Park *Journal of Inequalities and Applications* 2013, 2013:447
http://www.journalofinequalitiesandapplications.com/content/2013/1/447

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Additive functional inequalities in 2-Banach spaces

Choonkil Park*

*Correspondence:
baak@hanyang.ac.kr
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul, 133-791, Korea

Abstract

We prove the Hyers-Ulam stability of the Cauchy functional inequality and the Cauchy-Jensen functional inequality in 2-Banach spaces.

Moreover, we prove the superstability of the Cauchy functional inequality and the Cauchy-Jensen functional inequality in 2-Banach spaces under some conditions.

MSC: 39B82; 39B52; 39B62; 46B99; 46A19

Keywords: Hyers-Ulam stability; linear 2-normed space; additive mapping; additive functional inequality; superstability

1 Introduction and preliminaries

In 1940, Ulam [1] suggested the stability problem of functional equations concerning the stability of group homomorphisms as follows: *Let* $(\mathcal{G}, \circ)$ *be a group and let* $(\mathcal{H}, \star, d)$ *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\varepsilon > 0$, *does there exist a* $\delta = \delta(\varepsilon) > 0$ *such that if a mapping* $f : \mathcal{G} \to \mathcal{H}$ *satisfies the inequality*

$$d\big(f(x \circ y), f(x) \star f(y)\big) < \delta$$

for all $x, y \in \mathcal{G}$, *then a homomorphism* $F : \mathcal{G} \to \mathcal{H}$ *exists with*

$$d\big(f(x), F(x)\big) < \varepsilon$$

for all $x \in \mathcal{G}$?

In 1941, Hyers [2] gave a first (partial) affirmative answer to the question of Ulam for Banach spaces. Thereafter, we call that type the Hyers-Ulam stability.

Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function.

Gähler [6, 7] introduced the concept of linear 2-normed spaces.

Definition 1.1 Let $\mathcal{X}$ be a real linear space with $\dim \mathcal{X} > 1$, and let $\|\cdot,\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ be a function satisfying the following properties:

(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

(b) $\|x, y\| = \|y, x\|$,

(c) $\|\alpha x, y\| = |\alpha| \|x, y\|$,
(d) $\|x, y + z\| \le \|x, y\| + \|x, z\|$

for all $x, y, z \in \mathcal{X}$ and $\alpha \in \mathbb{R}$. Then the function $\|\cdot, \cdot\|$ is called 2-*norm* on $\mathcal{X}$ and the pair $(\mathcal{X}, \|\cdot, \cdot\|)$ is called a *linear* 2-*normed space*. Sometimes condition (d) is called the *triangle inequality*.

See [8] for examples and properties of linear 2-normed spaces.

White [9, 10] introduced the concept of 2-Banach spaces. In order to define completeness, the concepts of Cauchy sequences and convergence are required.

Definition 1.2 A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *Cauchy sequence* if

$$\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$$

for all $y \in \mathcal{X}$.

Definition 1.3 A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *convergent sequence* if there is an $x \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in \mathcal{X}$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n \to \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.

The triangle inequality implies the following lemma.

Lemma 1.4 [11] *For a convergent sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$,*

$$\lim_{n\to\infty} \|x_n, y\| = \left\| \lim_{n\to\infty} x_n, y \right\|$$

for all $y \in \mathcal{X}$.

Definition 1.5 A linear 2-normed space, in which every Cauchy sequence is a convergent sequence, is called a 2-*Banach space*.

Eskandani and Găvruta [12] proved the Hyers-Ulam stability of a functional equation in 2-Banach spaces.

In [13], Gilányi showed that if f satisfies the functional inequality

$$\left\|2f(x) + 2f(y) - f\left(xy^{-1}\right)\right\| \le \left\|f(xy)\right\|, \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f\left(xy^{-1}\right).$$

See also [14]. Gilányi [15] and Fechner [16] proved the Hyers-Ulam stability of functional inequality (1.1).

Park *et al.* [17] proved the Hyers-Ulam stability of the following functional inequalities:

$$\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\|, \tag{1.2}$$

$$\|f(x)+f(y)+2f(z)\| \le \left\|2f\left(\frac{x+y}{2}+z\right)\right\|. \tag{1.3}$$

In this paper, we prove the Hyers-Ulam stability of Cauchy functional inequality (1.2) and Cauchy-Jensen functional inequality (1.3) in 2-Banach spaces.

Moreover, we prove the superstability of Cauchy functional inequality (1.2) and Cauchy-Jensen functional inequality (1.3) in 2-Banach spaces under some conditions.

Throughout this paper, let $\mathcal{X}$ be a normed linear space, and let $\mathcal{Y}$ be a 2-Banach space.

2 Hyers-Ulam stability of Cauchy functional inequality (1.2) in 2-Banach spaces

In this section, we prove the Hyers-Ulam stability of Cauchy functional inequality (1.2) in 2-Banach spaces.

Proposition 2.1 *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x)+f(y)+f(z), w\| \le \|f(x+y+z), w\| \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Then the mapping $f : \mathcal{X} \to \mathcal{Y}$ is additive.

Proof Letting $x = y = z = 0$ in (2.1), we get $3\|f(0), w\| \le \|f(0), w\|$ and so $\|f(0), w\| = 0$ for all $w \in \mathcal{Y}$. Hence $f(0) = 0$.

Letting $y = -x$ and $z = 0$ in (2.1), we get $\|f(x)+f(-x), w\| \le \|f(0), w\| = 0$ and so $\|f(x) + f(-x), w\| = 0$ for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Hence $f(x) + f(-x) = 0$ for all $x \in \mathcal{X}$.

Letting $z = -x - y$ in (2.1), we get

$$\|f(x)+f(y)+f(-x-y), w\| \le \|f(0), w\| = 0$$

and so

$$\|f(x)+f(y)+f(-x-y), w\| = 0$$

for all $x, y \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Hence

$$0 = f(x)+f(y)+f(-x-y) = f(x)+f(y)-f(x+y)$$

for all $x, y \in \mathcal{X}$. So, $f : \mathcal{X} \to \mathcal{Y}$ is additive. □

Theorem 2.2 *Let $\theta \in [0,\infty)$, $p, q, r \in (0,\infty)$ with $p+q+r < 1$, and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x)+f(y)+f(z), w\| \le \|f(x+y+z), w\| + \theta\|x\|^p\|y\|^q\|z\|^r\|w\| \tag{2.2}$$

for all $x, y, z \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Then there is a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\|f(x) - A(x), w\right\| \le \frac{2^r\theta}{2 - 2^{p+q+r}} \|x\|^{p+q+r} \|w\| \tag{2.3}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$.

Proof Letting $x = y = z = 0$ in (2.2), we get $3\|f(0), w\| \le \|f(0), w\|$ and so $\|f(0), w\| = 0$ for all $w \in \mathcal{Y}$. Hence $f(0) = 0$.

Letting $y = -x$ and $z = 0$ in (2.2), we get $\|f(x) + f(-x), w\| \le \|f(0), w\| = 0$ and so $\|f(x) + f(-x), w\| = 0$ for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Hence $f(x) + f(-x) = 0$ for all $x \in \mathcal{X}$.

Putting $y = x$ and $z = -2x$ in (2.2), we get

$$\left\|f(2x) - 2f(x), w\right\| \le \left\|f(0), w\right\| + 2^r\theta \|x\|^{p+q+r} \|w\| = 2^r\theta \|x\|^{p+q+r} \|w\| \tag{2.4}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. So, we get

$$\left\|f(x) - \frac{1}{2}f(2x), w\right\| \le \frac{2^r\theta}{2} \|x\|^{p+q+r} \|w\| \tag{2.5}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Replacing x by $2^j x$ in (2.5) and dividing by 2^j, we obtain

$$\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right), w\right\| \le 2^{(p+q+r-1)j+r-1}\theta \|x\|^{p+q+r} \|w\|$$

for all $x \in \mathcal{X}$, all $w \in \mathcal{Y}$ and all integers $j \ge 0$. For all integers l, m with $0 \le l < m$, we get

$$\left\|\frac{1}{2^l}f\left(2^l x\right) - \frac{1}{2^m}f\left(2^m x\right), w\right\| \le \sum_{j=l}^{m-1} 2^{(p+q+r-1)j+r-1}\theta \|x\|^{p+q+r} \|w\| \tag{2.6}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. So, we get

$$\lim_{l \to \infty} \left\|\frac{1}{2^l}f\left(2^l x\right) - \frac{1}{2^m}f\left(2^m x\right), w\right\| = 0$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Thus the sequence $\{\frac{1}{2^j}f(2^j x)\}$ is a Cauchy sequence in $\mathcal{Y}$ for each $x \in \mathcal{X}$. Since $\mathcal{Y}$ is a 2-Banach space, the sequence $\{\frac{1}{2^j}f(2^j x)\}$ converges for each $x \in \mathcal{X}$. So, one can define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{j \to \infty} \frac{1}{2^j}f\left(2^j x\right)$$

for all $x \in \mathcal{X}$. That is,

$$\lim_{j \to \infty} \left\|\frac{1}{2^j}f\left(2^j x\right) - A(x), w\right\| = 0$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$.

By (2.2), we get

$$\begin{aligned}\lim_{j\to\infty}&\left\|\frac{1}{2^j}\left(f\left(2^jx\right)+f\left(2^jy\right)+f\left(2^jz\right)\right),w\right\|\\&\le\lim_{j\to\infty}\left(\frac{1}{2^j}\left\|f\left(2^jx+2^jy+2^jz\right),w\right\|+\frac{2^{(p+q+r)j}}{2^j}\theta\|x\|^p\|y\|^q\|z\|^r\|w\|\right)\\&\le\lim_{j\to\infty}\frac{1}{2^j}\left\|f\left(2^jx+2^jy+2^jz\right),w\right\|\end{aligned}$$

for all $x,y,z\in\mathcal{X}$ and all $w\in\mathcal{Y}$. So,

$$\left\|A(x)+A(y)+A(z),w\right\|\le\left\|A(x+y+z),w\right\|$$

for all $x,y,z\in\mathcal{X}$ and all $w\in\mathcal{Y}$. By Proposition 2.1, $A:\mathcal{X}\to\mathcal{Y}$ is additive.

By Lemma 1.4 and (2.6), we have

$$\left\|f(x)-A(x),w\right\|=\lim_{m\to\infty}\left\|f(x)-\frac{1}{2^m}f\left(2^mx\right),w\right\|\le\frac{2^r\theta}{2-2^{p+q+r}}\|x\|^{p+q+r}\|w\|$$

for all $x\in\mathcal{X}$ and all $w\in\mathcal{Y}$.

Now, let $B:\mathcal{X}\to\mathcal{Y}$ be another additive mapping satisfying (2.3). Then we have

$$\begin{aligned}\left\|A(x)-B(x),w\right\|&=\frac{1}{2^j}\left\|A\left(2^jx\right)-B\left(2^jx\right),w\right\|\\&\le\frac{1}{2^j}\left[\left\|A\left(2^jx\right)-f\left(2^jx\right),w\right\|+\left\|f\left(2^jx\right)-B\left(2^jx\right),w\right\|\right]\\&\le\frac{2\cdot2^r\theta}{2-2^{p+q+r}}\|x\|^{p+q+r}\|w\|\cdot\frac{2^{(p+q+r)j}}{2^j},\end{aligned}$$

which tends to zero as $j\to\infty$ for all $x\in\mathcal{X}$ and all $w\in\mathcal{Y}$. By Definition 1.1, we can conclude that $A(x)=B(x)$ for all $x\in\mathcal{X}$. This proves the uniqueness of A. □

Theorem 2.3 *Let $\theta\in[0,\infty)$, $p,q,r\in(0,\infty)$ with $p+q+r>1$, and let $f:\mathcal{X}\to\mathcal{Y}$ be a mapping satisfying* (2.2). *Then there is a unique additive mapping $A:\mathcal{X}\to\mathcal{Y}$ such that*

$$\left\|f(x)-A(x),w\right\|\le\frac{2^r\theta}{2^{p+q+r}-2}\|x\|^{p+q+r}\|w\|$$

for all $x\in\mathcal{X}$ and all $w\in\mathcal{Y}$.

Proof It follows from (2.4) that

$$\left\|f(x)-2f\left(\frac{x}{2}\right),w\right\|\le\frac{\theta}{2^{p+q}}\|x\|^{p+q+r}\|w\| \tag{2.7}$$

for all $x\in\mathcal{X}$ and all $w\in\mathcal{Y}$. Replacing x by $\frac{x}{2^j}$ in (2.7) and multiplying by 2^j, we obtain

$$\left\|2^jf\left(\frac{x}{2^j}\right)-2^{j+1}f\left(\frac{x}{2^{j+1}}\right),w\right\|\le\frac{2^j\theta}{2^{p+q}\cdot2^{(p+q+r)j}}\|x\|^{p+q+r}\|w\|$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$ and all integers $j \geq 0$. For all integers l, m with $0 \leq l < m$, we get

$$\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), w \right\| \leq \sum_{j=l}^{m-1} \frac{2^j \theta}{2^{p+q} \cdot 2^{(p+q+r)j}} \|x\|^{p+q+r} \|w\|$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. So, we get

$$\lim_{l \to \infty} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), w \right\| = 0$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Thus the sequence $\{2^j f(\frac{x}{2^j})\}$ is a Cauchy sequence in $\mathcal{Y}$. Since $\mathcal{Y}$ is a 2-Banach space, the sequence $\{2^j f(\frac{x}{2^j})\}$ converges. So, one can define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{j \to \infty} 2^j f\left(\frac{x}{2^j}\right)$$

for all $x \in \mathcal{X}$. That is,

$$\lim_{j \to \infty} \left\| 2^j f\left(\frac{x}{2^j}\right) - A(x), w \right\| = 0$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$.

The further part of the proof is similar to the proof of Theorem 2.2. □

Now we prove the superstability of the Cauchy functional inequality in 2-Banach spaces.

Theorem 2.4 *Let $\theta \in [0, \infty)$, $p, q, r, t \in (0, \infty)$ with $t \neq 1$, and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x) + f(y) + f(z), w\| \leq \|f(x + y + z), w\| + \theta \|x\|^p \|y\|^q \|z\|^r \|w\|^t \tag{2.8}$$

for all $x, y, z \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Then $f : \mathcal{X} \to \mathcal{Y}$ is an additive mapping.

Proof Replacing w by sw in (2.8) for $s \in \mathbb{R} \setminus \{0\}$, we get

$$\|f(x) + f(y) + f(z), sw\| \leq \|f(x + y + z), sw\| + \theta \|x\|^p \|y\|^q \|z\|^r \|sw\|^t$$

and so

$$\|f(x) + f(y) + f(z), w\| \leq \|f(x + y + z), w\| + \theta \|x\|^p \|y\|^q \|z\|^r \|w\|^t \frac{|s|^t}{|s|} \tag{2.9}$$

for all $x, y, z \in \mathcal{X}$, all $w \in \mathcal{Y}$ and all $s \in \mathbb{R} \setminus \{0\}$.

If $t > 1$, then the right-hand side of (2.9) tends to $\|f(x + y + z), w\|$ as $s \to 0$.

If $t < 1$, then the right-hand side of (2.9) tends to $\|f(x + y + z), w\|$ as $s \to +\infty$.

Thus

$$\|f(x) + f(y) + f(z), w\| \leq \|f(x + y + z), w\|$$

for all $x, y, z \in \mathcal{X}$ and all $w \in \mathcal{Y}$. By Proposition 2.1, $f : \mathcal{X} \to \mathcal{Y}$ is additive. □

3 Hyers-Ulam stability of Cauchy-Jensen functional inequality (1.3) in 2-Banach spaces

In this section, we prove the Hyers-Ulam stability of Cauchy-Jensen functional inequality (1.3) in 2-Banach spaces.

Proposition 3.1 *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x) + f(y) + 2f(z), w\| \leq \left\|2f\left(\frac{x+y}{2} + z\right), w\right\| \tag{3.1}$$

for all $x, y, z \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Then the mapping $f : \mathcal{X} \to \mathcal{Y}$ is additive.

Proof Letting $x = y = z = 0$ in (3.1), we get $4\|f(0), w\| \leq 2\|f(0), w\|$ and so $\|f(0), w\| = 0$ for all $w \in \mathcal{Y}$. Hence $f(0) = 0$.

Letting $y = -x$ and $z = 0$ in (3.1), we get $\|f(x) + f(-x), w\| \leq 2\|f(0), w\| = 0$ and so $\|f(x) + f(-x), w\| = 0$ for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Hence $f(x) + f(-x) = 0$ for all $x \in \mathcal{X}$.

Letting $z = -\frac{x+y}{2}$ in (3.1), we get

$$\left\|f(x) + f(y) + 2f\left(-\frac{x+y}{2}\right), w\right\| \leq 2\|f(0), w\| = 0$$

and so

$$\left\|f(x) + f(y) + 2f\left(-\frac{x+y}{2}\right), w\right\| = 0$$

for all $x, y \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Hence

$$0 = f(x) + f(y) + 2f\left(-\frac{x+y}{2}\right) = f(x) + f(y) - 2f\left(\frac{x+y}{2}\right)$$

for all $x, y \in \mathcal{X}$. Since $f(0) = 0$, $f : \mathcal{X} \to \mathcal{Y}$ is additive. □

Theorem 3.2 *Let $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ with $p + q + r < 1$, and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x) + f(y) + 2f(z), w\| \leq \left\|2f\left(\frac{x+y}{2} + z\right), w\right\| + \theta\|x\|^p\|y\|^q\|z\|^r\|w\| \tag{3.2}$$

for all $x, y, z \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Then there is a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x), w\| \leq \frac{\theta}{2 - 2^{p+q+r}}\|x\|^{p+q+r}\|w\|$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$.

Proof Letting $x = y = z = 0$ in (3.2), we get $4\|f(0), w\| \leq 2\|f(0), w\|$ and so $\|f(0), w\| = 0$ for all $w \in \mathcal{Y}$. Hence $f(0) = 0$.

Letting $y = -x$ and $z = 0$ in (3.2), we get $\|f(x) + f(-x), w\| \leq 2\|f(0), w\| = 0$ and so $\|f(x) + f(-x), w\| = 0$ for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Hence $f(x) + f(-x) = 0$ for all $x \in \mathcal{X}$.

Letting $y = x$ and $z = -x$ in (3.2), we get

$$\|2f(x) - f(2x), w\| \le \|2f(0), w\| + \theta \|x\|^{p+q+r} \|w\| = \theta \|x\|^{p+q+r} \|w\| \tag{3.3}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Replacing x by $2^j x$ in (3.3) and dividing by 3^j, we obtain

$$\left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x), w \right\| \le \frac{2^{(p+q+r)j}}{2 \cdot 2^j} \theta \|x\|^{p+q+r} \|w\|$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$ and all integers $j \ge 0$. For all integers l, m with $0 \le l < m$, we get

$$\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x), w \right\| \le \sum_{j=l}^{m-1} \frac{2^{(p+q+r)j}}{2 \cdot 2^j} \theta \|x\|^{p+q+r} \|w\|$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. So, we get

$$\lim_{l \to \infty} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x), w \right\| = 0$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Thus the sequence $\{\frac{1}{2^j} f(2^j x)\}$ is a Cauchy sequence in $\mathcal{Y}$ for each $x \in \mathcal{X}$. Since $\mathcal{Y}$ is a 2-Banach space, the sequence $\{\frac{1}{2^j} f(2^j x)\}$ converges for each $x \in \mathcal{X}$. So, one can define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{j \to \infty} \frac{1}{2^j} f(2^j x) = \lim_{j \to \infty} \frac{1}{2^j} f(2^j x)$$

for all $x \in \mathcal{X}$. That is,

$$\lim_{j \to \infty} \left\| \frac{1}{2^j} f(2^j x) - A(x), w \right\| = \lim_{j \to \infty} \left\| \frac{1}{2^j} f(2^j x) - A(x), w \right\| = 0$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$.

The further part of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3 *Let* $\theta \in [0, \infty)$, $p, q, r \in (0, \infty)$ *with* $p + q + r > 1$, *and let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping satisfying* (3.2). *Then there is a unique additive mapping* $A : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x) - A(x), w\| \le \frac{\theta}{2^{p+q+r} - 2} \|x\|^{p+q+r} \|w\|$$

for all $x \in \mathcal{X}$ *and all* $w \in \mathcal{Y}$.

Proof It follows from (3.3) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right), w \right\| \le \frac{1}{2^{p+q+r}} \theta \|x\|^{p+q+r} \|w\| \tag{3.4}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Replacing x by $\frac{x}{2^j}$ in (3.4) and multiplying by 2^j, we obtain

$$\left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right), w \right\| \le \frac{2^j}{2^{(p+q+r)(j+1)}} \theta \|x\|^{p+q+r} \|w\|$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$ and all integers $j \geq 0$. For all integers l, m with $0 \leq l < m$, we get

$$\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), w \right\| \leq \sum_{j=l}^{m-1} \frac{2^j}{2^{(p+q+r)(j+1)}} \theta \|x\|^{p+q+r} \|w\|$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. So, we get

$$\lim_{l\to\infty} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right), w \right\| = 0$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Thus the sequence $\{2^j f(\frac{x}{2^j})\}$ is a Cauchy sequence in $\mathcal{Y}$ for each $x \in \mathcal{X}$. Since $\mathcal{Y}$ is a 2-Banach space, the sequence $\{2^j f(\frac{x}{2^j})\}$ converges for each $x \in \mathcal{X}$. So, one can define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{j\to\infty} 2^j f\left(\frac{x}{2^j}\right)$$

for all $x \in \mathcal{X}$. That is,

$$\lim_{j\to\infty} \left\| 2^j f\left(\frac{x}{2^j}\right) - A(x), w \right\| = 0$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$.

The further part of the proof is similar to the proof of Theorem 2.2. □

Now we prove the superstability of the Jensen functional equation in 2-Banach spaces.

Theorem 3.4 *Let $\theta \in [0, \infty)$, $p, q, r, t \in (0, \infty)$ with $t \neq 1$, and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x) + f(y) + 2f(z), w\| \leq \left\| 2f\left(\frac{x+y}{2} + z\right), w \right\| + \theta \|x\|^p \|y\|^q \|z\|^r \|w\|^t \tag{3.5}$$

for all $x, y, z \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Then $f : \mathcal{X} \to \mathcal{Y}$ is an additive mapping.

Proof Replacing w by sw in (3.5) for $s \in \mathbb{R} \setminus \{0\}$, we get

$$\|f(x) + f(y) + 2f(z), sw\| \leq \left\| 2f\left(\frac{x+y}{2} + z\right), sw \right\| + \theta \|x\|^p \|y\|^q \|z\|^r \|sw\|^t$$

and so

$$\|f(x) + f(y) + 2f(z), w\| \leq \left\| 2f\left(\frac{x+y}{2} + z\right), w \right\| + \theta \|x\|^p \|y\|^q \|z\|^r \|w\|^t \frac{|s|^t}{|s|}$$

for all $x, y, z \in \mathcal{X}$, all $w \in \mathcal{Y}$ and all $s \in \mathbb{R} \setminus \{0\}$.

The rest of the proof is similar to the proof of Theorem 2.4. □

Competing interests

The author declares that they have no competing interests.

Authors' contributions
CP conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Received: 26 February 2013 Accepted: 30 August 2013 Published: 04 Nov 2013

References

1. Ulam, SM: A Collection of the Mathematical Problems. Interscience, New York (1960)
2. Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
3. Aoki, T: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64-66 (1950)
4. Rassias, TM: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297-300 (1978)
5. Găvruta, P: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431-436 (1994)
6. Gähler, S: 2-metrische Räume und ihre topologische Struktur. Math. Nachr. **26**, 115-148 (1963)
7. Gähler, S: Lineare 2-normierte Räumen. Math. Nachr. **28**, 1-43 (1964)
8. Cho, Y, Lin, PSC, Kim, S, Misiak, A: Theory of 2-Inner Product Spaces. Nova Science Publishers, New York (2001)
9. White, A: 2-Banach spaces. Dissertation, St. Louis University (1968)
10. White, A: 2-Banach spaces. Math. Nachr. **42**, 43-60 (1969)
11. Park, W: Approximate additive mappings in 2-Banach spaces and related topics. J. Math. Anal. Appl. **376**, 193-202 (2011)
12. Eskandani, GZ, Găvruta, P: Hyers-Ulam-Rassias stability of Pexiderized Cauchy functional equation in 2-Banach spaces. J. Nonlinear Sci. Appl. **5**, 459-465 (2012)
13. Gilányi, A: Eine zur Parallelogrammgleichung äquivalente Ungleichung. Aequ. Math. **62**, 303-309 (2001)
14. Rätz, J: On inequalities associated with the Jordan-von Neumann functional equation. Aequ. Math. **66**, 191-200 (2003)
15. Gilányi, A: On a problem by K. Nikodem. Math. Inequal. Appl. **5**, 707-710 (2002)
16. Fechner, W: Stability of a functional inequalities associated with the Jordan-von Neumann functional equation. Aequ. Math. **71**, 149-161 (2006)
17. Park, C, Cho, Y, Han, M: Functional inequalities associated with Jordan-von Neumann-type additive functional equations. J. Inequal. Appl. **2007**, Article ID 41820 (2007)

10.1186/1029-242X-2013-447
Cite this article as: Park: Additive functional inequalities in 2-Banach spaces. *Journal of Inequalities and Applications* 2013, 2013:447