Chen and Chen *Journal of Inequalities and Applications* 2012, **2012**:310
http://www.journalofinequalitiesandapplications.com/content/2012/1/310
Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Recognizing $L_2(p)$ by its order and one special conjugacy class size

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Abstract
In the past thirty years, many authors investigated some quantitative characterizations of finite groups, especially finite simple groups, such as quantitative characterizations by group order and element orders, by the set of lengths of conjugacy classes, by dimensions of irreducible characters, *etc.* In this article the projective special linear group $L_2(p)$ is characterized by its order and one special conjugacy class size, where p is a prime. This work implies that Thompson's conjecture holds for $L_2(p)$.

MSC: 20D08; 20D60

Keywords: finite simple groups; conjugacy class size; prime graph; Thompson's conjecture

1 Introduction

All groups considered in this paper are finite, and simple groups are non-Abelian. For convenience, we use $\pi(n)$ and n_p to denote the set of prime divisors and p-part of the nature number n, respectively. For any group G, we also denote by $N(G)$ the set of conjugacy class sizes of G and by $\pi(G) = \pi(|G|)$.

In 1970s, a simple graph called *prime graph* of the group G was introduced: the vertex set of this graph is $\pi(G)$, two vertices p and q are joined by an edge if and only if G contains an element of order pq (see [1]). Denote the connected components of the prime graph of the group G by $T(G) = \{\pi_i(G)|1 \leq i \leq t(G)\}$, where $t(G)$ is the number of the prime graph components of G. If the order of G is even, we always assume that $2 \in \pi_1(G)$. Then $|G|$ can be expressed as a product of $m_1, m_2, \ldots, m_{t(G)}$, where m_i's are positive integers with $\pi(m_i) = \pi_i(G)$. These m_i's are called the *order components* of G. In particular, if m_i is an odd number, then we call it an odd-order component of G.

In 1988, Thompson posed the following conjecture (ref. to [2, Problem 12.38]).

Thompson's conjecture *Let G be a group with trivial central. If L is a simple group satisfying $N(G) = N(L)$, then $G \cong L$.*

In 1994, Chen proved in his Ph.D. dissertation [3] that Thompson's conjecture holds for all simple groups with a non-connected prime graph (also ref. to [4–6]). In 2009, Vasil'ev first dealt with the simple groups with a connected prime graph and proved that Thompson's conjecture holds for A_{10} and $L_4(4)$ (see [7]). Later on, Ahanjideh in [8] proved that Thompson's conjecture is true for some projective special linear groups. Recently, Chen and Li contributed their interests on Thompson's conjecture under a weak condition. They only used group order and one or two special conjugacy class sizes of simple groups and characterized successfully sporadic simple groups (see Li's Ph.D. dissertation [9]) and simple K_3-groups (a finite simple group is called a simple K_n*-group* if its order is divisible by exactly n distinct primes), by which they checked Thompson's conjecture for sporadic simple groups and simple K_3-groups. Hence, it is an interesting topic to characterize simple groups with their orders and few conjugacy class sizes. In this paper, we characterize the projective special linear group $L_2(p)$ by its order and one special conjugacy class length, where p is a prime. This work partially generalizes Chen and Ahanjideh's work [4, 8], which proved that Thompson's conjecture holds for all the projective special linear group $L_n(q)$.

For convenience, we use ϵ to denote ± 1. In addition, we denote by G_p and $\mathrm{Syl}_p(G)$ a Sylow p-subgroup of the group G and the set of all of its Sylow p-subgroups for $p \in \pi(G)$, respectively. We also denote by $A : B$ an extension of a normal subgroup A by another subgroup B. The other notation and terminologies in this paper are standard and the reader is referred to [10] and [11] if necessary.

2 Some lemmas

A group G is called a *2-Frobenius group* if there exists a normal series $1 \lhd H \lhd K \lhd G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Now, we quote some known lemmas which are useful in the sequel.

Lemma 2.1 [12, Theorem 1] *Suppose that* G *is a Frobenius group of even order and* H, K *are the Frobenius kernel and the Frobenius complement of* G, *respectively. Then* $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$ *and* G *has one of the following structures*:

(i) $2 \in \pi(H)$ *and all Sylow subgroups of* K *are cyclic*;

(ii) $2 \in \pi(K)$, H *is an Abelian group*, K *is a solvable group, the Sylow subgroups of* K *of odd order are cyclic groups, and the Sylow* 2*-subgroups of* K *are cyclic or generalized quaternion groups*;

(iii) $2 \in \pi(K)$, H *is Abelian, and there exists a subgroup* K_0 *of* K *such that*

$$|K : K_0| \le 2, \quad K_0 = Z \times \mathrm{SL}(2,5), \big(|Z|, 2 \times 3 \times 5\big) = 1,$$

and the Sylow subgroups of Z *are cyclic.*

Lemma 2.2 [12, Theorem 2] *Let* G *be a* 2*-Frobenius group of even order. Then* $t(G) = 2$ *and* G *has a normal series* $1 \lhd H \lhd K \lhd G$ *such that* $\pi(K/H) = \pi_2(G)$, $\pi(H) \cup \pi(G/K) = \pi_1(G)$, *the order of* G/K *divides the order of the automorphism group of* K/H, *and both* G/K *and* K/H *are cyclic. Especially,* $|G/K| < |K/H|$ *and* G *is solvable.*

Lemma 2.3 *Let* G *be a group of order* $2^s(2^s-1)(2^{s-1}-1)$, *where* 2^s-1 *is a Mersenne prime and* s *is a natural number. Assume that* G *is a* 2*-Frobenius group with a normal series* $1 \lhd H \lhd K \lhd G$, *where* H *is an elementary Abelian* 2*-group of order* 2^s *and* K/H *is a cyclic group of order* 2^s-1. *Then* $G \cong (Z_2 \times Z_2 \times Z_2 : Z_7) : Z_3$.

Proof By hypothesis, $|G/K| = 2^{s-1}-1$, and so $(|K|, |G/K|) = 1$. By the Zassenhaus theorem, there exists a complement subgroup C of K in G such that $G = KC$. Let B be a complement subgroup of H in K. Then $G = HBC$, where H and HB are normal subgroups of G; HB and BC are Frobenius groups with kernels H, B and complements B, C, respectively. Let $F =$

$GF(2^s)$ and so H is the additive group of F. Also, $|B| = 2^s - 1$ and so B is the multiplicative group of F. Now, C acts by conjugation on H and similarly C acts by conjugation on B, and this action is faithful. Therefore, C keeps the structure of the field F and so C is isomorphic to a subgroup of the automorphism group of F. Hence, $|C| = (2^{s-1} - 1) \mid |\operatorname{Aut}(F)| = s$. Since $2^s - 1$ is a Mersenne prime, s is a prime. It follows that $s = 2$ or 3. If $s = 2$, then G is a group of order 12. By the structures of groups of order 12, we have that any group of order 12 is not a 2-Frobenius group, a contradiction. Hence, $s = 3$, and then G is a group of order 168 that contains eight subgroups of order seven. By [13], we have that the total number of groups of order 168 with eight subgroups of order seven is three, which are $(Z_2 \times Z_2 \times Z_2 : Z_7) : Z_3$, $(Z_2 \times Z_2 \times Z_2 : Z_7) \times Z_3$, and $L_2(7)$. Since G is a 2-Frobenius group, we obtain that $G \cong (Z_2 \times Z_2 \times Z_2 : Z_7) : Z_3$. □

Lemma 2.4 [1, Theorem A] *Let G be a group with more than one prime graph component. Then G is one of the following*:

(i) *a Frobenius or* 2-*Frobenius group*;

(ii) *G has a normal series* $1 \subseteq H \subseteq K \subseteq G$, *where H is a nilpotent* π_1-*group*, K/H *is a simple group and* G/K *is a* π_1-*group such that* $|G/K|$ *divides the order of the outer automorphism group of* K/H. *Besides, each odd-order component of G is also an odd-order component of* K/H.

Lemma 2.5 [11, Theorem 4.5.3] *Let G be a p-group with order* p^n, $n \geq 1$, *and d is the number of minimal generators of G. Then* $|\operatorname{Aut}(G)| \mid p^{d(n-d)}(p^d - 1)(p^d - p) \cdots (p^d - p^{d-1})$.

Lemma 2.6 *Let G be a simple group with a disconnected prime graph. Then its order components are exhibited in Tables* 1-3, *where p is an odd prime and q is a prime power.*

Proof By [1, 14, 15] and the definition of order component, we easily get order components of G in Tables 1-3. Note that some mistakes and misprints in [1, 14, 15] are amended in this paper. □

3 Characterization of $L_2(p)$ by its order and one special conjugacy class size

By Lemma 2.3, we know that $L_2(7)$ is some special such that we have to choose a different way from other cases to deal with it. In fact, Chen and Li have characterized $L_2(7)$ by its order and the smallest conjugacy class size larger than one in their unpublished paper which characterized simple K_3-groups. The following theorem is their one result.

Theorem 3.1 *Let G be a group with* $|G| = 2^3 \cdot 3 \cdot 7$. *Then* $G \cong L_2(7)$ *if and only if G has a conjugacy class size of* 21.

Proof The necessity of the theorem can be checked easily, so we only need to prove the sufficiency.

By hypothesis, there exists an element x of G such that $|x^G| = |G : C_G(x)| = 3 \cdot 7$. In view of $Z(G) \leq C_G(x)$, one has that $Z(G)$ is a proper subgroup of G, and $3, 7 \notin \pi(Z(G))$. We assert that every minimal normal subgroup of $\overline{G} = G/Z(G)$ is non-solvable. Let S be any minimal normal subgroup of $\overline{G}$. Suppose that S is solvable. Then S is an elementary Abelian group, from which we get the preimage T of S in G is a nilpotent group. If $|S| = r^t$, then the Sylow r-subgroup R of T is normal in G. Moreover, R cannot be contained in $Z(G)$. Thus, there exists an element y of R which is not contained in $Z(G)$ such that $1 < |y^G| \leq |R| < 3 \cdot 7$, a

Table 1 The order components of simple groups *G* with $t(G) = 2$

G	Restrictions on *G*	m_1	m_2
A_n	$6 < n = p, p+1, p+2$ and one of $n, n-2$ is not a prime	$\frac{n!}{2p}$	p
$A_{p-1}(q)$	$(p,q) \neq (3,2),(3,4)$	$q^{\frac{p(p-1)}{2}} \prod_{i=1}^{p-1}(q^i-1)$	$\frac{q^p-1}{(q-1)(p,q-1)}$
$A_p(q)$	$(q-1)\mid(p+1)$	$q^{\frac{p(p+1)}{2}}(q^{p+1}-1)\prod_{i=1}^{p-1}(q^i-1)$	$\frac{q^p-1}{q-1}$
$^2A_{p-1}(q)$		$q^{\frac{p(p-1)}{2}} \prod_{i=1}^{p-1}(q^i-(-1)^i)$	$\frac{q^p+1}{(q+1)(p,q+1)}$
$^2A_p(q)$	$(q+1)\mid(p+1)$ and $(p,q) \neq (3,3),(5,2)$	$q^{\frac{p(p+1)}{2}}(q^p-1)\prod_{i=1}^{p-1}(q^i-1)$	$\frac{q^p+1}{q+1}$
$B_n(q)$	$n = 2^m \geq 4$ and q is odd	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{2}$
$B_p(3)$		$3^{p^2}(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p-1}{2}$
$C_n(q)$	$n = 2^m \geq 2$	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q-1)}$
$C_p(q)$	$q = 2,3$	$q^{p^2}(q^p+1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{(2,q-1)}$
$D_p(q)$	$p \geq 5$ and $q = 2,3,5$	$q^{p(p-1)}\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{q-1}$
$D_{p+1}(q)$	$q = 2,3$	$\frac{1}{(2,q-1)}q^{p(p+1)}(q^p+1)$ $(q^{p+1}-1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{q^p-1}{(2,q-1)}$
$^2D_n(q)$	$n = 2^m \geq 4$	$q^{n(n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{q^n+1}{(2,q+1)}$
$^2D_n(q)$	$n = 2^m+1 \geq 5$ if $q = 2$ and $n = 2^m+1 \geq 9$ if $q = 3$	$\frac{1}{(2,q-1)}q^{n(n-1)}(q^n+1)$ $(q^n-1)\prod_{i=1}^{n-2}(q^{2i}-1)$	$\frac{q^{n-1}+1}{(2,q-1)}$
$^2D_p(3)$	$p \neq 2^m+1$ and $p \geq 5$	$3^{p(p-1)}\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{3^p+1}{4}$
$G_2(q)$	$q \equiv \epsilon \pmod 3$ and $q > 2$	$q^6(q^3-\epsilon)(q^2-1)(q+\epsilon)$	$q^2-\epsilon q+1$
$^3D_4(q)$		$q^{12}(q^6-1)(q^2-1)(q^4+q^2+1)$	q^4-q^2+1
$F_4(q)$	q is odd	$q^{24}(q^8-1)(q^6-1)^2(q^4-1)$	q^4-q^2+1
$E_6(q)$		$q^{36}(q^{12}-1)(q^8-1)$ $(q^6-1)(q^5-1)(q^3-1)(q^2-1)$	$\frac{q^6+q^3+1}{(3,q-1)}$
$^2E_6(q)$	$q > 2$	$q^{36}(q^{12}-1)(q^8-1)$ $(q^6-1)(q^5+1)(q^3+1)(q^2-1)$	$\frac{q^6-q^3+1}{(3,q-1)}$
$^2A_3(2)$		$2^6 \cdot 3^4$	5
$^2F_4(2)'$		$2^{11} \cdot 3^3 \cdot 5^2$	13
M_{12}		$2^6 \cdot 3^3 \cdot 5$	11
J_2		$2^7 \cdot 3^3 \cdot 5^2$	7
Ru		$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13$	29
He		$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3$	17
M^cL		$2^7 \cdot 3^6 \cdot 5^3 \cdot 7$	11
Co_1		$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13$	23
Co_3		$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11$	23
F_{22}		$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11$	13
HN		$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11$	19

contradiction. Hence, every minimal normal subgroup of $\overline{G}$ is non-solvable, as desired. It follows that $M \leq \overline{G} \leq \mathrm{Aut}(M)$, where $M = S_1 \times \cdots \times S_k$ and S_i is a direct product of isomorphic non-Abelian simple groups for $i = 1, 2, \ldots, k$.

It is clear that 3 and 7 must belong to $\pi(M)$. Otherwise, M is solvable, a contradiction. Hence, M is a simple K_3-group and $\pi(M) = \{2, 3, 7\}$. By checking possible order of M, M must be isomorphic to $L_2(7)$, which implies that $G \cong L_2(7)$, as claimed. □

Theorem 3.2 *Let G be a group. Then* $G \cong L_2(p)$ *if and only if* $|G| = p(p^2-1)/(2,p-1)$ *and G has a special conjugacy class size of* $(p^2-1)/(2,p-1)$, *where p is a prime and not equal to seven*.

Proof Since the necessity of the theorem can be checked easily, we only need to prove the sufficiency.

Table 2 The order components of simple groups G with $t(G) = 3$

G	Restrictions on G	m_1	m_2	m_3
A_p	$6 < p$ and $p, p-2$ both are primes	$(p-3)!/2$	p	$p-2$
$A_1(q)$	$3 < q \equiv \epsilon (\mathrm{mod}\, 4)$	$q - \epsilon$	q	$\frac{q+\epsilon}{2}$
$A_1(q)$	$q > 4$ is even	q	$(q-1)$	$q+1$
${}^2D_p(3)$	$p = 2^m + 1 \geq 5$	$2 \cdot 3^{p(p-1)}(3^{p-1}-1)\prod_{i=1}^{p-2}(3^{2i}-1)$	$\frac{3^{p-1}+1}{2}$	$\frac{3^p+1}{4}$
$G_2(q)$	$q \equiv 0 (\mathrm{mod}\, 3)$	$q^6(q^2-1)^2$	q^2+q+1	q^2-q+1
${}^2G_2(q)$	$p = 3^{2m+1} > 3$	$q^3(q^2-1)$	$q+\sqrt{3q}+1$	$q-\sqrt{3q}+1$
$F_4(q)$	$2 < q$ is even	$q^{24}(q^6-1)^2(q^4-1)^2$	q^4+1	q^4-q^2+1
${}^2F_4(q)$	$2 < q = 2^{2m+1}$	$q^{12}(q^4-1)$ $(q^3+1)(q^2+1)(q-1)$	$q^2-\sqrt{2q^3}+$ $q'-\sqrt{2q}+1$	$q^2+\sqrt{2q^3}+$ $q+\sqrt{2q}+1$
${}^2A_5(2)$		$2^{15} \cdot 3^6 \cdot 5$	7	11
$E_7(2)$		$2^{63} \cdot 3^{11} \cdot 5^2 \cdot 7^3 \cdot$ $11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43$	73	127
$E_7(3)$		$2^{23} \cdot 3^{63} \cdot 5^2 \cdot 7^3 \cdot$ $11^2 \cdot 13^3 \cdot 19 \cdot 37 \cdot 41 \cdot 61 \cdot 73 \cdot 547$	757	1,093
M_{11}		$2^4 \cdot 3^2$	5	11
M_{23}		$2^7 \cdot 3^2 \cdot 5 \cdot 7$	11	23
M_{24}		$2^{10} \cdot 3^3 \cdot 5 \cdot 7$	11	23
J_3		$2^7 \cdot 3^5 \cdot 5$	17	19
HS		$2^9 \cdot 3^2 \cdot 5^3$	7	11
Sz		$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7$	11	13
Co_2		$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7$	11	23
F_{23}		$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	17	23
$F_2 = B$		$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot$ $11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	31	47
$F_3 = Th$		$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13$	19	31

Table 3 The order components of simple groups G with $t(G) \geq 4$

$t(G)$	G	Restrictions on G	m_1	m_2	m_3	m_4	m_5	m_6
4	${}^2B_2(q)$	$2 < q = 2^{2m+1}$	q^2	$q-1$	$q+\sqrt{2q}+1$	$q-\sqrt{2q}+1$		
	$E_8(q)$	$q \equiv 2, 3 (\mathrm{mod}\, 5)$	$q^{120}(q^{20}-1)$ $(q^{18}-1)(q^{14}-1)$ $(q^{12}-1)(q^8-1)$ $(q^4+1)(q^4+q^2+1)$	q^8+q^7- q^5-q^4- q^2+q+1	q^8-q^7+ q^5-q^4+ q^2-q+1	q^8-q^4+1		
	$A_2(4)$		2^6	5	7	9		
	M_{22}		$2^7 \cdot 3^2$	5	7	11		
	J_1		$2^3 \cdot 3 \cdot 5$	7	11	19		
	ON		$2^9 \cdot 3^4 \cdot 5 \cdot 7^3$	11	19	31		
	Ly		$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11$	31	37	67		
	F'_{24}		$2^{21} \cdot 3^{16} \cdot$ $5^2 \cdot 7^3 \cdot 11 \cdot 13$	17	23	29		
	$F_1 = M$		$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot$ $11^2 \cdot 13^3 \cdot$ $17 \cdot 19 \cdot 23 \cdot$ $29 \cdot 31 \cdot 47$	41	59	71		
5	$E_8(q)$	$q \equiv 0, 1, 4 (\mathrm{mod}\, 5)$	$q^{120}(q^{18}-1)$ $(q^{14}-1)(q^{12}-1)^2$ $(q^{10}-1)^2(q^8-1)^2$ (q^4+q^2+1)	q^8+q^7- q^5-q^4- q^2+q+1	q^8-q^7+ q^5-q^4+ q^2-q+1	q^8-q^6+ q^4-q^2+ 1	q^8-q^4+1	
6	J_4		$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3$	23	29	31	37	43

If $p = 2, 3$, then G is a group of order six or twelve and not an element of order six. By the structures of the groups of order six and twelve, one has that $G \cong S_3$ or A_4, where S_3 is a symmetric group of degree three and A_4 is an alternating group of degree four. Note that $L_2(2) \cong S_3$ and $L_2(3) \cong A_4$, as desired.

Let $p \geq 5$ but not equal to seven. By hypothesis, there exists an element x of order p in G such that $C_G(x) = \langle x \rangle$ and $C_G(x)$ is a Sylow p-subgroup of G. By the Sylow theorem, we have that $C_G(y) = \langle y \rangle$ for any element y in G of order p. So, $\{p\}$ is a prime graph component of G and $t(G) \geq 2$. Therefore, G has one of the structures in Lemma 2.4. In addition, p is the maximal prime divisor of $|G|$ and an odd-order component of G.

Suppose that G is a Frobenius group with a kernel H and a complement K. Then $|K| \mid (|H| - 1)$. If $p \in \pi(H)$, then by Lemma 2.1, $|H| = p$ and $|K| = (p^2 - 1)/2$. It follows that $\frac{p^2-1}{2} \mid (p - 1)$, and thus $p = 1$, a contradiction. If $p \in \pi(K)$, then $|K| = p$ and $|H| = (p^2 - 1)/2$ by Lemma 2.1, and so $p \mid \frac{p^2-3}{2}$. Since p is odd, we have that $p \mid (p^2 - 3)$, which implies $p = 3$, a contradiction. Hence, G is not a Frobenius group.

Assume that G is a 2-Frobenius group. By Lemma 2.2, we have that G has a normal series $1 \lhd H \lhd K \lhd G$ such that $\pi(K/H) = \{p\} = \pi_2(G)$, $\pi(H) \cup \pi(G/K) = \pi_1(G)$, and $|G/K| \mid (p - 1)$. Then we have that K/H is of order p and $\pi((p + 1)/2) \subseteq \pi(H)$.

(a) Let p be not a Mersenne number. Then there exists an odd prime $r \in \pi((p + 1)/2)$ such that $|H_r| < p$. By Lemma 2.5, $(p, |\operatorname{Aut}(H_r)|) = 1$, which implies that an element of order p of G can trivially act on H_r. In other words, r can be connected to p in the prime graph of G, a contradiction.

(b) Let p be a Mersenne number, and $p = 2^s - 1$, $s \geq 2$. Then s is a prime and $(p - 1)/2$ is an odd number. Recall that $|G/K| \mid (p - 1)$.

If $|G/K| = p - 1$, then $|H| = (p + 1)/2 = 2^{s-1} < p$ such that $(p, |\operatorname{Aut}(H)|) = 1$ by Lemma 2.5. It follows that 2 and p connect in the prime graph of G, a contradiction.

If $|G/K| = (p - 1)/2$, then $|H| = (p + 1) = 2^s$. If H is not an elementary Abelian 2-group, then $(p, |\operatorname{Aut}(H)|) = 1$ by Lemma 2.5, a contradiction. If H is an elementary Abelian 2-group, then by Lemma 2.3, $s = 3$, and thus $p = 7$, a contradiction.

Regarding other cases of $|G/K|$, we always can find an odd prime $r \in \pi(H)$ and $r \mid \frac{p-1}{2}$ such that $(p, |\operatorname{Aut}(H_r)|) = 1$, which implies that G has an element of order pr, a contradiction. Therefore, G is not a 2-Frobenius group either.

Now, G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a simple group, G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H and each odd-order component of G is also an odd-order component of K/H. It follows that p is an odd-order component of K/H, and $t(K/H) \geq t(G) \geq 2$, and also $K/H \leq G/H \leq \operatorname{Aut}(K/H)$. We now proceed with the following proof in five steps by the possible number of the prime graph component of K/H in the Tables 1-3. Note that $t(K/H) = 2, 3, 4, 5$ or 6.

Step 1. Assume that $t(K/H) = 2$. Then K/H is isomorphic to one of simple groups in Table 1. We assert that it is impossible.

1.1. If $K/H \cong A_{n'}$, where $6 < n' = p', p' + 1, p' + 2$, and one of n', $n' - 2$ is not a prime, then $p = p'$ and $\frac{n'!}{2} \mid \frac{p(p^2-1)}{2}$. If $n' = p'$, then $2(p - 2) < p + 1$, and thus $p < 5$, a contradiction. The cases $n' = p' + 1$ or $p' + 2$ can be ruled out similarly.

1.2. If $K/H \cong A_{p'-1}(q')$, where $(p', q') \neq (3, 2), (3, 4)$, then $p = \frac{q'^{p'}-1}{(q'-1)(p',q'-1)}$ and $q'^{\frac{p'(p'-1)}{2}} \times \prod_{i=1}^{p'-1}(q'^i - 1) \mid \frac{p^2-1}{2}$. Since p' is an odd prime and q' is a prime power, we have that $p^2 = \frac{(q'^{p'}-1)^2}{[(q'-1)(p',q'-1)]^2} < q'^{2p'}$ and $q'^{\frac{p'(p'-1)}{2}} < \frac{p^2-1}{2} < p^2$. It follows that $q'^{\frac{p'(p'-1)}{2}} < q'^{2p'}$, and so $\frac{p'(p'-1)}{2} <$

$2p'$, which implies that $p' = 3$. Hence, $p^2 = \frac{(q'^3-1)^2}{[(q'-1)(3,q'-1)]^2}$ and $q'^3(q'^2-1)(q'-1) \leq \frac{p^2-1}{2}$. Since $p^2 \geq 2q'^3(q'^2-1)$ and $4q'^4 = q'^2(2q')^2 > q'^2(q'+2)^2 > (q'^2+q'+1)^2 \geq \frac{(q'^3-1)^2}{[(q'-1)(3,q'-1)]^2} = p^2$, we get that $4q'^4 > 2q'^3(q'^2-1)$, and so $q'^2-2q'-1 < 0$. Therefore, $q' = 2$ if $p' = 3$, a contradiction.

1.3. If $K/H \cong A_{p'}(q')$, where $(q'-1)|(p'+1)$, then $p = \frac{q'^{p'}-1}{q'-1}$ and $q'^{\frac{p'(p'+1)}{2}}(q'^{p'+1}-1) \times \prod_{i=1}^{p'-1}(q'^i-1)|\frac{p^2-1}{2}$. Since $p^2 \leq q'^{2p'}$ and $q'^{\frac{p'(p'+1)}{2}} < p^2$, we have that $\frac{p'(p'+1)}{2} < 2p'$, and thus $p' < 3$, a contradiction.

1.4. If $K/H \cong {}^2A_{p'-1}(q')$, then $p = \frac{q'^{p'}+1}{(q'+1)(p',q'+1)}$ and $q'^{\frac{p'(p'-1)}{2}}\prod_{i=1}^{p'-1}(q'^i-(-1)^i)|\frac{p^2-1}{2}$. Since $p^2 < q'^{2(p'+1)}$ and $q'^{\frac{p'(p'-1)}{2}} < p^2$, we get that $\frac{p'(p'-1)}{2} < 2(p'+1)$, and thus $p'^2-5p'-4 < 0$. It follows that $p' = 3$ or 5. If $p' = 3$, then $p^2 = \frac{(q'^3+1)^2}{[(q'+1)(3,q'+1)]^2}$ and $q'^3(q'^2-1)(q'+1)|\frac{p^2-1}{2}$. Therefore $p^2 \leq (q'^2-q'+1)^2 \leq q'^4$ and $q'^4 < q'^3(q'+1) < p^2$, a contradiction. Similarly, if $p' = 5$, then $p^2 \leq q'^8$ and $q'^{10} < p^2$, a contradiction.

1.5. If $K/H \cong {}^2A_{p'}(q')$, where $(q'+1)|(p'+1)$ and $(p',q') \neq (3,3),(5,2)$, then $p = \frac{q'^{p'}+1}{q'+1}$ and $q'^{\frac{p'(p'+1)}{2}}(q'^{p'}-1)\prod_{i=1}^{p'-1}(q'^i-1)|\frac{p^2-1}{2}$. Hence, $p^2 \leq q'^{2(p'+1)}$ and $q'^{\frac{p'(p'+1)}{2}} < p^2$, and thus $q'^{\frac{p'(p'+1)}{2}} < q'^{2(p'+1)}$. It follows that $\frac{p'(p'+1)}{2} < 2(p'+1)$, which means that $p' = 3$. Therefore, $q'^6 < p^2 = (\frac{q'^3+1}{q'+1})^2 = (q'^2-q'+1)^2 \leq q'^4$, a contradiction.

1.6. If $K/H \cong B_{n'}(q')$, where $n' = 2^{m'} \geq 4$ and q' is odd, then $p = \frac{q'^{n'}+1}{2}$ and $q'^{n'^2}(q'^{n'}-1) \times \prod_{i=1}^{n'-1}(q'^{2i}-1)|\frac{p^2-1}{2}$. Since $p^2 < q'^{2(n'+1)}$ and $q'^{n'^2} < p^2$, we have that $n'^2 < 2(n'+1)$, and thus $n'^2-2n'-2 < 0$, which implies that $n' < 3$, a contradiction.

1.7. If $K/H \cong B_{p'}(3)$, then $p = \frac{3^{p'}-1}{2}$ and $3^{p'^2}(3^{p'}+1)\prod_{i=1}^{p'-1}(3^{2i}-1)|\frac{p^2-1}{2}$. Thus, $p^2 < 3^{2p'}$ and $3^{p'^2} < p^2$, and so $p' < 2$, a contradiction.

1.8. If $K/H \cong C_{n'}(q')$, where $n' = 2^{m'} \geq 2$, then $p = \frac{q'^{n'}+1}{(2,q'-1)}$ and $q'^{n'^2}(q'^{n'}-1)\prod_{i=1}^{n'-1}(q'^{2i}-1)|\frac{p^2-1}{2}$. Since $p^2 < q'^{2(n'+1)}$ and $q'^{n'^2} < p^2$, we have that $n'^2 < 2(n'+1)$, and thus $n'^2-2n'-2 < 0$, which implies that $n' = 2$. Therefore, $p'^2 \leq (q'^2+1)^2 < 4q'^4$ and $q'^4(q'^2-1)^2 < p'^2$, and thus $(q'^2-1)^2 < 4$. It follows that $q' = 1$, a contradiction.

1.9. If $K/H \cong C_{p'}(q')$, where $q' = 2$ or 3, then $p = \frac{q'^{p'}-1}{(2,q'-1)}$ and $q'^{p'^2}(q'^{p'}+1)\prod_{i=1}^{p'-1}(q'^{2i}-1)|\frac{p^2-1}{2}$. Hence, $p^2 < q'^{2p'}$ and $q'^{p'^2} < p^2$, and so $p' < 2$, a contradiction.

1.10. If $K/H \cong D_{p'}(q')$, where $p' \geq 5$ and $q' = 2, 3$, or 5, then $p = \frac{q'^{p'}-1}{q'-1}$ and $q'^{p'(p'-1)} \times \prod_{i=1}^{p'-1}(q'^{2i}-1)|\frac{p^2-1}{2}$. Hence, $p^2 < q'^{2p'}$ and $q'^{p'(p'-1)} < p^2$, and so $p'(p'-1) < 2p'$, which implies that $p' < 3$, a contradiction.

1.11. If $K/H \cong D_{p'+1}(q')$, where $q' = 2$ or 3, then $p = \frac{q'^{p'}-1}{(2,q'-1)}$ and $\frac{1}{(2,q'-1)}q'^{p'(p'+1)}(q'^{p'}+1) \times (q'^{p'+1}-1)\prod_{i=1}^{p'-1}(q'^{2i}-1)|\frac{p^2-1}{2}$. Since $p^2 < q'^{2p'}$ and $q'^{p'(p'+1)} < p^2$, we get that $p'(p'+1) < 2p'$, and thus $0 < p' < 1$, a contradiction.

1.12. If $K/H \cong {}^2D_{n'}(q')$, where $n' = 2^{m'} \geq 4$, then $p = \frac{q'^{n'}+1}{(2,q'+1)}$ and $q'^{n'(n'-1)}\prod_{i=1}^{n'-1}(q'^{2i}-1)|\frac{p^2-1}{2}$. Therefore, $p^2 < q'^{2(n'+1)}$ and $q'^{n'(n'-1)} < p^2$. Then we have that $n'(n'-1) < 2(n'+1)$, and thus $n'^2-3n'-2 < 0$, which implies that $n' = 2$, a contradiction.

1.13. If $K/H \cong {}^2D_{n'}(q')$, where $n' = 2^{m'}+1 \geq 5$ if $q' = 2$ and $n' \geq 9$ if $q' = 3$, then $p = \frac{q'^{n'-1}+1}{(2,q'-1)}$ and $\frac{1}{(2,q'-1)}q'^{n'(n'-1)}(q'^{n'}+1)(q'^{n'}-1)\prod_{i=1}^{n'-2}(q'^{2i}-1)|\frac{p^2-1}{2}$. Because $p^2 < q'^{2n'}$ and $q'^{n'(n'-1)} < p^2$, we have that $n'(n'-1) < 2n'$, and so $0 < n' < 3$, a contradiction.

1.14. If $K/H \cong {}^2D_{p'}(3)$, where $p' \neq 2^{m'}+1$ and $p' \geq 5$, then $p = \frac{3^{p'}+1}{4}$ and $3^{p'(p'-1)}\prod_{i=1}^{p'-1}(3^{2i}-1)|\frac{p^2-1}{2}$. Hence, $p^2 < 3^{2(p'+1)}$ and $3^{p'(p'-1)} < p^2$. It follows that $p'(p'-1) < 2(p'+1)$, and so $0 < p' < 4$, a contradiction.

1.15. If $K/H \cong G_2(q')$, where $q' \equiv \epsilon (\mathrm{mod}\, 3)$ and $q' > 2$, then $p = q'^2 - \epsilon q' + 1$ and $q'^6(q'^3 - \epsilon)(q'^2 - 1)(q' + \epsilon) | \frac{p^2-1}{2}$. It follows that $p^2 = (q'^2 - \epsilon q' + 1)^2 < q'^6$ and $q'^6 < p^2$, a contradiction.

1.16. If $K/H \cong {}^3D_4(q')$, then $p = q'^4 - q'^2 + 1$ and $q'^{12}(q'^6 - 1)(q'^2 - 1)(q'^4 + q'^2 + 1) | \frac{p^2-1}{2}$. Therefore, $p^2 = (q'^4 - q'^2 + 1)^2 < q'^8$ and $q'^{12} < p^2$, a contradiction.

1.17. If $K/H \cong F_4(q')$, where q' is odd, then $p = q'^4 - q'^2 + 1$ and $q'^{24}(q'^8 - 1)(q'^6 - 1)^2(q'^4 - 1) | \frac{p^2-1}{2}$. Thus, $p^2 < q'^8$ and $q'^{24} < p^2$, a contradiction.

1.18. If $K/H \cong E_6(q')$, then $p = \frac{q'^6 + q'^3 + 1}{(3, q'-1)}$ and $q'^{36}(q'^{12} - 1)(q'^8 - 1)(q'^6 - 1)(q'^5 - 1)(q'^3 - 1)(q'^2 - 1) | \frac{p^2-1}{2}$. Since $p^2 \leq (q'^6 + q'^3 + 1)^2 < q'^{18}$ and $q'^{36} < p^2$, a contradiction can be obtained.

1.19. If $K/H \cong {}^2E_6(q')$, where $q' > 2$, then $p = \frac{q'^6 - q'^3 + 1}{(3, q'-1)}$ and $q'^{36}(q'^{12} - 1)(q'^8 - 1)(q'^6 - 1)(q'^5 + 1)(q'^3 + 1)(q'^2 - 1) | \frac{p^2-1}{2}$. Hence, $p^2 \leq (q'^6 - q'^3 + 1)^2 < q'^{12}$ and $q'^{36} < p^2$, a contradiction.

1.20. If K/H is isomorphic to one of ${}^2A_3(2)$, ${}^2F_4(2)'$, M_{12}, J_2, Ru, He, M^cL, Co_1, Co_3, F_{22}, and HN, then $p = 5, 7, 11, 13, 17, 19, 23$, or 29, and $|K/H|_2 | \frac{p^2-1}{2}$. By [10], we have that $|K/H|_2 \geq 2^6$, but $(\frac{p^2-1}{2})_2 \leq 2^4$, a contradiction.

Step 2. Suppose that $t(K/H) = 3$. Then K/H is isomorphic to one of simple groups in Table 2. We assert that it is impossible except $A_1(p)$, where $p \geq 5$ is an odd prime and not equal to seven.

2.1. If $K/H \cong A_{p'}$, where $p' > 6$ such that p' and $p' - 2$ are primes, then $p = p'$ and $\frac{p'!}{2} | \frac{p(p^2-1)}{2}$. It follows that $2(p - 2) < p + 1$, and thus $p < 5$, a contradiction.

2.2. If $K/H \cong A_1(q')$, where $q' \equiv \epsilon (\mathrm{mod}\, 4)$ and $q' > 3$, then $p = q', \frac{q'+\epsilon}{2}$.

Let $p = q'$. Then $K/H \cong A_1(p) = L_2(p)$. Recall that $K/H \leq G/H$ and $|G| = \frac{p(p^2-1)}{2} = |L_2(p)|$, we have that $G \cong L_2(p)$, as desired.

Let $p = \frac{q'+1}{2}$. Then $q' = 2p - 1$ and $q'(q' - 1) | \frac{p^2-1}{2}$, and thus $4(2p - 1) \leq p + 1$. Hence we get that $p < 1$, a contradiction.

Let $p = \frac{q'-1}{2}$. Then $q' = 2p + 1$ and $q'(q' + 1) | \frac{p^2-1}{2}$. It follows that $4(2p + 1) \leq p - 1$, and so it is impossible.

2.3. If $K/H \cong A_1(q')$, where $q' > 4$ is even, then $p = q' + \epsilon$ and $q'(q' - \epsilon) | \frac{p^2-1}{2}$, and thus $4(2p - \epsilon) \leq p + \epsilon$. Therefore, $p = 5$, which means that $q' = 4$, a contradiction.

2.4. If $K/H \cong {}^2D_{p'}(3)$, where $p' = 2^{m'} + 1$ and $m' \geq 2$, then $p = \frac{3^{p'-1}+1}{2}$ or $\frac{3^{p'}+1}{4}$.

Let $p = \frac{3^{p'-1}+1}{2}$. Then $\frac{3^{p'}+1}{2} 3^{p'(p'-1)}(3^{(p'-1)} - 1) \prod_{i=1}^{p'-2}(3^{2i} - 1) | \frac{p^2-1}{2}$. Since $p^2 < 3^{2p'}$ and $3^{p'(p'-1)} < p^2$, we have that $p'(p' - 1) < 2p'$, and so $p' < 3$, a contradiction.

Let $p = \frac{3^{p'}+1}{4}$. Then $(3^{p'-1} + 1) 3^{p'(p'-1)}(3^{(p'-1)} - 1) \prod_{i=1}^{p'-2}(3^{2i} - 1) | \frac{p^2-1}{2}$. Hence, $p^2 < 3^{2(p'+1)}$ and $3^{p'(p'-1)} < p^2$, and thus $p'(p' - 1) < 2(p' + 1)$. It follows that $p' = 3$ and $p = 7$, contradicting $p^2 < 3^{2(p'+1)}$.

2.5. If $K/H \cong G_2(q')$, where $q' \equiv 0 (\mathrm{mod}\, 3)$, then $p = q'^2 - \epsilon q' + 1$ and $(q'^2 + \epsilon q' + 1) q'^6 (q'^2 - 1) | \frac{p^2-1}{2}$. It follows that $p^2 = (q'^2 - \epsilon q' + 1)^2 < q'^6$ and $q'^6 < p^2$, a contradiction.

2.6. If $K/H \cong {}^2G_2(q')$, where $q' = 3^{2m'+1} > 3$, then $p = q' - \epsilon \sqrt{3q'} + 1$ and $(q' + \epsilon \sqrt{3q'} + 1) q'^3 (q'^2 - 1) | \frac{p^2-1}{2}$. Therefore, $p^2 = (q' - \epsilon \sqrt{3q'} + 1)^2 < q'^4$ and $q'^3(q'^2 - 1) < p^2$, and thus $q'^2 - 1 < q'$, a contradiction.

2.7. If $K/H \cong F_4(q')$, where $q' > 2$ is even, then $p = q'^4 + 1$ or $q'^4 - q'^2 + 1$. Assume $p = q'^4 + 1$. Then $q'^{24}(q'^6 - 1)^2(q'^4 - 1)^2(q'^4 - q'^2 + 1) | \frac{p^2-1}{2}$. It follows that $p^2 = (q'^4 + 1)^2 < q'^{10}$ and $q'^{24} < p^2$, a contradiction. Similarly, if $p = q'^4 - q'^2 + 1$, then $q'^{24}(q'^6 - 1)^2(q'^4 - 1)^2(q'^4 + 1) | \frac{p^2-1}{2}$, and so $p^2 = (q'^4 - q'^2 + 1)^2 < q'^8$ and $q'^{24} < p^2$, a contradiction.

2.8. If $K/H \cong {}^2F_4(q')$, where $q' = 2^{2n'+1}$ and $q' > 2$, then $p = q'^2 + \epsilon \sqrt{2q'^3} + q' + \epsilon \sqrt{2q'} + 1$ and $q'^{12}(q'^4 - 1)(q'^3 + 1)(q'^2 + 1)(q' - 1)(q'^2 - \epsilon \sqrt{2q'^3} + q' - \epsilon \sqrt{2q'} + 1) | \frac{p^2-1}{2}$. Hence, $p^2 =$

$(q'^2 + \epsilon\sqrt{2q'^3} + q' + \epsilon\sqrt{2q'} + 1)^2 < (2q'^2 + 2q' + 2)^2 < 4(q'^3 - 1)^2 < 4q'^6 < q'^8$ and $q'^{12} < p^2$, a contradiction.

2.9. If K/H is isomorphic to one of M_{11}, M_{23}, and J_3, then $p = 11, 19$, or 23 and $|K/H|_2|\frac{p^2-1}{2}$. By [10], we have that $2^7 \geq |K/H|_2 \geq 2^4$, but $(\frac{p^2-1}{2})_2 \leq 2^3$, a contradiction.

2.10. If K/H is isomorphic to one of ${}^2A_5(2), E_7(2), E_7(3), M_{24}, HS, Sz, Co_2, F_{23}, F_2$, and F_3, then $p = 11, 13, 19, 23, 31, 47, 127$, or 1,093, and $|K/H|_2|\frac{p^2-1}{2}$. By [10], we have that $|K/H|_2 > 2^7$, but $(\frac{p^2-1}{2})_2 \leq 2^7$, a contradiction.

Step 3. Assume that $t(K/H) = 4$. Then K/H is isomorphic to one of simple groups in Table 3 except $E_8(q')$, $q' \equiv 0, 1, 4 (\mathrm{mod}\, 5)$ and J_4. We assert that it is impossible.

3.1. If $K/H \cong J_1$, then $p = 19$ and $11|\frac{p^2-1}{2}$, but $\frac{p^2-1}{2} = 180$, a contradiction.

3.2. If K/H is isomorphic to one of $A_2(4)$, ${}^2E_6(2)$, M_{22}, ON, Ly, F'_{24}, and F_1, then $p = 7, 11, 19, 29, 31, 67$, or 71 and $|K/H|_2|\frac{p^2-1}{2}$. By [10], we have that $|K/H|_2 \geq 2^6$, but $(\frac{p^2-1}{2})_2 \leq 2^5$, a contradiction.

3.3. If $K/H \cong {}^2B_2(q')$, where $q' = 2^{2m'+1}$ and $q' > 2$, then $p = q' - 1$ or $q' + \epsilon\sqrt{2q'} + 1$.

Let $p = q' - 1$. Then $q'^2(q' - \sqrt{2q'} + 1)(q' + \sqrt{2q'} + 1)|\frac{p^2-1}{2}$. It follows that $q'^2 < p^2 = (q' - 1)^2 < q'^2$, a contradiction.

Let $p = q' - \sqrt{2q'} + 1$. Then $q'^2(q' - 1)(q' + \sqrt{2q'} + 1)|\frac{p^2-1}{2}$. Therefore, $p^2 < q'^2$ and $q'^3 < p^2$, a contradiction.

Let $p = q' + \sqrt{2q'} + 1$. Then $q'^2(q' - 1)(q' - \sqrt{2q'} + 1)|\frac{p^2-1}{2}$. Therefore, $p = 2^{2m'+1} + 2^{m'+1} + 1$ and $\frac{p^2-1}{2} = 2^{m'+1}(2^{m'} + 1)(2^{2m'} + 2^{m'} + 1)$. Since $|K/H|_2|(\frac{p^2-1}{2})_2$, we have that $2^{2(2m'+1)}|2^{m'+1}$, and thus $2(2m' + 1) \leq m' + 1$, a contradiction.

3.4. If $K/H \cong E_8(q')$, where $q' \equiv 2, 3 (\mathrm{mod}\, 5)$, then $p = q'^8 - q'^4 + 1, q'^8 - q'^7 + q'^5 - q'^4 + q'^2 - q' + 1$ or $q'^8 + q'^7 - q'^5 - q'^4 - q'^2 + q' + 1$. Assume that $p = q'^8 - q'^4 + 1$. Then $p^2 < q'^{16}$, but $q'^{120} < p^2$, a contradiction. Similarly, for other cases of p, we can always obtain a contradiction.

Step 4. Assume that $t(K/H) = 5$. Then $K/H \cong E_8(q')$ in Table 3, where $q' \equiv 0, 1, 4 (\mathrm{mod}\, 5)$, and thus $p = q'^8 - q'^4 + 1, q^8 - q'^6 + q'^4 - q'^2 + 1, q'^8 - q'^7 + q'^5 - q'^4 + q'^2 - q' + 1$ or $q'^8 + q'^7 - q'^5 - q'^4 - q'^2 + q' + 1$. By the way of 3.4, for all cases of p, we can always obtain a contradiction. Hence, we omit it.

Step 5. Assume that $t(K/H) = 6$. Then $K/H \cong J_4$ in Table 3. It follows that $p = 43$ and $37|\frac{p^2-1}{2}$, but $\frac{p^2-1}{2} = 2^2 \cdot 3 \cdot 7 \cdot 11$, a contradiction.

Therefore, we have that $G \cong L_2(p)$, where p is a prime and not equal to 7, as desired. □

By Theorems 3.1 and 3.2, the following corollary holds.

Corollary 3.3 *Thompson's conjecture holds for the projective special linear group* $L_2(p)$, *where p is a prime.*

Proof Let G be a group with trivial central and $N(G) = N(L_2(p))$. Then it is proved in [4] that $|G| = |L_2(p)|$. Hence, the corollary follows from Theorems 3.1 and 3.2. □

Remark 3.4 By 2.2 of Theorem 3.2 and Lemma 2.3, the following conclusion is true.

Let G be a group with $|G| = 2^3 \cdot 3 \cdot 7$. *Then G has a class size of 24 if and only if* $G \cong L_2(7)$ *or* $(Z_2 \times Z_2 \times Z_2 : Z_7) : Z_3$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Acknowledgements
The authors are very grateful to the referee for carefully reading the earlier version of this manuscript and providing valuable suggestions and useful comments. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11271301, 11171364, 11001226), Science and Technology Foundation of Chongqing Education Committee (No. KJ111107), Youth Foundation of Chongqing Three Gorges University (Grant Nos. 12QN-22, 12QN-23).

Received: 10 September 2012 Accepted: 15 December 2012 Published: 28 December 2012

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doi:10.1186/1029-242X-2012-310
Cite this article as: Chen and Chen: **Recognizing $L_2(p)$ by its order and one special conjugacy class size.** *Journal of Inequalities and Applications* 2012 **2012**:310.