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Explicit criteria for stability of fractional h-difference two-dimensional systems

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Abstract In the paper the explicit conditions for stability of linear fractional order h-difference systems with the Grünwald–Letnikov-type operator are presented. The state variables of the considered systems are taken from the plane. As the tool the $\mathcal{Z}$-transform, which can be considered as an effective method for the stability analysis of linear systems, is used. The main result gives the sufficient and necessary condition for the asymptotic stability of the considered system according to the entries of the given matrix associated with the system.

Keywords Stability · Fractional operator · $\mathcal{Z}$-Transform

1 Introduction

The stability analysis is one of the most essential problems in dynamical systems as in control theory. Recently, it has been investigated for the fractional systems in some papers, see for example [1–3] for continuous-time case and [4–6] for the discrete-time case. Contrary to the continuous case, the stability theory of fractional difference equations is less developed. In [7–9] $\mathcal{Z}$-transform is used as an effective method for stability analysis of linear discrete-time fractional order systems. The explicit stability conditions for a linear fractional difference system with the Caputo-type operator of order $\alpha \in (0, 1)$ are presented in [10]. The alternative (less convenient for practical purposes compared to [10]) stability conditions for the Caputo difference systems are presented in [11]. Additionally, in [10] the discussion concerning the stability behaviour of systems with the Riemann–Liouville-type difference operator is given. In [10] the stability conditions are considered as a direct extension of the classical results given in [12] for the difference systems.

Our main goal is to formulate the explicit stability conditions for the two–dimensional h-difference systems with the Grünwald–Letnikov-type operator since in many applications one needs explicit criteria on the entries of the matrix associated with the considered system. The formulated alternative stability conditions could be considered as an extension of Theorem 2.37 in [12] to the fractional systems.

The main results are stated in Theorem 1, where there are given the exact conditions on elements of the matrix of the considered system. The proposed method is not pretended to be better than the other's method available in the literature, but it is stated as exact conditions on values of the matrix. This method is restricted to two-dimensional system as it is particularly often that some models are needed only for two-dimensional systems.

It should be also stressed that the Grünwald–Letnikov-type fractional h-difference operator is used in papers connected with applications of fractional differences in circuit systems, see for example [13].

The paper is organized as follows. In Sect. 2 the basic definitions of h-difference fractional order operator are given. Section 3 provides the properties of the $\mathcal{Z}$-transform acting on fractional operators, especially on the Grünwald–Letnikov-type operator. In Sect. 4 the problem of stability of linear multi-parameter fractional difference control sys-

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tems with the Grünwald–Letnikov h-difference operator is considered. Finally, Sect. 5 provides brief conclusions.

2 Preliminaries

Firstly, we recall some necessary definitions and notations used in the sequel therein the paper. Let $a \in \mathbb{R}$. Then $(h\mathbb{N})_a := \{a, a+h, a+2h, \ldots\}$. Let x denote a real function defined on $(h\mathbb{N})_a$, i.e. $x : (h\mathbb{N})_a \to \mathbb{R}$. Let us recall the definition of the Grünwald–Letnikov-type difference operators, see for example [14–16] for cases $h = 1$ and extended for general case $h > 0$ in [17]. Here we present basic results for the case when $h > 1$.

Definition 1 Let $\alpha \in \mathbb{R}$. The *Grünwald–Letnikov-type h-difference operator* ${}_a\Delta_h^\alpha$ *of order* α *for a function* $x : (h\mathbb{N})_a \to \mathbb{R}$ is defined by

$$\left({}_a\Delta_h^\alpha x\right)(t) := h^{-\alpha} \sum_{s=0}^{\frac{t}{h}-a} c_s^{(\alpha)} x(t - sh), \tag{1}$$

where $t \in (h\mathbb{N})_a$ and $c_s^{(\alpha)} = (-1)^s \binom{\alpha}{s}$ with

$$\binom{\alpha}{s} = \begin{cases} 1 & \text{for } s = 0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha-s+1)}{s!} & \text{for } s \in \mathbb{N}. \end{cases}$$

The Grünwald–Letnikov-type h-difference operator can be extended to vector valued sequences in the componentwise manner, i.e. for $x = (x_1, x_2) : (h\mathbb{N})_a \to \mathbb{R}^2$ we have ${}_a\Delta_h^\alpha x = \left({}_a\Delta_h^\alpha x_1, {}_a\Delta_h^\alpha x_2\right)$. If $a = 0$, then we will write: $\Delta_h^\alpha :={}_0\Delta_h^\alpha$.

Let us recall that the $\mathcal{Z}$-transform of a sequence $\{y(n)\}_{n\in\mathbb{N}_0}$ is a complex function given by

$$Y(z) := \mathcal{Z}[y](z) = \sum_{k=0}^{\infty} \frac{y(k)}{z^k},$$

where $z \in \mathbb{C}$ denotes a complex number for which this series converges absolutely. More about one-sided $\mathcal{Z}$-transform can be found in [18]. The $\mathcal{Z}$-transform can be extended to vector valued sequences in the componentwise manner, i.e. for $y = (y_1, y_2) : \mathbb{N}_0 \to \mathbb{R}^2$ we have $\mathcal{Z}[y] = (\mathcal{Z}[y_1], \mathcal{Z}[y_2])$. Then the inverse $\mathcal{Z}$-transform addresses the reverse problem, i.e., given a function $Y(z)$ and a region of convergence, find the signal $y(n)$ whose $\mathcal{Z}$-transform is $Y(z)$ and has the specified region of convergence. The presented $\mathcal{Z}$-transform involves, by definition, only the values of $y(n)$ for $n \geq 0$. Similarly as in the case of the classical $\mathcal{Z}$-transform, the sequence $\{y(n)\}_{n\in\mathbb{N}_0}$ can be obtain from the function $Y(z)$ by a process called the inverse $\mathcal{Z}$-transform. This process is symbolically denoted as $y(n) = \mathcal{Z}^{-1}[Y(z)](n)$.

Proposition 1 ([7]) *For* $a \in \mathbb{R}$, $\alpha \in (0, 1]$ *and* $x : (h\mathbb{N})_a \to \mathbb{R}^2$ *let us define* $y(k) := \left({}_a\Delta_h^\alpha x\right)(t)$, *where* $t \in (h\mathbb{N})_a$ *and* $t = a + kh$, $k \in \mathbb{N}_0$. *Then*

$$\mathcal{Z}[y](z) = h^{-\alpha} \left(\frac{z}{z-1}\right)^{-\alpha} X(z), \tag{2}$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ *and* $\overline{x}(k) := x(a + kh)$.

3 Systems

In this section we investigate the stability of the linear nonautonomous difference system with the Grünwald–Letnikov-type h-difference operator given by

$$\left(\Delta_h^\alpha x\right)((n+1)h) = Ax(nh), \quad n \in \mathbb{N}_0, \tag{3}$$

where $x : (h\mathbb{N})_0 \to \mathbb{R}^2$ and A is a 2×2 matrix, with the initial condition

$$x(0) = x_0 \in \mathbb{R}^2. \tag{4}$$

For the case $h = 1$ we write $\Delta^\alpha := \Delta_1^\alpha$. Since $\left(\Delta_h^\alpha x\right)(sh) = h^{-\alpha}\left(\Delta^\alpha \overline{x}\right)(s)$, the system (3) can be rewritten as follows:

$$\left(\Delta^\alpha \overline{x}\right)(n+1) = h^\alpha A\overline{x}(n), \tag{5}$$

where $\overline{x}(n) := x(nh)$.

In many applications one needs explicit criteria on the entries of the matrix for the zeros of the corresponding characteristic equation to lie inside the unit disk. Therefore consider the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ whose characteristic polynomial is given by

$$\begin{aligned} p_A(\lambda) &:= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \\ &= \lambda^2 - (\operatorname{tr} A)\lambda + \det A \end{aligned} \tag{6}$$

and

$$p_{h^\alpha A}(\lambda) = h^{2\alpha}\left[\left(\frac{\lambda}{h^\alpha}\right)^2 - (\operatorname{tr} A)\frac{\lambda}{h^\alpha} + \det A\right], \tag{7}$$

where $\operatorname{tr} A := a_{11} + a_{22}$ is the trace of matrix A and $\det A$ is the determinant of A. Moreover, let

$$\begin{aligned} p_{(A,\alpha,h)}(z) &:= p_{h^\alpha A}\left(z\left(1 - \frac{1}{z}\right)^\alpha\right) \\ &= h^{2\alpha} p_A\left(z\left(\frac{z-1}{hz}\right)^\alpha\right). \end{aligned} \tag{8}$$

Then $p_{(A,\alpha,h)}(z) = \det\left(Iz\left(1-\frac{1}{z}\right)^{\alpha} - h^{\alpha}A\right)$. The function $p_{(A,\alpha,h)}(z)$ is called the *fractional characteristic function of matrixAfor orderαand step* $h > 0$. And the equation

$$p_{(A,\alpha,h)}(z) = 0 \tag{9}$$

is named as the *fractional characteristic equation of matrix Afor order* αem and step $h > 0$. Using the results given for the Grünwald–Letnikov difference systems of the form 3 presented in [7] we can state the following proposition:

Proposition 2 *Let* $\alpha \in (0, 1]$. *Then solution of (3) with initial condition* (4) *is given by*

$$x(nh) = \mathcal{Z}^{-1}\left[\left[Iz\left(1-\frac{1}{z}\right)^{\alpha} - h^{\alpha}A\right]^{-1} x_0\right](n),$$

where I *is the identity matrix and* $n \in \mathbb{N}_1$.

4 Stability

At the beginning let us recall that the constant vector $x^{\text{eq}} = \left(x_1^{\text{eq}}, x_2^{\text{eq}}\right)$ is an *equilibrium point* of the fractional difference system (3) if and only if

$$\left(\Delta_h^{\alpha} x^{\text{eq}}\right)((n+1)h) = Ax^{\text{eq}}$$

for all $n \in \mathbb{N}_0$. Note that the trivial solution $x \equiv 0$ is an equilibrium point of system (3). Of course, if the determinant of the matrix A is nonzero, then system (3) has only one equilibrium point $x^{\text{eq}} = 0$.

Definition 2 The equilibrium point $x^{\text{eq}} = 0$ of system (3) is said to be

(a) *stable* if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(nh)\| < \epsilon$, for all $n \in \mathbb{N}_0$.
(b) *attractive* if there exists $\delta > 0$ such that $\|x_0\| < \delta$ implies $\lim_{n\to\infty} x(nh) = 0$.
(c) *asymptotically stable* if it is stable and attractive.

The fractional difference system (3) is called *stable/asymptotically stable* if their equilibrium points $x^{\text{eq}} = 0$ are stable/asymptotically stable.

Proposition 3 ([7]) *Let* R *be the set of all roots of the equation*

$$p_{(A,\alpha,h)}(z) = 0, \tag{10}$$

where A *is the square matrix in system* (3). *Then the following items are satisfied.*

(a) If all elements from R *are strictly inside the unit circle, then system* (3) *is asymptotically stable.*
(b) If there is $z \in R$ *such that* $|z| > 1$, *then system* (3) *is not stable.*

Proposition 4 *System* (3) *is asymptotically stable if and only if*

$$\varphi_i \in \left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right] \wedge |\lambda_i| < \left(2h\left|\sin\frac{\varphi_i - \alpha\frac{\pi}{2}}{2-\alpha}\right|\right)^{\alpha} \tag{11}$$

for $i = 1, 2$ *and where* $|\lambda_i|$ *and* φ_i *are the modulus and argument of the corresponding eigenvalue of the matrix* A.

Proof The result is based on those presented in [9]. Here we only use them for matrix $h^{\alpha}A$.

The main result that connects entries of the matrix A and the stability of the system is stated in the following theorem.

Theorem 1 *All elements from* R *are strictly inside the unit circle if and only if one of the set of conditions holds:*

1) $\begin{cases} -2^{\alpha+1} < h^{\alpha}\operatorname{tr} A < 0 \\ 0 < \det A \le \frac{1}{4}\operatorname{tr}^2 A \\ 4^{\alpha} + (2h)^{\alpha}\operatorname{tr} A + h^{2\alpha}\det A > 0 \end{cases}$,

2) $\begin{cases} \operatorname{tr} A \neq 0 \\ \frac{1}{4}\operatorname{tr}^2 A < \det A < \left(2\left|\sin\frac{\psi - \alpha\frac{\pi}{2}}{2-\alpha}\right|\right)^{2\alpha} \end{cases}$,

where $\psi = \arctan\frac{\sqrt{4\det A - \operatorname{tr}^2 A}}{\operatorname{tr} A}$ *or* $\psi = \pi + \arctan\frac{\sqrt{4\det A - \operatorname{tr}^2 A}}{\operatorname{tr} A}$,

3) $\begin{cases} \operatorname{tr} A = 0 \\ 0 < \det A < \left(2\left|\cos\frac{\pi}{2(2-\alpha)}\right|\right)^{2\alpha} \end{cases}$.

Proof The proof is based on the cases that for real roots of $p_{(A,\alpha,h)}(\lambda) = 0$ elements of R are strictly inside the unit circle if and only if $\lambda = z\left(\frac{z-1}{hz}\right)^{\alpha}$ are from the interval $(-2^{\alpha}, 0)$. Hence we need to find the solution of the systems of inequalities: $p_1^2 - 4p_2 \ge 0$, $-2^{\alpha} < \frac{-p_1 \pm \sqrt{p_1^2 - 4p_2}}{2} < 0$, with $p_1 = -h^{\alpha} tr A$, $p_2 = h^{2\alpha} det A$ that gives the set 1). The parts 2) and 3) are the version of Proposition 4.

The interesting and less difficult statement we receive for the order $\alpha = \frac{1}{2}$ and $h = 1$. Then

Corollary 1 *Let* $\alpha = \frac{1}{2}$ *and* $h = 1$. *Then all elements from* R *are strictly inside the unit circle if and only if one of the set of conditions holds:*

1) $\begin{cases} -2\sqrt{2} < \operatorname{tr} A < 0 \\ 0 < \det A \le \frac{1}{4}\operatorname{tr}^2 A \\ 2 + \sqrt{2}\operatorname{tr} A + \det A > 0 \end{cases}$,

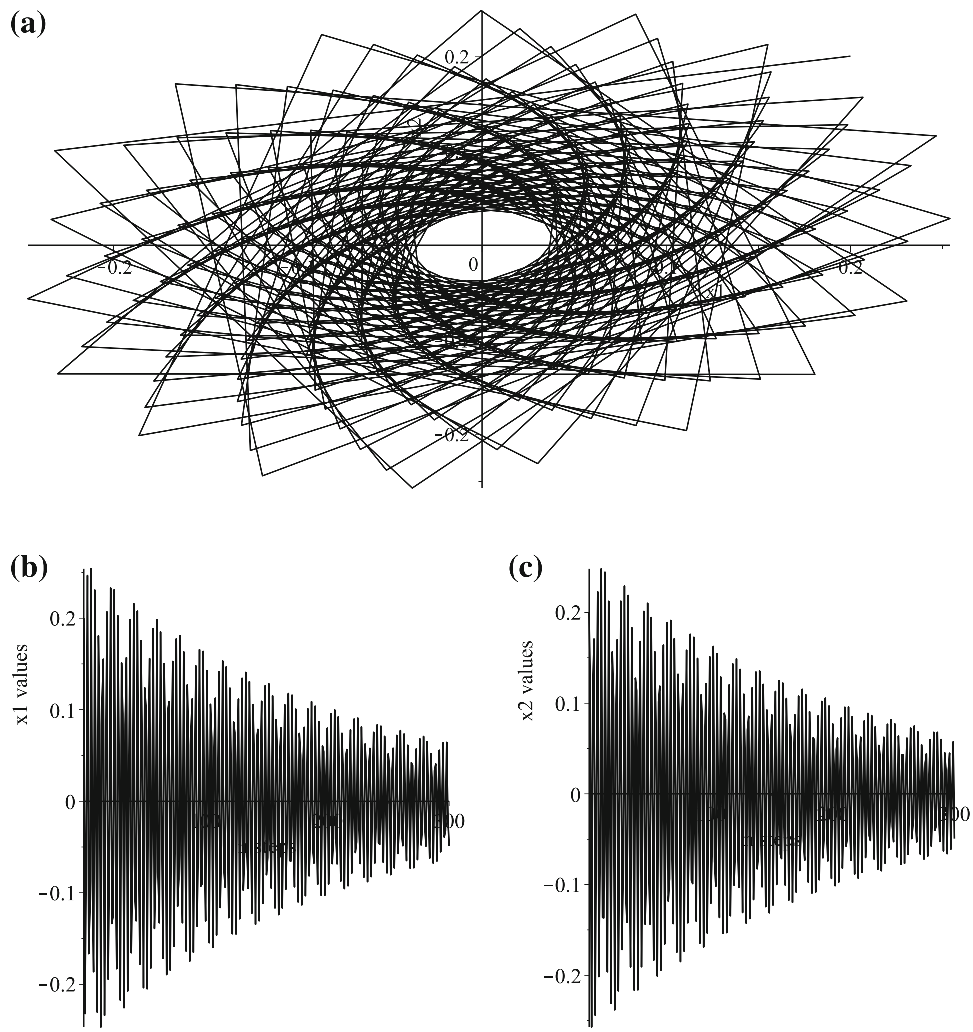

Fig. 1 The solution of the initial value problem for the (asymptotically stable) systems (3) with $a = 0.66$ and $\alpha = 0.5$, $x_0 = (0.2; 0.2)$. **a** The phase trajectory of solution for $n = 300$ steps. **b** The graph of x_1 for $n = 300$ steps. **c** The graph of x_2 for $n = 300$ steps

2) $\begin{cases} \operatorname{tr} A \neq 0 \\ \frac{1}{4}\operatorname{tr}^2 A < \det A < 2\left|\sin\left(2\psi/3 - \frac{\pi}{6}\right)\right| \end{cases}$,
where $\psi = \arctan \frac{\sqrt{4\det A - \operatorname{tr}^2 A}}{\operatorname{tr} A}$ *or* $\psi = \pi + \arctan \frac{\sqrt{4\det A - \operatorname{tr}^2 A}}{\operatorname{tr} A}$,

3) $\begin{cases} \operatorname{tr} A = 0 \\ 0 < \det A < 1 \end{cases}$.

Remark 1 It is known that for $\alpha = 1$ the definition of the fractional operator on the right hand side of Eq. (3) $(\Delta^\alpha x)(n+1) = \sum_{s=0}^{n+1} c_s^{(1)} x(n+1-s) = x(n+1) - x(n)$, as $c_0^{(1)} = 1$, $c_1^{(1)} = -1$ and $c_s^{(1)} = 0$ for $s > 1$. Moreover, it is easy to notice that conditions 1), for real case, from Proposition 1 coincide with those proposed in the book [12] for classical difference equation, i.e. $x(n+1) = (I+A)x(n)$.

Example 1 Let us consider the system with order $\alpha = \frac{1}{2}$, $h = 1$ and matrix $A = \begin{bmatrix} -a & -1 \\ 1 & -1 \end{bmatrix}$. Then $\operatorname{tr} A = -a - 1$ and $\det A = a + 1$. For $0 < a < 0.67$ the corresponding systems are asymptotically stable and for $a > 0.67$ they are unstable. In Fig. 1 there are presented the phase trajectory and graphs of two coordinates of solutions that are associated with the stable systems while one can see the graphs for unstable systems in Fig. 2. Note that for $a = 0.66$ we have that the smaller value $2\left|\sin\left(2\psi/3 - \frac{\pi}{6}\right)\right| = 1.672330968$ and $\frac{1}{4}\operatorname{tr}^2 A = 0.688900$. Then $\det A = 1.66$ lies in the interval from the point 3) in Proposition 1. Moreover for $a = 0.68$ we have that $2\left|\sin\left(2\psi/3 - \frac{\pi}{6}\right)\right| = 1.676028757$. Then $\det A = 1.68 > 2\left|\sin\left(2\psi/3 - \frac{\pi}{6}\right)\right|$ and consequently, the system is unstable.

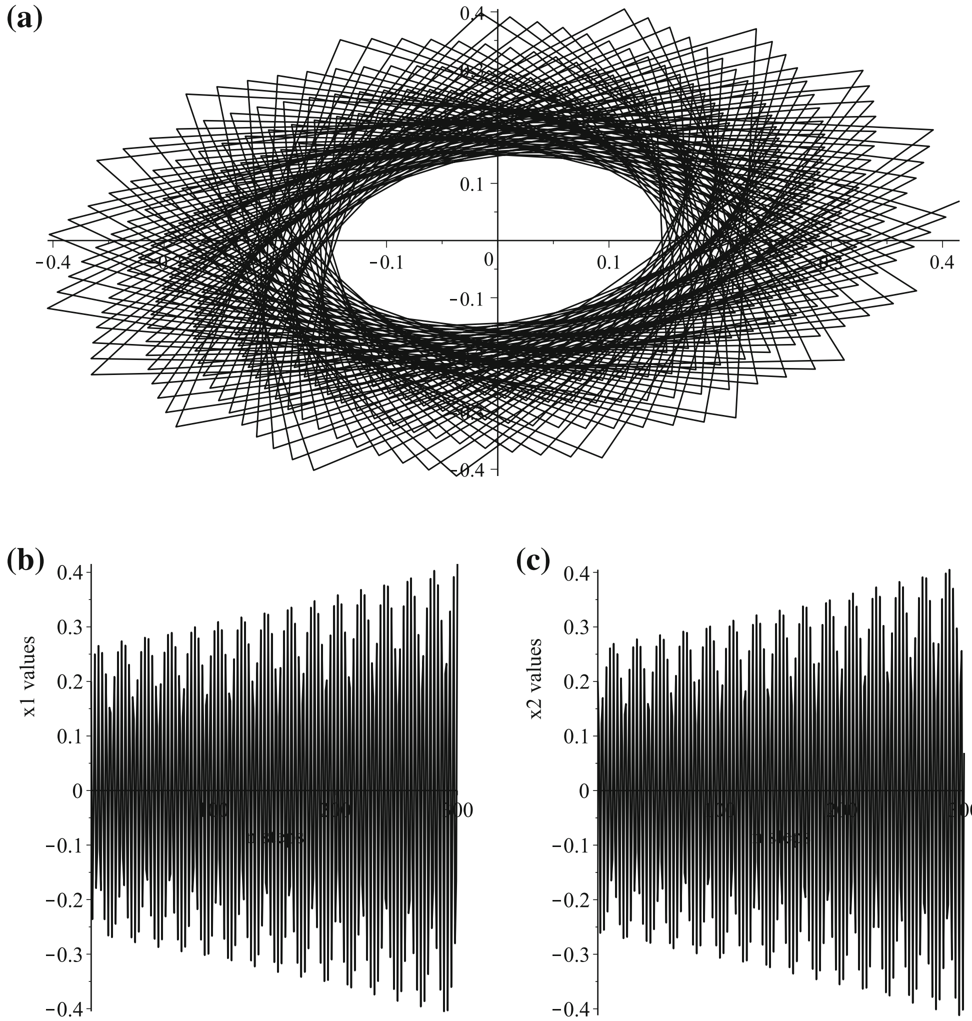

Fig. 2 The solution of the initial value problem for the (unstable) systems (3) with $a = 0.68$ and $\alpha = 0.5$, $x_0 = (0.2; 0.2)$. **a** The phase trajectory of solution for $n = 300$ steps. **b** The graph of x_1 for $n = 300$ steps. **c** The graph of x_2 for $n = 300$ steps

5 Conclusion

The paper describes sufficient and necessary conditions for the asymptotic stability of fractional difference two-dimensional systems with the Grünwald–Letnikov operator. These conditions depend on the entries of the given matrix associated with the considered system.

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