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# A note on the Von Staudt-Clausen?s theorem for the weighted $q$-Genocchi numbers

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**Abstract**

Recently, the Von Staudt-Clausen theorem for $q$-Euler numbers was introduced by Kim (Russ. J. Math. Phys. 20(1):33-38, 2013) and Araci *et al.* have also studied this theorem for $q$-Genocchi numbers (see Araci *et al.* in Appl. Math. Comput. 247:780-785, 2014) based on the work of Kim *et al.* In this paper, we give the corresponding Von Staudt-Clausen theorem for the weighted $q$-Genocchi numbers and also prove the Kummer-type congruences for the generated weighted $q$-Genocchi numbers.

**MSC:** 11B68; 11S40

**Keywords:** Genocchi number; weighted $q$-Genocchi number; weighted $q$-Euler number; Von Staudt-Clausen theorem

## 1 Introduction and preliminaries

As is well known, a theorem including the fractional part of Bernoulli numbers, which is called the Von Staudt-Clausen theorem, was introduced by Karl Von Staudt and Thomas Clausen (see [1]). In [2], Kim has studied the Von Staudt-Clausen theorem for the $q$-Euler numbers and Araci *et al.* have introduced the Von Staudt-Clausen theorem associated with $q$-Genocchi numbers.

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure $\mathbb{Q}_p$. Let us assume that $q$ is an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{1-p}}$ where $|\cdot|_p$ is a $p$-adic norm. The $q$-extension of $x$ is defined by $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. For $f \in C(\mathbb{Z}_p)$ = the space of all continuous functions on $\mathbb{Z}_p$, the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$\int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \quad \text{(see [2–6])}. \tag{1}$$

From (1), we note that

$$q\int_{\mathbb{Z}_p} f(x+1)\,d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = [2]_q f(0). \tag{2}$$

From $n \in \mathbb{N}$, we have

$$q^n \int_{\mathbb{Z}_p} f(x+n)\,d\mu_{-q}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = [2]_q \sum_{l=0}^{n-1} f(l)(-1)^{n-l-1} q^l \quad \text{(see [4])}. \tag{3}$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and $(p,d) = 1$. Then we set

$$x = x_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \qquad X^* = \bigcup_{0<a<dp,(a,p)=1} a + dp\mathbb{Z}_p$$

and $a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{dp^N}\}$ where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. It is well known that

$$\int_X f(x)\,d\mu_{-q}(x) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x), \quad \text{where } f \in C(\mathbb{Z}_p) \text{ (see [2–6])}. \tag{4}$$

Recently, the weighted $q$-Euler numbers were introduced by the generating function to be

$$\sum_{n=0}^{\infty} E_{n,q}^{(\alpha)} \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[x]_{q^\alpha} t}\,d\mu_{-q}(x) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n\,d\mu_{-q}(x) \right) \frac{t^n}{n!} \quad \text{(see [5, 7])}. \tag{5}$$

Thus, by (5), we get

$$E_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n\,d\mu_{-q}(x) \quad \text{(see [5, 8])},$$

where $\alpha \in \mathbb{C}_p$. Many researchers have studied the weighted $q$-Euler numbers and $q$-Genocchi numbers in the recent decade (see [1–16]).

From (5), Araci defined the weighted $q$-Genocchi numbers as follows:

$$\sum_{n=0}^{\infty} G_{n,q}^{(\alpha)} \frac{t^n}{n!} = t \int_{\mathbb{Z}_p} e^{[x]_{q^\alpha} t}\,d\mu_{-q}(x) = \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n\,d\mu_{-q}(x) \right) \frac{t^{n+1}}{n!}. \tag{6}$$

By (6), we get

$$\frac{G_{n+1,q}^{(\alpha)}}{n+1} = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n\,d\mu_{-q}(x), \qquad G_{0,q}^{(\alpha)} = 0. \tag{7}$$

The weighted $q$-Genocchi polynomials are also defined by

$$\sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x) \frac{t^n}{n!} = t \int_{\mathbb{Z}_p} e^{[x+y]_{q^\alpha} t}\,d\mu_{-q}(x). \tag{8}$$

Thus, by (8), we have

$$\frac{G_{n+1,q}^{(\alpha)}(x)}{n+1} = \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n\,d\mu_{-q}(y) \quad (n \ge 0). \tag{9}$$

Let us assume that $\chi$ is a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then we defined the generalized weighted $q$-Genocchi numbers attached to $\chi$ as follows:

$$\frac{G_{n+1,q,\chi}^{(\alpha)}}{n+1} = \int_X \chi(x)[x]_{q^\alpha}^n \, d\mu_{-q}(x). \tag{10}$$

From (10), we have

$$\begin{aligned} \frac{G_{n+1,q,\chi}^{(\alpha)}}{n+1} &= \int_X \chi(x)[x]_{q^\alpha}^n \, d\mu_{-q}(x) \\ &= \lim_{N\to\infty} \frac{1}{[dp^N]_{-q}} \sum_{x=0}^{dp^N-1} \chi(x)(-1)^x [x]_{q^\alpha}^n \\ &= \frac{[d]_{q^\alpha}^n}{[d]_{-q}} \sum_{k=0}^{d-1} (-1)^k \chi(k) q^k \left( \lim_{N\to\infty} \frac{1}{[p^N]_{-q^d}} \sum_{x=0}^{p^N-1} \left[ x + \frac{k}{d} \right]_{q^{d\alpha}} (-1)^x q^{dx} \right) \\ &= \frac{[d]_{q^\alpha}^n}{[d]_{-q}} \sum_{k=0}^{d-1} (-1)^k \chi(k) q^k \frac{G_{n+1,q^d}^{(\alpha)}(\frac{k}{d})}{n+1}. \end{aligned} \tag{11}$$

**Theorem 1.1** *Let* $\chi$ *be the Dirichlet character with conductor* $d \in \mathbb{N}$ *with* $d \equiv 1 \pmod 2$. *For* $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$, *we have*

$$G_{n,q,\chi}^{(\alpha)} = \frac{[d]_{q^\alpha}^n}{[d]_{-q}} \sum_{k=0}^{d-1} (-1)^k \chi(k) q^k G_{n,q^d}^{(\alpha)}\left(\frac{k}{d}\right).$$

Next we give a familiar theorem, which is known as the Von Staudt-Clausen theorem.

**Lemma 1.2** (Von Staudt-Clausen theorem) *Let* $n$ *be an even and positive integer. Then*

$$B_n + \sum_{p-1|n, p:\text{prime}} \frac{1}{p} \in \mathbb{Z}.$$

Notice that $pB_n$ is a $p$-adic integer where $p$ is an arbitrary prime number, $n$ is an arbitrary integer and also $B_n$ is a Bernoulli number as in [1]. The purpose of this paper is to show that the weighted $q$-Genocchi numbers can be described by a Von Staudt-Clausen-type theorem. Finally, we prove a Kummer-type congruence for the generated weighted $q$-Genocchi numbers.

## 2 Von Staudt-Clausen theorems

From (10), we have

$$\frac{G_{n+1,q}^{(\alpha)}}{n+1} = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n \, d\mu_{-q}(x) = \frac{[2]_q}{2} \int_{\mathbb{Z}_p} q^x [x]_{q^\alpha}^n \, d\mu_{-1}(x). \tag{12}$$

Thus, by (12), we have

$$\lim_{q\to 1} \frac{G_{n+1,q}^{(\alpha)}}{n+1} = \frac{G_{n+1}}{n+1} = \int_{\mathbb{Z}_p} x^n \, d\mu_{-1}(x) \quad \text{(see [2–6, 15])}.$$

In [2], Kim introduced the following inequality:

$$\left|\sum_{j=0}^{p-1}(-1)^j[j]_{q^\alpha}q^j\right| \leq 1. \tag{13}$$

Let us define the following equality: for $k \geq 1$,

$$L_{n-1}^{(\alpha)}(k) = [0]_{q^\alpha}^{n-1} - q[1]_{q^\alpha}^{n-1} + \cdots + \left[p^k - 1\right]_{q^\alpha}^{n-1} q^{p^k-1}. \tag{14}$$

From (3), we note that

$$q^d \frac{G_{n+1,q^d}^{(\alpha)}(d)}{n+1} + \frac{G_{n+1,q^d}^{(\alpha)}}{n+1} = [2]_q \sum_{l=0}^{d-1}[l]_{q^d}^n(-1)^l q^l, \tag{15}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. By (14) and (12), we get

$$\lim_{k\to\infty} nL_{n-1}^{(\alpha)}(k) = \frac{2}{[2]_q}G_{n,q}^{(\alpha)}.$$

By (14), we get

$$\begin{aligned}
&L_{n-1}^{(\alpha)}(k+1)\\
&\quad= \sum_{a=0}^{p^{k+1}-1}(-1)^a q^a [a]_{q^\alpha}^{n-1}\\
&\quad= \sum_{a=0}^{p^k-1}\sum_{j=0}^{p-1}(-1)^{a+jp^k} q^{a+jp^k}\left[a+jp^k\right]_{q^\alpha}^{n-1}\\
&\quad= \sum_{a=0}^{p^k-1}\sum_{j=0}^{p-1}(-1)^{a+jp^k} q^{a+jp^k}\left([a]_{q^\alpha} + q^{\alpha a}\left[jp^k\right]_{q^\alpha}\right)^{n-1}\\
&\quad= \sum_{a=0}^{p^k-1}\sum_{j=0}^{p-1}\sum_{l=0}^{n-1}\binom{n-1}{l}[a]_{q^\alpha}^{n-1-l}(-1)^{a+j}q^{a\alpha l}\left[jp^k\right]_{q^\alpha}^l q^{a+jpk}\\
&\quad= \sum_{a=0}^{p^k-1}\sum_{j=0}^{p-1}\sum_{l=0}^{n-1}\binom{n-1}{l}[a]_{q^\alpha}^{n-1-l}(-1)^{a+j}q^{a(\alpha l+1)+jp^k}\left[p^k\right]_{q^\alpha}^l[j]_{q^\alpha p^k}^l\\
&\quad= \sum_{a=0}^{p^k-1}(-1)^a q^a [a]_{q^\alpha}^{n-1}\frac{[2]_{q^{p^{2k}}}}{[2]_{q^{p^k}}}\\
&\qquad+ \sum_{a=0}^{p^k-1}\sum_{j=0}^{p-1}\sum_{l=1}^{n-1}\binom{n-1}{l}[a]_{q^\alpha}^{n-1-l}(-1)^{a+j}q^{a(\alpha l+1)+jp^k}\left[p^k\right]_{q^\alpha}^l[j]_{q^\alpha p^k}^l\\
&\quad= \sum_{a=0}^{p^k-1}\sum_{j=0}^{p-1}\sum_{l=0}^{n-1}\binom{n-1}{l}[a]_{q^\alpha}^{n-1-l}(-1)^{a+j}q^{a(\alpha+l)+jp^k}\left[p^k\right]_{q^\alpha}^l[j]_{q^\alpha p^k}^l
\end{aligned}$$

$$= \sum_{a=0}^{p^k-1}(-1)^a q^a [a]_{q^\alpha}^{n-1} \frac{[2]_{q^{p^{2k}}}}{[2]_{q^{p^k}}} + \sum_{a=0}^{p^k-1}\sum_{j=0}^{p-1}\sum_{l=0}^{n-1}\binom{n-1}{l}[a]_{q^\alpha}^{n-1-l}(-1)^{a+j}q^{a(\alpha l+1)+jp^k}\left[p^k\right]_{q^\alpha}^l [j]_{q^{\alpha p^k}}^l. \tag{16}$$

Thus, by (16), we get

$$L_{n-1}^{(\alpha)}(k+1) \equiv \sum_{a=0}^{p^k-1}[a]_{q^\alpha}^{n-1}(-1)^a q^a \left(\mathrm{mod}\left[p^k\right]_{q^\alpha}\right). \tag{17}$$

From (16), we have

$$\begin{aligned}
&\sum_{a=0}^{p^{k+1}-1}(-1)^a [a]_{q^\alpha}^{n-1} q^a \\
&= \sum_{a=0}^{p-1}\sum_{j=0}^{p^k-1}(-1)^{a+pj}[a+pj]_{q^\alpha}^{n-1} q^{a+pj} \\
&= \sum_{a=0}^{p-1}(-1)^a q^a \sum_{j=0}^{p^k-1}(-1)^j q^{pj}\left([a]_{q^\alpha} + q^{\alpha a}[p]_{q^\alpha}[j]_{q^{\alpha p}}\right)^{n-1} \\
&= \sum_{a=0}^{p-1}\sum_{j=0}^{p^k-1}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^{a+j}q^{a+pj}[a]_{q^\alpha}^{n-1-l}q^{\alpha a l}[p]_{q^\alpha}^l [j]_{q^{p\alpha}}^l \\
&= \sum_{a=0}^{p-1}(-1)^a q^a [a]_{q^\alpha}^{n-1}\frac{[2]_{q^{p^{k+1}}}}{[2]_{q^p}} \\
&\quad + \sum_{a=0}^{p-1}\sum_{j=0}^{p^k-1}\sum_{l=1}^{n-1}\binom{n-1}{l}(-1)^{a+j}q^{a+pj+\alpha a l}[a]_{q^\alpha}^{n-1-l}[p]_{q^\alpha}^l [j]_{q^{p\alpha}}^l \\
&= \sum_{a=0}^{p-1}(-1)^a q^a [a]_{q^\alpha}^{n-1} \left(\mathrm{mod}\,[p]_{q^\alpha}\right).
\end{aligned} \tag{18}$$

Therefore, by (17) and (18), we obtain the following theorem.

**Theorem 2.1** *Let* $L_{n-1}^{(\alpha)}(k) = \sum_{a=0}^{p^k-1}(-1)^a [a]_{q^\alpha}^{n-1}$. *Then we have*

$$L_{n-1}^{(\alpha)}(k+1) = \sum_{a=0}^{p^k-1}[a]_{q^\alpha}^{n-1}(-1)^a q^a.$$

*Furthermore*

$$\sum_{a=0}^{p^k-1}[a]_{q^a}^{n-1}(-1)^a q^a \alpha \left(\mathrm{mod}\left[p^k\right]_{q^\alpha}\right) \equiv \sum_{a=0}^{p-1}(-1)^a q^a [a]_{q^\alpha}^{n-1} \left(\mathrm{mod}\,[p]_{q^\alpha}\right).$$

By Theorem 2.1, we get

$$\sum_{a=0}^{p-1}(-1)^a n[a]_{q^\alpha}^{n-1} q^a = \int_X [x]_{q^\alpha}^{n-1}\, d\mu_{-q}(x) \equiv G_{n,q}^{(\alpha)} \pmod{[p]_q}. \tag{19}$$

Therefore, by (19), we have the following theorem.

**Theorem 2.2** *For $n \geq 1$, we have*

$$\sum_{a=0}^{p-1}(-1)^a n[a]_{q^\alpha}^{n-1} = G_{n,q}^{(\alpha)} \pmod{[p]_q}.$$

From (17) and (19), we note that

$$G_{n+1,q}^{(\alpha)} + n\sum_{a=0}^{p-1}(-1)^{a+1}[a]_{q^\alpha}^{n-1} q^a \in \mathbb{Z}_p \quad (n \geq 1).$$

**Corollary 2.3** *For $n \geq 1$, we have*

$$G_{n+1,q}^{(\alpha)} + n\sum_{a=0}^{p-1}(-1)^{a+1}[a]_{q^\alpha}^{n-1} q^a \in \mathbb{Z}_p.$$

Let $n \geq 1$. Then we observe that

$$\begin{aligned} \left|\frac{G_{n+1,q}^{(\alpha)}}{n+1}\right|_p &= \left|\frac{G_{n+1,q}^{(\alpha)}}{n+1} - \sum_{a=0}^{p-1}(-1)^a [a]_{q^\alpha}^n q^a + \sum_{a=0}^{p-1}(-1)^a q^a [a]_{q^\alpha}^n\right|_p \\ &\leq \max\left\{\left|\frac{G_{n+1,q}^{(\alpha)}}{n+1} - \sum_{a=0}^{p-1}(-1)^a [a]_{q^\alpha}^n\right|_p, \left|\sum_{a=0}^{p-1}(-1)^a q^a [a]_{q^\alpha}^n\right|_p\right\} \leq 1. \end{aligned} \tag{20}$$

Therefore, we obtain the following theorem.

**Theorem 2.4** *For $n \geq 1$, we have*

$$\frac{G_{n+1,q}^{(\alpha)}}{n+1} \in \mathbb{Z}_p.$$

Let $\chi$ be the Dirichlet character $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. The generalized weighted $q$-Genocchi numbers attached to $\chi$ are introduced as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,q,\chi}^{(\alpha)} \frac{t^n}{n!} &= [2]_q t \sum_{m=0}^{\infty}(-1)^m \chi(m) e^{[m]_{q^\alpha} t} \\ &= t\int_X \chi(x) e^{[x]_{q^\alpha} t}\, d\mu_{-q}(x). \end{aligned} \tag{21}$$

Let $\overline{f} = [f,p]$ be the least common multiple of the conductor $f$ of $\chi$ and $p$. By (21), we get

$$G_{n,q,\chi}^{(\alpha)} = n\int_X \chi(x)[x]_{q^\alpha}^{n-1}\, d\mu_{-q}(x) = n\lim_{n\to\infty}\sum_{x=0}^{fp^N-1}\chi(x)(-1)^x[x]_{q^\alpha}^{n-1}. \tag{22}$$

Thus, we have

$$\begin{aligned} G_{n,q,\chi}^{(\alpha)} &= n \lim_{\rho\to\infty} \sum_{1\le a\le \bar{f}p^{\rho},(a,p)=1} \chi(a)(-1)^a q^a [a]_{q^\alpha}^{n-1} \\ &\quad + n[p]_{q^\alpha}^{n-1}\chi(p) \lim_{\rho\to\infty} \sum_{1\le a\le \bar{f}p^{\rho},(a,p)=1}^{\bar{f}p^{\rho}-1} \chi(a)(-1)^a q^{ap}[a]_{q^{\alpha}p}^{n-1} \\ &= n \lim_{\rho\to\infty} \sum_{1\le a\le \bar{f}p^{p},(a,p)=1} \chi(a)(-1)^a q^a [a]_{q^\alpha}^{n-1} + a[p]_{q^\alpha}^{n-1}\chi(p) G_{n,q^p,\chi}^{(\alpha)}. \end{aligned} \tag{23}$$

Therefore, by (23), we obtain the following theorem.

**Theorem 2.5** *For $n \ge 1$, we have*

$$n \lim_{\rho\to\infty} \sum_{1\le a\le \bar{f}p^{\rho},(a,p)=1} \chi(a)(-1)^a q^a [a]_{q^\alpha}^{n-1} = G_{n,q,\chi}^{(\alpha)} - [p]_{q^\alpha}^{n-1}\chi(p) G_{n,q^p,\chi}^{(\alpha)}. \tag{24}$$

Assume that $w$ is the Teichmüller character by mod $p$. For $a \in X^*$, set $\langle a\rangle_\alpha = \langle a : q\rangle_\alpha = \frac{[a]_{q^\alpha}}{w(a)}$. Note that $|\langle a\rangle_\alpha - 1|_p < p^{\frac{1}{p-1}}$, where $\langle a\rangle^s = \exp(s\log\langle a\rangle)$ for $s \in \mathbb{Z}_p$. For $s \in \mathbb{Z}_p$, we define the weighted $p$-adic $l$-function associated with $G_{n,q,\chi}^{(\alpha)}$ as follows:

$$l_{p,q}^{(\alpha)}(s,\chi) = \lim_{\rho\to\infty} \sum_{1\le a\le \bar{f}p^{\rho},(a,p)=1} \chi(a)(-1)^a \langle a\rangle_\alpha^{-s} q^a = \int_{X^*} \chi(x)\langle x\rangle_\alpha^{-s}\, d\mu_{-q}(x).$$

For $k \ge 1$,

$$\begin{aligned} &kl_{p,q}\bigl(1-k,\chi w^{k-1}\bigr) \\ &\quad = k \lim_{\rho\to\infty} \sum_{1\le a\le \bar{f}p^{\rho}} \chi(a)(-1)^a q^a [a]_{q^\alpha}^{k-1} \\ &\quad = k \int_X \chi(x)[x]_{q^\alpha}^{k-1}\, d\mu_{-q}(x) - k\int_{pX} \chi(x)[x]_{q^\alpha}^{k-1}\, d\mu_{-q}(x) \\ &\quad = k \int_X \chi(x)[x]_{q^\alpha}^{k-1}\, d\mu_{-q}(x) - \frac{k[2]_q\chi(p)}{[2]_{q^p}}[p]_{q^\alpha}^{k-1}\int_X \chi(x)[x]_{q^{\alpha}p}^{k-1}\, d\mu_{-q^p}(x) \\ &\quad = G_{x,q,\chi}^{(\alpha)} - \frac{[2]_q}{[2]_{q^p}}\chi(p)[p]_{q^\alpha}^{k-1} G_{k,q^p,\chi}^{(\alpha)}. \end{aligned}$$

It is easy to show that

$$\begin{aligned} \langle a\rangle_\alpha^{p^n} &= \exp\bigl(p^n \log\langle a\rangle_\alpha\bigr) \\ &= 1 + p^n\log\langle a\rangle_\alpha + \frac{(p^n\log_p\langle a\rangle_\alpha)^2}{2!} + \cdots \\ &\equiv 1 \pmod{p^n}. \end{aligned}$$

So, by the definition of $l_{p,q}^{(\alpha)}(1-k,x)$, we get

$$l_{p,q}^{(\alpha)}(-k,\chi)=\lim_{\rho\to\infty}\sum_{1\le a\le \bar{f}p^{\rho},(a,p)=1}\chi(a)(-1)^a q^a\langle a\rangle_{\alpha}^{k}$$
$$\equiv\lim_{\rho\to\infty}\sum_{1\le a\le \bar{f}p^{\rho},(a,p)=1}\chi(a)(-1)^a q^a\langle a\rangle_{\alpha}^{k'}\ \left(\bmod\ p^n\right),$$

where $k\equiv k'\ (\bmod\ p^n(p-1))$. Namely, we have

$$l_{p,q}^{(\alpha)}\left(-k,\chi w^k\right)\equiv l_{p,q}^{(\alpha)}\left(-k',\chi w^{k'}\right)\ \left(\bmod\ p^n\right).$$

**Theorem 2.6** *For $k\equiv k'\ (\bmod\ p^n(p-1))$, we have*

$$\frac{G_{k+1,q,\chi}^{(\alpha)}}{k+1}-\frac{[2]_q}{[2]_{q^p}}\frac{G_{k+1,q^p,\chi}^{(\alpha)}}{k+1}\equiv\frac{G_{k'+1,q,\chi}^{(\alpha)}}{k'+1}-\frac{[2]_q}{[2]_{q^p}}\frac{G_{k'+1,q^p,\chi}^{(\alpha)}}{k'+1}\ \left(\bmod\ p^n\right).$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors? contributions**
All authors contributed equally to this work. All authors read and approved the final manuscript.

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