The symmetric orbifold of $\mathcal{N} = 2$ minimal models

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ABSTRACT: The large level limit of the $\mathcal{N} = 2$ minimal models that appear in the duality with the $\mathcal{N} = 2$ supersymmetric higher spin theory on AdS$_3$ is shown to be a natural subsector of a certain symmetric orbifold theory. We study the relevant decompositions in both the untwisted and the twisted sector, and analyse the structure of the higher spin representations in the twisted sector in some detail. These results should help to identify the string background of which the higher spin theory is expected to describe the leading Regge trajectory in the tensionless limit.

Contents

1 Introduction

During the last few years fairly concrete evidence has emerged for the idea that Vasiliev higher spin theories [1] arise as classically consistent subtheories of string theory in the tensionless limit, as had been anticipated many years ago [2–4]. In particular, a relation of this kind was suggested for the case of AdS$_4$ in [5], while for AdS$_3$ a somewhat different proposal was made in [6]. In the latter case, the $\mathcal{N} = 4$ superconformal generalisation [7] of the original bosonic minimal model holography of [8], relating a higher spin theory on AdS$_3$ [9, 10] to the large N limit of a family of minimal model CFTs, was shown to define a subtheory of the CFT dual of string theory. More specifically, this was only shown for the background of the form AdS$_3 \times$ S$^3 \times \mathbb{T}^4$, where the CFT dual of string theory is believed to be described by the symmetric orbifold of $\mathbb{T}^4$, see [11] for a review. The CFT duals of the $\mathcal{N} = 4$ higher spin theories on AdS$_3$ are described by the so-called Wolf space cosets, see [12–17] for some early literature on this subject; in the limit where the torus background is approached — this is the case where the level k of the cosets is taken to infinity — these cosets simplify to become the theory of $4(N + 1)$ free bosons and fermions, subject to a U(N) singlet constraint. They then form a natural subsector of the untwisted sector of the symmetric orbifold where the same free theory is only subjected to a singlet constraint under the permutation group $S_{N+1} \subset$ U(N).

It is obviously tempting to believe that this sort of relation is not just restricted to the maximally supersymmetric setting, but that the less-supersymmetric higher spin — CFT

dualities may also be related naturally to string theory. One particularly interesting case is the $\mathcal{N} = 2$ version of the duality [18, 19], for which the dual 2d CFTs are Kazama-Suzuki (KS) models [20, 21] that have an additional parameter and may therefore allow for a matrix-like construction as in [5], see [22] for an attempt in this direction. In this paper we follow a different route by trying to imitate the analysis of [6] for the $\mathcal{N} = 2$ case: following on from our earlier work [23] (see also [24, 25]), where we showed that the large level limit of the relevant KS models can be described as the continuous orbifold of a free theory, we discuss how this (constrained) free theory is related to a symmetric orbifold construction. This symmetric orbifold is quite plausibly dual to string theory on AdS$_3$, following the general philosophy of [26], see also [27–29] for subsequent work.

The paper is organised as follows. In section 2 we define the symmetric orbifold in question, and explain how the large level limit of the relevant KS models describe a sub-sector of this theory. In particular, we study the embedding in detail for the untwisted sector, where we can give very concrete decompositions in terms of the representations of the $\mathcal{N} = 2$ $s\mathcal{W}_\infty$ algebra. Section 3 is devoted to understanding how the twisted sector states of the symmetric orbifold can be similarly described in terms of these representations; we study in detail the (2)-cycle, as well as the $(2)^2$-cycle twisted sector, for which we give detailed decomposition formulae; we also explain how the structure of a general twisted sector can be understood in similar terms. Finally, we undertake (in section 3.4) first steps towards characterising the higher spin representations that are relevant for the description of the twisted sector, generalising the recent discussion of [30] to the $\mathcal{N} = 2$ case. We end with some conclusions, and there is an appendix where a self-contained description of the low-lying bosonic generators of the $s\mathcal{W}_\infty$ algebra in terms of the KS cosets is given. (This analysis is an important ingredient for the identification of the higher spin algebra representations, but it may also be useful in other contexts.)

2   The untwisted sector of the symmetric orbifold

It was shown in [23] that the $\mathcal{N} = 2$ superconformal cosets that appear in the duality to the $\mathcal{N} = 2$ supersymmetric higher spin theory on AdS$_3$ can be expressed as a continuous orbifold of a free field theory in the limit where the level $k \to \infty$. More precisely, in this limit the coset (see [19] for our conventions)

$$\frac{\mathfrak{su}(N+1)^{(1)}_{k+N+1}}{\mathfrak{su}(N)^{(1)}_{k+N+1} \oplus \mathfrak{u}(1)^{(1)}_\kappa} \cong \frac{\mathfrak{su}(N+1)_k \oplus \mathfrak{so}(2N)_1}{\mathfrak{su}(N)_{k+1} \oplus \mathfrak{u}(1)_\kappa} \tag{2.1}$$

was shown to agree with an orbifold theory of N free bosons and fermions by the continuous orbifold group $U(N)$. A similar approach was applied to the $\mathcal{N} = 4$ Wolf space cosets in [6], where it was shown that the corresponding coset algebra is a natural subalgebra of the chiral algebra of the symmetric orbifold; in turn the symmetric orbifold is believed to be dual to string theory on AdS$_3$, thus exhibiting how the higher spin theory is embedded into string theory. In this paper we want to analyse how the $\mathcal{N} = 2$ cosets (2.1) can be related to an $\mathcal{N} = 2$ symmetric orbifold. This should be a first step towards understanding the string theory interpretation of the corresponding $\mathcal{N} = 2$ higher spin theory.

The continuous orbifold describes the theory of N free complex bosons and fermions transforming in the fundamental (and anti-fundamental) representation of U(N). Thus it can be represented as the orbifold $(\mathbb{T}^2)^N/\mathrm{U}(N)$.[1] The untwisted sector consists of the states that are invariant under the action of U(N). The full orbifold theory includes also a twisted sector for each conjugacy class of U(N). The conjugacy classes can be labelled by ascending N-tuples $[\xi_1, \ldots, \xi_N]$ where $-1/2 \leq \xi_1 \leq \cdots \leq \xi_N < 1/2$; the relevant conjugacy class contains then the diagonal matrix with eigenvalues $\exp(2\pi i \xi_l)$. The ξ_l, $l = 1, \ldots, N$, can be interpreted as the twists of the N free bosons and fermions.

As in the $\mathcal{N} = 4$ case one may then consider, instead of the U(N) action, the permutation action of $S_{N+1} \subset \mathrm{U}(N)$. To explain this, it is natural to start with a theory of $N+1$ free bosons and fermions, on which S_{N+1} acts by permutations. This action is not irreducible since the sum of all bosons (or fermions) is invariant under the permutation action,

$$N + 1 \cong N \oplus 1 . \tag{2.2}$$

Here and in the following, normal font is used to denote representations of S_{N+1}, while bold font is reserved for representations of U(N). The N-dimensional representation on the right hand side is irreducible and is called the standard representation of S_{N+1}. In a suitable basis this representation acts on only N copies of $\mathbb{T}^2$, so the orbifold of $N + 1$ copies decomposes in fact as

$$(\mathbb{T}^2)^{N+1}/S_{N+1} \cong (\mathbb{T}^2)^N/S_{N+1} \oplus \mathbb{T}^2 . \tag{2.3}$$

Since in the $\mathcal{N} = 2$ case the theory in question only involves N complex bosons and fermions, we need to remove the last factor that describes the diagonal torus (which transforms as a singlet under S_{N+1}). This is a complication relative to the $\mathcal{N} = 4$ case discussed in [6], where the large level limit of the coset theory involved $2(N + 1)$ complex bosons and fermions, and hence the diagonal torus was also part of the coset theory. We shall therefore, in the following, study the orbifold $(\mathbb{T}^2)^N/S_{N+1}$.

In order to see the relation to the KS models we recall that the standard representation ρ of S_{N+1} acting on the N tori maps permutations to unitary (actually even orthogonal) $N \times N$ matrices. Thus we can view $\rho(S_{N+1})$ as a finite subgroup of U(N), and since the standard representation is faithful, that subgroup is isomorphic to S_{N+1}. Furthermore, as discussed in [6], the fundamental (and anti-fundamental) representation of U(N) branches down to the standard representation of S_{N+1}. Thus the U(N)-invariant states of the free theory form a consistent subsector of the S_{N+1} invariant states, and hence the untwisted sector of the continuous orbifold is a subsector of the untwisted sector of the symmetric orbifold.

In the rest of this section we shall analyse the untwisted sector of the symmetric orbifold from the viewpoint of the continuous orbifold. The twisted sectors of the symmetric orbifold will be discussed in the following section.

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[1]Strictly speaking the relevant orbifold is $(\mathbb{R}^2)^N/\mathrm{U}(N)$, since the U($N$) action is not compatible with discrete momenta. However, we shall usually refer to it as the torus orbifold since the zero momentum sector (which is what we shall be considering) is independent of the radius of the torus.

2.1 Perturbative decomposition of the untwisted sector

The untwisted sector of the symmetric orbifold by S_{N+1} contributes to the partition function as

$$Z_{\mathrm{U}}(q, \bar{q}, y, \bar{y}) = |\mathcal{Z}_{\mathrm{vac}}(q, y)|^2 + \sum_R \left| \mathcal{Z}_R^{(\mathrm{U})}(q, y) \right|^2 , \qquad (2.4)$$

where $\mathcal{Z}_{\mathrm{vac}}$ denotes the vacuum character, and R labels the non-trivial irreducible representations of S_{N+1}. In order to avoid having to write repeatedly $N + 1$, we now change notation and replace the N from (2.1), (2.2) and (2.3) by $\tilde{N}$, and define $N \equiv \tilde{N} + 1$; in any case, we shall always be considering the large N (and hence large $\tilde{N}$ limit) for which this distinction is immaterial. In their analysis [31], Dijkgraaf, Moore, Verlinde and Verlinde computed the partition function of the symmetric orbifold X^N/S_N in the R-R sector with insertion of $(-1)^{F+\bar{F}}$, which reads

$$\sum_{N=0}^{\infty} p^N \tilde{Z}_{\mathrm{R}}(S^N(X)) = \prod_{m=1}^{\infty} \prod_{\Delta, \bar{\Delta}, \ell, \bar{\ell}} \frac{1}{\left(1 - p^m q^{\frac{\Delta}{m}} \bar{q}^{\frac{\bar{\Delta}}{m}} y^\ell \bar{y}^{\bar{\ell}}\right)^{c(\Delta, \bar{\Delta}, \ell, \bar{\ell})}} . \qquad (2.5)$$

Here we have indicated by the tilde that we have inserted a factor of $(-1)^{F+\bar{F}}$, and $c(\Delta, \bar{\Delta}, \ell, \bar{\ell})$ are the expansion coefficients of the R-R partition function (with insertion of $(-1)^{F+\bar{F}}$) of the base manifold X,

$$\tilde{Z}_{\mathrm{R}}(X) = \sum_{\Delta, \bar{\Delta}, \ell, \bar{\ell}} c(\Delta, \bar{\Delta}, \ell, \bar{\ell}) \, q^\Delta \bar{q}^{\bar{\Delta}} y^\ell \bar{y}^{\bar{\ell}} . \qquad (2.6)$$

In our case, $X = \mathbb{T}^2$ and the partition function factorises into its chiral parts, with $c(\Delta, \bar{\Delta}, \ell, \bar{\ell}) = c(\Delta, \ell) c(\bar{\Delta}, \bar{\ell})$. The chiral partition function reads (as in [6] we will be ignoring the momentum and winding states)

$$\begin{aligned}
\tilde{Z}_{\mathrm{R}}^{(\mathrm{chiral})}(\mathbb{T}^2) &= i \frac{\vartheta_1(z|\tau)}{\eta^3(\tau)} = -\left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right) \prod_{n=1}^{\infty} \frac{(1 - yq^n)(1 - y^{-1}q^n)}{(1 - q^n)^2} \\
&= -y^{\frac{1}{2}} + y^{-\frac{1}{2}} + q\left(y^{\frac{3}{2}} - 3y^{\frac{1}{2}} + 3y^{-\frac{1}{2}} - y^{-\frac{3}{2}}\right) \\
&\quad + 3q^2\left(y^{\frac{3}{2}} - 3y^{\frac{1}{2}} + 3y^{-\frac{1}{2}} - y^{-\frac{3}{2}}\right) \\
&\quad + q^3\left(-y^{\frac{5}{2}} + 9y^{\frac{3}{2}} - 22y^{\frac{1}{2}} + 22y^{-\frac{1}{2}} - 9y^{-\frac{3}{2}} + y^{-\frac{5}{2}}\right) + \mathcal{O}\left(q^4\right) ,
\end{aligned} \qquad (2.7)$$

where

$$\vartheta_1(z|\tau) = i\left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right) q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1 - q^n)(1 - y q^n)(1 - y^{-1} q^n) . \qquad (2.8)$$

In our analysis we will only be concerned with the NS-NS sector. The partition function in that sector can be obtained from (2.5) by spectral flow

$$y \to y \, q^{\frac{1}{2}} , \qquad \bar{y} \to \bar{y} \, \bar{q}^{\frac{1}{2}} , \qquad p \to p \, q^{\frac{1}{8}} \bar{q}^{\frac{1}{8}} y^{\frac{1}{2}} \bar{y}^{\frac{1}{2}} . \qquad (2.9)$$

This leads to an overall factor of $(q\bar{q})^{-\frac{N}{8}} = (q\bar{q})^{-\frac{c}{24}}$, which we will suppress throughout this paper for better readability. (Effectively, this is equivalent to multiplying the right-hand

side of the last replacement in (2.9) by an additional factor of $(q\bar{q})^{\frac{1}{8}}$.) We then obtain the symmetric orbifold generating function in the NS-NS sector (without a $(-1)^{F+\bar{F}}$ insertion)

$$\sum_{N=0}^{\infty} p^N Z\left(S^N(X)\right) = \prod_{m=1}^{\infty} \prod_{\substack{\Delta,\bar{\Delta} \\ \ell,\bar{\ell}}} \frac{1}{\left(1 - (-1)^{\ell+\bar{\ell}+1} p^m q^{\frac{\Delta}{m}+\frac{\ell}{2}+\frac{m}{4}} \bar{q}^{\frac{\bar{\Delta}}{m}+\frac{\bar{\ell}}{2}+\frac{m}{4}} y^{\ell+\frac{m}{2}} \bar{y}^{\bar{\ell}+\frac{m}{2}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}} \cdot \tag{2.10}$$

Now the generating function of the untwisted sector corresponds to the $m = 1$ factor of (2.10),

$$\sum_{N=0}^{\infty} p^N Z^{(\mathrm{U})}\left(S^N(X)\right) = \prod_{\substack{\Delta,\bar{\Delta} \\ \ell,\bar{\ell}}} \frac{1}{\left(1 - (-1)^{\ell+\bar{\ell}+1} p\, q^{\Delta+\frac{\ell}{2}+\frac{1}{4}} \bar{q}^{\bar{\Delta}+\frac{\bar{\ell}}{2}+\frac{1}{4}} y^{\ell+\frac{1}{2}} \bar{y}^{\bar{\ell}+\frac{1}{2}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}} \,, \tag{2.11}$$

and the chiral vacuum character (the partition function of the $\mathcal{W}$ algebra) of the orbifold $(\mathbb{T}^2)^{\tilde{N}+1}/S_{\tilde{N}+1}$ can be found from (2.11) by setting $\bar{\Delta} = 0, \bar{\ell} = -\frac{1}{2}$ and taking N large enough so that the coefficients stabilise; it is given by

$$\mathcal{Z}'_{\mathrm{vac}} = 1 + q^{\frac{1}{2}}(y + y^{-1}) + 4q + 6q^{\frac{3}{2}}(y + y^{-1}) + 4q^2(y^2 + 6 + y^{-2})$$
$$+ q^{\frac{5}{2}}(y^3 + 37y + 37y^{-1} + y^{-3}) + 7q^3(4y^2 + 17 + 4y^{-2}) + \mathcal{O}\left(q^{\frac{7}{2}}\right) \,. \tag{2.12}$$

In order to obtain the vacuum character of the orbifold $(\mathbb{T}^2)^{\tilde{N}}/S_{\tilde{N}+1} \equiv (\mathbb{T}^2)^{N-1}/S_N$, we have to divide this by the chiral partition function of $\mathbb{T}^2$, which means we neglect the torus that transforms as a singlet under $S_N \equiv S_{\tilde{N}+1}$ and corresponds to the trivial factor in the permutation representation of $S_{\tilde{N}+1}$, see eq. (2.2). Since this torus partition function is given by

$$Z_{\mathrm{NS}}^{(\mathrm{chiral})}(\mathbb{T}^2) = \prod_{n=1}^{\infty} \frac{(1 + y q^{n-1/2})(1 + y^{-1} q^{n-1/2})}{(1 - q^n)^2} \,, \tag{2.13}$$

where we have once more suppressed the prefactor $q^{-\frac{1}{8}}$, we obtain the modified vacuum character

$$\begin{aligned}
\mathcal{Z}_{\mathrm{vac}}(q, y) &= \frac{\mathcal{Z}'_{\mathrm{vac}}}{Z_{\mathrm{NS}}^{(\mathrm{chiral})}(\mathbb{T}^2)} \\
&= 1 + q + 2q^{\frac{3}{2}}(y + y^{-1}) + q^2(y^2 + 8 + y^{-2}) + 10q^{\frac{5}{2}}(y + y^{-1}) \\
&\quad + q^3(5y^2 + 32 + 5y^{-2}) + q^{\frac{7}{2}}(2y^3 + 47y + 47y^{-1} + 2y^{-3}) \\
&\quad + q^4(y^4 + 37y^2 + 142 + 37y^{-2} + y^{-4}) + \mathcal{O}\left(q^{\frac{9}{2}}\right) \,. \tag{2.14}
\end{aligned}$$

This vacuum character counts the chiral states that transform trivially under S_N, and hence includes, in particular, the character of the $\mathcal{N} = 2$ coset $s\mathcal{W}_{\infty}$ algebra (in the limit $k \to \infty$). Thus the vacuum sector should decompose into the coset characters as

$$\mathcal{Z}_{\mathrm{vac}}(q, y) = \sum_{\Lambda} n(\Lambda) \chi_{(0;\Lambda)}(q, y) \,. \tag{2.15}$$

Indeed, by comparing both sides of the equation order by order in q, we find explicitly

$$
\begin{aligned}
\mathcal{Z}_{\text{vac}}(q,y) = {} & \chi_{(0;0)}(y,q) + \chi_{(0;[2,0,...,0])}(q,y) + \chi_{(0;[0,0,...,0,2])}(q,y) \\
& + \chi_{(0;[3,0,...,0,0])}(q,y) + \chi_{(0;[0,0,0,...,0,3])}(q,y) \\
& + \chi_{(0;[2,0,...,0,1])}(q,y) + \chi_{(0;[1,0,0,...,0,2])}(q,y) \\
& + 2 \cdot \chi_{(0;[4,0,...,0,0])}(q,y) + 2 \cdot \chi_{(0;[0,0,0,...,0,4])}(q,y) \\
& + \chi_{(0;[0,2,0,...0,0])}(q,y) + \chi_{(0;[0,0,...0,2,0])}(q,y) \\
& + \chi_{(0;[3,0,...,0,1])}(q,y) + \chi_{(0;[1,0,0,...,0,3])}(q,y) \\
& + 2 \cdot \chi_{(0;[2,0,0,...,0,2])}(q,y) \\
& + \chi_{(0;[2,1,0,...,0,1])}(q,y) + \chi_{(0;[1,0,...,0,1,2])}(q,y) \\
& + \chi_{(0;[0,2,0,...,0,1])}(q,y) + \chi_{(0;[1,0,...,0,2,0])}(q,y) \\
& + 3 \cdot \chi_{(0;[3,0,...,0,2])}(q,y) + 3 \cdot \chi_{(0;[2,0,...,0,3])}(q,y) \\
& + \chi_{(0;[1,1,0,...,0,2])}(q,y) + \chi_{(0;[2,0,...,0,1,1])}(q,y) \\
& + \chi_{(0;[3,1,0,...,0])}(q,y) + \chi_{(0;[0,...,0,1,3])}(q,y) \\
& + 2 \cdot \chi_{(0;[4,0,...,0,1])}(q,y) + 2 \cdot \chi_{(0;[1,0,...,0,4])}(q,y) \\
& + \chi_{(0;[2,1,0,...,0,1,0])}(q,y) + \chi_{(0;[0,1,0,...,0,1,2])}(q,y) \\
& + \chi_{(0;[1,1,0,...,0,1,1])}(q,y) + \mathcal{O}\left(q^{\frac{9}{2}}\right) \ . \quad (2.16)
\end{aligned}
$$

As in [6], this is precisely of the form (2.15), with $n(\Lambda)$ denoting the multiplicity of the S_N singlet representation in the $\mathrm{U}(N-1)$ representation Λ, where we think of Λ as a $S_{\tilde{N}+1} \equiv S_N$ representation using the embedding $S_{\tilde{N}+1} \subset \mathrm{U}(\tilde{N})$.[2] Although the coefficients $n(\Lambda)$ appearing in (2.16) and the analogous $\mathcal{N} = 4$ decomposition, see eq. (4.10) of [6], agree precisely, one should emphasise that the computation of this decomposition does not directly follow from the $\mathcal{N} = 4$ case since the explicit cosets are quite different. In addition, in the case at hand, it is the modified vacuum character (2.14) (where the invariant torus partition function is divided out) that appears on the left-hand-side, while in the $\mathcal{N} = 4$ case the corresponding identity involved the full symmetric orbifold vacuum character.

Furthermore, as in [32], we can identify the single particle generators that generate this extended $\mathcal{W}$-algebra; if we had not divided out by the diagonal $\mathbb{T}^2$, the generating function of the single particle generators would have been (see [32])

$$
\begin{aligned}
\sum_{s,l} \tilde{d}(s,l) q^s y^l &= (1-q) \left[Z_{\text{NS}}^{(\text{chiral})}(\mathbb{T}^2)(q,y) - 1 \right] \\
&= (1-q) \left[\prod_{n=1}^{\infty} \frac{(1+yq^{n-1/2})(1+y^{-1}q^{n-1/2})}{(1-q^n)^2} - 1 \right], \quad (2.17)
\end{aligned}
$$

where the factor of $(1-q)$ removes the derivatives, and $\tilde{d}(s,l)$ are the number of single particle generators of spin s and charge l. Dividing out by the diagonal torus removes just the contribution coming from the two free fermions and bosons; thus the actual generating

function equals

$$\sum_{s,l} d(s,l) q^s y^l = (1-q) \left[Z_{\text{NS}}^{\text{(chiral)}} \left(\mathbb{T}^2 \right) (q,y) - \left(1 + \frac{q^{\frac{1}{2}}(y + y^{-1})}{(1-q)} + 2 \frac{q^1}{(1-q)} \right) \right]$$

$$= q + 2q^{\frac{3}{2}}(y + y^{-1}) + q^2 (6 + y^2 + y^{-2}) + 6q^{\frac{5}{2}}(y + y^{-1}) + \cdots . \qquad (2.18)$$

These single particle generators generate then the $\mathcal{W}$ algebra in the sense that

$$\mathcal{Z}_{\text{vac}}(q,y) = \prod_{s,l} \prod_{n=0}^{\infty} \frac{1}{(1 - y^l q^{s+n})^{(-1)^{2s} d(s,l)}} . \qquad (2.19)$$

They should sit in wedge representations of the $\mathcal{N} = 2$ $s\mathcal{W}_\infty$ algebra, and one finds, analogously to [32], that we have the decomposition

$$\sum_{s,l} d(s,l) q^s y^l = (1-q) \sideset{}{'}\sum_{m,n=0}^{\infty} \chi_{(0;[m,0,0,\ldots,0,0,n])}^{\text{(wedge)}}(q,y) , \qquad (2.20)$$

where the prime indicates that the terms with $(m,n) = (0,0), (1,0), (0,1)$ are not included in the sum. Note that the term with $m = n = 1$ accounts precisely for the generators of the original $s\mathcal{W}_\infty$ algebra. We have checked these identities up to order q^{15}, and it should be straightforward to prove them using the techniques of [32].

We can similarly extract the characters corresponding to the second sum in (2.4). For example, the representation that contains, among others, the coset states

$$(0; \text{f}) , \qquad (0; \bar{\text{f}}) , \qquad (2.21)$$

is associated to R being the standard representation of S_N. The corresponding character $\mathcal{Z}_1$ is obtained from the coefficient of $\bar{q}(\bar{y} + \bar{y}^{-1})$ in $Z^{(\text{U})}$, from which one has to subtract the contribution from $|\mathcal{Z}'_{\text{vac}}|^2$ and then divide by the torus partition function again. This character turns out to be given by

$$\mathcal{Z}_1 = \frac{\mathcal{Z}'_{\text{vac}} \left(Z_{\text{NS}}^{\text{(chiral)}}(\mathbb{T}^2) - 1 \right)}{Z_{\text{NS}}^{\text{(chiral)}}(\mathbb{T}^2)} = \mathcal{Z}'_{\text{vac}} - \mathcal{Z}_{\text{vac}}$$

$$= q^{\frac{1}{2}}(y + y^{-1}) + 3q + 4q^{\frac{3}{2}}(y + y^{-1}) + q^2 (3y^2 + 16 + 3y^{-2})$$

$$+ q^{\frac{5}{2}}(y^3 + 27y + 27y^{-1} + y^{-3}) + q^3 (23y^2 + 87 + 23y^{-2})$$

$$+ 5q^{\frac{7}{2}}(2y^3 + 29y + 29y^{-1} + 2y^{-3}) + q^4 (3y^4 + 141y^2 + 433 + 141y^{-2} + 3y^{-4})$$

$$+ \mathcal{O}\left(q^{\frac{9}{2}} \right) . \qquad (2.22)$$

It can be decomposed into coset characters in the $k \to \infty$ limit according to

$$
\begin{aligned}
\mathcal{Z}_1(q,y) = {} & \chi_{(0;[1,0,...,0])}(q,y) + \chi_{(0;[0,...,0,1])}(q,y) \\
& + \chi_{(0;[2,0,...,0])}(q,y) + \chi_{(0;[0,...,0,2])}(q,y) \\
& + \chi_{(0;[1,0,...,0,1])}(q,y) \\
& + 2 \cdot \chi_{(0;[3,0,...,0])}(q,y) + 2 \cdot \chi_{(0;[0,...,0,3])}(q,y) \\
& + \chi_{(0;[1,1,0,...,0])}(q,y) + \chi_{(0;[0,...,0,1,1])}(q,y) \\
& + 2 \cdot \chi_{(0;[2,0,...,0,1])}(q,y) + 2 \cdot \chi_{(0;[1,0,...,0,2])}(q,y) \\
& + 3 \cdot \chi_{(0;[4,0,...,0])}(q,y) + 3 \cdot \chi_{(0;[0,...,0,4])}(q,y) \\
& + 2 \cdot \chi_{(0;[2,1,0,...,0])}(q,y) + 2 \cdot \chi_{(0;[0,...,0,1,2])}(q,y) \\
& + \chi_{(0;[0,2,0,...,0])}(q,y) + \chi_{(0;[0,...,0,2,0])}(q,y) \\
& + 4 \cdot \chi_{(0;[3,0,...,0,1])}(q,y) + 4 \cdot \chi_{(0;[1,0,...,0,3])}(q,y) \\
& + 2 \cdot \chi_{(0;[1,1,0,...,0,1])}(q,y) + 2 \cdot \chi_{(0;[1,0,...,0,1,1])}(q,y) \\
& + 5 \cdot \chi_{(0;[2,0,...,0,2])}(q,y) \\
& + \chi_{(0;[2,0,...,0,1,0])}(q,y) + \chi_{(0;[0,1,0...,0,2])}(q,y) \\
& + 4 \cdot \chi_{(0;[3,1,0,...,0])}(q,y) + 4 \cdot \chi_{(0;[0,...,0,1,3])}(q,y) \\
& + \chi_{(0;[0,1,1,0,...,0])}(q,y) + \chi_{(0;[0,...,0,1,1,0])}(q,y) \\
& + 7 \cdot \chi_{(0;[4,0,...,0,1])}(q,y) + 7 \cdot \chi_{(0;[1,0,...,0,4])}(q,y) \\
& + 4 \cdot \chi_{(0;[2,1,0,...,0,1])}(q,y) + 4 \cdot \chi_{(0;[1,0,...,0,1,2])}(q,y) \\
& + 3 \cdot \chi_{(0;[0,2,0,...,0,1])}(q,y) + 3 \cdot \chi_{(0;[1,0,...,0,2,0])}(q,y) \\
& + 9 \cdot \chi_{(0;[3,0,...,0,2])}(q,y) + 9 \cdot \chi_{(0;[2,0,...,0,3])}(q,y) \\
& + 2 \cdot \chi_{(0;[3,0,...,0,1,0])}(q,y) + 2 \cdot \chi_{(0;[0,1,0,...,0,3])}(q,y) \\
& + 5 \cdot \chi_{(0;[1,1,0,...,0,2])}(q,y) + 5 \cdot \chi_{(0;[2,0,...,0,1,1])}(q,y) \\
& + \chi_{(0;[1,1,0,...,0,1,0])}(q,y) + \chi_{(0;[0,1,0,...,0,1,1])}(q,y) \\
& + \chi_{(0;[2,0,1,0...,0,1])}(q,y) + \chi_{(0;[1,0,...,0,1,0,2])}(q,y) \\
& + 3 \cdot \chi_{(0;[2,1,0,...,0,1,0])}(q,y) + 3 \cdot \chi_{(0;[0,1,0,...,0,1,2])}(q,y) \\
& + \chi_{(0;[1,0,1,0,...,0,2])}(q,y) + \chi_{(0;[2,0,...,0,1,0,1])}(q,y) \\
& + 6 \cdot \chi_{(0;[1,1,0,...,0,1,1])}(q,y) + \mathcal{O}\left(q^{\frac{9}{2}}\right) .
\end{aligned}
\tag{2.23}
$$

This time, the coefficients of the coset characters $\chi_{(0;\Lambda)}$ correspond precisely to the multiplicity of the $(N-1)$-dimensional standard representation of S_N inside Λ.[3] This is obviously in line with the fact that the $\tilde{N} = N - 1$ boson and fermion fields (that give rise to the representations (2.21)) transform precisely in this representation of the permutation group.

2.2 The building blocks of the untwisted sector

Having identified the lowest two representations of S_N by explicitly evaluating the orbifold partition function order by order in q, we will now turn to a more systematic analysis of

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[3]Once more we thank Marco Baggio for helping us compute these multiplicities.

the untwisted sector. We will show that it organises itself in terms of multi-particle powers of the 'minimal representation' $\mathcal{Z}_1$, in parallel to what was observed in [30].

Let us first introduce the wedge character χ_1 pertaining to $\mathcal{Z}_1$ by stripping off the modes outside of the wedge,

$$\mathcal{Z}_1 = \mathcal{Z}_{\text{vac}} \cdot \chi_1 \qquad \text{or} \qquad \chi_1 = Z_{\text{NS}}^{(\text{chiral})}(\mathbb{T}^2) - 1 \, , \tag{2.24}$$

where $\mathcal{Z}_{\text{vac}}$ is the vacuum character (that counts the modes outside the wedge); explicitly, we have

$$
\begin{aligned}
\chi_1(q, y) &= \sum_{(\Delta, \ell) \neq \left(0, -\frac{1}{2}\right)} |c(\Delta, \ell)| \, q^{\Delta + \frac{\ell}{2} + \frac{1}{4}} \, y^{\ell + \frac{1}{2}} \\
&= q^{\frac{1}{2}} \left(y + y^{-1}\right) + 3q + 3q^{\frac{3}{2}} \left(y + y^{-1}\right) + q^2 \left(y^2 + 9 + y^{-2}\right) + \mathcal{O}\left(q^{\frac{5}{2}}\right) \, .
\end{aligned} \tag{2.25}
$$

Then we claim that the full partition function of the untwisted sector for $N \to \infty$ can be written as

$$Z^{(\text{U})}(q, \bar{q}, y, \bar{y}) = |\mathcal{Z}_{\text{vac}}(q, y)|^2 \left(1 + \sum_\Lambda |\chi_\Lambda(q, y)|^2\right) \, , \tag{2.26}$$

where Λ runs over all Young diagrams, and $\chi_\Lambda(q, y)$ is the Λ-symmetrised power of $\chi_1(q, y)$ given by (see e.g. [33])

$$\chi_\Lambda(q, y) = \frac{1}{m!} \sum_{\rho \in S_m} \chi_m^\Lambda(\rho) \prod_{k=1}^m \mathcal{F}^{k-1} \chi_1\left(q^k, y^k\right)^{a_k(\rho)} \, . \tag{2.27}$$

Here $m = |\Lambda|$ is the number of boxes of Λ, $\chi_m^\Lambda(\rho)$ is the character of Λ seen as an S_m-representation, $a_k(\rho)$ is the number of k-cycles in the permutation ρ, and $\mathcal{F}$ is the involutive mapping that acts on a character or partition function by insertion of $(-1)^{F+\bar{F}}$. So denoting $\mathcal{F}\chi_1$ by $\tilde{\chi}_1$, the first few characters read

$$
\begin{aligned}
\chi_\square(q, y) &= \chi_1(q, y) \, , \\
\chi_{\square\square}(q, y) &= \frac{1}{2} \left(\chi_1(q, y)^2 + \tilde{\chi}_1(q^2, y^2)\right) \, , \\
\chi_{\substack{\square\\\square}}(q, y) &= \frac{1}{2} \left(\chi_1(q, y)^2 - \tilde{\chi}_1(q^2, y^2)\right) \, , \\
\chi_{\square\square\square}(q, y) &= \frac{1}{6} \left(\chi_1(q, y)^3 + 3\chi_1(q, y)\tilde{\chi}_1(q^2, y^2) + 2\chi_1(q^3, y^3)\right) \, , \\
\chi_{\substack{\square\\\square\\\square}}(q, y) &= \frac{1}{6} \left(\chi_1(q, y)^3 - 3\chi_1(q, y)\tilde{\chi}_1(q^2, y^2) + 2\chi_1(q^3, y^3)\right) \, , \\
\chi_{\substack{\square\square\\\square}}(q, y) &= \frac{1}{3} \left(\chi_1(q, y)^3 - \chi_1(q^3, y^3)\right) \, .
\end{aligned} \tag{2.28}
$$

A proof of (2.26) will be given at the end of section 3.3. We have checked agreement of eqs. (2.11) and (2.26) for up to three boxes and up to order $\mathcal{O}(q^2)\mathcal{O}(\bar{q}^2)$, which is the lowest order to which the Young diagrams Λ with four boxes contribute.

3 The twisted sector

The twisted sectors are labelled by conjugacy classes $[g]$ of S_N, and consist of those states which are invariant under C^g, the centraliser of g in S_N. The conjugacy classes of S_N can be labelled by cycle structures

$$(1)^{N_1}(2)^{N_2}(3)^{N_3}\cdots(m)^{N_m}\,, \qquad \text{where } \sum_{i=1}^{m} N_i = N\,. \tag{3.1}$$

The conjugacy class labelled by such a string consists of all elements of S_N that can be decomposed into N_2 2-cycles, N_3 3-cycles, etc. The centraliser of this conjugacy class is then

$$C^{(1)^{N_1}(2)^{N_2}\cdots(m)^{N_m}} \cong S_{N_1} \times \left(S_{N_2} \ltimes \mathbb{Z}_2^{N_2}\right) \times \cdots \times \left(S_{N_m} \ltimes \mathbb{Z}_m^{N_m}\right)\,. \tag{3.2}$$

The n free fermions and bosons corresponding to an n-cycle have twists of i/n, for $i = 1,\ldots,n$, and the corresponding $\mathbb{Z}_n$ acts by the usual phases on them. On the other hand, the S_{N_n} factors in the semi-direct products permute the N_n different n-cycles among each other.

Since states are tensor products of left- and right-moving states, the action of the centraliser on these chiral states need not be trivial (only the combined action on left- and right-movers must be). The partition function of the $[g]$-twisted sector will thus have the structure

$$Z^{[g]} = \sum_R \left| \mathcal{Z}_R^{[g]} \right|^2\,, \tag{3.3}$$

where R labels the different irreducible representations of the centraliser $C^{[g]}$. We will see examples of this below.

3.1 The 2-cycle twisted sector

We will start our analysis of the twisted sector with the subsector corresponding to a 2-cycle twist, which is the simplest example. The partition function of the 2-cycle twisted sector in the ordinary symmetric orbifold can be obtained from the generating function; more specifically, the R-R sector expression can be extracted from the $m = 1$ and $m = 2$ factors of (2.5),

$$\sum_{N=0}^{\infty} p^N \tilde{Z}_{\mathrm{R}}^{(2)}\left(S^N \mathbb{T}^2\right) = p^2 \sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}' c(\Delta,\bar{\Delta},\ell,\bar{\ell})q^{\frac{\Delta}{2}}\bar{q}^{\frac{\bar{\Delta}}{2}}y^\ell \bar{y}^{\bar{\ell}} \cdot \prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}} \frac{1}{\left(1 - p\,q^\Delta \bar{q}^{\bar{\Delta}} y^\ell \bar{y}^{\bar{\ell}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\,, \tag{3.4}$$

where the prime at the sum indicates that $\Delta - \bar{\Delta}$ has to be even. Flowing to the NS-NS sector and considering the stabilising limit of large N we find for the partition function without $(-1)^{F+\bar{F}}$ insertion

$$Z^{(2)}(S^N \mathbb{T}^2) = \sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}' |c(\Delta,\bar{\Delta},\ell,\bar{\ell})|\, q^{\frac{1}{2}(\Delta+\ell+1)}\bar{q}^{\frac{1}{2}(\bar{\Delta}+\bar{\ell}+1)}y^{\ell+1}\bar{y}^{\bar{\ell}+1}$$

$$\times \prod_{\substack{(\Delta,\bar{\Delta},\ell,\bar{\ell}) \\ \neq(0,0,-1/2,-1/2)}} \frac{1}{\left(1 - (-1)^{\ell+\bar{\ell}+1}q^{\Delta+\frac{\ell}{2}+\frac{1}{4}}\bar{q}^{\bar{\Delta}+\frac{\bar{\ell}}{2}+\frac{1}{4}}y^{\ell+\frac{1}{2}}\bar{y}^{\bar{\ell}+\frac{1}{2}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\,. \tag{3.5}$$

We then obtain the partition function we are interested in by dividing by the left- and right-moving torus partition function $Z_{\text{NS}}(\mathbb{T}^2) = |Z_{\text{NS}}^{(\text{chiral})}(\mathbb{T}^2)|^2$,

$$
\begin{aligned}
Z^{(2)}(q, \bar{q}, y, \bar{y}) = \frac{q^{\frac{1}{4}} \bar{q}^{\frac{1}{4}}}{y^{\frac{1}{2}} \bar{y}^{\frac{1}{2}}} \Big[& 1 + y\bar{y} + (y\bar{y}^2 + 3y + 3\bar{y} + \bar{y}^{-1})\bar{q}^{\frac{1}{2}} \\
& + (y^2\bar{y} + y + \bar{y} + y^{-1})q^{\frac{1}{2}} \\
& + 2(y^2\bar{y}^2 + 2y^2 + 5y\bar{y} + 2\bar{y}^2 + 5 + 2y\bar{y}^{-1} + 2y^{-1}\bar{y} + y^{-1}\bar{y}^{-1})q^{\frac{1}{2}}\bar{q}^{\frac{1}{2}} \\
& + \cdots \Big] .
\end{aligned} \tag{3.6}
$$

Since the centraliser of this sector (ignoring the $N - 2$ sectors that are not affected by the twist — invariance with respect to this subgroup will just guarantee that the remaining factors give rise to a factor equal to the untwisted sector $Z^{(\text{U})}$ for large N) is simply $S_2 \cong \mathbb{Z}_2$, there are two representations that contribute, namely

$$
\begin{aligned}
\mathcal{Z}_+(q, y) &= \mathcal{Z}_{\text{vac}} \cdot \sum_{\Delta \text{ even}, \ell} |c(\Delta, \ell)| \, q^{\frac{1}{2}(\Delta + \ell + 1)} y^{\ell + 1} \\
&= y^{\frac{1}{2}} q^{\frac{1}{4}} + \left(y^{\frac{3}{2}} + 3y^{-\frac{1}{2}} \right) q^{\frac{3}{4}} + \left(10 y^{\frac{1}{2}} + 3y^{-\frac{3}{2}} \right) q^{\frac{5}{4}} \\
&\quad + \left(12 y^{\frac{3}{2}} + 27 y^{-\frac{1}{2}} + y^{-\frac{5}{2}} \right) q^{\frac{7}{4}} + \mathcal{O}\left(q^{\frac{9}{4}} \right) ,
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
\mathcal{Z}_-(q, y) &= \mathcal{Z}_{\text{vac}} \cdot \sum_{\Delta \text{ odd}, \ell} |c(\Delta, \ell)| \, q^{\frac{1}{2}(\Delta + \ell + 1)} y^{\ell + 1} \\
&= y^{-\frac{1}{2}} q^{\frac{1}{4}} + \left(3 y^{\frac{1}{2}} + y^{-\frac{3}{2}} \right) q^{\frac{3}{4}} + \left(3 y^{\frac{3}{2}} + 10 y^{-\frac{1}{2}} \right) q^{\frac{5}{4}} \\
&\quad + \left(y^{\frac{5}{2}} + 27 y^{\frac{1}{2}} + 12 y^{-\frac{3}{2}} \right) q^{\frac{7}{4}} + \mathcal{O}\left(q^{\frac{9}{4}} \right) .
\end{aligned} \tag{3.8}
$$

Defining the wedge characters $\chi_\pm^{(2)}$ by

$$
\mathcal{Z}_\pm = \mathcal{Z}_{\text{vac}} \cdot \chi_\pm^{(2)} , \tag{3.9}
$$

the whole sector can then simply be written as

$$
\begin{aligned}
Z^{(2)} &= Z^{(\text{U})} \cdot \left(|\chi_+^{(2)}|^2 + |\chi_-^{(2)}|^2 \right) \\
&= |\mathcal{Z}_{\text{vac}}|^2 \cdot \left(1 + \sum_\Lambda |\chi_\Lambda(q, y)|^2 \right) \cdot \left(|\chi_+^{(2)}|^2 + |\chi_-^{(2)}|^2 \right) .
\end{aligned} \tag{3.10}
$$

The two wedge characters $\chi_\pm$ have the same leading q behaviour, and their lowest terms are described by the coset representations [23]

$$
([k/2, 0, \ldots, 0]; [k/2, 0, \ldots, 0]) \qquad \text{and} \qquad ([k/2, 0, \ldots, 0]; [k/2 + 1, 0, \ldots, 0]) \tag{3.11}
$$

for large k, respectively, i.e., have twist $\xi = [-1/2, 0, \ldots, 0]$ in the continuous orbifold picture. One of these states can be obtained from the other by acting on it with a fermionic

zero-mode. In fact, both $\chi_{\pm}$ can be written in terms of coset representations (for $k \to \infty$), and we have checked that up to order q^2 we have

$$
\begin{aligned}
\mathcal{Z}_+(q,y) =\ & \chi_{([k/2,0,\ldots,0];[k/2+1,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2-1,0,\ldots,0])}(q,y) \\
& + \chi_{([k/2,0,\ldots,0];[k/2+3,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2,1,0,\ldots,0])}(q,y) \\
& + \chi_{([k/2,0,\ldots,0];[k/2+1,0,\ldots,0,1])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2,1,0,\ldots,0,1])}(q,y) \\
& + \chi_{([k/2,0,\ldots,0];[k/2-2,1,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+3,0,\ldots,0,1])}(q,y) \\
& + \chi_{([k/2,0,\ldots,0];[k/2-1,0,\ldots,0,1])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+2,1,0,\ldots,0])}(q,y) \\
& + 2 \cdot \chi_{([k/2,0,\ldots,0];[k/2-1,2,0,\ldots,0])}(q,y) + 2 \cdot \chi_{([k/2,0,\ldots,0];[k/2+1,0,\ldots,0,2])}(q,y) \\
& + 2 \cdot \chi_{([k/2,0,\ldots,0];[k/2-2,1,0,\ldots,0,1])}(q,y) + \mathcal{O}\left(q^{\frac{9}{4}}\right), \\
\mathcal{Z}_-(q,y) =\ & \chi_{([k/2,0,\ldots,0];[k/2,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+2,0,\ldots,0])}(q,y) \\
& + \chi_{([k/2,0,\ldots,0];[k/2,0,\ldots,0,1])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2-1,1,0,\ldots,0])}(q,y) \\
& + \chi_{([k/2,0,\ldots,0];[k/2-1,1,0,\ldots,0,1])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+1,1,0,\ldots,0])}(q,y) \\
& + \chi_{([k/2,0,\ldots,0];[k/2+2,0,\ldots,0,1])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+1,1,0,\ldots,0,1])}(q,y) \\
& + \chi_{([k/2,0,\ldots,0];[k/2-2,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2+4,0,\ldots,0])}(q,y) \\
& + 2 \cdot \chi_{([k/2,0,\ldots,0];[k/2,0,\ldots,0,2])}(q,y) + 2 \cdot \chi_{([k/2,0,\ldots,0];[k/2-2,2,0,\ldots,0])}(q,y) \\
& + \chi_{([k/2,0,\ldots,0];[k/2-3,1,0,\ldots,0])}(q,y) + \chi_{([k/2,0,\ldots,0];[k/2-2,0,\ldots,0,1])}(q,y) + \mathcal{O}\left(q^{\frac{9}{4}}\right).
\end{aligned}
\tag{3.12}
$$

As in [6], we can understand the multiplicities in these decompositions systematically: $\mathcal{Z}_{\pm}$ contains all those coset representations

$$
([k/2, 0, \ldots, 0]; [k/2 + l_0, \Lambda']) \tag{3.13}
$$

for which $l_0 + \sum_i \Lambda'_i$ is odd or even, respectively.[4] This is due to the fact that $l_0 + \sum_i \Lambda'_i$ counts the number of twisted modes by which the ground state

$$
(\Lambda_+; \Lambda_-) = ([k/2, 0, \ldots, 0]; [k/2, 0, \ldots, 0]) \tag{3.14}
$$

has been excited. Each of these twisted modes has odd parity under the $\mathbb{Z}_2$ in the centraliser. In addition, each state has to be invariant under the S_{N-2} factor of the centraliser — the states that are not invariant are accounted for by the middle factor in (3.10). For the boxes in the first row of Λ_-, this is automatically true, so the overall multiplicity with which $(\Lambda_+; \Lambda_-)$ contributes to $\mathcal{Z}_{\pm}$ is determined by the multiplicity of the trivial S_{N-2} representation inside the $\mathrm{SU}(N-2)$ representation Λ'. Using the (by now) standard embedding $S_{N-2} \subset \mathrm{U}(N-3) \subset \mathrm{SU}(N-2)$, we obtain the decompositions

$$
(\mathbf{N-2})_{\mathrm{SU}(N-2)} \to (N-3)_{S_{N-2}} \oplus 1_{S_{N-2}}, \quad \overline{(\mathbf{N-2})}_{\mathrm{SU}(N-2)} \to (N-3)_{S_{N-2}} \oplus 1_{S_{N-2}}. \tag{3.15}
$$

---

[4]Here we sum only over the first few Dynkin labels of Λ', such that anti-boxes and their tensor powers do not contribute to the $\mathbb{Z}_2$ parity. Actually, we should treat Λ' as a $\mathrm{U}(N-2)$ rather than $\mathrm{SU}(N-2)$ representation, since an anti-box of $\mathrm{U}(N-2)$ differs from $[0, \ldots, 0, 1]$ of $\mathrm{SU}(N-2)$ by its $\mathrm{U}(1)$ charge, which we have suppressed in our notation.

Hence states with $\Lambda' = \square$ or $\Lambda' = \overline{\square}$ have multiplicity 1. Moreover, the symmetric product of two boxes contains two S_{N-2} singlets, whereas the antisymmetric product contains none. This explains why states with $\Lambda' = \boxminus$ do not appear in the decomposition, whereas states with $\Lambda' = \square\square$ appear with multiplicity 2. The tensor product of a box with an antibox, $\overline{\square} \otimes \square$, contains two singlets, but one of them corresponds to the $s\mathcal{W}_\infty$ generators and hence does not give rise to a new representation; the resulting multiplicity in the coset decomposition is therefore again 1.

3.2 The twisted sector with two 2-cycles

The next, slightly more complicated step is to study the sector whose twist corresponds to the conjugacy class of permutations which have two 2-cycles. This means that two of the free bosons and fermions are twisted, while all the others are untwisted. We are interested in this sector because it contains the operators corresponding to exactly marginal deformations of the theory, which should, in particular, allow us to study the behaviour upon switching on the string coupling constant, compare [34]. By the same reasoning as before, we can obtain the generating function of the partition function from (2.10)

$$\sum_{N=0}^{\infty} p^N Z^{(2)^2}\left(S^N \mathbb{T}^2\right) = \frac{p^4}{2}\left[\left(\sideset{}{'}\sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}} |c(\Delta,\bar{\Delta},\ell,\bar{\ell})| q^{\frac{1}{2}(\Delta+\ell+1)}\bar{q}^{\frac{1}{2}(\bar{\Delta}+\bar{\ell}+1)}y^{\ell+1}\bar{y}^{\bar{\ell}+1}\right)^2\right.$$
$$+ \sideset{}{'}\sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}} c(\Delta,\bar{\Delta},\ell,\bar{\ell}) q^{\Delta+\ell+1}\bar{q}^{\bar{\Delta}+\bar{\ell}+1}y^{2\ell+2}\bar{y}^{2\bar{\ell}+2}\Bigg]$$
$$\times \prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}} \frac{1}{\left(1 - pq^{\Delta+\frac{\ell}{2}+\frac{1}{4}}\bar{q}^{\bar{\Delta}+\frac{\bar{\ell}}{2}+\frac{1}{4}}(-y)^{\ell+\frac{1}{2}}(-\bar{y})^{\bar{\ell}+\frac{1}{2}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}} \ . \tag{3.16}$$

In the first term, a factor of $(-1)^{\ell+\bar{\ell}+1}$ has again been absorbed into $|c(\Delta,\bar{\Delta},\ell,\bar{\ell})|$, whereas the second term contains a factor of $(-1)^{2(\ell+\bar{\ell}+1)} = 1$. As before, the partition function for our symmetric orbifold can be obtained by taking N large, and dividing by the partition function of the free $\mathbb{T}^2$ theory. We thus obtain

$$Z^{(2)^2} = q^{\frac{1}{2}}\bar{q}^{\frac{1}{2}}(1 + y\bar{y} + y^{-1}\bar{y}^{-1}) + q^{\frac{1}{2}}\bar{q}^1\left(y\bar{y} + y^{-1}\bar{y}^{-1} + 3(y + y^{-1}) + 4(\bar{y} + \bar{y}^{-1})\right)$$
$$+ q^1\bar{q}^{\frac{1}{2}}\left(y\bar{y} + y^{-1}\bar{y}^{-1} + 4(y + y^{-1}) + 3(\bar{y} + \bar{y}^{-1})\right)$$
$$+ q^1\bar{q}^1\left(38 + 3(y^2\bar{y}^2 + y^{-2}\bar{y}^{-2}) + 17(y + y^{-1})(\bar{y} + \bar{y}^{-1}) + 7(y^2 + \bar{y}^2 + \bar{y}^{-2} + y^{-2})\right)$$
$$+ \cdots . \tag{3.17}$$

The centraliser of this sector is

$$C^{(2)^2} = S_{N-4} \times (S_2 \ltimes \mathbb{Z}_2^2) \ . \tag{3.18}$$

Again, we can ignore the action of the S_{N-4} part — this will only ensure that the $N-4$ untwisted bosons and fermions from the directions that are unaffected by the twist reproduce again the contribution from the untwisted sector. The remaining group $S_2 \ltimes \mathbb{Z}_2^2 \cong D_8$ (the

dihedral group of order 8) has five irreducible representations, four of dimension 1, and one of dimension 2. In order to describe them, we first note that the abelian $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup has 4 different one-dimensional representations that are characterised by the eigenvalues $(\pm, \pm)$ of the two non-trivial $\mathbb{Z}_2$ generators. In D_8, both $(+, +)$ and $(-, -)$ give rise to two one-dimensional representations each that differ by the sign under the exchange of S_2 — this accounts for the 4 one-dimensional representations. The two-dimensional representation of D_8 is spanned by the two states with mixed charges $(\pm, \mp)$ that are exchanged under the action of S_2.

The simplest way to describe the contribution of these representations to the twisted sector is in multi-particle form. It follows from the derivation from eq. (3.16) that the $(2)^2$ sector has the partition function

$$Z^{(\mathrm{U})} \cdot \frac{1}{2} \left[\left(\Big| \sum_{\substack{\Delta \text{ even,} \\ \ell}} |c(\Delta, \ell)| \, q^{\frac{1}{2}(\Delta + \ell + 1)} y^{\ell + 1} \Big|^2 + \Big| \sum_{\substack{\Delta \text{ odd,} \\ \ell}} |c(\Delta, \ell)| \, q^{\frac{1}{2}(\Delta + \ell + 1)} y^{\ell + 1} \Big|^2 \right)^2 \right.$$
$$\left. + \Big| \sum_{\substack{\Delta \text{ even,} \\ \ell}} c(\Delta, \ell) q^{\Delta + \ell + 1} y^{2\ell + 2} \Big|^2 + \Big| \sum_{\substack{\Delta \text{ odd,} \\ \ell}} c(\Delta, \ell) q^{\Delta + \ell + 1} y^{2\ell + 2} \Big|^2 \right]. \qquad (3.19)$$

Since the wedge characters of the 2-cycle twisted sector, see eq. (3.9), are given by

$$\chi_{\pm}^{(2)} = \sum_{\substack{\Delta \text{ even/odd,} \\ \ell}} |c(\Delta, \ell)| \, q^{\frac{1}{2}(\Delta + \ell + 1)} y^{\ell + 1}, \qquad (3.20)$$

the above $(2)^2$ sector partition function can then be written as

$$Z^{(2)^2} = Z^{(\mathrm{U})} \cdot \left[\left| \left(\chi_+^{(2)} \right)_{\square\square} \right|^2 + \left| \left(\chi_+^{(2)} \right)_{\boxminus} \right|^2 + \left| \left(\chi_-^{(2)} \right)_{\square\square} \right|^2 + \left| \left(\chi_-^{(2)} \right)_{\boxminus} \right|^2 + \left| \chi_+^{(2)} \chi_-^{(2)} \right|^2 \right], \quad (3.21)$$

where

$$(\chi_{\pm}^{(2)})_{\square\square/\boxminus}(q, y) = \frac{1}{2} \left(\chi_{\pm}^{(2)}(q, y)^2 \pm \tilde{\chi}_{\pm}(q^2, y^2) \right)$$
$$= \frac{1}{2} \left[\left(\sum_{\substack{\Delta \text{ even/odd,} \\ \ell}} |c(\Delta, \ell)| \, q^{\frac{1}{2}(\Delta + \ell + 1)} y^{\ell + 1} \right)^2 \pm \sum_{\substack{\Delta \text{ even/odd,} \\ \ell}} c(\Delta, \ell) \, q^{\Delta + \ell + 1} y^{2\ell + 2} \right].$$
$$(3.22)$$

Each of the terms in (3.21) corresponds to one of the five irreducible representations of D_8, and can be organised in terms of coset representations. In order to describe this in detail, let us start from the ground state that has the eigenvalues $(+, +)$ with respect to the two $\mathbb{Z}_2$ factors; it appears in the $(\chi_-^{(2)})_{\boxminus}$ sector,[5] is an element of the coset representation

$$\left(\Lambda_+; \Lambda_- \right) = \left([0, k/2, 0, \ldots, 0]; [0, k/2, 0, \ldots, 0] \right), \qquad (3.23)$$

---

[5]Our convention for the definition of $\chi_{\pm}^{(2)}$ follows [6], and is motivated by the fact that $\pm$ corresponds to even/odd in eq. (3.20); this then leads to the somewhat strange (but inevitable) conclusion that the corresponding $\mathbb{Z}_2$ eigenvalue is $\mp$, see also eq. (7.17) and (7.18) of [6].

and therefore has the twist $\xi = [-1/2, -1/2, 0, \ldots, 0]$. All other states of the $(2)^2$ twisted sector can be obtained by adding boxes to Λ_- (while leaving Λ_+ invariant), yielding

$$\Lambda_- = [l_1, k/2 + l_2, \Lambda'], \tag{3.24}$$

where $l_1, l_2 \in \mathbb{Z}$, and Λ' denotes the remaining $N - 4$ Dynkin labels. For example, $l_1 = 0$, $l_2 = 1$ contains the ground state transforming as $(-, -)$ with respect to the two $\mathbb{Z}_2$ factors — it appears in the sector $(\chi_+^{(2)})_{\square\square}$ — while $l_1 = 1$, $l_2 = 0$ contains the ground state with eigenvalues $(+, -)$, which appears in the sector $\chi_+^{(2)}\chi_-^{(2)}$. The other two dihedral representations only contribute at order q^1; in terms of coset representations we have the decompositions

$$
\begin{aligned}
\mathcal{Z}_{\text{vac}} \cdot \left(\chi_+^{(2)}\right)_{\square\square} &= \chi_{([0,k/2,0,\ldots,0];[0,k/2+1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,0])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[0,k/2,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2+1,0,\ldots,0,1])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[2,k/2+1,0,\ldots,0])} + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[2,k/2-2,1,0,\ldots,0])} \\
&\quad + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,0,1])} + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[0,k/2,1,0,\ldots,0,1])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,0,1])} + \mathcal{O}(q^2) , \\
\mathcal{Z}_{\text{vac}} \cdot \left(\chi_+^{(2)}\right)_{\boxminus} &= \chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,0])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[0,k/2,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2+1,0,\ldots,0,1])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[2,k/2+1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-2,0,\ldots,0])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[0,k/2,0,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2+1,0,\ldots,0,1,0])} \\
&\quad + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[0,k/2,1,0,\ldots,0,1])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,0,1])} \\
&\quad + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[2,k/2-2,1,0,\ldots,0])} + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,0,1])} \\
&\quad + \mathcal{O}(q^2) , \\
\mathcal{Z}_{\text{vac}} \cdot \left(\chi_-^{(2)}\right)_{\square\square} &= \chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,0])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[0,k/2-1,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2,0,\ldots,0,1])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[2,k/2+1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-2,0,\ldots,0])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[0,k/2-1,0,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2,0,\ldots,0,1,0])} \\
&\quad + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,1,0,\ldots,0])} + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,0,1])} \\
&\quad + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[0,k/2-1,1,0,\ldots,0,1])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[2,k/2-2,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,0,\ldots,0,1])} \\
&\quad + \mathcal{O}(q^2) , \\
\mathcal{Z}_{\text{vac}} \cdot \left(\chi_-^{(2)}\right)_{\boxminus} &= \chi_{([0,k/2,0,\ldots,0];[0,k/2,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[2,k/2,0,\ldots,0])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[0,k/2-1,1,0,\ldots,0])} + \chi_{([0,k/2,0,\ldots,0];[0,k/2,0,\ldots,0,1])} \\
&\quad + \chi_{([0,k/2,0,\ldots,0];[2,k/2-2,0,\ldots,0])} + 2 \cdot \chi_{([0,k/2,0,\ldots,0];[2,k/2-1,1,0,\ldots,0])}
\end{aligned}
$$

$$+ 2 \cdot \chi_{([0,k/2,0,...,0];[2,k/2,0,...,0,1])} + 2 \cdot \chi_{([0,k/2,0,...,0];[0,k/2-1,1,0,...,0,1])}$$
$$+ \chi_{([0,k/2,0,...,0];[2,k/2-2,1,0,...,0])} + \chi_{([0,k/2,0,...,0];[2,k/2-1,0,...,0,1])} + \mathcal{O}(q^2) \,,$$

$$\mathcal{Z}_{\text{vac}} \cdot \chi_+^{(2)} \chi_-^{(2)} = \chi_{([0,k/2,0,...,0];[1,k/2,0,...,0])}$$
$$+ 2 \cdot \chi_{([0,k/2,0,...,0];[1,k/2-1,1,0,...,0])} + 2 \cdot \chi_{([0,k/2,0,...,0];[1,k/2,0,...,0,1])}$$
$$+ \chi_{([0,k/2,0,...,0];[1,k/2-1,0,...,0])} + \chi_{([0,k/2,0,...,0];[1,k/2+1,0,...,0])}$$
$$+ 2 \cdot \chi_{([0,k/2,0,...,0];[1,k/2-2,1,0,...,0])} + 2 \cdot \chi_{([0,k/2,0,...,0];[1,k/2-1,0,...,0,1])}$$
$$+ 4 \cdot \chi_{([0,k/2,0,...,0];[1,k/2-1,1,0,...,0,1])}$$
$$+ \chi_{([0,k/2,0,...,0];[1,k/2-1,0,1,0,...,0])} + \chi_{([0,k/2,0,...,0];[1,k/2,0,...,0,1,0])}$$
$$+ 2 \cdot \chi_{([0,k/2,0,...,0];[1,k/2,1,0,...,0])} + 2 \cdot \chi_{([0,k/2,0,...,0];[1,k/2+1,0,...,0,1])}$$
$$+ 2 \cdot \chi_{([0,k/2,0,...,0];[3,k/2,0,...,0])} + 2 \cdot \chi_{([0,k/2,0,...,0];[3,k/2-1,0,...,0])}$$
$$+ 2 \cdot \chi_{([0,k/2,0,...,0];[3,k/2-2,0,...,0])} + \mathcal{O}(q^2) \,. \tag{3.25}$$

The systematics of the decompositions are analogous to the 2-cycle twist case, see the discussion following eq. (3.14) above, but are somewhat more complicated. Each box appended to the first two rows of Λ_- of the ground state (3.23) has odd parity under one of the two $\mathbb{Z}_2$'s. As a consequence, the states that appear in the mixed sector $\chi_+^{(2)} \chi_-^{(2)}$ are precisely those that have an odd number of them, i.e., for which l_1 is odd. Conversely, the other four representations contain the states with l_1 even, but the selection rules among them are more subtle, and indeed, the same coset representation can appear in different D_8 decompositions. For example, the lowest state in the representation

$$\Lambda_- = [2, k/2, 0, \dots, 0] \tag{3.26}$$

can be constructed as an excitation of the twisted sector ground state with a fermionic zero-mode and a bosonic $\left(-\frac{1}{2}\right)$-mode involving the same twisted coordinate. Then the state has $(+, +)$ charge under $\mathbb{Z}_2^2$, and we can either symmetrise or anti-symmetrise it with respect to the S_2 factor. That is why this state appears both in $\left(\chi_-^{(2)}\right)_{\square\square}$ and in $\left(\chi_-^{(2)}\right)_{\boxminus}$. But we can also construct the lowest state of (3.26) by exciting the twisted sector ground state with a fermionic zero-mode from one twisted coordinate, and a bosonic $\left(-\frac{1}{2}\right)$-mode from the other, and symmetrise with respect to S_2.[6] In this case the charge is $(-, -)$ under $\mathbb{Z}_2^2$, and the state is even under the S_2; thus the representation (3.26) also appears in the decomposition of $\left(\chi_+^{(2)}\right)_{\boxminus}$.

3.3 Sectors of arbitrary twist

While the detailed description of the decompositions into $s\mathcal{W}_\infty$ characters becomes more and more combersome, some aspects of the twisted sector can be described quite generally. In particular, the partition function of any twisted sector can be written in 'multiparticle'

---

[6]The antisymmetric combination is actually a supersymmetric descendant of the excitation by the two fermionic zero-modes and is therefore part of $([0, k/2, 0, \dots, 0]; [0, k/2 + 1, 0, \dots, 0])$.

form, generalising eq. (3.21).[7] Let us first explain this for the twisted sectors $(2)^n$ corresponding to multiple 2-cycle twists. By the DMVV formula (2.5), the generating function for this part of the partition function in the R-R sector equals

$$\sum_{N=0}^{\infty} p^N Z_{\mathrm{R}}^{(2)^n}\left(S^N \mathbb{T}^2\right) = {\prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}}}' \frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1}p^2\, q^{\frac{\Delta}{2}}\bar{q}^{\frac{\bar{\Delta}}{2}}y^\ell\bar{y}^{\bar{\ell}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\Bigg|_{p^{2n}}$$
$$\times\, p^{2n}\prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}} \frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1}p\, q^{\Delta}\bar{q}^{\bar{\Delta}}y^\ell\bar{y}^{\bar{\ell}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\ . \tag{3.27}$$

We recognise the second factor as the untwisted partition function of S_{N-2n}, which is indistinguishable from the untwisted partition function of S_N as $N \to \infty$. The first factor, on the other hand, can be expressed in terms of sums of squares of all possible symmetrisations of the elementary characters $\chi_{\pm}^{(2)}$. To see this, let us write

$${\prod_{\Delta,\bar{\Delta},\ell,\bar{\ell}}}' \frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1}p^2 q^{\frac{\Delta}{2}}\bar{q}^{\frac{\bar{\Delta}}{2}}y^\ell\bar{y}^{\bar{\ell}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\Bigg|_{p^{2n}} =$$

$$= \exp\left[-{\sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}}' c(\Delta,\bar{\Delta},\ell,\bar{\ell})\log\left(1-(-1)^{\ell+\bar{\ell}+1}p^2 q^{\frac{\Delta}{2}}\bar{q}^{\frac{\bar{\Delta}}{2}}y^\ell\bar{y}^{\bar{\ell}}\right)\right]\Bigg|_{p^{2n}}$$

$$= \exp\left[{\sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}}' c(\Delta,\bar{\Delta},\ell,\bar{\ell})\sum_{k=1}^{\infty}\frac{p^{2k}}{k}(-1)^{k(\ell+\bar{\ell}+1)}q^{\frac{k\Delta}{2}}\bar{q}^{\frac{k\bar{\Delta}}{2}}y^{k\ell}\bar{y}^{k\bar{\ell}}\right]\Bigg|_{p^{2n}}\ .$$

Changing the order of summation and flowing to the NS-NS sector, this becomes

$$\exp\left[\sum_{k=1}^{\infty}\frac{p^{2k}}{k}{\sum_{\Delta,\bar{\Delta},\ell,\bar{\ell}}}' c(\Delta,\bar{\Delta},\ell,\bar{\ell})(-1)^{k(\ell+\bar{\ell}+1)}q^{\frac{k}{2}(\Delta+\ell+1)}\bar{q}^{\frac{k}{2}(\bar{\Delta}+\bar{\ell}+1)}y^{k(\ell+1)}\bar{y}^{k(\bar{\ell}+1)}\right]\Bigg|_{p^{2n}}$$

$$= \exp\left[\sum_{k=1}^{\infty}\frac{p^{2k}}{k}\left(\left|\mathcal{F}^{k-1}\chi_{+}^{(2)}(q^k,y^k)\right|^2 + \left|\mathcal{F}^{k-1}\chi_{-}^{(2)}(q^k,y^k)\right|^2\right)\right]\Bigg|_{p^{2n}}$$

$$= \sum_{m=0}^{n}\exp\left[\sum_{k=1}^{m}\frac{p^{2k}}{k}\left|\mathcal{F}^{k-1}\chi_{+}^{(2)}(q^k,y^k)\right|^2\right]\Bigg|_{p^{2m}}\cdot\exp\left[\sum_{k=1}^{n-m}\frac{p^{2k}}{k}\left|\mathcal{F}^{k-1}\chi_{-}^{(2)}(q^k,y^k)\right|^2\right]\Bigg|_{p^{2(n-m)}}\ .$$

---

[7]This observation was recently also made in [35], and in fact the following analysis is not specific to the specific theory we are considering here. In particular, it also applies similarly to the $\mathcal{N}=4$ case that was studied in [6].

Next we note that

$$
\exp\left[\sum_{j=1}^{m} \frac{p^{2j}}{j} \left|\mathcal{F}^{j-1}\chi(q^j,y^j)\right|^2\right]\Bigg|_{p^{2m}} = \sum_{\substack{k_1,\dots,k_m\geq 0 \\ \sum_j jk_j=m}} \frac{1}{\prod_{i=1}^{m} i^{k_i}k_i!}\prod_{j=1}^{m}\left|\mathcal{F}^{j-1}\chi(q^j,y^j)\right|^{2k_j}
$$

$$
= \frac{1}{m!}\sum_{\rho\in S_m}\prod_{j=1}^{m}\left|\mathcal{F}^{j-1}\chi(q^j,y^j)\right|^{2a_j(\rho)}
$$

$$
= \sum_{\Lambda\in Y_m}\left|\chi_\Lambda(q,y)\right|^2 . \tag{3.28}
$$

In the second equality, we have used that $m!/\prod_{i=1}^{m} i^{k_i}k_i!$ is the number of elements in the conjugacy class $C_{k_1,\dots,k_m}$ of S_m, which consist of k_i cycles of length i. On the other hand, the last equality follows from (2.27) and the column orthogonality of S_m characters,

$$
\sum_{\Lambda\in Y_m}\left(\chi_m^\Lambda(\rho)\right)^2 = \frac{m!}{|C_\rho|} \qquad \text{for any } \rho\in S_m . \tag{3.29}
$$

Here the sum is over all Young diagrams of m boxes or all irreducible representations of S_m. It follows that

$$
Z^{(2)^n}\!\left(S^N\mathbb{T}^2\right) = Z^{(\mathrm{U})}\!\left(S^{N-2n}\mathbb{T}^2\right)\cdot\sum_{k=0}^{n}\sum_{\Lambda_1\in Y_k}\left|(\chi_+^{(2)})_{\Lambda_1}(q,y)\right|^2\sum_{\Lambda_2\in Y_{n-k}}\left|(\chi_-^{(2)})_{\Lambda_2}(q,y)\right|^2 , \tag{3.30}
$$

thus generalising (3.21) to the case $n>2$.

So far we have only dealt with multiple 2-cycles, but the analysis is fairly analogous for the twist $(m)^n$ consisting of n non-overlapping m-cycles. The analogue of eq. (3.27) for $m\geq 2$ is now

$$
\sum_{N=0}^{\infty}p^N Z_{\mathrm{R}}^{(m)^n}\left(S^N\mathbb{T}^2\right) = \prod_{\substack{\Delta,\bar\Delta,\ell,\bar\ell \\ m|(\Delta-\bar\Delta)}}\frac{1}{\left(1-(-1)^{\ell+\bar\ell+1}p^m\,q^{\frac{\Delta}{m}}\bar q^{\frac{\bar\Delta}{m}}y^\ell\bar y^{\bar\ell}\right)^{c(\Delta,\bar\Delta,\ell,\bar\ell)}}\Bigg|_{p^{mn}}
$$

$$
\times\, p^{mn}\prod_{\Delta,\bar\Delta,\ell,\bar\ell}\frac{1}{\left(1-(-1)^{\ell+\bar\ell+1}p\,q^\Delta\bar q^{\bar\Delta}y^\ell\bar y^{\bar\ell}\right)^{c(\Delta,\bar\Delta,\ell,\bar\ell)}} . \tag{3.31}
$$

The analysis goes through essentially unmodified, and we find that we can express the partition function of this sector in terms of the elementary characters

$$
\chi_i^{(m)}(q,y) = \sum_{\substack{\Delta,\ell \\ \Delta\equiv i\,(\mathrm{mod}\,m)}}|c(\Delta,\ell)|\,q^{\frac{\Delta}{m}+\frac{\ell}{2}+\frac{m}{4}}\,y^{\ell+\frac{m}{2}} \quad \text{for } i=1,\dots,m \tag{3.32}
$$

as

$$
Z^{(m)^n}\left(S^N\mathbb{T}^2\right) = Z^{(\mathrm{U})}\left(S^{N-mn}\mathbb{T}^2\right)\cdot\sum_{\substack{k_1,\dots,k_m\geq 0 \\ \sum_j k_j=n}}\prod_{i=1}^{m}\sum_{\Lambda\in Y_{k_i}}\left|(\chi_i^{(m)})_\Lambda(q,y)\right|^2 . \tag{3.33}
$$

In particular, in the sector whose twist is just one cycle of length m, we have $n = 1$ and thus

$$Z^{(m)}\left(S^N\mathbb{T}^2\right) = Z^{(\text{U})}\left(S^{N-m}\mathbb{T}^2\right) \cdot \sum_{i=1}^{m}\left|\chi_i^{(m)}(q,y)\right|^2 . \tag{3.34}$$

It remains to combine these statements to cover the general case of a twist with cycle structure $(1)^{N_1}(2)^{N_2}\cdots(n)^{N_n}$, i.e., N_i cycles of length i for $i = 1, \ldots, n$, where $\sum_i iN_i = N$. By the DMVV formula (2.5), the R-R partition function factorises into n components pertaining to the different cycle lengths

$$Z_{\text{R}}^{(1)^{N_1}\cdots(n)^{N_n}}(S^N\mathbb{T}^2) = \prod_{m=1}^{n}\prod_{\substack{\Delta,\bar{\Delta},\ell,\bar{\ell} \\ m|(\Delta-\bar{\Delta})}}\frac{1}{\left(1-(-1)^{\ell+\bar{\ell}+1}p^m\,q^{\frac{\Delta}{m}}\bar{q}^{\frac{\bar{\Delta}}{m}}y^\ell\bar{y}^\ell\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}\Bigg|_{p^{mN_m}}, \tag{3.35}$$

and correspondingly for the NS-NS sector. Plugging in our results from above, we obtain

$$Z^{(1)^{N_1}\cdots(n)^{N_n}}\left(S^N\mathbb{T}^2\right) = \prod_{m=1}^{n}\sum_{\substack{k_1,\ldots,k_m\geq 0 \\ \sum_j k_j = N_m}}\prod_{i=1}^{m}\sum_{\Lambda\in Y_{k_i}}\left|(\chi_i^{(m)})_\Lambda(q,y)\right|^2$$

$$= Z^{(\text{U})}\left(S^{N_1}\mathbb{T}^2\right)\cdot\prod_{m=2}^{n}\sum_{\substack{k_1,\ldots,k_m\geq 0 \\ \sum_j k_j = N_m}}\prod_{i=1}^{m}\sum_{\Lambda\in Y_{k_i}}\left|(\chi_i^{(m)})_\Lambda(q,y)\right|^2 . \tag{3.36}$$

Thus we can think of the entire twisted sector as consisting of the 'multiparticle' contributions of the fundamental building blocks (3.32).

As was already alluded to before, essentially the same techniques also allow us to prove the identity eq. (2.26) for the untwisted sector partition function. Since $\chi_1^{(1)}(q,y) = Z_{\text{NS}}^{(\text{chiral})}(\mathbb{T}^2)(q,y) = 1 + \chi_1(q,y)$ and

$$\mathcal{Z}'_{\text{vac}}(q,y) = \prod_{\substack{(\Delta,\ell) \\ \neq(0,-\frac{1}{2})}}\frac{1}{\left(1-q^{\Delta+\frac{\ell}{2}+\frac{1}{4}}(-y)^{\ell+\frac{1}{2}}\right)^{c(\Delta,\ell)}} , \tag{3.37}$$

we get, for $N \to \infty$,

$$Z^{(\text{U})}(S^N\mathbb{T}^2)(q,\bar{q},y,\bar{y}) = \prod_{\substack{(\Delta,\bar{\Delta},\ell,\bar{\ell}) \\ \neq(0,0,-\frac{1}{2},-\frac{1}{2})}}\frac{1}{\left(1-q^{\Delta+\frac{\ell}{2}+\frac{1}{4}}\bar{q}^{\bar{\Delta}+\frac{\bar{\ell}}{2}+\frac{1}{4}}(-y)^{\ell+\frac{1}{2}}(-\bar{y})^{\bar{\ell}+\frac{1}{2}}\right)^{c(\Delta,\bar{\Delta},\ell,\bar{\ell})}}$$

$$= |\mathcal{Z}'_{\text{vac}}(q,y)|^2 \cdot \exp\sum_{k=1}^{\infty}\frac{1}{k}\left|\mathcal{F}^{k-1}\chi_1(q^k,y^k)\right|^2$$

$$= |\mathcal{Z}'_{\text{vac}}(q,y)|^2\sum_{m=0}^{\infty}\sum_{\Lambda\in Y_m}|\chi_\Lambda(q,y)|^2 , \tag{3.38}$$

which reproduces (2.26) upon dividing by $Z_{\text{NS}}(\mathbb{T}^2)$, see eq. (2.14).

3.4 Twisted representations of the wedge algebra

Given the multiparticle structure of the entire twisted sector, see eq. (3.36), it only remains to understand the structure of the building blocks $\chi_i^{(m)}$ (that account for the individual 'particles'). These wedge characters count states that sit in representations of the wedge subalgebra $\mathrm{shs}[\mu]$ of $s\mathcal{W}_\infty[\mu]$. In this section we undertake first steps to understand the structure of these higher spin representations. This should shed light on the 'particle' structure of the stringy extension of the higher spin theory; in [30] the relevant analysis was done for the bosonic toy model consisting of a single boson, here we explain the $\mathcal{N} = 2$ generalisation.

As was explained at the beginning of this section, the m-cycle twisted sector is generated by complex fermions and bosons of twist $\xi_i = \frac{i}{m}$, where $i = 1, \ldots, m$. Since the $s\mathcal{W}_\infty$ generators are neutral bilinears in the currents (and since their mode numbers continue to be integers or half-integers depending on the statistics), the contribution coming from the individual twisted (complex) bosons and fermions decouple from one another, and we can think of the representation as consisting of an m-fold tensor product of the individual twist ξ_i contributions. Apart from one untwisted component corresponding to $i = m$ — this does not contribute to the wedge character — the other $(m-1)$ components all lead to representations whose wedge character is of the form (see also [30])

$$\chi_\xi(q, y) = q^h \prod_{n=1}^{\infty} \frac{(1 + zyq^{n-\frac{1}{2}-\xi})(1 + z^{-1}y^{-1}q^{n-\frac{1}{2}+\xi})}{(1 - zq^{n-\xi})(1 - z^{-1}q^{n-1+\xi})} \bigg|_{z^p} . \tag{3.39}$$

Here we have assumed that $0 < \xi < \frac{1}{2}$, and z keeps track of the twist, i.e., the terms with a given power of z^p pick up the same phase under the cyclic group $\mathbb{Z}_m$ in the centraliser. In the following, we shall concentrate on the z^0 case, for which the states transforms trivially under $\mathbb{Z}_m$. The q-expansion of this character is

$$\chi_\xi(q, y) = q^h \left(1 + yq^{\frac{1}{2}} + 2q + (3y + y^{-1})q^{\frac{3}{2}} + (y^2 + 6)q^2 + (8y + 3y^{-1})q^{\frac{5}{2}} + \ldots \right) . \tag{3.40}$$

For $\xi < \frac{1}{2} < 1$ there is a similar answer where y is replaced by $y \mapsto y^{-1}$; the case $\xi = \frac{1}{2}$ is a bit special since there are then fermionic zero modes.

Each such representation has a single descendant at level $1/2$, and is therefore a special case of what one may like to call a 'level-$1/2$ representation', compare the terminology of [30]. Thus we can learn about the structure of the twisted sector by studying general level-$1/2$ representations, and this is what we shall be doing in the following.

Suppose ϕ is the ground state of a level-$1/2$ representation. Let us assume for definiteness that ϕ is annihilated by $G_{-1/2}^-$ (rather than $G_{-1/2}^+$), i.e.,

$$G_{-1/2}^- \phi = 0 , \tag{3.41}$$

as well as by all the other negative charge fermionic spin s supercharges, i.e.,

$$W_{-1/2}^{s-} \phi = 0 \qquad \text{for } s = 2, 3, \ldots . \tag{3.42}$$

(This is the situation that is relevant for (3.40); the conjugate solution arises for $\frac{1}{2} < \xi < 1$.) Here we have denoted the generators of the spin s multiplet by (see e.g., [36])

$$W^{s0}, \qquad W^{s\pm}, \qquad W^{s1} \tag{3.43}$$

of spin s, $s + \frac{1}{2}$, and $s + 1$, respectively. The corresponding modes then transform in a representation of the superconformal algebra

$$
\begin{aligned}
[G_r^\pm, W_n^{s0}] &= \mp W_{r+n}^{s\pm} \\
\{G_r^\pm, W_r^{s\pm}\} &= 0 \\
\{G_r^\pm, W_r^{s\mp}\} &= \pm\big((2s-1)r - t\big) W_{r+t}^{s0} + 2W_{r+t}^{s1} \\
[G_r^\pm, W_n^{s1}] &= \big(sr - \tfrac{1}{2}n\big) W_{r+n}^{s\pm} .
\end{aligned} \tag{3.44}
$$

Let us denote the eigenvalues of the zero-modes W_0^{s0} and W_0^{s1} on the ground state ϕ by w^{s0} and w^{s1}, respectively. Then it follows from (3.42) that

$$0 = G_{1/2}^+ W_{-1/2}^{s-} \phi = \big(sw^{s0} + 2w^{s1}\big) \phi \tag{3.45}$$

and hence

$$w^{s1} = -\frac{1}{2} s w^{s0} . \tag{3.46}$$

Note that for $s = 1$ this reduces to the familiar chiral primary condition, namely that $h = -\frac{1}{2}q$, where $q = w^{10}$ is the U(1) charge with respect to the spin 1 field in the $\mathcal{N} = 2$ supermultiplet, and $h = w^{11}$ is the conformal dimension.

The other condition that follows from the level-1/2 condition is that all the states generated by the $W_{-1/2}^{s+}$ modes from the ground state are proportional to $G_{-1/2}^+ \phi$, i.e.,

$$W_{-1/2}^{s+} \phi = \alpha(s)\, G_{-1/2}^+ \phi . \tag{3.47}$$

Applying $G_{1/2}^-$ to this relation and using the above commutation relations, we find that

$$\alpha(s) = -\frac{sw^{s0}}{2h} , \tag{3.48}$$

where we have used (3.46).

In order to obtain a relation between the different quantum numbers $\alpha(s)$, we finally apply the W_0^{20} mode to both sides of eq. (3.47). For example, for the case where $s = 2$ and using the $[W_m^{20}, W_r^{2+}]$ commutation relation, we conclude that

$$\alpha(3) = -\frac{8q_3 \left(5\nu^2 - 8\sqrt{3}\nu\alpha(2) - 15(8\alpha(2)^2 + 3)\right)}{9(\nu - 5)} , \tag{3.49}$$

where $\nu = 2\mu - 1$ and q_3 is a normalisation constant of W^{30}. This determines $\alpha(3)$ as a function of $\alpha(2)$. Continuing in this manner, we obtain a recursion relation for all $\alpha(s)$. This shows that all higher quantum numbers w^{s0} and w^{s1} are recursively determined. Thus the assumption that there is a single state at level 1/2 implies that the most general level-1/2 representation is characterised by only two quantum numbers

$$h \equiv w^{11} \quad \text{and} \quad \alpha(2) \equiv -\frac{w^{20}}{h} . \tag{3.50}$$

3.5 A relation between the parameters

As in the bosonic analysis of [30], it seems that the actual ξ-twisted representation is a special type of level-1/2 representation, and has in fact one fewer state at level 3/2 than a generic level-1/2 representation.[8] One should therefore expect that it is characterised by a special relation between the two eigenvalues in (3.50). In order to work out what this relation should be, we can use that the ξ-twisted representation is described, in the coset language, by the large k limit of the coset representation $([\xi k, 0, \ldots, 0]; [\xi k, 0, \ldots, 0])$ [23]. In order to evaluate the eigenvalues of L and W^{20} on this coset state, we have worked out the form of the spin 2 fields in the coset; this is dicussed, in some detail, in the appendix. With the notation of the appendix, in particular, (A.19), (A.21), (A.23) and (A.26), we find that in the (large c and $\nu = -1$) 't Hooft limit

$$L = L_{\rm b} + L_{\rm f} + \frac{3}{2c} J^2 \,,$$
$$W^{20} = \frac{1}{\sqrt{3}} \left(-L_{\rm b} + 2 \, L_{\rm f} \right) \,. \tag{3.51}$$

The mode expansions of the stress tensor of a single free boson and fermion are given by (the fermion has NS boundary conditions)

$$(L_{\rm b})_m = \sum_{n \in \mathbb{Z}}^{\infty} {:}\bar{\alpha}_{m-n} \alpha_n{:} \,,$$
$$(L_{\rm f})_m = \frac{1}{2} \sum_{r \in \mathbb{Z}+1/2}^{\infty} (2r - m) {:}\bar{\psi}_{m-r} \psi_r{:} \,. \tag{3.52}$$

Here the bosonic and fermionic modes satisfy the usual commutation relations

$$[\alpha_m, \alpha_n] = 0 = [\bar{\alpha}_m, \bar{\alpha}_n] \,, \qquad [\alpha_m, \bar{\alpha}_n] = m \delta_{m,-n} \,,$$
$$\{\psi_r, \psi_s\} = 0 = \{\bar{\psi}_r, \bar{\psi}_s\} \,, \qquad \{\psi_r, \bar{\psi}_s\} = \delta_{r,-s} \,. \tag{3.53}$$

In the ξ-twisted sector, the boson and fermion mode numbers get shifted, and the zero mode of the stress tensor picks up a normal-ordering contribution

$$(L_{\rm b})_0 = \sum_{r \in \mathbb{Z}+\xi} {:}\bar{\alpha}_{-r} \alpha_r{:} \ + \ \frac{1}{2} \xi \, (1 - \xi) \,,$$
$$(L_{\rm f})_0 = \sum_{s \in \mathbb{Z}+\frac{1}{2}+\xi} s \, {:}\bar{\psi}_{-s} \psi_s{:} \ + \ \frac{\xi^2}{2} \,. \tag{3.54}$$

For large c we then find for the eigenvalues of L_0 and W_0^{20}

$$h = \frac{\xi}{2} \,, \qquad w^{20} = \frac{1}{2\sqrt{3}} \xi(3\xi - 1) \,. \tag{3.55}$$

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[8]This will be discussed in more detail in [37].

Eliminating ξ from the above equations yields

$$\alpha(2) = -\frac{w^{20}}{h} = -\frac{1}{\sqrt{3}}\left(3\xi - 1\right) = -\frac{6h - 1}{\sqrt{3}}\ . \tag{3.56}$$

This is therefore the additional relation which characterises the special level-1/2 representations that arise in the twisted sector.

4 Conclusions

In this paper we have analysed the embedding of the $\mathcal{N} = 2$ cosets that appear in the duality with the $\mathcal{N} = 2$ supersymmetric higher spin theory on AdS$_3$ into the symmetric orbifold of $\mathbb{T}^2$. This is the $\mathcal{N} = 2$ analogue of the $\mathcal{N} = 4$ construction of [6] where the relevant symmetric orbifold is known to be dual to string theory on AdS$_3 \times$ S$^3 \times \mathbb{T}^4$. It is therefore tempting to believe, in particular given the recent discussions of [26–29], that also the symmetric orbifold of $\mathbb{T}^2$ should be dual to some string theory on AdS$_3$. For example, a candidate background could be the (warped) product of the form

$$\text{AdS}_3 \times \text{S}^3 \times \left(\mathbb{T}^2 \times \mathbb{T}^2\right)/S_2\,, \tag{4.1}$$

where the S_2 exchanges the two $\mathbb{T}^2$'s — this is not too dissimilar to the background with $(4, 2)$ superconformal symmetry found in [38].[9] Alternatively, one may want to replace the symmetric orbifold with an orbifold with respect to a smaller group, e.g.,

$$\left(\mathbb{T}^2\right)^{2N}/(S_2^N \rtimes S_N)\,, \tag{4.2}$$

where $S_2^N \rtimes S_N$ is the so-called wreath product, i.e., the semidirect product which contains S_2^N as a subgroup on which S_N acts in the obvious manner. Since the wreath product is a subgroup of the full permutation group, $S_2^N \rtimes S_N \subset S_{2N}$, the corresponding conformal field theory defines an even further extension of the symmetric orbifold we have considered above. In particular, it therefore contains the $\mathcal{N} = 2$ Kazama-Suzuki models that are dual to the higher spin theory on AdS$_3$.

Part of the motivation for studying the $\mathcal{N} = 2$ version of the duality is that the Kazama-Suzuki models that appear in the dual of the higher spin theory [18, 19] correspond to the special family of $\mathcal{N} = 2$ cosets

$$\frac{\mathfrak{su}(N + M)^{(1)}_{k+N+M}}{\mathfrak{su}(N)^{(1)}_{k+N+M} \oplus \mathfrak{su}(M)^{(1)}_{k+N+M} \oplus \mathfrak{u}(1)^{(1)}_\kappa} \tag{4.3}$$

with $M = 1$. The cosets therefore allow for a 'matrix-like' extension ($M > 1$), similar to what was considered for the case of AdS$_4$ in [5], and it would be very interesting to understand the correct AdS$_3$ description of this construction. First steps in this direction were already undertaken in [22], but it would be very instructive to repeat the analysis of

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[9]As far as we are aware, no supergravity background with $(2, 2)$ superconformal symmtry is explicitly known, although such backgrounds probably exist. We thank Jerome Gauntlett for a correspondence about this point.

the present paper for these more general cosets, and see how the results fit together with permutation orbifold theories that may have a fairly direct stringy interpretation.

We have also analysed the representations of the higher spin algebra that arise in the twisted sector; a good understanding of these representations will be key for characterising the stringy extension from a higher spin viewpoint. While some aspects of the description were rather similar to the bosonic analysis of [30], it seems that there are also interesting and subtle differences; these will be explored further in [37].

Acknowledgments

A   The coset analysis

In this appendix we explain in some detail the construction of the spin 1 and 2 currents of the coset

$$\frac{\mathfrak{su}(N+1)^{(1)}_{k+N+1}}{\mathfrak{su}(N)^{(1)}_{k+N+1} \oplus \mathfrak{u}(1)^{(1)}_{\kappa}} \tag{A.1}$$

with $\kappa = N(N+1)(N+k+1)$. We will closely follow the analysis of [39] in the $\mathcal{N} = 4$ case and [40]. The numerator consists of $N(N+2)$ bosonic currents $\mathcal{J}^A$ and free fermions ψ^A transforming in the adjoint representation of $\mathfrak{su}(N+1)$. Given a hermitian orthonormal basis t^A_{ij} of $\mathfrak{su}(N+1)$ satisfying

$$[t^A, t^B] = if^{ABC}t^C \qquad \text{and} \qquad \text{Tr}(t^A t^B) = \delta^{AB}, \tag{A.2}$$

which we order in such a way that t^a for $a = 1, \ldots, N^2 - 1$ form a hermitian orthonormal basis of $\mathfrak{su}(N)$, the numerator fields satisfy the commutation relations

$$\begin{aligned}
[\mathcal{J}^A_m, \mathcal{J}^B_n] &= if^{ABC}\mathcal{J}^C_{m+n} + (k+N+1)\delta^{AB}\delta_{m,-n}, \\
[\mathcal{J}^A_m, \psi^B_r] &= if^{ABC}\psi^C_{m+r}, \\
\{\psi^A_r, \psi^B_s\} &= \delta^{AB}\delta_{r,-s}.
\end{aligned} \tag{A.3}$$

Restricting the adjoint representation to the denominator subalgebra, it decomposes as

$$\mathfrak{su}(N+1) \rightarrow \mathfrak{su}(N) \oplus \mathfrak{u}(1) \oplus \mathbf{N} \oplus \bar{\mathbf{N}}. \tag{A.4}$$

We can decouple the currents from the fermions by defining

$$J^A = \mathcal{J}^A + \frac{i}{2}f^{ABC}(\psi^B\psi^C) \tag{A.5}$$

in the numerator or

$$\widetilde{\mathcal{J}}^a = \mathcal{J}^a + \frac{i}{2} f^{abc}(\psi^b \psi^c) \tag{A.6}$$

in the denominator, where again lower-case indices from the beginning of the alphabet range from 1 to $N^2 - 1$ only. These currents and the fermion bilinears give rise to the bosonic coset

$$\frac{\mathfrak{su}(N+1)_k \oplus \mathfrak{so}(2N)_1}{\mathfrak{su}(N)_{k+1} \oplus \mathfrak{u}(1)_\kappa} . \tag{A.7}$$

From the $N(N+2)$ fermions in the numerator we subtract the N^2 fermions in the denominator. The $2N$ surviving fermions can be defined by

$$\psi^i = t^A_{N+1,i} \psi^A , \qquad \bar{\psi}^i = t^A_{i,N+1} \psi^A , \tag{A.8}$$

satisfying

$$\{\psi^i_r, \bar{\psi}^j_s\} = \delta^{ij}\delta_{r,-s} ,$$
$$\{\psi^i_r, \psi^j_s\} = \{\bar{\psi}^i_r, \bar{\psi}^j_s\} = 0 . \tag{A.9}$$

The bosonic currents in the numerator can be split up in J^a for $a = 1, \ldots, N^2 - 1$, J^i and $\bar{J}^i$, for $i = 1, \ldots, N$, and K, where we define

$$J^i = t^A_{N+1,i} J^A , \qquad \bar{J}^i = t^A_{i,N+1} J^A , \qquad K = (N+1) \, t^A_{N+1,N+1} J^A . \tag{A.10}$$

Here, $\bar{J}^i$ and J^i correspond to the $\mathbf{N}$ and $\bar{\mathbf{N}}$ of $\mathfrak{su}(N)$, respectively, while K is the $\mathfrak{u}(1)$ current embedded into $\mathfrak{su}(N+1)$. The $\mathfrak{u}(1)$ embedding into $\mathfrak{so}(2N)$ can be written as

$$j = -(N+1)(\psi^i \bar{\psi}^i) . \tag{A.11}$$

The total $\mathfrak{u}(1)$ current is then equal to $K + j$. It will be useful to express the decoupled $\mathfrak{su}(N)_{k+1}$ currents in terms of the decoupled $\mathfrak{su}(N+1)_k$ currents:

$$\widetilde{J}^a = J^a + t^a_{ij}(\psi^i \bar{\psi}^j) , \tag{A.12}$$

where we have assumed, without loss of generality, that the matrices t^A for $A = N^2, \ldots, N(N+2) - 1$ are of the form

$$t^A = \left(\begin{array}{c|c} 0_N & * \\ \hline * & 0 \end{array} \right) , \qquad A = N^2, \ldots, N(N+2) - 1 , \tag{A.13}$$

and that $t^{N(N+2)}$ is diagonal. We also define the unique spin-1 primary of the coset, which is also the lowest field in the superconformal algebra, as

$$J = \frac{1}{N+k+1} \left(K - \frac{k}{N+1} j \right) . \tag{A.14}$$

Then the stress-energy tensor of the coset theory is given by the difference of the numerator and denominator Sugawara tensors:

$$L = L_{\mathfrak{su}(N+1)} - L_{\mathfrak{su}(N)} - L_{\mathfrak{u}(1)} + L_{\text{free fermions}}$$

$$= \frac{1}{2(N+k+1)}\Bigg((J^i \bar{J}^i) + (\bar{J}^i J^i) + k\left((\partial\psi^i\bar{\psi}^i) - (\psi^i\partial\bar{\psi}^i)\right)$$

$$- 2\,t_{ij}^a \left(J^a(\psi^i\bar{\psi}^j)\right) - \frac{2}{N(N+1)}\,(Kj) \Bigg), \tag{A.15}$$

where we have used that

$$(J^A J^A) = (J^a J^a) + (J^i \bar{J}^i) + (\bar{J}^i J^i) + \frac{1}{N(N+1)}\,(KK)\,. \tag{A.16}$$

We can split up the stress-energy tensor into three mutually commuting stress-energy tensors given by

$$L_{\mathrm{b}} = \frac{1}{2(N+k+1)}\Bigg((J^i\bar{J}^i) + (\bar{J}^i J^i) - \frac{1}{N+k}(J^a J^a) - \frac{1}{Nk}(KK) \Bigg),$$

$$L_{\mathrm{f}} = \frac{k}{2(N+k+1)}\Bigg((\partial\psi^i\bar{\psi}^i) - (\psi^i\partial\bar{\psi}^i) - \frac{2}{k}t_{ij}^a\left(J^a(\psi^i\bar{\psi}^j)\right) + \frac{1}{k(N+k)}(J^a J^a)$$

$$- \frac{1}{N(N+1)^2}(jj) \Bigg),$$

$$L_{\mathrm{(JJ)}} = \frac{N+k+1}{2Nk}(JJ)\,, \tag{A.17}$$

with central charges

$$c_{\mathrm{b}} = \frac{N(k-1)(N+2k+1)}{(N+k)(N+k+1)}\,,$$

$$c_{\mathrm{f}} = \frac{k(N-1)(k+2N+1)}{(N+k)(N+k+1)}\,,$$

$$c_{\mathrm{(JJ)}} = 1\,, \tag{A.18}$$

such that the total stress energy tensor reads

$$L = L_{\mathrm{b}} + L_{\mathrm{f}} + L_{\mathrm{(JJ)}} \tag{A.19}$$

with total central charge

$$c = c_{\mathrm{b}} + c_{\mathrm{f}} + c_{\mathrm{(JJ)}} = \frac{3Nk}{N+k+1}\,. \tag{A.20}$$

There is another elementary primary field of conformal dimension 2, which was called W^{20} in [36]. We can make an ansatz

$$W^{20} = \alpha\,L_{\mathrm{b}} + \beta\,L_{\mathrm{f}} + \gamma\,L_{\mathrm{(JJ)}}\,. \tag{A.21}$$

From the analysis in [36], we know that W^{20} satisfies the OPE

$$W^{20} \star W^{20} \sim n_2 \, 1 + c_{22,2} \, W^{20} + \frac{4 \, n_2}{c-1} \left(L - \frac{3}{2c} \, (JJ) \right) \, . \tag{A.22}$$

Demanding this as well as a vanishing central term in the OPE $L \star W^{20}$, we obtain

$$\alpha = -\sqrt{\frac{2k(N-1)(2N+k+1)(N+k+1) \, n_2}{N(k-1)(N+2k+1)(3Nk-(N+k+1))}} \, ,$$

$$\beta = -\frac{N(k-1)(N+2k+1)}{k(N-1)(k+2N+1)} \, \alpha$$

$$= \sqrt{\frac{2N(k-1)(N+2k+1)(N+k+1) \, n_2}{k(N-1)(2N+k+1)(3Nk-(N+k+1))}} \, ,$$

$$\gamma = 0 \, . \tag{A.23}$$

This then also reproduces correctly the form of $(c_{22,2})^2$ as predicted by eq. (3.27) of [36]. For the normalisation of W^{20} we choose the convention

$$n_2 = -\frac{c}{6}(\nu+3)(\nu-3) \, , \tag{A.24}$$

where

$$\nu = 2\mu - 1 = \frac{N-k-1}{N+k+1} \, ,$$

$$c = \frac{3Nk}{N+k+1} \, . \tag{A.25}$$

In the $c \to \infty$ limit, the parameters then become

$$\alpha \to -\frac{\nu+3}{2\sqrt{3}} \, , \qquad \beta \to -\frac{\nu-3}{2\sqrt{3}} \, . \tag{A.26}$$

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