following hold*:

(a) *for each $x \in H$, $T_r^{(F,\varphi)} x \neq \emptyset$*;

(b) *$T_r^{(F,\varphi)}$ is single-valued*;

(c) *$T_r^{(F,\varphi)}$ is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\bigl\|T_r^{(F,\varphi)} x - T_r^{(F,\varphi)} y\bigr\|^2 \leq \bigl\langle T_r^{(F,\varphi)} x - T_r^{(F,\varphi)} y, x - y\bigr\rangle;$$

(d) *for all $s, t > 0$ and $x \in H$,*

$$\bigl\|T_s^{(F,\varphi)} x - T_t^{(F,\varphi)} x\bigr\|^2 \leq \frac{s - t}{s}\bigl\langle T_s^{(F,\varphi)} x - x, T_s^{(F,\varphi)} x - T_t^{(F,\varphi)} x\bigr\rangle;$$

(e) $F(T_r^{(F,\varphi)}) = \mathrm{MEP}(F, \varphi)$;

(f) $\mathrm{MEP}(F, \varphi)$ *is closed and convex.*

Remark 2.1 It is easy to see from conclusions (c) and (d) in Proposition 2.2 that

$$\| T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y \| \leq \|x - y\|, \quad \forall r > 0, x, y \in H$$

and

$$\| T_s^{(F,\varphi)}x - T_t^{(F,\varphi)}x \| \leq \frac{|s-t|}{s} \| T_s^{(F,\varphi)}x - x \|, \quad \forall s, t > 0, x \in H.$$

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1 *Let X be a real inner product space. Then the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Lemma 2.2 *Let H be a real Hilbert space. Then the following hold*:
(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H$;
(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ *for all* $x, y \in H$ *and* $\lambda, \mu \in [0,1]$ *with* $\lambda + \mu = 1$;
(c) *If* $\{x_n\}$ *is a sequence in H such that* $x_n \rightharpoonup x$, *it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

We have the following crucial lemmas concerning the W-mappings defined by (1.13).

Lemma 2.3 (see [21, Lemma 3.2]) *Let* $\{T_n\}_{n=1}^\infty$ *be a sequence of nonexpansive self-mappings on C such that* $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$, *and let* $\{\lambda_n\}$ *be a sequence in* $(0, b]$ *for some* $b \in (0,1)$. *Then, for every* $x \in C$ *and* $k \geq 1$, *the limit* $\lim_{n\to\infty} U_{n,k}x$ *exists, where* $U_{n,k}$ *is defined by* (1.13).

Remark 2.2 (see [6, Remark 3.1]) It can be known from Lemma 2.3 that if D is a nonempty bounded subset of C, then for $\epsilon > 0$ there exists $n_0 \geq k$ such that for all $n > n_0$,

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \leq \epsilon.$$

Remark 2.3 (see [6, Remark 3.2]) Utilizing Lemma 2.3, we define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C.$$

Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$,

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \leq \|x - y\|.$$

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 1\}$. Hence, it is clear from Remark 2.2 that for an arbitrary $\epsilon > 0$, there exists $N_0 \geq 1$ such that for all $n > N_0$,

$$\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D} \|U_{n,1}x - U_1 x\| \leq \epsilon.$$

This implies that

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0.$$

Lemma 2.4 (see [21, Lemma 3.3]) *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.5 (see [22, demiclosedness principle]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C with $F(T) \neq \emptyset$. Then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here I is the identity operator of H.*

Lemma 2.6 *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem, the characterization of the projection* (*see Proposition* 2.1(i)) *implies*

$$u \in \mathrm{VI}(C,A) \quad \Leftrightarrow \quad u = P_C(u - \lambda A u), \quad \forall \lambda > 0.$$

Lemma 2.7 (see [19]) *Let A be a $\overline{\gamma}$-strongly positive bounded linear operator on H and assume $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.8 (see [23]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \sigma_n \gamma_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $[0,1]$ and $\{\sigma_n\}$ is a real sequence such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n \gamma_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

We will introduce and analyze a general iterative algorithm for finding a common solution of MEP (1.2), GSVI (1.5) and the fixed point problem of infinitely many nonexpansive mappings in a real Hilbert space. Under appropriate conditions imposed on the parameter sequences, we will prove strong convergence of the proposed algorithm.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying conditions* (A1)-(A4), *and let $\varphi : C \to R$ be a lower semicontinuous and convex function with restriction* (B1) *or* (B2). *Let the mapping $B_i : C \to H$ be ζ_i-inverse strongly monotone for $i = 1,2$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C, and let $\{\lambda_n\}$ be a sequence in $(0,b]$ for some $b \in (0,1)$. Let A be a $\overline{\gamma}$-strongly positive bounded linear operator on H and $f : H \to H$ be an l-Lipschitz continuous mapping with $\gamma l < \overline{\gamma}$. Let W_n be the W-mapping defined by* (1.13). *Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{MEP}(F,\varphi) \cap \Gamma \neq \emptyset$, where Γ is the fixed point set of the mapping $G = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ with $\mu_i \in (0, 2\zeta_i)$ for $i = 1,2$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$ and $\{r_n\}$ be a sequence in $(0,\infty)$ such that*:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$;
(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ *and* $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$;
(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$.

Given $x_1 \in H$ *arbitrarily, the sequences* $\{x_n\}$ *and* $\{u_n\}$ *generated iteratively by*

$$\begin{cases} F(u_n,y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \ge 0, & \forall y \in C, \\ x_{n+1} = P_C[\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n G u_n], & \forall n \ge 1, \end{cases} \tag{3.1}$$

converge strongly to $x^* \in \Omega$, *which is the unique solution of the VIP*

$$\langle (A-\gamma f)x^*, x^* - x\rangle \le 0, \quad \forall x \in \Omega, \tag{3.2}$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega}\frac{1}{2}\langle Ax, x\rangle - h(x), \tag{3.3}$$

where h *is a potential function for* γf.

Proof Taking into account that $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, we may assume, without loss of generality, that $\alpha_n \le (1-\beta_n)\|A\|^{-1}$ for all $n \ge 1$. Since A is a $\overline{\gamma}$-strongly positive bounded linear operator on H, we know that $\|A\| = \sup\{\langle Au, u\rangle : u \in H, \|u\| = 1\} \ge \overline{\gamma}$,

$$\|I - A\| = \sup\{\langle (I-A)u, u\rangle : u \in H, \|u\| = 1\} \le 1 - \overline{\gamma},$$

and for $u \in H$ with $\|u\| = 1$,

$$\begin{aligned} \langle ((1-\beta_n)I - \alpha_n A)u, u\rangle &= 1 - \beta_n - \alpha_n\langle Au, u\rangle \\ &\ge 1 - \beta_n - \alpha_n\|A\| \ge 0, \end{aligned}$$

that is, $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned} \|(1-\beta_n)I - \alpha_n A\| &= \sup\{\langle ((1-\beta_n)I - \alpha_n A)u, u\rangle : u \in H, \|u\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Au, u\rangle : u \in H, \|u\| = 1\} \\ &\le 1 - \beta_n - \alpha_n\overline{\gamma}. \end{aligned}$$

We observe that $P_\Omega(\gamma f + (I-A))$ is a contraction. Indeed, for all $x, y \in H$, we have

$$\begin{aligned} &\|P_\Omega(\gamma f + (I-A))x - P_\Omega(\gamma f + (I-A))y\| \\ &\quad\le \|(\gamma f + (I-A))x - (\gamma f + (I-A))y\| \\ &\quad\le \gamma\|f(x) - f(y)\| + \|I-A\|\|x-y\| \\ &\quad\le \gamma l\|x-y\| + (1-\overline{\gamma})\|x-y\| \\ &\quad= (1 - (\overline{\gamma} - \gamma l))\|x-y\|. \end{aligned}$$

By the Banach contraction principle, we deduce that $P_\Omega(\gamma f + (I - A))$ has a unique fixed point $x^* \in H$. That is, $x^* = P_\Omega(\gamma f + (I - A))x^*$. In addition, by Proposition 2.2 we have $u_n = T_{r_n}^{(F,\varphi)} x_n$ for all $n \geq 1$.

We divide the rest of the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, take $p \in \Omega$ arbitrarily. Since $p = T_{r_n}^{(F,\varphi)} p$ and $p = Gp = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)p$, B_i is ζ_i-inverse-strongly monotone for $i = 1, 2$, by Proposition 2.2(c) we deduce from $0 \leq \mu_i \leq 2\zeta_i$, $i = 1, 2$, that for any $n \geq 1$,

$$\begin{aligned}
\|Gu_n - p\|^2 &= \big\|P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)u_n - P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)p\big\|^2 \\
&\leq \big\|(I - \mu_1 B_1)P_C(I - \mu_2 B_2)u_n - (I - \mu_1 B_1)P_C(I - \mu_2 B_2)p\big\|^2 \\
&= \big\|\big[P_C(I - \mu_2 B_2)u_n - P_C(I - \mu_2 B_2)p\big] \\
&\quad - \mu_1\big[B_1 P_C(I - \mu_2 B_2)u_n - B_1 P_C(I - \mu_2 B_2)p\big]\big\|^2 \\
&\leq \big\|P_C(I - \mu_2 B_2)u_n - P_C(I - \mu_2 B_2)p\big\|^2 \\
&\quad + \mu_1(\mu_1 - 2\zeta_1)\big\|B_1 P_C(I - \mu_2 B_2)u_n - B_1 P_C(I - \mu_2 B_2)p\big\|^2 \\
&\leq \big\|P_C(I - \mu_2 B_2)u_n - P_C(I - \mu_2 B_2)p\big\|^2 \\
&\leq \big\|(I - \mu_2 B_2)u_n - (I - \mu_2 B_2)p\big\|^2 \\
&= \big\|(u_n - p) - \mu_2(B_2 u_n - B_2 p)\big\|^2 \\
&\leq \|u_n - p\|^2 + \mu_2(\mu_2 - 2\zeta_2)\|B_2 u_n - B_2 p\|^2 \\
&\leq \|u_n - p\|^2 = \big\|T_{r_n}^{(F,\varphi)} x_n - T_{r_n}^{(F,\varphi)} p\big\|^2 \leq \|x_n - p\|^2.
\end{aligned} \tag{3.4}$$

(This shows that G is nonexpansive.) Thus, from (3.1), (3.4) and $W_n p = p$, we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \big\|P_C\big[\alpha_n \gamma f(x_n) + \beta_n x_n + \big((1 - \beta_n)I - \alpha_n A\big)W_n G u_n\big] - p\big\| \\
&\leq \big\|\alpha_n \gamma f(x_n) + \beta_n x_n + \big((1 - \beta_n)I - \alpha_n A\big)W_n G u_n - p\big\| \\
&= \big\|\alpha_n\big(\gamma f(x_n) - Ap\big) + \beta_n(x_n - p) + \big((1 - \beta_n)I - \alpha_n A\big)(W_n G u_n - p)\big\| \\
&\leq \big\|(1 - \beta_n)I - \alpha_n A\big\|\|W_n G u_n - p\| + \beta_n\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - Ap\big\| \\
&\leq (1 - \beta_n - \alpha_n\overline{\gamma})\|Gu_n - p\| + \beta_n\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - Ap\big\| \\
&\leq (1 - \alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n\big(\gamma\big\|f(x_n) - f(p)\big\| + \big\|\gamma f(p) - Ap\big\|\big) \\
&\leq (1 - \alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n\big(\gamma l\|x_n - p\| + \big\|\gamma f(p) - Ap\big\|\big) \\
&= \big[1 - (\overline{\gamma} - \gamma l)\alpha_n\big]\|x_n - p\| + \alpha_n\big\|\gamma f(p) - Ap\big\| \\
&= \big[1 - (\overline{\gamma} - \gamma l)\alpha_n\big]\|x_n - p\| + (\overline{\gamma} - \gamma l)\alpha_n\frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma l} \\
&\leq \max\left\{\|x_n - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma l}\right\}.
\end{aligned}$$

By induction, we get

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \gamma l}\right\}.$$

Therefore, $\{x_n\}$ is bounded and so are the sequences $\{u_n\}$, $\{f(x_n)\}$ and $\{W_n G u_n\}$.

Step 2. We show that $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$. Indeed, we write $y_n = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_nGu_n$. Then $x_{n+1} = P_C y_n$ for each $n \geq 1$. Define $y_n = \beta_n x_n + (1-\beta_n)w_n$ for each $n \geq 1$. Then from the definition of w_n, we obtain

$$\begin{aligned}
w_{n+1} - w_n &= \frac{y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{y_n - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)W_{n+1}Gu_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma f(x_n) + ((1-\beta_n)I - \alpha_n A)W_nGu_n}{1-\beta_n} \\
&= \gamma\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1}) - \frac{\alpha_n}{1-\beta_n}f(x_n)\right) \\
&\quad + \left(I - \frac{\alpha_{n+1}A}{1-\beta_{n+1}}\right)W_{n+1}Gu_{n+1} - \left(I - \frac{\alpha_n A}{1-\beta_n}\right)W_nGu_n \\
&= \gamma\left[\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)f(x_{n+1}) + \frac{\alpha_n}{1-\beta_n}\big(f(x_{n+1}) - f(x_n)\big)\right] \\
&\quad + \left(\left(I - \frac{\alpha_{n+1}A}{1-\beta_{n+1}}\right) - \left(I - \frac{\alpha_n A}{1-\beta_n}\right)\right)W_{n+1}Gu_{n+1} \\
&\quad + \left(I - \frac{\alpha_n A}{1-\beta_n}\right)(W_{n+1}Gu_{n+1} - W_nGu_n) \\
&= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)\big(\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\big) + \frac{\alpha_n}{1-\beta_n}\gamma\big(f(x_{n+1}) - f(x_n)\big) \\
&\quad + \left(I - \frac{\alpha_n}{1-\beta_n}A\right)(W_{n+1}Gu_{n+1} - W_nGu_n).
\end{aligned}$$

It follows from Lemma 2.7 that

$$\begin{aligned}
\|w_{n+1} - w_n\| &\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \big\|\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\big\| \\
&\quad + \frac{\alpha_n}{1-\beta_n}\gamma\big\|f(x_{n+1}) - f(x_n)\big\| + \left\|\left(I - \frac{\alpha_n}{1-\beta_n}A\right)(W_{n+1}Gu_{n+1} - W_nGu_n)\right\| \\
&\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| \big\|\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\big\| + \frac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1} - x_n\| \\
&\quad + \left\|I - \frac{\alpha_n}{1-\beta_n}A\right\| \|W_{n+1}Gu_{n+1} - W_nGu_n\| \\
&\leq \left|\frac{(\alpha_{n+1} - \alpha_n)(1-\beta_n) + \alpha_n(\beta_{n+1} - \beta_n)}{(1-\beta_n)(1-\beta_{n+1})}\right| \big\|\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\big\| \\
&\quad + \frac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1} - x_n\| \\
&\quad + \left\|I - \frac{\alpha_n}{1-\beta_n}A\right\| \big(\|W_{n+1}Gu_{n+1} - W_{n+1}Gu_n\| + \|W_{n+1}Gu_n - W_nGu_n\|\big) \\
&\leq \frac{|\alpha_{n+1} - \alpha_n| + |\beta_{n+1} - \beta_n|}{(1-d)(1-\beta_n)} \big\|\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\big\| \\
&\quad + \frac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1} - x_n\|
\end{aligned}$$

$$
\begin{aligned}
&+ \left(1 - \frac{\alpha_n}{1-\beta_n}\overline{\gamma}\right)\bigl(\|Gu_{n+1} - Gu_n\| + \|W_{n+1}Gu_n - W_nGu_n\|\bigr)\\
&\le \frac{|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n|}{(1-d)(1-\beta_n)}\bigl\|\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\bigr\|\\
&+ \frac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1}-x_n\|\\
&+ \left(1 - \frac{\alpha_n}{1-\beta_n}\overline{\gamma}\right)\bigl(\|u_{n+1} - u_n\| + \|W_{n+1}Gu_n - W_nGu_n\|\bigr).
\end{aligned} \tag{3.5}
$$

From (1.13), since W_n, T_n and $U_{n,i}$ are all nonexpansive, we have

$$
\begin{aligned}
\|W_{n+1}Gu_n - W_nGu_n\| &= \|\lambda_1 T_1 U_{n+1,2}Gu_n - \lambda_1 T_1 U_{n,2}Gu_n\|\\
&\le \lambda_1\|U_{n+1,2}Gu_n - U_{n,2}Gu_n\|\\
&= \lambda_1\|\lambda_2 T_2 U_{n+1,3}Gu_n - \lambda_2 T_2 U_{n,3}Gu_n\|\\
&\le \lambda_1\lambda_2\|U_{n+1,3}Gu_n - U_{n,3}Gu_n\|\\
&\le \cdots\\
&\le \lambda_1\lambda_2\cdots\lambda_n\|U_{n+1,n+1}Gu_n - U_{n,n+1}Gu_n\|\\
&\le M\prod_{i=1}^{n}\lambda_i,
\end{aligned} \tag{3.6}
$$

where $\sup_{n\ge 1}\{\|U_{n+1,n+1}Gu_n\| + \|U_{n,n+1}Gu_n\|\} \le M$ for some $M > 0$. Furthermore, we estimate $\|u_{n+1} - u_n\|$. Taking into account that $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ and $\liminf_{n\to\infty} r_n > 0$, we may assume, without loss of generality, that $\{\beta_n\} \subset [c,d] \subset (0,1)$ and $\{r_n\} \subset [\epsilon,\infty)$ for some $\epsilon > 0$. Utilizing Remark 2.1, we get

$$
\begin{aligned}
\|u_{n+1} - u_n\| &= \bigl\|T^{(F,\varphi)}_{r_{n+1}}x_{n+1} - T^{(F,\varphi)}_{r_n}x_n\bigr\|\\
&\le \bigl\|T^{(F,\varphi)}_{r_{n+1}}x_{n+1} - T^{(F,\varphi)}_{r_{n+1}}x_n\bigr\| + \bigl\|T^{(F,\varphi)}_{r_{n+1}}x_n - T^{(F,\varphi)}_{r_n}x_n\bigr\|\\
&\le \|x_{n+1}-x_n\| + \frac{|r_{n+1}-r_n|}{r_{n+1}}\bigl\|T^{(F,\varphi)}_{r_{n+1}}x_n - x_n\bigr\|\\
&\le \|x_{n+1}-x_n\| + \frac{|r_{n+1}-r_n|}{\epsilon}\bigl\|T^{(F,\varphi)}_{r_{n+1}}x_n - x_n\bigr\|\\
&\le \|x_{n+1}-x_n\| + M_1|r_{n+1}-r_n|,
\end{aligned} \tag{3.7}
$$

where $\sup_{n\ge 1}\{\frac{1}{\epsilon}\|T^{(F,\varphi)}_{r_{n+1}}x_n - x_n\|\} \le M_1$ for some $M_1 > 0$.

Note that

$$
y_{n+1} - y_n = \beta_n(x_{n+1}-x_n) + (\beta_{n+1}-\beta_n)(x_{n+1}-w_{n+1}) + (1-\beta_n)(w_{n+1}-w_n).
$$

Hence it follows from (3.5)-(3.7) that

$$
\begin{aligned}
&\|x_{n+2} - x_{n+1}\|\\
&\quad = \|P_C y_{n+1} - P_C y_n\| \le \|y_{n+1} - y_n\|\\
&\quad \le \beta_n\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_{n+1}-w_{n+1}\| + (1-\beta_n)\|w_{n+1}-w_n\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \beta_n\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_{n+1}-w_{n+1}\| \\
&\quad + (1-\beta_n)\left\{\frac{|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n|}{(1-d)(1-\beta_n)}\|\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\|\right. \\
&\quad \left. + \frac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1}-x_n\| + \left(1-\frac{\alpha_n}{1-\beta_n}\overline{\gamma}\right)\|u_{n+1}-u_n\| + \|W_{n+1}Gu_n - W_nGu_n\|\right\} \\
&\leq \beta_n\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_{n+1}-w_{n+1}\| \\
&\quad + (1-\beta_n)\left\{\frac{|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n|}{(1-d)(1-\beta_n)}\|\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\|\right. \\
&\quad + \frac{\alpha_n}{1-\beta_n}\gamma l\|x_{n+1}-x_n\| + \left(1-\frac{\alpha_n}{1-\beta_n}\overline{\gamma}\right)\big[\|x_{n+1}-x_n\| \\
&\quad \left. + M_1|r_{n+1}-r_n| + \|W_{n+1}Gu_n - W_nGu_n\|\big]\right\} \\
&\leq \beta_n\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_{n+1}-w_{n+1}\| \\
&\quad + (1-\beta_n)\left\{\frac{|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n|}{(1-d)(1-\beta_n)}\|\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\|\right. \\
&\quad \left. + \left(1-\frac{\alpha_n}{1-\beta_n}(\overline{\gamma}-\gamma l)\right)\|x_{n+1}-x_n\| + M_1|r_{n+1}-r_n| + \|W_{n+1}Gu_n - W_nGu_n\|\right\} \\
&\leq \beta_n\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_{n+1}-w_{n+1}\| \\
&\quad + \frac{|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n|}{1-d}\|\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\| \\
&\quad + \big(1-\beta_n-\alpha_n(\overline{\gamma}-\gamma l)\big)\|x_{n+1}-x_n\| + M_1|r_{n+1}-r_n| + \|W_{n+1}Gu_n - W_nGu_n\| \\
&\leq \big(1-\alpha_n(\overline{\gamma}-\gamma l)\big)\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_{n+1}-w_{n+1}\| \\
&\quad + \frac{|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n|}{1-d}\|\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\| \\
&\quad + M_1|r_{n+1}-r_n| + \|W_{n+1}Gu_n - W_nGu_n\| \\
&\leq \big(1-\alpha_n(\overline{\gamma}-\gamma l)\big)\|x_{n+1}-x_n\| + |\beta_{n+1}-\beta_n|\|x_{n+1}-w_{n+1}\| \\
&\quad + \frac{|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n|}{1-d}\|\gamma f(x_{n+1}) - AW_{n+1}Gu_{n+1}\| + M_1|r_{n+1}-r_n| + M\prod_{i=1}^{n}\lambda_i \\
&\leq \big(1-\alpha_n(\overline{\gamma}-\gamma l)\big)\|x_{n+1}-x_n\| + M_2\big(|\alpha_{n+1}-\alpha_n| + |\beta_{n+1}-\beta_n| + |r_{n+1}-r_n| + b^n\big),
\end{aligned}$$

where $\sup_{n\geq 1}\{\frac{1}{1-d}\|\gamma f(x_n) - AW_nGu_n\| + \|x_n - w_n\| + M_1 + M\} \leq M_2$ for some $M_2 > 0$. Since $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n+1}-\beta_n| < \infty$ and $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$, from $b \in (0,1)$ and Lemma 2.8 we conclude that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.8}$$

Step 3. We show that $\lim_{n\to\infty}\|u_n - Gu_n\| = 0$. Indeed, for simplicity, we write $\tilde{u}_n = P_C(I - \mu_2B_2)u_n$, $v_n = P_C(I-\mu_1B_1)\tilde{u}_n$ and $\tilde{p} = P_C(I-\mu_2B_2)p$. Then $v_n = Gu_n$ and

$$p = P_C(I-\mu_1B_1)\tilde{p} = P_C(I-\mu_1B_1)P_C(I-\mu_2B_2)p = Gp.$$

From (3.1), (3.4) and Proposition 2.1(i) and Lemma 2.2(b), we obtain that for $p \in \Omega$,

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \big\|\alpha_n\big(\gamma f(x_n) - AW_nGu_n\big) + \beta_n(x_n - p) + (1-\beta_n)(W_nGu_n - p)\big\|^2 \\
&\quad = \big\|\beta_n(x_n - p) + (1-\beta_n)(W_nGu_n - p)\big\|^2 \\
&\qquad + 2\alpha_n\big\langle\big(\gamma f(x_n) - AW_nGu_n\big), \beta_n(x_n - p) + (1-\beta_n)(W_nGu_n - p)\big\rangle \\
&\qquad + \alpha_n^2\big\|\gamma f(x_n) - AW_nGu_n\big\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|W_nGu_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - W_nGu_n\|^2 \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big[2\big\|\beta_n(x_n - p) + (1-\beta_n)(W_nGu_n - p)\big\| \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big] \\
&\quad \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|Gu_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - W_nGu_n\|^2 \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big[2\big(\beta_n\|x_n - p\| + (1-\beta_n)\|u_n - p\|\big) \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big] \\
&\quad \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|Gu_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - W_nGu_n\|^2 \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\quad \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\big[\|u_n - p\|^2 + \mu_2(\mu_2 - 2\zeta_2)\|B_2u_n - B_2p\|^2 \\
&\qquad + \mu_1(\mu_1 - 2\zeta_1)\|B_1\tilde{u}_n - B_1\tilde{p}\|^2\big] - \beta_n(1-\beta_n)\|x_n - W_nGu_n\|^2 \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\quad \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\big[\|x_n - p\|^2 + \mu_2(\mu_2 - 2\zeta_2)\|B_2u_n - B_2p\|^2 \\
&\qquad + \mu_1(\mu_1 - 2\zeta_1)\|B_1\tilde{u}_n - B_1\tilde{p}\|^2\big] - \beta_n(1-\beta_n)\|x_n - W_nGu_n\|^2 \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\quad = \|x_n - p\|^2 - (1-\beta_n)\big[\mu_2(2\zeta_2 - \mu_2)\|B_2u_n - B_2p\|^2 \\
&\qquad + \mu_1(2\zeta_1 - \mu_1)\|B_1\tilde{u}_n - B_1\tilde{p}\|^2\big] - \beta_n(1-\beta_n)\|x_n - W_nGu_n\|^2 \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big),
\end{aligned} \tag{3.9}$$

which immediately implies that

$$\begin{aligned}
&(1-d)\big[\mu_2(2\zeta_2 - \mu_2)\|B_2u_n - B_2p\|^2 + \mu_1(2\zeta_1 - \mu_1)\|B_1\tilde{u}_n - B_1\tilde{p}\|^2\big] \\
&\qquad + c(1-d)\|x_n - W_nGu_n\| \\
&\quad \le (1-\beta_n)\big[\mu_2(2\zeta_2 - \mu_2)\|B_2u_n - B_2p\|^2 + \mu_1(2\zeta_1 - \mu_1)\|B_1\tilde{u}_n - B_1\tilde{p}\|^2\big] \\
&\qquad + \beta_n(1-\beta_n)\|x_n - W_nGu_n\| \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\quad \le \|x_n - x_{n+1}\|\big(\|x_n - p\| + \|x_{n+1} - p\|\big) \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big).
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\mu_i \in (0, 2\zeta_i)$, $i = 1, 2$, we deduce from the boundedness of $\{x_n\}$, $\{f(x_n)\}$ and $\{W_nGu_n\}$ that

$$\begin{aligned} &\lim_{n\to\infty}\|B_2u_n - B_2p\| = 0, \qquad \lim_{n\to\infty}\|B_1\tilde{u}_n - B_1\tilde{p}\| = 0 \quad \text{and} \\ &\lim_{n\to\infty}\|x_n - W_nGu_n\| = 0. \end{aligned} \tag{3.10}$$

Also, in terms of the firm nonexpansivity of P_C and the ζ_i-inverse strong monotonicity of B_i for $i = 1, 2$, we obtain from $\mu_i \in (0, 2\zeta_i)$, $i \in \{1, 2\}$ and (3.4) that

$$\begin{aligned} \|\tilde{u}_n - \tilde{p}\|^2 &= \big\|P_C(I - \mu_2B_2)u_n - P_C(I - \mu_2B_2)p\big\|^2 \\ &\le \big\langle (I - \mu_2B_2)u_n - (I - \mu_2B_2)p, \tilde{u}_n - \tilde{p}\big\rangle \\ &= \frac{1}{2}\Big[\big\|(I - \mu_2B_2)u_n - (I - \mu_2B_2)p\big\|^2 + \|\tilde{u}_n - \tilde{p}\|^2 \\ &\quad - \big\|(I - \mu_2B_2)u_n - (I - \mu_2B_2)p - (\tilde{u}_n - \tilde{p})\big\|^2\Big] \\ &\le \frac{1}{2}\Big[\|u_n - p\|^2 + \|\tilde{u}_n - \tilde{p}\|^2 - \big\|(u_n - \tilde{u}_n) - \mu_2(B_2u_n - B_2p) - (p - \tilde{p})\big\|^2\Big] \\ &\le \frac{1}{2}\Big[\|x_n - p\|^2 + \|\tilde{u}_n - \tilde{p}\|^2 - \big\|(u_n - \tilde{u}_n) - (p - \tilde{p})\big\|^2 \\ &\quad + 2\mu_2\big\langle (u_n - \tilde{u}_n) - (p - \tilde{p}), B_2u_n - B_2p\big\rangle - \mu_2^2\|B_2u_n - B_2p\|^2\Big] \end{aligned}$$

and

$$\begin{aligned} \|v_n - p\|^2 &= \big\|P_C(I - \mu_1B_1)\tilde{u}_n - P_C(I - \mu_1B_1)\tilde{p}\big\|^2 \\ &\le \big\langle (I - \mu_1B_1)v_n - (I - \mu_1B_1)\tilde{p}, v_n - p\big\rangle \\ &= \frac{1}{2}\Big[\big\|(I - \mu_1B_1)\tilde{u}_n - (I - \mu_1B_1)\tilde{p}\big\|^2 + \|v_n - p\|^2 \\ &\quad - \big\|(I - \mu_1B_1)\tilde{u}_n - (I - \mu_1B_1)\tilde{p} - (v_n - p)\big\|^2\Big] \\ &\le \frac{1}{2}\Big[\|\tilde{u}_n - \tilde{p}\|^2 + \|v_n - p\|^2 - \big\|(\tilde{u}_n - v_n) + (p - \tilde{p})\big\|^2 \\ &\quad + 2\mu_1\big\langle B_1\tilde{u}_n - B_1\tilde{p}, (\tilde{u}_n - v_n) + (p - \tilde{p})\big\rangle - \mu_1^2\|B_1\tilde{u}_n - B_1\tilde{p}\|^2\Big] \\ &\le \frac{1}{2}\Big[\|x_n - p\|^2 + \|v_n - p\|^2 - \big\|(\tilde{u}_n - v_n) + (p - \tilde{p})\big\|^2 \\ &\quad + 2\mu_1\big\langle B_1\tilde{u}_n - B_1\tilde{p}, (\tilde{u}_n - v_n) + (p - \tilde{p})\big\rangle\Big]. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|\tilde{u}_n - \tilde{p}\|^2 &\le \|x_n - p\|^2 - \big\|(u_n - \tilde{u}_n) - (p - \tilde{p})\big\|^2 \\ &\quad + 2\mu_2\big\langle (u_n - \tilde{u}_n) - (p - \tilde{p}), B_2u_n - B_2p\big\rangle - \mu_2^2\|B_2u_n - B_2p\|^2 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \|v_n - p\|^2 &\le \|x_n - p\|^2 - \big\|(\tilde{u}_n - v_n) + (p - \tilde{p})\big\|^2 \\ &\quad + 2\mu_1\|B_1\tilde{u}_n - B_1\tilde{p}\|\,\big\|(\tilde{u}_n - v_n) + (p - \tilde{p})\big\|. \end{aligned} \tag{3.12}$$

Consequently, from (3.4), (3.9) and (3.11) it follows that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \beta_n\|x_n-p\| + (1-\beta_n)\|Gu_n-p\|^2 - \beta_n(1-\beta_n)\|x_n - W_nGu_n\| \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n-p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\quad\le \beta_n\|x_n-p\| + (1-\beta_n)\|\tilde{u}_n - \tilde{p}\|^2 \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n-p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\quad\le \beta_n\|x_n-p\| + (1-\beta_n)\big[\|x_n-p\|^2 - \big\|(u_n-\tilde{u}_n) - (p-\tilde{p})\big\|^2 \\
&\qquad + 2\mu_2\big\|(u_n-\tilde{u}_n) - (p-\tilde{p})\big\|\|B_2u_n - B_2p\|\big] \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n-p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\quad\le \|x_n-p\|^2 - (1-\beta_n)\big\|(u_n-\tilde{u}_n) - (p-\tilde{p})\big\|^2 \\
&\qquad + 2\mu_2\big\|(u_n-\tilde{u}_n) - (p-\tilde{p})\big\|\|B_2u_n - B_2p\| \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n-p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big),
\end{aligned}$$

which yields

$$\begin{aligned}
&(1-d)\big\|(u_n-\tilde{u}_n) - (p-\tilde{p})\big\|^2 \\
&\quad\le (1-\beta_n)\big\|(u_n-\tilde{u}_n) - (p-\tilde{p})\big\|^2 \\
&\quad\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2\mu_2\big\|(u_n-\tilde{u}_n) - (p-\tilde{p})\big\|\|B_2u_n - B_2p\| \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n-p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\quad\le \|x_n - x_{n+1}\|\big(\|x_n-p\| + \|x_{n+1}-p\|\big) + 2\mu_2\big\|(u_n-\tilde{u}_n) - (p-\tilde{p})\big\|\|B_2u_n - B_2p\| \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n-p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big).
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and $\lim_{n\to\infty}\|B_2u_n - B_2p\| = 0$, we deduce from the boundedness of $\{x_n\}$, $\{u_n\}$, $\{\tilde{u}_n\}$, $\{f(x_n)\}$ and $\{W_nGu_n\}$ that

$$\lim_{n\to\infty}\big\|(u_n-\tilde{u}_n) - (p-\tilde{p})\big\| = 0. \tag{3.13}$$

Furthermore, from (3.4), (3.9) and (3.12) it follows that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \beta_n\|x_n-p\| + (1-\beta_n)\|Gu_n-p\|^2 \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n-p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\quad\le \beta_n\|x_n-p\| + (1-\beta_n)\|v_n-p\|^2 \\
&\qquad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n-p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\quad\le \beta_n\|x_n-p\| + (1-\beta_n)\big[\|x_n-p\|^2 - \big\|(\tilde{u}_n-v_n) + (p-\tilde{p})\big\|^2 \\
&\qquad + 2\mu_1\|B_1\tilde{u}_n - B_1\tilde{p}\|\big\|(\tilde{u}_n-v_n) + (p-\tilde{p})\big\|\big]
\end{aligned}$$

$$
\begin{aligned}
&\quad + \alpha_n \big\| \gamma f(x_n) - AW_nGu_n \big\| \big(2\|x_n - p\| + \alpha_n \big\| \gamma f(x_n) - AW_nGu_n \big\| \big) \\
&\le \|x_n - p\|^2 - (1-\beta_n)\big\| (\tilde{u}_n - v_n) + (p - \tilde{p}) \big\|^2 + 2\mu_1 \|B_1\tilde{u}_n - B_1\tilde{p}\| \big\| (\tilde{u}_n - v_n) + (p - \tilde{p}) \big\| \\
&\quad + \alpha_n \big\| \gamma f(x_n) - AW_nGu_n \big\| \big(2\|x_n - p\| + \alpha_n \big\| \gamma f(x_n) - AW_nGu_n \big\| \big),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&(1-d)\big\| (\tilde{u}_n - v_n) + (p - \tilde{p}) \big\|^2 \\
&\quad \le (1-\beta_n)\big\| (\tilde{u}_n - v_n) + (p - \tilde{p}) \big\|^2 \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\mu_1 \|B_1\tilde{u}_n - B_1\tilde{p}\| \big\| (\tilde{u}_n - v_n) + (p - \tilde{p}) \big\| \\
&\qquad + \alpha_n \big\| \gamma f(x_n) - AW_nGu_n \big\| \big(2\|x_n - p\| + \alpha_n \big\| \gamma f(x_n) - AW_nGu_n \big\| \big) \\
&\quad \le \|x_n - x_{n+1}\| \big(\|x_n - p\| + \|x_{n+1} - p\| \big) + 2\mu_1 \|B_1\tilde{u}_n - B_1\tilde{p}\| \big\| (\tilde{u}_n - v_n) + (p - \tilde{p}) \big\| \\
&\qquad + \alpha_n \big\| \gamma f(x_n) - AW_nGu_n \big\| \big(2\|x_n - p\| + \alpha_n \big\| \gamma f(x_n) - AW_nGu_n \big\| \big).
\end{aligned}
$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|B_1\tilde{u}_n - B_1\tilde{p}\| = 0$, we deduce from the boundedness of $\{x_n\}$, $\{v_n\}$, $\{\tilde{u}_n\}$, $\{f(x_n)\}$ and $\{W_nGu_n\}$ that

$$
\lim_{n\to\infty} \big\| (\tilde{u}_n - v_n) + (p - \tilde{p}) \big\| = 0. \tag{3.14}
$$

Note that

$$
\|u_n - v_n\| \le \big\| (u_n - \tilde{u}_n) - (p - \tilde{p}) \big\| + \big\| (\tilde{u}_n - v_n) + (p - \tilde{p}) \big\|.
$$

Hence from (3.13) and (3.14) we get

$$
\lim_{n\to\infty} \|u_n - v_n\| = \lim_{n\to\infty} \|u_n - Gu_n\| = 0. \tag{3.15}
$$

Step 4. We show that $\lim_{n\to\infty} \|u_n - x_n\| = 0$ and $\lim_{n\to\infty} \|u_n - Wu_n\| = 0$. Indeed, by Proposition 2.2(c) we have

$$
\begin{aligned}
\|u_n - p\|^2 &= \big\| T_{r_n}^{(F,\varphi)} x_n - T_{r_n}^{(F,\varphi)} p \big\|^2 \\
&\le \langle u_n - p, x_n - p \rangle \\
&= \frac{1}{2}\big[\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 \big].
\end{aligned}
$$

That is,

$$
\|u_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - x_n\|^2. \tag{3.16}
$$

From (3.4), (3.9) and (3.16) it follows that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \beta_n \|x_n - p\| + (1-\beta_n)\|Gu_n - p\|^2 \\
&\quad + \alpha_n \big\| \gamma f(x_n) - AW_nGu_n \big\| \big(2\|x_n - p\| + \alpha_n \big\| \gamma f(x_n) - AW_nGu_n \big\| \big)
\end{aligned}
$$

$$\begin{aligned}
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|u_n - p\|^2 \\
&\quad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\big[\|x_n - p\|^2 - \|u_n - x_n\|^2\big] \\
&\quad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&= \|x_n - p\|^2 - (1-\beta_n)\|u_n - x_n\|^2 \\
&\quad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big).
\end{aligned}$$

So, we get

$$\begin{aligned}
(1-d)\|u_n - x_n\|^2 &\le (1-\beta_n)\|u_n - x_n\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big) \\
&\le \|x_n - x_{n+1}\|\big(\|x_n - p\| + \|x_{n+1} - p\|\big) \\
&\quad + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big(2\|x_n - p\| + \alpha_n\big\|\gamma f(x_n) - AW_nGu_n\big\|\big).
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we deduce from the boundedness of $\{x_n\}$, $\{f(x_n)\}$ and $\{W_nGu_n\}$ that

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.17}$$

In the meantime, we observe that

$$\|Gu_n - W_nGu_n\| \le \|Gu_n - u_n\| + \|u_n - x_n\| + \|x_n - W_nGu_n\|.$$

From (3.10), (3.15) and (3.17) it follows that

$$\lim_{n\to\infty}\|Gu_n - W_nGu_n\| = 0. \tag{3.18}$$

Also, note that

$$\begin{aligned}
\|u_n - Wu_n\| &\le \|u_n - Gu_n\| + \|Gu_n - W_nGu_n\| + \|W_nGu_n - W_nu_n\| + \|W_nu_n - Wu_n\| \\
&\le 2\|u_n - Gu_n\| + \|Gu_n - W_nGu_n\| + \|W_nu_n - Wu_n\|.
\end{aligned}$$

From (3.15), (3.18), Remark 2.3 and the boundedness of $\{u_n\}$ we immediately obtain

$$\lim_{n\to\infty}\|u_n - Wu_n\| = 0. \tag{3.19}$$

Step 5. We show that

$$\limsup_{n\to\infty}\big\langle(\gamma f - A)x^*, x_n - x^*\big\rangle \le 0,$$

where x^* uniquely solves the minimization problem (3.3).

Indeed, as previously, we have proven that x^* is the unique fixed point of the mapping $P_\Omega(\gamma f + (I - A))$ (*i.e.*, $x^* = P_\Omega(\gamma f + (I - A))x^*$). That is, x^* is the unique solution of VIP (3.2). Equivalently, x^* is the unique solution of the minimization problem (3.3).

First, we observe that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle(\gamma f - A)x^*, x_{n_i} - x^*\rangle. \tag{3.20}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some w. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup w$. From (3.17), we have that $u_{n_i} \rightharpoonup w$. By (3.15) and (3.19) we have that $\|Gu_n - u_n\| \to 0$ and $\|Wu_n - u_n\| \to 0$ as $n \to \infty$. Utilizing similar arguments to those of (3.4), we know that G is nonexpansive. Hence, by Lemma 2.5 we obtain $w \in \mathrm{Fix}(G) = \Gamma$ and $w \in \mathrm{Fix}(W) = \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ (due to Lemma 2.4). Next, we prove that $w \in \mathrm{MEP}(F,\varphi)$. As a matter of fact, from $u_n = T_{r_n}^{(F,\varphi)}(I - r_nA)x_n$, we know that

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, y - u_n\rangle \geq 0, \quad \forall y \in C.$$

From (A2) it follows that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle u_n - x_n, y - u_n\rangle \geq F(y, u_n), \quad \forall y \in C.$$

Replacing n by n_i, we have

$$\varphi(y) - \varphi(u_{n_i}) + \left\langle \frac{u_{n_i} - x_{n_i}}{r_{n_i}}, y - u_{n_i}\right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{3.21}$$

Put $u_t = ty + (1 - t)w$ for all $t \in (0,1]$ and $y \in C$. Then, from (3.21) we have

$$0 \geq -\varphi(u_t) + \varphi(u_{n_i}) - \left\langle \frac{u_{n_i} - x_{n_i}}{r_{n_i}}, u_t - u_{n_i}\right\rangle + F(u_t, u_{n_i}).$$

So, from (A4), the weak lower semicontinuity of φ, $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, we have

$$0 \geq -\varphi(u_t) + \varphi(w) + F(u_t, w) \quad \text{as } i \to \infty. \tag{3.22}$$

From (A1), (A4) and (3.22) we also have

$$\begin{aligned}
0 &= F(u_t, u_t) + \varphi(u_t) - \varphi(u_t) \\
&\leq tF(u_t, y) + (1 - t)F(u_t, w) + t\varphi(y) + (1 - t)\varphi(w) - \varphi(u_t) \\
&= t\big[F(u_t, y) + \varphi(y) - \varphi(u_t)\big] + (1 - t)\big[F(u_t, w) + \varphi(w) - \varphi(u_t)\big] \\
&\leq t\big[F(u_t, y) + \varphi(y) - \varphi(u_t)\big],
\end{aligned}$$

and hence

$$0 \leq F(u_t, y) + \varphi(y) - \varphi(u_t).$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \le F(w,y) + \varphi(y) - \varphi(w).$$

This implies that $w \in \mathrm{MEP}(F,\varphi)$. Therefore, $w \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{MEP}(F,\varphi) \cap \Gamma := \Omega$. Consequently, from (3.2) and (3.20) we have

$$\limsup_{n\to\infty} \langle (\gamma f - A)x^*, x_n - x^* \rangle = \lim_{i\to\infty} \langle (\gamma f - A)x^*, w - x^* \rangle \le 0. \tag{3.23}$$

Step 6. Finally, we show that $x_n \to x^* \in \Omega$ as $n \to \infty$.

Indeed, taking into account that $x_{n+1} = P_C y_n$ and $y_n = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n G u_n$, we obtain from (3.4) and Proposition 2.1(i) that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \langle P_C y_n - y_n, P_C y_n - x^* \rangle + \langle y_n - x^*, x_{n+1} - x^* \rangle \\
&\quad \le \langle y_n - x^*, x_{n+1} - x^* \rangle \\
&\quad = \langle \alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(W_n G u_n - x^*), x_{n+1} - x^* \rangle \\
&\quad = \langle \alpha_n \gamma (f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(W_n G u_n - x^*), x_{n+1} - x^* \rangle \\
&\qquad + \alpha_n \langle (\gamma f - A)x^*, x_{n+1} - x^* \rangle \\
&\quad \le \| \alpha_n \gamma (f(x_n) - f(x^*)) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(W_n G u_n - x^*) \| \|x_{n+1} - x^*\| \\
&\qquad + \alpha_n \langle (\gamma f - A)x^*, x_{n+1} - x^* \rangle \\
&\quad \le [\alpha_n \gamma \|f(x_n) - f(x^*)\| + \beta_n \|x_n - x^*\| + \|(1-\beta_n)I - \alpha_n A\| \|W_n G u_n - x^*\|] \\
&\qquad \times \|x_{n+1} - x^*\| + \alpha_n \langle (\gamma f - A)x^*, x_{n+1} - x^* \rangle \\
&\quad \le [\alpha_n \gamma l \|x_n - x^*\| + \beta_n \|x_n - x^*\| + (1 - \beta_n - \alpha_n \overline{\gamma}) \|u_n - x^*\|] \|x_{n+1} - x^*\| \\
&\qquad + \alpha_n \langle (\gamma f - A)x^*, x_{n+1} - x^* \rangle \\
&\quad \le [\alpha_n \gamma l \|x_n - x^*\| + \beta_n \|x_n - x^*\| + (1 - \beta_n - \alpha_n \overline{\gamma}) \|x_n - x^*\|] \|x_{n+1} - x^*\| \\
&\qquad + \alpha_n \langle (\gamma f - A)x^*, x_{n+1} - x^* \rangle \\
&\quad = (1 - \alpha_n(\overline{\gamma} - \gamma l)) \|x_n - x^*\| \|x_{n+1} - x^*\| + \alpha_n \langle (\gamma f - A)x^*, x_{n+1} - x^* \rangle \\
&\quad \le \frac{1}{2}(1 - \alpha_n(\overline{\gamma} - \gamma l))(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \alpha_n \langle (\gamma f - A)x^*, x_{n+1} - x^* \rangle,
\end{aligned}$$

which immediately implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{1 - \alpha_n(\overline{\gamma} - \gamma l)}{1 + \alpha_n(\overline{\gamma} - \gamma l)} \|x_n - x^*\|^2 + \frac{\alpha_n}{1 + \alpha_n(\overline{\gamma} - \gamma l)} \langle (\gamma f - A)x^*, x_{n+1} - x^* \rangle \\
&= \left(1 - \frac{2\alpha_n(\overline{\gamma} - \gamma l)}{1 + \alpha_n(\overline{\gamma} - \gamma l)}\right) \|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n(\overline{\gamma} - \gamma l)}{1 + \alpha_n(\overline{\gamma} - \gamma l)} \cdot \frac{1}{2(\overline{\gamma} - \gamma l)} \langle (\gamma f - A)x^*, x_{n+1} - x^* \rangle \\
&= (1 - \gamma_n) \|x_n - x^*\|^2 + \sigma_n \gamma_n,
\end{aligned} \tag{3.24}$$

where $\gamma_n = \frac{2\alpha_n(\overline{\gamma}-\gamma l)}{1+\alpha_n(\overline{\gamma}-\gamma l)}$ and $\sigma_n = \frac{1}{2(\overline{\gamma}-\gamma l)}\langle(\gamma f - A)x^*, x_{n+1} - x^*\rangle$. Note that $\sum_{n=1}^{\infty}\alpha_n = \infty$ implies $\sum_{n=1}^{\infty}\gamma_n \geq \frac{2(\overline{\gamma}-\gamma l)}{1+(\overline{\gamma}-\gamma l)} \cdot \sum_{n=1}^{\infty}\alpha_n = \infty$ and that (3.23) leads to

$$\limsup_{n\to\infty}\sigma_n = \limsup_{n\to\infty}\frac{1}{2(\overline{\gamma}-\gamma l)}\big\langle(\gamma f - A)x^*, x_{n+1} - x^*\big\rangle \leq 0.$$

Applying Lemma 2.8 to (3.24), we infer that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Putting $T_n \equiv I$ the identity mapping, and $l = \alpha \in (0,1)$ in Theorem 3.1, we have the following result.

Corollary 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying conditions* (A1)-(A4), *and let $\varphi : C \to R$ be a lower semicontinuous and convex function with restriction* (B1) *or* (B2). *Let the mapping $B_i : C \to H$ be ζ_i-inverse strongly monotone for $i = 1,2$. Let A be a $\overline{\gamma}$-strongly positive bounded linear operator on H and $f : H \to H$ be an α-contraction with $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that $\Omega :=$* MEP$(F,\varphi) \cap \Gamma \neq \emptyset$, *where Γ is the fixed point set of the mapping $G = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ with $\mu_i \in (0, 2\zeta_i)$ for $i = 1,2$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in* $(0,1)$ *and $\{r_n\}$ be a sequence in* $(0,\infty)$ *such that*:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ *and* $\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$;
(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$.

Given $x_1 \in H$ arbitrarily, the sequences $\{x_n\}$ and $\{u_n\}$ generated iteratively by

$$\begin{cases} F(u_n,y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = P_C[\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)Gu_n], & \forall n \geq 1, \end{cases} \tag{3.25}$$

converge strongly to $x^* \in \Omega$, which is the unique solution of the VIP

$$\big\langle(A - \gamma f)x^*, x^* - x\big\rangle \leq 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega}\frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf.

Putting $\varphi \equiv 0$ and $l = \alpha \in (0,1)$ in Theorem 3.1, we have the following result.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying conditions* (A1)-(A4). *Let the mapping $B_i : C \to H$ be ζ_i-inverse strongly monotone for $i = 1,2$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in* $(0,b]$ *for some* $b \in (0,1)$. *Let A be a $\overline{\gamma}$-strongly positive bounded linear operator on H and $f : H \to H$ be an α-contraction with $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let W_n be the W-mapping defined by* (1.13). *Assume that* $\Omega := \bigcap_{n=1}^{\infty}F(T_n) \cap$ EP$(F) \cap \Gamma \neq \emptyset$, *where Γ is the fixed point set of the mapping $G = P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$ with $\mu_i \in (0, 2\zeta_i)$*

for $i = 1, 2$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0, 1)$ and $\{r_n\}$ be a sequence in $(0, \infty)$ such that:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Given $x_1 \in H$ arbitrarily, the sequences $\{x_n\}$ and $\{u_n\}$ generated iteratively by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)W_n G u_n], & \forall n \geq 1, \end{cases} \tag{3.26}$$

converge strongly to $x^* \in \Omega$, which is the unique solution of the VIP

$$\langle (A - \gamma f)x^*, x^* - x\rangle \leq 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf.

Putting $B_1 \equiv B_2 \equiv 0$ in Corollary 3.1, we have the following result.

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying conditions* (A1)-(A4). *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Let A be a $\overline{\gamma}$-strongly positive bounded linear operator on H and $f : H \to H$ be an α-contraction with $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let W_n be the W-mapping defined by* (1.13). *Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap \mathrm{EP}(F) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0, 1)$ and $\{r_n\}$ be a sequence in $(0, \infty)$ such that*:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Given $x_1 \in H$ arbitrarily, the sequences $\{x_n\}$ and $\{u_n\}$ generated iteratively by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)W_n u_n], & \forall n \geq 1, \end{cases} \tag{3.27}$$

converge strongly to $x^* \in \Omega$, which is the unique solution of the VIP

$$\langle (A - \gamma f)x^*, x^* - x\rangle \leq 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgements
All the authors were partially supported by the National Science Foundation of China (11071169) and PhD Program Foundation of the Ministry of Education of China (20123127110002).

Received: 15 October 2013 Accepted: 11 March 2014 Published: 26 Mar 2014

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10.1186/1687-1812-2014-80
Cite this article as: Wang et al.: **General iterative algorithms for mixed equilibrium problems, variational inequalities and fixed point problems.** *Fixed Point Theory and Applications* 2014, 2014:80