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# Fractional instantons and bions in the principal chiral model on $\mathbb{R}^2 \times S^1$ with twisted boundary conditions

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**ABSTRACT:** Bions are multiple fractional instanton configurations with zero instanton charge playing important roles in quantum field theories on a compactified space with a twisted boundary condition. We classify fractional instantons and bions in the  $SU(N)$  principal chiral model on  $\mathbb{R}^2 \times S^1$  with twisted boundary conditions. We find that fractional instantons are global vortices wrapping around  $S^1$  with their  $U(1)$  moduli twisted along  $S^1$ , that carry  $1/N$  instanton (baryon) numbers for the  $\mathbb{Z}_N$  symmetric twisted boundary condition and irrational instanton numbers for generic boundary condition. We work out neutral and charged bions for the  $SU(3)$  case with the  $\mathbb{Z}_3$  symmetric twisted boundary condition. We also find for generic boundary conditions that only the simplest neutral bions have zero instanton charges but instanton charges are not canceled out for charged bions. A correspondence between fractional instantons and bions in the  $SU(N)$  principal chiral model and those in Yang-Mills theory is given through a non-Abelian Josephson junction.

**KEYWORDS:** Solitons Monopoles and Instantons, Sigma Models, Field Theories in Lower Dimensions

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**1 Introduction**

Recently, much attention have been paid to quantum field theories on compactified spaces  $\mathbb{R}^d \times S^1$  with twisted boundary conditions, such as QCD with adjoint fermions on  $\mathbb{R}^3 \times S^1$  and nonlinear sigma models on  $\mathbb{R}^1 \times S^1$ , that admit fractional instantons and bions, i.e. multiple fractional instanton configurations with vanishing instanton charge [1]–[24]. Magnetic (charged) bions carry a magnetic charge and are conjectured to lead semiclassical confinement in QCD on  $\mathbb{R}^3 \times S^1$  [5], while neutral bions carry no magnetic charge and are identified as the infrared renormalons [7]–[16] (see refs. [25–27] for earlier works) playing an essential role in self-consistent semiclassical definition of quantum field theories through the resurgence [28].

In lower dimensions, fractional instantons were found in the  $CP^{N-1}$  model [29, 30] (see also refs. [31–34] for subsequent study) and the Grassmann sigma model [35, 36] on  $\mathbb{R}^1 \times S^1$  with twisted boundary conditions. Bions and their role in the resurgence have been extensively studied in the  $CP^{N-1}$  model [9–11, 18, 19, 24] and the Grassmann sigma model [20, 21] on  $\mathbb{R}^1 \times S^1$ . The former admits only neutral bions while the latter admits both neutral and charged bions [20]. Fractional instantons and bions in the  $O(N)$  nonlinear

sigma model on  $\mathbb{R}^{N-2} \times S^1$  have been studied recently with general twisted boundary conditions in which arbitrary number of fields changes signs [37]. The  $O(3)$  model is equivalent to the  $\mathbb{C}P^1$  model studied before [9–11, 17–19]. The  $O(4)$  model is equivalent to a principal chiral model with a group  $SU(2)$  (or a Skyrme model if four derivative term is added [38, 39]), for which the case of the boundary condition with two fields changing their signs is equivalent to the  $\mathbb{Z}_2$  (center) symmetric boundary condition. In this case, fractional instantons are vortices winding around  $S^1$  with  $U(1)$  moduli twisted half along  $S^1$ .

In this paper, we study the  $SU(N)$  principal chiral model on  $\mathbb{R}^2 \times S^1$  with twisted boundary conditions. Previously the principal chiral models were studied in two dimensions [13, 16] for which instantons do not exist. We study the principal chiral model in three dimensions, where instantons exist with the instanton number defined by the third homotopy group  $\pi_3$  that is also known as baryon number in the context of the Skyrme model [38, 39]. We show that this case allows  $N - 1$  kinds of global vortices accompanied by  $U(1)$  moduli, and fractional instantons are vortices wrapping around the  $S^1$  direction, with  $U(1)$  moduli twisted along  $S^1$  by the angle  $2\pi/N$  (or its complement) for the  $\mathbb{Z}_N$  center symmetric twisted boundary condition and by generic angle for generic boundary conditions. They carry  $1/N$  instanton (baryon) numbers for the  $\mathbb{Z}_N$  symmetric twisted boundary condition and irrational instanton numbers for generic boundary condition. We classify neutral and charged bions for the  $SU(3)$  case with the  $\mathbb{Z}_3$  symmetric twisted boundary condition. We also point out that for the cases with generic boundary conditions only the simplest neutral bions, composed of a set of a fractional instanton and fractional anti-instanton, have zero instanton charges but instanton charges are not canceled out for charged bions. We further discuss a correspondence between fractional instantons and bions in the  $SU(N)$  principal chiral model and those in Yang-Mills theory; the latter become the former if reside inside a non-Abelian domain wall [40–42] (non-Abelian Josephson junction [43]) in the Higgs phase [44].

This paper is organized as follows. In section 2, we first give the  $SU(N)$  principal chiral model. In section 3 we review fractional instantons and bions in the  $SU(2)$  principal chiral model on  $\mathbb{R}^2 \times S^1$  with the center symmetric twisted boundary conditions. We find charged bions that were not studied before. In section 4 we work out fractional instantons and bions in the  $SU(3)$  principal chiral model on  $\mathbb{R}^2 \times S^1$  with the center symmetric twisted boundary conditions. In section 5, we briefly discuss the  $SU(N)$  principal chiral model. In section 6, generic boundary conditions are discussed for  $SU(N)$ . In section 7, we discuss the relation between fractional instanton and bions in the  $SU(N)$  principal chiral model and those in the  $SU(N)$  Yang-Mills theory. Section 8 is devoted to a summary and discussion.

## 2 The principal chiral model on $\mathbb{R}^2 \times S^1$ with twisted boundary conditions

Let  $U(x)$  be scalar fields taking a value in the group  $G = SU(N)$ . Then, the Lagrangian of the  $SU(N)$  principal chiral model is given as

$$\mathcal{L} = f_\pi^2 \text{tr} (\partial_\mu U^\dagger \partial^\mu U) \tag{2.1}$$

with a decay constant  $f_\pi$ . The symmetry of the Lagrangian is  $G = SU(N)_L \times SU(N)_R$  ( $U \rightarrow U' = g_L U g_R^\dagger$ ), that is spontaneously broken down to  $H \simeq SU(N)_{L+R}$  ( $U \rightarrow U' = g U g^\dagger$ ).

The target space is  $M \equiv G/H \simeq \text{SU}(N) \ni U(x)$ . The instanton number  $B$  in  $d = 3 + 0$  dimensions (or equivalently the baryon number or Skyrme charge in  $d = 3 + 1$  dimensions), taking a value in the third homotopy group  $B \in \pi_3(M)$ , is defined as ( $i = 1, 2, 3$ )

$$\begin{aligned} B &= -\frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{tr} \left( U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U \right) \\ &= \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{tr} \left( U^\dagger \partial_i U \partial_j U^\dagger \partial_k U \right). \end{aligned} \quad (2.2)$$

We consider the space  $\mathbb{R}^2 \times S^1$  with non-trivially twisted boundary conditions along  $S^1$ . The  $\mathbb{Z}_N$  symmetric twisted boundary condition for the  $\text{SU}(N)$  principal chiral model is defined by

$$\begin{aligned} U(x^1, x^2, x^3 + R) &= WU(x^1, x^2, x^3)W^\dagger, \\ W = \text{diag.}(1, \omega, \omega^2, \dots, \omega^{N-1}) &= \exp \left[ \frac{2\pi i}{N} \text{diag.}(0, 1, \dots, N-1) \right], \quad \omega = e^{\frac{2\pi i}{N}}. \end{aligned} \quad (2.3)$$

The  $\mathbb{Z}_2$  twisted boundary condition for the  $\text{SU}(2)$  case is

$$U(x^1, x^2, x^3 + R) = WU(x^1, x^2, x^3)W^\dagger, \quad W = \sigma_3 = \text{diag.}(1, -1). \quad (2.4)$$

The  $\text{SU}(2)$  principal chiral model is equivalent to the  $O(4)$  nonlinear sigma model. If we define four real scalar fields  $n_A(x)$  ( $A = 1, 2, 3, 4$ ) from the  $\text{SU}(2)$ -valued field  $U(x)$  by

$$U = i \sum_{a=1,2,3} n_a \sigma_a + n_4 \mathbf{1}_2 \quad (2.5)$$

with the Pauli matrices  $\sigma_a$  and the constraint  $\mathbf{n} \cdot \mathbf{n} = 1$  equivalent to  $U^\dagger U = \mathbf{1}_2$ , the boundary condition (2.4) becomes  $(n_1, n_2, n_3, n_4)(x^1, x^2, x^3 + R) = (-n_1, -n_2, n_3, n_4)(x^1, x^2, x^3)$  that we called  $(-, -, +, +)$  [37].

We also consider more general twisted boundary condition

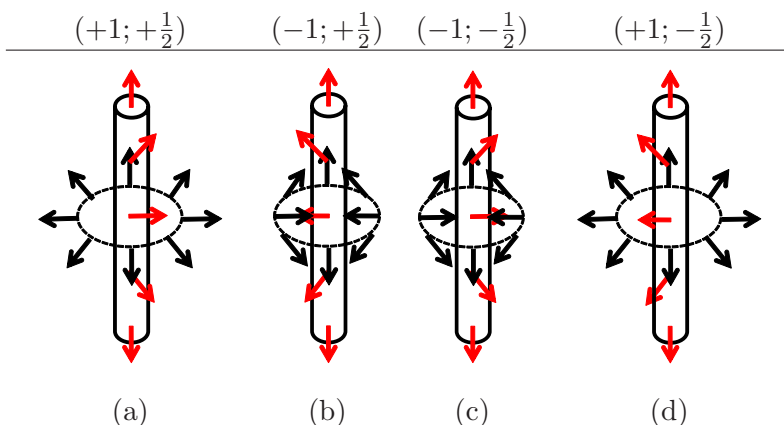
$$\begin{aligned} U(x^1, x^2, x^3 + R) &= WU(x^1, x^2, x^3)W^\dagger, \\ W = \text{diag.}(e^{im_1}, e^{im_2}, \dots, e^{im_N}), \quad m_1 \leq m_2 \leq \dots \leq m_N. \end{aligned} \quad (2.6)$$

In this paper, we first focus on the  $\mathbb{Z}_N$  symmetric twisted boundary condition in eq. (2.3), that corresponds to  $m_a = 2\pi(a-1)/N$ . We also consider the generic non-degenerate case later.

In small compactification radius limit, the Scherk-Schwarz dimensional reduction is effectively induced, to yield a potential term (twisted mass) [43]

$$V = f_\pi^2 \text{tr} ([M, U]^\dagger [M, U]) \quad (2.7)$$

with  $M \equiv (m_1, m_2, \dots, m_N)$ .



**Figure 1.** Fractional instantons in the  $SU(2)$  principal chiral model (figures are taken from ref. [37]). The first lines indicate the topological charges (homotopy groups)  $(\pi_1(\mathcal{N}); \pi_3(M))$  for the vortices and instantons (Skyrmions). The black arrows denote the  $U(1)$  moduli of the vacua while the red arrows denote the  $U(1)$  moduli of the vortices. Fractional (anti-)instantons can constitute following composite structures: (a)+(b) instanton, (c)+(d) anti-instanton, (a)+(c), (b)+(d) neutral bions, (a)+(d), (b)+(c) charged bions.

### 3 Fractional instantons and bions in the $SU(2)$ principal chiral model

In this section, we consider the  $SU(2)$  principal chiral model with the  $\mathbb{Z}_2$  symmetric boundary condition. This section is mostly rewriting the results in ref. [37] in terms of the principal chiral field  $U(x)$  because the  $SU(2)$  principal chiral model is equivalent to the  $O(4)$  sigma model, but we will find it useful for warming up to study the  $SU(N)$  principal chiral model. Charged bions in the third subsection is a new result that were not studied before.

#### 3.1 Fractional instantons

Fractional instantons in the principal chiral model were classified into four kinds, as illustrated in figure 1. These can be obtained as follows. The fixed manifold  $\mathcal{N}$  under the action that acts on the boundary condition is

$$\mathcal{N} \simeq U(1) \simeq S^1 \quad (3.1)$$

that is generated by  $\sigma_3$ . Therefore, it has a nontrivial first homotopy group

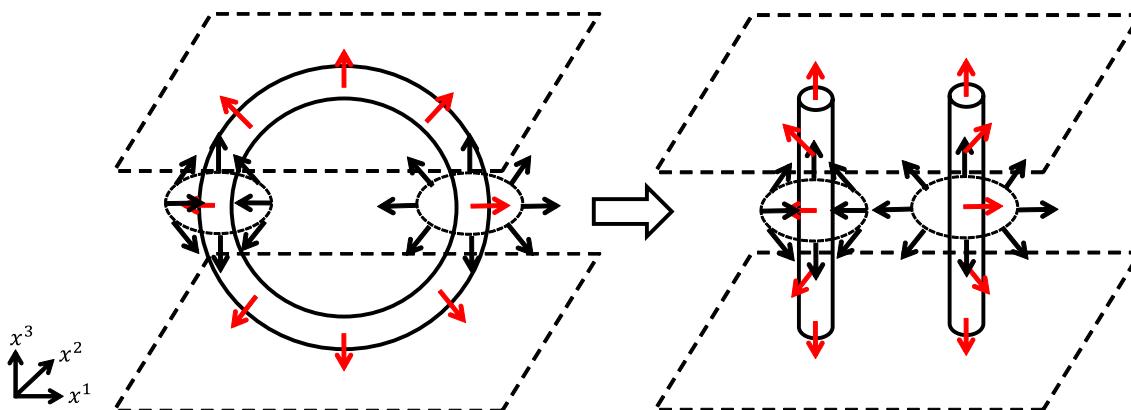
$$\pi_1(S^1) \simeq \mathbb{Z}. \quad (3.2)$$

Let us place a vortex along the  $z = x^3$  direction. The ansatz for a vortex configuration can be given as

$$U(r, \theta, z) = \begin{pmatrix} \cos f(r)e^{+i\theta} & -\sin f(r)e^{+i\alpha(z)} \\ \sin f(r)e^{-i\alpha(z)} & \cos f(r)e^{-i\theta} \end{pmatrix}, \quad (3.3)$$

where  $(r, \theta, z)$  are cylindrical coordinates  $f$  is a profile function satisfying the boundary conditions

$$f \rightarrow 0 \text{ for } r \rightarrow \infty, \quad f = \pi/2 \text{ for } r = 0. \quad (3.4)$$



**Figure 2.** Decay of a twisted closed vortex string of the size of the compact direction into two fractional instantons in the SU(2) principal chiral model (figures are taken from ref. [37]). The notations are the same with figure 1. The dotted planes denote the boundary at  $z = 0$  and  $z = R$  where the fields are twisted. When a closed vortex touches to itself through the compact direction  $z$ , a reconnection of the two parts of the string occurs to be split into two fractional (anti-)instantons, that is, vortices winding around  $S^1$  with the half twisted U(1) moduli.

An anti-vortex can be obtained as  $U(r, -\theta, z)$ . In eq. (3.3),  $\alpha$  is a U(1) modulus of the vortex, that is constant if the vortex does not wind around  $S^1$ . When the vortex winds around  $S^1$  with the twisted boundary condition in eq. (2.4), the modulus  $\alpha$  has to satisfy the boundary condition

$$\alpha(z + R) = \alpha(z) \pm \pi. \quad (3.5)$$

The following  $z$ -dependence of  $\alpha$  satisfy the boundary condition;

$$\alpha(z) = \alpha_0 \pm \frac{\pi}{R} z \quad (3.6)$$

that we denote  $\alpha^+$  and  $\alpha^-$ , respectively.

The topological instanton charge (baryon number) can be calculated as

$$B = \frac{1}{16\pi^2} \int d^3x \frac{1}{r} \sin(f) f_r \alpha_z = \pm \frac{1}{2\pi} \int_0^R dz \partial_z \alpha = \pm \frac{1}{2\pi} [\alpha]_{z=0}^{z=R} = \begin{cases} \pm \frac{1}{2} & \text{for } \alpha = \alpha^+ \\ \mp \frac{1}{2} & \text{for } \alpha = \alpha^- \end{cases}, \quad (3.7)$$

where the upper and lower signs correspond to a vortex and anti-vortex, respectively. More generally, a vortex string with the winding number  $Q$ , along which the U(1) modulus is twisted  $P$  times, has the instanton number  $B = PQ$  [45] (which was obtained in ref. [46] to calculate the Hopf number for Hopfions by lifting up  $\pi_3(S^2)$  to  $\pi_3(S^3)$ ). The topological charges of fractional (anti-)instantons with the  $\mathbb{Z}_2$  symmetric twisted boundary condition are summarized in table 1.

It is known that a single instanton (Skyrmion) can be represented by (a global analog of) a vorton [47–51], i.e., a closed vortex string along which a U(1) modulus is twisted [52–58], which was first found in the context of Bose-Einstein condensates (see also [59]), and

$(v; B)$	$\pi_1(\mathcal{N})$	$\pi_1(\mathcal{M})$	$B \in \pi_3(M)$	Figure
$(+1; +1/2)$	+1	+1/2	+1/2	Figure 1 (a)
$(-1; +1/2)$	-1	-1/2	+1/2	Figure 1 (b)
$(-1; -1/2)$	-1	+1/2	-1/2	Figure 1 (c)
$(+1; -1/2)$	+1	-1/2	-1/2	Figure 1 (d)

**Table 1.** Homotopy groups of fractional (anti-)instantons in the SU(2) principal chiral model with the  $\mathbb{Z}_2$  symmetric twisted boundary condition. The columns represent the homotopy groups of a vortex  $\pi_1$ , a U(1) modulus  $\pi_1$ , and the instanton  $\pi_3$  from left to right.

stable solutions in a Skyrme model were also constructed in refs. [45, 60, 61]. A single instanton as a vorton is shown in the left panel in figure 2. When the size of the closed vortex string is of the same with that of the compactification scale  $R$ , the closed vortex string touches itself through the compact  $x^3$  direction with the twisted boundary condition. Subsequently a reconnection of two fractions of the closed string occurs (see ref. [62] for a reconnection of strings with moduli). Then, the closed string is split into two vortex strings winding around the compact direction, and subsequently they are separated into the  $x$ - $y$  plane, as illustrated in the right panel of figure 2. The U(1) modulus is twisted half along each string, resulting in a fractional (anti-)instanton. We thus find four kinds of fractional (anti-)instantons, as summarized in figure 1(a)–(d).

The Skyrme term is not needed for the stability even though fractional instantons are Skyrmions as was demonstrated in ref. [45], in which stable configurations of (half) Skyrmions inside a vortex string were constructed without the Skyrme term (on  $\mathbb{R}^3$  without twisted boundary condition).

All vortices are global vortices having the divergent energy

$$E \sim \log \Lambda/\xi \tag{3.8}$$

at large distance, apart from finite contribution from the core. Here,  $\Lambda$  is the system size in the  $x$ - $y$  plane, and  $\xi \sim m^{-1}$  is the size of the vortex core.

Fractional instantons are global vortices in the  $x$ - $y$  plane so that the interaction between them is

$$E_{\text{int}} \sim \mp \log \rho/\xi, \quad F_{\text{int}} = -\frac{\partial E_{\text{int}}}{\partial \rho} \sim \pm 1/\rho \tag{3.9}$$

with distance  $\rho$  for large separation  $\rho \gg \xi$ , where the upper signs are for a pair of (anti-)vortices (repulsion) and the lower signs are for a pair of a vortex and anti-vortex (attraction). The interaction between two vortices at short distance  $\rho \sim \xi$  depends on the moduli  $\alpha$  in the cores, but we do not discuss it in this paper.

### 3.2 Neutral bions

Neutral bions in the SU(2) principal chiral model were discussed before in ref. [37]. Neutral bions are configurations with zero instanton charges and zero vortex charges:

$$\sum_i (v_i; B_i) = (0; 0). \quad (3.10)$$

Neutral bions composed of two fractional (anti-)instantons can be constructed from fractional instantons with the *opposite* vortex charges with the *same* winding of the U(1) modulus along  $z$ , that is, a configuration composed of (a) and (c) or (b) and (d) in figure 1.

The interaction between fractional instantons constituting a neutral bion is attractive, because they are a pair of a global vortex and global anti-vortex:

$$E_{\text{int}} \sim +\log \rho/\xi, \quad F_{\text{int}} \sim -1/\rho \quad (3.11)$$

with distance  $\rho$  for large separation.

### 3.3 Charged bions

Charged bions were not discussed before in the SU(2) principal chiral model. Charged bions are configurations with zero instanton charges and non-zero vortex charges:

$$\sum_i (v_i; B_i) = (v; 0), \quad v \neq 0. \quad (3.12)$$

For charged bions composed of two fractional (anti-)instantons, one prepares fractional instantons with the *same* vortex charges with the *opposite* winding of the U(1) modulus along  $z$ , that is, the configurations  $(v; B) = (2; 0)$  for (a) and (d) in figure 1,  $(v; B) = (-2; 0)$  for (b) and (c) in figure 1.

The interaction between fractional instantons constituting a charged bion is repulsive because they are a pair of global vortices:

$$E_{\text{int}} \sim -\log \rho/\xi, \quad F_{\text{int}} \sim +1/\rho \quad (3.13)$$

with distance  $\rho$  for large separation.

## 4 Fractional instantons and bions in the SU(3) principal chiral model

### 4.1 Fractional instantons

We consider the  $\mathbb{Z}_3$  symmetric twisted boundary condition:

$$W = \text{diag.}(1, \omega, \omega^2) = \exp \left[ \frac{2\pi i}{3} \text{diag.}(0, 1, 2) \right], \quad \omega = e^{\frac{2\pi i}{3}}. \quad (4.1)$$

The fixed manifold  $\mathcal{N}$  is

$$U = \text{diag.}(e^{i\alpha}, e^{i\beta}, e^{-i\alpha-i\beta}), \quad \mathcal{N} \simeq \text{U}(1)^2. \quad (4.2)$$



The non-trivial first homotopy group

$$\pi_1(\mathcal{N}) \simeq \mathbb{Z} \oplus \mathbb{Z} \quad (4.3)$$

admits homotopically distinct two kinds of vortices. The fundamental, *i.e.*, minimum winding vortices in the SU(3) principal chiral model can be obtained by embedding the one in eq. (3.3) of the SU(2) principal chiral model to the SU(3) matrix:

$$\begin{aligned} U_1(r, \theta, z) &= \begin{pmatrix} \cos f(r)e^{+i\theta} & -\sin f(r)e^{+i\alpha(z)} & 0 \\ \sin f(r)e^{-i\alpha(z)} & \cos f(r)e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ U_2(r, \theta, z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos f(r)e^{+i\theta} & -\sin f(r)e^{+i\alpha(z)} \\ 0 & \sin f(r)e^{-i\alpha(z)} & \cos f(r)e^{-i\theta} \end{pmatrix}, \\ U_3(r, \theta, z) &= \begin{pmatrix} \cos f(r)e^{-i\theta} & 0 & +\sin f(r)e^{-i\alpha(z)} \\ 0 & 1 & 0 \\ -\sin f(r)e^{+i\alpha(z)} & 0 & \cos f(r)e^{+i\theta} \end{pmatrix}. \end{aligned} \quad (4.4)$$

Anti-vortices can be obtained as  $U_a(r, -\theta, z)$  with  $a = 1, 2, 3$ . These three are not homotopically independent of each other. We take the first and second as the independent basis of the first homotopy group in eq. (4.2), in which the topological vortex charges in eq. (4.4) are

$$u_1 = (1, 0), \quad u_2 = (0, 1), \quad u_3 = (-1, -1) \in \mathbb{Z} \oplus \mathbb{Z} \simeq \pi_1(\mathcal{N}), \quad (4.5)$$

respectively. The third one is not independent of the rests as can be seen from the fact that the vortex charges are canceled out when all three are present together:

$$\sum_{a=1,2,3} u_a = (0, 0). \quad (4.6)$$

In eq. (4.4),  $\alpha$  is a U(1) modulus of a vortex that is constant if the vortex does not wrap the  $S^1$  direction.

When a vortex wraps the  $S^1$  direction,  $\alpha$  must change along the vortex world-line due to the twisted boundary condition in eq. (2.3):

$$\exp i\alpha(z + R) = \exp \left[ i\alpha(z) + \frac{2\pi i}{3} \right], \quad (4.7)$$

where  $z$  is the coordinate along  $S^1$  with the period  $R$ . This boundary condition can be satisfied by the two different minimum paths with the following  $z$ -dependence of  $\alpha$ :

$$\alpha(z) = \begin{cases} \alpha^+(z) = \alpha_0 + \frac{2\pi}{3} \frac{z}{R} \\ \alpha^-(z) = \alpha_0 - \frac{4\pi}{3} \frac{z}{R} \end{cases}, \quad (4.8)$$

$(v_1, v_2; B)$	$\pi_1(\mathcal{N})$	$U$	$\pi_1(\mathcal{M})$	$\alpha$	$B \in \pi_3(M)$
$(+1, 0; +1/3)$	$(+1, 0)$	$U_1(\theta)$	$+1/3$	$\alpha^+$	$+1/3$
$(0, +1; +1/3)$	$(0, +1)$	$U_2(\theta)$	$+1/3$	$\alpha^+$	$+1/3$
$(-1, -1; +1/3)$	$(-1, -1)$	$U_3(\theta)$	$+1/3$	$\alpha^+$	$+1/3$
$(-1, 0; +2/3)$	$(-1, 0)$	$U_1(-\theta)$	$-2/3$	$\alpha^-$	$+2/3$
$(0, -1; +2/3)$	$(0, -1)$	$U_2(-\theta)$	$-2/3$	$\alpha^-$	$+2/3$
$(+1, +1; +2/3)$	$(+1, +1)$	$U_3(-\theta)$	$-2/3$	$\alpha^-$	$+2/3$
$(-1, 0; -1/3)$	$(-1, 0)$	$U_1(-\theta)$	$+1/3$	$\alpha^+$	$-1/3$
$(0, -1; -1/3)$	$(0, -1)$	$U_2(-\theta)$	$+1/3$	$\alpha^+$	$-1/3$
$(+1, +1; -1/3)$	$(+1, +1)$	$U_3(-\theta)$	$+1/3$	$\alpha^+$	$-1/3$
$(+1, 0; -2/3)$	$(+1, 0)$	$U_1(\theta)$	$-2/3$	$\alpha^-$	$-2/3$
$(0, +1; -2/3)$	$(0, +1)$	$U_2(\theta)$	$-2/3$	$\alpha^-$	$-2/3$
$(-1, -1; -2/3)$	$(-1, -1)$	$U_3(\theta)$	$-2/3$	$\alpha^-$	$-2/3$

**Table 2.** We label configurations by  $(v_1, v_2; B) \in (\pi_1(\mathcal{N}); \pi_3(M))$ .

with a constant  $\alpha_0$ . Correspondingly, each of them carries fractional instanton (baryon) number:

$$B = \pm \frac{1}{2\pi} [\alpha]_{z=0}^{z=R} = \begin{cases} \pm \frac{1}{3} & \text{for } \alpha = \alpha^+ \\ \mp \frac{2}{3} & \text{for } \alpha = \alpha^- \end{cases}, \quad (4.9)$$

where the upper and lower signs correspond to a vortex and anti-vortex, respectively. We thus have found six (four independent) types of elementary fractional instantons as well as six (four independent) types of elementary fractional anti-instantons, as summarized in table 2. We label all configurations by the first homotopy group of the fixed point manifold  $\mathcal{N}$  and the instanton (baryon) number  $B$ :

$$(v_1, v_2; B) \in (\pi_1(\mathcal{N}); \pi_3(M)). \quad (4.10)$$

The first homotopy group  $\pi_1(\mathcal{M})$  of the moduli space does not give independent information, so we omit it from the label. The number of elementary vortices in the SU(3) case is twice of that of the SU(2) case, just because of the two independent vortices in eq. (4.4) compared with one for the SU(2) case.

As the case of SU(2), all vortices are global vortices having the divergent energy

$$E \sim \log \Lambda/\xi \quad (4.11)$$

at large distance, apart from finite contribution from the core. In the SU(3) case, the size  $\xi$  of the vortex core is  $\xi \sim m_1^{-1}, m_2^{-1}, m_3^{-1}$  depending on species.

The interaction between vortices at large distance  $\rho \gg \xi$  depends only on their winding numbers:

$$E_{\text{int}} = \mp 2 \log \rho / \xi, \quad F_{\text{int}} = \pm \frac{2}{\rho} \quad (4.12)$$

for vortices of the same kind  $U_a(\theta)$  and  $U_a(\theta)$  (the upper sign; repulsion), and for a vortex  $U_a(\theta)$  and an anti-vortex  $U_a(-\theta)$  (the lower sign; attraction) of the same kind. It is, however, opposite for different kinds of vortices:

$$E_{\text{int}} = \pm \log \rho / \xi, \quad F_{\text{int}} = \mp \frac{1}{\rho}, \quad (4.13)$$

for a vortex  $U_a(\theta)$  and a vortex  $U_b(\theta)$  ( $a \neq b$ ) (the upper sign; attraction) and a vortex  $U_a(\theta)$  and an anti-vortex  $U_b(\theta)$  ( $a \neq b$ ) (the lower sign; repulsion). The interaction between two vortices at short distance  $\rho \sim \xi$  depends on the moduli  $\alpha^\pm$  in the cores, but we do not discuss it.

A unit (anti-)instanton  $(0, 0; \pm 1)$  can be  $\mathbb{Z}_3$  symmetrically decomposed into three  $\pm 1/3$  instantons in eq. (4.4) with  $\alpha = \alpha^+$  in eq. (4.8):

$$\begin{aligned} (0, 0; +1) &= (+1, 0; +1/3) + (0, +1; +1/3) + (-1, -1; +1/3), \\ (0, 0; -1) &= (-1, 0; -1/3) + (0, +1; -2/3) + (+1, +1; -1/3). \end{aligned} \quad (4.14)$$

However, the decompositions from left to right are energetically unfavorable at least for large radius  $R$ , because of the absence of vortices in the left and the presence of three vortices in the right. This can be also verified from the interactions in eq. (4.13) between two among them are attractive at large separations.

A unit (anti-)instanton  $(0, 0; \pm 1)$  also can be decomposed asymmetrically into two fractions as

$$\begin{aligned} (0, 0; +1) &= (+1, 0; +1/3) + (-1, 0; +2/3), \\ (0, 0; +1) &= (0, +1; +1/3) + (0, -1; +2/3), \\ (0, 0; +1) &= (-1, -1; +1/3) + (+1, +1; +2/3), \\ (0, 0; -1) &= (-1, 0; -1/3) + (+1, 0; -2/3), \\ (0, 0; -1) &= (0, -1; -1/3) + (0, +1; -2/3), \\ (0, 0; -1) &= (+1, +1; -1/3) + (-1, -1; -2/3). \end{aligned} \quad (4.15)$$

These decompositions are also energetically unfavorable at least for large radius  $R$  of  $S^1$ .

The charge two (anti-)instanton  $(0, 0; \pm 2)$  also can decay in several ways such as

$$\begin{aligned} (0, 0; +2) &= (-1, 0; +2/3) + (0, -1; +2/3) + (+1, +1; +2/3), \\ (0, 0; -2) &= (+1, 0; -2/3) + (0, +1; -2/3) + (-1, -1; -2/3). \end{aligned} \quad (4.16)$$

or

$$\begin{aligned} (0, 0; +2) &= 2(+1, 0; +1/3) + 2(0, +1; +1/3) + 2(-1, -1; +1/3), \\ (0, 0; -2) &= 2(-1, 0; -1/3) + 2(0, +1; -2/3) + 2(+1, +1; -1/3). \end{aligned} \quad (4.17)$$

As one can expect, the fundamental fractional instantons with the instanton charge  $\pm 2/3$  can be decomposed, at least homotopically, into two fractional instantons with the instanton charge  $\pm 1/3$  as:

$$\begin{aligned}
 (-1, 0; +2/3) &= (0, +1; +1/3) + (-1, -1; +1/3), \\
 (0, -1; +2/3) &= (-1, -1; +1/3) + (+1, 0; +1/3), \\
 (+1, +1; +2/3) &= (+1, 0; +1/3) + (0, +1; +1/3), \\
 (+1, 0; -2/3) &= (0, -1; -1/3) + (+1, +1; -1/3), \\
 (0, +1; -2/3) &= (+1, +1; -1/3) + (-1, 0; -1/3), \\
 (-1, -1; -2/3) &= (-1, 0; -1/3) + (0, -1; -1/3).
 \end{aligned}
 \tag{4.18}$$

These decompositions are energetically unfavorable because the numbers of vortices are one in the left and two in the right. Unlike the case of decomposition of unit instantons, we regard configurations with  $B = 2/3$  are elementary for the moment, because the vortex winding numbers are the minimum.

In order to satisfy the twisted boundary condition in eq. (4.7), one may consider a configuration with a more rapid  $z$ -dependence modulo  $2\pi$  instead of eq. (4.8), such as  $8\pi/3 = 2\pi + 2\pi/3$ . However, it can be decomposed through a self-reconnection into a closed-line configuration with an integer  $B$  that does not reach the boundary and a fraction given above. In this sense, such configurations are not elementary.

A comment is in order here. The decompositions of the unit instanton and  $B = 2/3$  instantons are very similar to those of vortices (color flux tubes) in dense QCD [63–68]. The unit instanton corresponds to a U(1) superfluid vortex without color flux and  $B = 1/3$  and  $2/3$  fractional instantons correspond to  $M_1$  and  $M_2$  non-Abelian vortices having color fluxes, respectively.

## 4.2 Neutral bions

As the SU(2) case, neutral bions are configurations with zero instanton charges and zero vortex charges:

$$\sum_i (v_{1,i}, v_{2,i}; B_i) = (0, 0; 0).
 \tag{4.19}$$

Let us define the order of neutral bions as the maximum instanton charge of a subgroup of constituents.

The lowest order of neutral bions is  $1/3$  (the total instanton charge is therefore  $B = 1/3 - 1/3$ ):

$$\begin{aligned}
 (+1, 0; +1/3) + (-1, 0; -1/3) &= (0, 0; 0), \\
 (0, +1; +1/3) + (0, -1; -1/3) &= (0, 0; 0), \\
 (-1, -1; +1/3) + (+1, +1; -1/3) &= (0, 0; 0).
 \end{aligned}
 \tag{4.20}$$

The neutral bions in the left sides can annihilate in a pair to the vacuum.

There exist two kinds of neutral bions of the order  $2/3$  ( $B = 2/3 - 2/3$ ). The simplest ones are composed of two fractional (anti-)instantons:

$$\begin{aligned}
 (-1, 0; +2/3) + (+1, 0; -2/3) &= (0, 0; 0), \\
 (0, -1; +1/3) + (0, +1; -2/3) &= (0, 0; 0), \\
 (+1, +1; +2/3) + (-1, -1; -2/3) &= (0, 0; 0).
 \end{aligned}
 \tag{4.21}$$

They can annihilate in pair to the vacuum. More nontrivial ones are made of three fractional instantons:

$$\begin{aligned}
 (+1, 0; +1/3) + (0, +1; +1/3) + (-1, -1; -2/3) &= (0, 0; 0), \\
 (0, +1; +1/3) + (-1, -1; +1/3) + (+1, 0; -2/3) &= (0, 0; 0), \\
 (-1, -1; +1/3) + (+1, 0; +1/3) + (0, +1; -2/3) &= (0, 0; 0), \\
 (-1, 0; -1/3) + (0, -1; -1/3) + (+1, +1; -2/3) &= (0, 0; 0), \\
 (0, -1; -1/3) + (+1, +1; -1/3) + (-1, 0; -2/3) &= (0, 0; 0), \\
 (+1, +1; -1/3) + (-1, 0; +1/3) + (0, -1; -2/3) &= (0, 0; 0).
 \end{aligned}
 \tag{4.22}$$

Since each set of them is not a simple pair of fractional and anti-fractional instantons, it does not have to annihilate to the vacuum. Instead it may constitute a stable bound state.

Interesting is that we have neutral bions of the order one ( $B = 1 - 1$ ) that is *not* a pair of instanton and anti-instanton:

$$\begin{aligned}
 &[(+1, 0; +1/3) + (0, -1; +2/3)] + [(-1, 0; -1/3) + (0, +1; -2/3)] \\
 &= (+1, -1; +1) + (-1, +1; -1) = (0, 0; 0), \\
 &[(0, +1; +1/3) + (+1, +1; +2/3)] + [(0, -1; -1/3) + (-1, -1; -2/3)] \\
 &= (+1, +2; +1) + (-1, -2; -1) = (0, 0; 0), \\
 &[(-1, -1; +1/3) + (-1, 0; +2/3)] + [(+1, +1; -1/3) + (+1, 0; -2/3)] \\
 &= (-2, -1; +1) + (+2, +1; -1) = (0, 0; 0), \\
 &[(+1, 0; +1/3) + (+1, +1; +2/3)] + [(-1, 0; -1/3) + (-1, -1; -2/3)] \\
 &= (+2, +1; +1) + (-2, -1; -1) = (0, 0; 0), \\
 &[(0, +1; +1/3) + (+1, +1; +2/3)] + [(0, -1; -1/3) + (-1, -1; -2/3)] \\
 &= (+1, +2; +1) + (-1, -2, -1) = (0, 0; 0), \\
 &[(-1, -1; +1/3) + (-1, 0; +2/3)] + [(+1, +1; -1/3) + (+1, 0; -2/3)] \\
 &= (-2, -1; +1) + (+2, +1; -1) = (0, 0; 0).
 \end{aligned}
 \tag{4.23}$$

More surprisingly, there are neutral bions of the order greater than one, that do not contain instanton or anti-instanton. For instance, the following is of the order  $4/3$ :

$$\begin{aligned}
 &[(+1, 0; +1/3) + (0, +1; +1/3) + (+1, +1; +2/3)] \\
 &+ [(-1, 0; -1/3) + (0, -1; -1/3) + (-1, -1; +2/3)] \\
 &= (+2, +2; +4/3) + (-2, -2; -4/3) = (0, 0; 0).
 \end{aligned}
 \tag{4.24}$$

### 4.3 Charged bions

In the same way, charged bions are configurations with zero instanton charges and non-zero vortex charges:

$$\sum_i (v_{1,i}, v_{2,i}; B_i) = (v_1, v_2, 0), \quad (v_1, v_2) \neq (0, 0). \quad (4.25)$$

We define the order of charged bions as the same way with that of neutral bions. Charged bions of the order  $1/3$  are composed of two fractional (anti-)instantons with the topological charges  $B = \pm 1/3$  are

$$\begin{aligned} (+1, 0; +1/3) + (0, -1; -1/3) &= (1, -1; 0), \\ (+1, 0; +1/3) + (+1, +1; -1/3) &= (2, +1; 0), \\ (0, +1; +1/3) + (-1, 0; -1/3) &= (-1, +1; 0), \\ (0, +1; +1/3) + (+1, +1; -1/3) &= (+1, +2; 0), \\ (-1, -1; +1/3) + (-1, 0; -1/3) &= (-2, -1; 0), \\ (-1, -1; +1/3) + (0, -1; -1/3) &= (-1, -2; 0). \end{aligned} \quad (4.26)$$

We may call them “mesons.”

Similarly to this, charged bions of the order  $2/3$ , which are composed of two fractional (anti-)instantons with the topological charges  $B = \pm 2/3$  are:

$$\begin{aligned} (-1, 0; +2/3) + (0, +1; -2/3) &= (-1, +1; 0), \\ (-1, 0; +2/3) + (-1, -1; -2/3) &= (-2, -1; 0), \\ (0, -1; -1/3) + (+1, 0; -2/3) &= (+1, -1; 0), \\ (0, -1; -1/3) + (-1, -1; -2/3) &= (-1, -2; 0), \\ (+1, +1; -1/3) + (+1, 0; -2/3) &= (+2, +1; 0), \\ (+1, +1; -1/3) + (0, +1; -2/3) &= (+1, +2; 0). \end{aligned} \quad (4.27)$$

In addition, there are charged bions of the order  $2/3$  ( $B = 2/3 - 2/3$ ), that are composed of three distinct fractional (anti-)instantons:

$$\begin{aligned} (+1, 0; +1/3) + (0, +1; +1/3) + (+1, 0; -2/3) &= (+2, +1; 0), \\ (+1, 0; +1/3) + (0, +1; +1/3) + (0, +1; -2/3) &= (+1, +2; 0), \\ (0, +1; +1/3) + (-1, -1; +1/3) + (+1, 0; -2/3) &= (+1, +1; 0), \\ (0, +1; +1/3) + (-1, -1; +1/3) + (-1, -1; -2/3) &= (-2, -1; 0), \\ (-1, -1; +1/3) + (+1, 0; +1/3) + (-1, -1; -2/3) &= (-1, -2; 0), \\ (-1, -1; +1/3) + (+1, 0; +1/3) + (+1, 0; -2/3) &= (+1, -1; 0), \end{aligned} \quad (4.28)$$

and those composed of the two same and one distinct (anti-)fractional instantons:

$$\begin{aligned}
 2(+1, 0; +1/3) + (+1, 0; -2/3) &= (+3, 0; 0), \\
 2(0, +1; +1/3) + (0, +1; -2/3) &= (0, +3; 0), \\
 2(-1, -1; +1/3) + (-1, -1; -2/3) &= (-3, -3; 0), \\
 2(+1, 0; +1/3) + (0, +1; -2/3) &= (+2, +1; 0), \\
 2(+1, 0; +1/3) + (-1, -1; -2/3) &= (+1, -1; 0), \\
 2(0, +1; +1/3) + (+1, 0; -2/3) &= (+1, +2; 0), \\
 2(0, +1; +1/3) + (-1, -1; -2/3) &= (-1, +1; 0), \\
 2(-1, -1; +1/3) + (+1, 0; -2/3) &= (-1, -2; 0), \\
 2(-1, -1; +1/3) + (0, +1; -2/3) &= (-2, -1; 0).
 \end{aligned} \tag{4.29}$$

## 5 Generalization to the $SU(N)$ principal chiral model

It is straightforward to generalize our results to the  $SU(N)$  principal chiral model with the center symmetric twisted boundary condition in eq. (2.3). The fixed manifold  $\mathcal{N}$  is

$$U = \text{diag.}(e^{i\alpha_1}, e^{i\alpha_2}, \dots, e^{-i\sum_{a=1}^{N-1} \alpha_a}), \quad \mathcal{N} \simeq U(1)^{N-1}. \tag{5.1}$$

The non-trivial first homotopy group

$$\pi_1(\mathcal{N}) \simeq \mathbb{Z}^{N-1} \tag{5.2}$$

admits homotopically distinct  $N - 1$  kinds of vortices. The fundamental vortices in the  $SU(N)$  principal chiral model can be obtained by embedding the one in eq. (3.3) of the  $SU(2)$  principal chiral model to the  $SU(N)$  matrix:

$$\begin{aligned}
 U_1(r, \theta, z) &= \begin{pmatrix} \cos f(r)e^{+i\theta} & -\sin f(r)e^{+i\alpha(z)} & 0 \\ \sin f(r)e^{-i\alpha(z)} & \cos f(r)e^{-i\theta} & 0 \\ 0 & 0 & \mathbf{1}_{N-2} \end{pmatrix}, \\
 U_2(r, \theta, z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos f(r)e^{+i\theta} & -\sin f(r)e^{+i\alpha(z)} \\ 0 & \sin f(r)e^{-i\alpha(z)} & \cos f(r)e^{-i\theta} \\ & & & \mathbf{1}_{N-3} \end{pmatrix}, \\
 &\vdots \\
 U_N(r, \theta, z) &= \begin{pmatrix} \cos f(r)e^{-i\theta} & 0 & +\sin f(r)e^{-i\alpha(z)} \\ 0 & \mathbf{1}_{N-2} & 0 \\ -\sin f(r)e^{+i\alpha(z)} & 0 & \cos f(r)e^{+i\theta} \end{pmatrix}.
 \end{aligned} \tag{5.3}$$

Anti-vortices can be obtained as  $U_a(r, -\theta)$  with  $a = 1, \dots, N - 1$ . These are not homotopically independent of each other. We take the first  $N - 1$  as the independent basis of

the first homotopy group in eq. (4.2), in which the topological vortex charges in eq. (4.4) are  $(N - 1)$  vectors)

$$\begin{aligned} u_1 &= (1, 0, \dots, 0), & u_2 &= (0, 1, 0, \dots, 0), & \dots, \\ u_{N-1} &= (0, 0, 0, \dots, 0, 1), & u_N &= (-1, -1, \dots, -1) \in \mathbb{Z}^{N-1} \simeq \pi_1(\mathcal{N}), \end{aligned} \quad (5.4)$$

respectively. The last one is not independent of the rests as can be seen from the fact that the vortex charges are canceled out when all three are present together:

$$\sum_{a=1}^N u_a = (0, 0, \dots, 0). \quad (5.5)$$

In eq. (5.3),  $\alpha$  is a U(1) modulus of a vortex that is constant if the vortex does not wrap the  $S^1$  direction.

When a vortex wraps the  $S^1$  direction, the modulus  $\alpha$  must change along the vortex enforced by the twisted boundary condition in eq. (2.3):

$$\exp i\alpha(z + R) = \exp \left[ i\alpha(z) + \frac{2\pi i}{N} \right], \quad (5.6)$$

where  $z$  is coordinate along  $S^1$  with the period  $R$ . This boundary condition can be satisfied by the two different minimum paths with the following  $z$ -dependence of  $\alpha$ :

$$\alpha(z) = \begin{cases} \alpha^+(z) = \alpha_0 + \frac{2\pi}{N} \frac{z}{R} \\ \alpha^-(z) = \alpha_0 - \frac{2\pi(N-1)}{N} \frac{z}{R} \end{cases}. \quad (5.7)$$

Correspondingly, each of them carries fractional instanton (baryon) number:

$$B = \pm \frac{1}{2\pi} [\alpha]_{z=0}^{z=R} = \begin{cases} \pm \frac{1}{N} & \text{for } \alpha = \alpha^+ \\ \mp \frac{N-1}{N} & \text{for } \alpha = \alpha^- \end{cases}, \quad (5.8)$$

where the upper and lower signs correspond to a vortex or anti-vortex, respectively. We thus have found  $2N$  ( $2N - 2$  independent) types of elementary fractional instantons as well as  $2N$  ( $2N - 2$  independent) types of elementary fractional anti-instantons, labeled by  $(v_1, \dots, v_{N-1}; B)$ .

Neutral bions and charged bions can be constructed in the same way with the SU(3) case, but the number of combinations grows drastically. We need a more systematic analysis that we leave for a future study.

## 6 More general twisted boundary conditions

The more general twisted boundary condition for the SU( $N$ ) principal chiral model was given in eq. (2.6). In this case, the boundary condition on the U(1) moduli are

$$\exp i\alpha_a(z + R) = \exp [i\alpha_a(z) + i(m_{a+1} - m_a)], \quad (6.1)$$



with  $m_{N+1} \equiv m_1 + 2\pi$ . This can be satisfied by the following  $z$ -dependence of the moduli:

$$\alpha(z) = \begin{cases} \alpha^+(z) = +(m_{a+1} - m_a) \frac{z}{R} \\ \alpha^-(z) = -(2\pi - m_{a+1} + m_a) \frac{z}{R} \end{cases} . \quad (6.2)$$

Correspondingly, each of them carries fractional instanton (baryon) number that are not rational number anymore:

$$B_a = \pm \frac{1}{2\pi} [\alpha_a]_{z=0}^{z=R} = \begin{cases} \pm(m_{a+1} - m_a) & \text{for } \alpha = \alpha^+ \\ \mp(2\pi - m_{a+1} + m_a) & \text{for } \alpha = \alpha^- \end{cases} . \quad (6.3)$$

The sum of all fractions is of course unity:

$$\sum_{a=1}^N B_a = \frac{m_{N+1} - m_1}{2\pi} = 1. \quad (6.4)$$

In the previous sections, we have defined bions as configurations with zero instanton charges, that is enough for the  $\mathbb{Z}_N$  symmetric twisted boundary conditions. In refs. [8, 9], an invariant definition of bions using the Lie algebra was given. Here, we follow their definition of bions. Instead of using the Lie algebra, it is enough to regard bions in the previous sections remain bions for generic boundary conditions. In this case, cancellation of instanton charges for bions becomes very restrictive and is drastically simplified, since fractions of instanton charges are not rational. For non-degenerate and generic  $m_a$  without any particular relation such as  $m_a + m_b = m_c$ , fractional instanton charges cannot be canceled out among different types of fractional instantons. Consequently, only neutral bions of the minimum orders composed of a vortex and an anti-vortex of the same kind with the same  $U(1)$  moduli shift along the  $S^1$  direction have zero instanton charges. We conclude the existence of  $2N$  neutral bions of the minimum order, but higher order ones, having zero instanton charges for the  $\mathbb{Z}_N$  symmetric boundary condition, have instanton charges.

On the other hand, charged bions have instanton charges in general, because different vortices cannot cancel their instanton charges unless a particular relation such as  $m_a + m_b = m_c$  exists.

A partially degenerated case is interesting since vortices would carry non-Abelian moduli. We will return to this case in a future.

## 7 Relation to Yang-Mills fractional instantons and bions

A  $\mathbb{C}P^{N-1}$  instanton with the twisted boundary condition is decomposed into a set of  $N$  fractional instantons which are half twisted domain walls. The same relation holds between a Yang-Mills instanton and a BPS monopole. In ref. [29, 30, 80],  $\mathbb{C}P^{N-1}$  fractional instantons were realized as fractional Yang-Mills instantons trapped inside a vortex in a  $U(N)$  gauge theory. This explains a correspondence between fractional instantons and bions in the  $\mathbb{C}P^{N-1}$  on  $\mathbb{R}^1 \times S^1$  and those in  $SU(N)$  Yang-Mills in on  $\mathbb{R}^3 \times S^1$ . In this

section, we point out fractional instantons and bions in the  $SU(N)$  principal chiral model also correspond to  $SU(N)$  Yang-Mills fractional instantons and bions.

Let us consider the  $U(N)$  gauge theory in  $d = 4 + 1$  dimensions ( $A, B = 0, \dots, 4$ ). The matter contents are  $U(N)$  gauge field  $A_A(x)$ ,  $2N$  complex scalar fields  $H(x) = (H_1(x), H_2(x))$  (decomposed into two  $N \times N$  matrices of scalar fields) in the fundamental representation charged under  $U(1)$ , an  $N \times N$  matrix  $\Sigma(x)$  of scalar fields in the adjoint representation neutral under  $U(1)$ . The Lagrangian is given by [40–42]

$$\mathcal{L} = -\frac{1}{4g^2} \text{tr} F_{AB} F^{AB} + \frac{1}{g^2} \text{tr} (D_A \Sigma)^2 + \text{tr} |D_A H|^2 - V \quad (7.1)$$

$$V = \frac{g^2}{4} \text{tr} (HH^\dagger - v^2 \mathbf{1}_N)^2 + \text{tr} |\Sigma H - HM|^2, \quad (7.2)$$

where the covariant derivatives are given by  $D_A H = \partial_A H - iA_A H$  and  $D_A \Sigma = \partial_A \Sigma - i[A_A, \Sigma]$ ,  $g$  is the gauge coupling constant that we take common for the  $U(1)$  and  $SU(N)$  factors of  $U(N)$ ,  $v$  is a real constant giving the vacuum expectation value of  $H$ ,  $M$  is a  $2N$  by  $2N$  mass matrix given as  $M = \text{diag.}(m\mathbf{1}_N, -m\mathbf{1}_N)$ . The  $U(N)$  gauge (color) symmetry and the flavor symmetry act on fields as

$$A_A \rightarrow g A_A g^{-1} + i g \partial_A g^{-1}, \quad \Sigma \rightarrow g \Sigma g^{-1}, \quad H \rightarrow g H, \quad g \in U(N)_C, \quad (7.3)$$

$$H_1 \rightarrow H_1 U_L e^{+i\alpha}, \quad H_2 \rightarrow H_2 U_R e^{-i\alpha}, \quad U_{L,R} \in SU(N)_{L,R}. \quad (7.4)$$

The model admits two disjoint vacua

$$\begin{aligned} H &= (v\mathbf{1}_N, \mathbf{0}_N), & \Sigma &= -m\mathbf{1}_N : & SU(N)_{C+L}, \\ H &= (\mathbf{0}_N, v\mathbf{1}_N), & \Sigma &= +m\mathbf{1}_N : & SU(N)_{C+R} \end{aligned} \quad (7.5)$$

with the unbroken color-flavor locked *global* symmetries  $g = U_L$  and  $g = U_R$ , respectively.

The model admits a non-Abelian domain wall solution interpolating between the two vacua in eq. (7.5), that is obtained by embedding the  $\mathbb{C}P^1$  domain wall [69–72]. The solution perpendicular to the  $x^4$  coordinate can be given by [40–42, 73–77]

$$\begin{aligned} H_{\text{wall}} &= V H_{\text{wall},0} \begin{pmatrix} V^\dagger & 0 \\ 0 & V \end{pmatrix} = \frac{v}{\sqrt{1 + e^{\mp 2m(x^4 - X)}}} \left( \mathbf{1}_N, e^{\mp m(x^4 - X)} U \right), \\ \Sigma_{\text{wall}} &= V \Sigma_{\text{wall},0} V^\dagger, & A_{4,\text{wall}} &= V A_{4,\text{wall},0} V^\dagger, \end{aligned} \quad (7.6)$$

with  $V \in SU(N)$ , and we have defined the moduli  $U \equiv V^2 e^{i\varphi} \in U(N)$  of the domain wall:

$$\mathcal{M}_{\text{wall}} \simeq \mathbb{R} \times U(N) \ni (X, U). \quad (7.7)$$

The width of the domain wall is  $m^{-1}$ .

The low-energy dynamics of the non-Abelian domain wall can be described by the effective theory within the moduli approximation [78, 79], by promoting the moduli parameters  $X$  and  $U$  to moduli fields  $X(x^\mu)$  and  $U(x^\mu)$ , respectively ( $\mu = 0, 1, 2, 3$ ) on the

world volume of the domain wall, and by performing integration over the codimension. We thus obtain the effective theory [40–42]:

$$\mathcal{L}_{\text{wall}} = \frac{v^2}{2m} \partial_\mu X \partial^i X - f_\pi^2 \text{tr} \left( U^\dagger \partial_\mu U U^\dagger \partial^\mu U \right), \quad f_\pi^2 \equiv \frac{v^2}{4m}, \quad (7.8)$$

that is a  $U(N)$  principal chiral model we are discussing.

It was shown in ref. [41] that Yang-Mills instantons inside the domain wall are described by Skyrmions in the principal chiral model on the domain wall. This setting physically realizes the Atiyah-Manton construction of Skyrmions from instanton holonomy [81, 82]. Furthermore, it has been recently found in ref. [37] that magnetic monopoles inside the domain wall are described by vortices in the principal chiral model with the twisted mass term in eq. (2.7). In fact, the instanton charge  $\pi_3$  in eq. (6.3) is proportional to the monopole charge. Interpolating solutions for the  $SU(2)$  case were already constructed numerically in the Skyrme model with the twisted boundary condition [83]. In the limit of  $m \rightarrow 0$ , the width of the domain wall diverges and further taking  $v \rightarrow 0$ , the system leaves from the Higgs phase recovering the pure Yang-Mills theory. The fractional instantons and bions become those in Yang-Mills theory. Therefore, this setting offers a correspondence between fractional instantons and bions in the  $SU(N)$  principal chiral model and those in Yang-Mills theory.

## 8 Summary and discussion

We have classified fractional instantons and bions in the  $SU(N)$  principal chiral model on  $\mathbb{R}^2 \times S^1$  with twisted boundary conditions. This model allows  $N$  kinds of global vortices with  $U(1)$  moduli, among which  $N - 1$  kinds are independent. Fractional instantons are  $N$  ( $N - 1$  independent) kinds of global vortices wrapping around  $S^1$  with  $U(1)$  moduli twisted with the angle  $2\pi/N$  along the world-line  $S^1$ . We have found that they carry  $1/N$  instanton (baryon) numbers for the  $\mathbb{Z}_N$  symmetric twisted boundary condition and irrational instanton numbers for the generic boundary conditions with the general phases  $m_a$ . We have classified neutral and charged bions for the  $SU(3)$  case with the  $\mathbb{Z}_3$  symmetric twisted boundary condition. We have found for the generic boundary conditions that there are only the simplest neutral bions composed of a pair of (anti-)fractional instantons have zero instanton charges and instanton charges are not canceled out for charged bions unless some particular relation holds among  $m_a$ . We have also discussed a correspondence between fractional instantons and bions in the  $SU(N)$  principal chiral model and those in Yang-Mills theory through a non-Abelian Josephson junction.

We have studied  $\mathbb{Z}_N$  center symmetric twisted boundary condition and non-degenerate ( $m_a \neq m_b$  for  $a \neq b$ ) boundary conditions. A partially degenerated case will be interesting since vortices would carry non-Abelian moduli in this case. We will return to this case in a future.

In this paper, we have put the twisted boundary conditions by hand, but the  $\mathbb{Z}_N$  symmetric boundary condition was chosen for the  $\mathbb{C}P^{N-1}$  model from the effective potential in quantum theory [9, 10]. The same analysis should be done in our case. Although the

principal chiral model is not renormalizable in perturbation in three dimensions, they are renormalizable at large  $N$ . We will return to this problem in the future.

In order to generalize to the principal chiral model with arbitrary groups, the description of bions in terms of the Lie algebra and the root system given in refs. [8, 9] should be useful.

In the context of the Skyrme model, Skyrme chains on  $\mathbb{R}^2 \times S^1$  with a twisted boundary condition were studied before [83], in which a vortex structure was found. If we add the Skyrme term to our model, we can consider  $SU(N)$  Skyrme chains.

The fractional instantons in the principal chiral model are all global vortices and the interaction between them is long range,  $E_{\text{int}} \sim \pm \log \rho$  with the distance  $\rho$ . Therefore, they are confined. If we gauge the  $U(1)^{N-1}$  center action, vortices become local vortices, i.e., of the Abrikosov-Nielsen-Olesen type [84, 85], for which the interaction is exponentially suppressed with respect to the distance. While this case will be interesting on its own, fractional instantons are also local and the total action is the sum of actions of individual fractional instantons so that they would be useful for resurgence of the quantum field theory.

Supersymmetric extension is possible by generalizing the target space to the cotangent bundle,  $T^*SU(N)$ , that is Kähler.

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