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Bubbling phenomena in calculus of variations

Abstract This paper is a survey on bubbling phenomena occurring in some geometric problems. We present here a few problems from conformal geometry, gauge theory and contact geometry and we give the main ideas of the proofs and important results. We focus in particular on the Yamabe type problems and the Weinstein conjecture, where A. Bahri made a huge contribution by introducing new methods in variational theory.

Mathematics Subject Classification   53A30 · 58E30 · 53D10

الملخص

هذه الورقة دراسة استقصائية عن ظاهرة الاحتدام التي تحدث على بعض المسائل الهندسية. نعرض بعض المسائل من الهندسة المحافظة، ونظرية المعيار وهندسة التماس، ونعطي الأفكار الرئيسية في الإثباتات والنتائج المهمة. نركز بشكل خاص على مسائل من نوع يامبي، وعلى حدس واينشتاين حيث كان لعباس بحري إسهام عظيما في تقديم طرق جديدة في نظرية التغيرية.

1 Introduction

In this survey, we discuss few problems from conformal and contact geometry that share a phenomenon of lack of compactness. By lack of compactness, we mean the absence of the Palais–Smale condition. These problems have a variational structure, that is, the solutions that we look for are critical points of a certain energy functional defined on an adequate Sobolev space. Recall that a C^1 functional $F : X \to \mathbb{R}$, where X is a Banach–Finsler manifold, satisfies the Palais–Smale condition or (PS) if $F(u_n) \to c$ and $\nabla F(u_n) \to 0$ imply the existence of a convergent subsequence. This property is fundamental in the theory of calculus of variations since it allows the construction of a deformation flow between level sets of the functional giving us a tool to compare their topology. The violation of this condition is usually due to the action of a non-compact group, in most cases it is the conformal group. This can also be seen as the non-compactness of the embedding of a certain Sobolev space into an L^p space. So, in order to study the variational problem, one needs to first understand the bubbling phenomena and see the expansion of the energy along a Palais–Smale sequence. In general, there is a quantization of the energy along the sequence, that is,

$$F(u_n) = F(u_\infty) + \sum_{i=1}^{k} F_\infty(U_i) + o(1),$$

In memory of Prof. Abbas Bahri, a great mathematician from whom I am still learning.

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where u_∞ is the weak limit of the (PS) sequence and also a critical point of F, F_∞ is what we call the functional at infinity, and the U_i are critical points to the functional F_∞. One thing to note from this expansion is that in addition to the critical point u_∞ that we hope to find, there is an interference with other critical points. However, these are not from the original functional and not even in the space of variations.

Morse theory is a powerful tool in the study of critical points. It largely consists of finding links between the critical points and the topology of the space of variations. In order to follow a Morse theoretical approach, one needs some compactness or a compactification procedure. For instance, in the presence of bubbling phenomena, this becomes more complicated and one needs to include the asymptotes of the functional in order to have some sort of compactness. But notice that this enlarges the set of possible critical points, making it hard to differentiate between a genuine critical point and a critical point "at infinity", that is, an asymptote. Therefore, one needs to estimate the contribution of critical points at infinity to the change of the topology in order to possibly achieve a positive existence result.

This survey is broken down into two main parts. The first deals with Yamabe type problems. Indeed, we start by giving the geometric background that led to the study of such PDEs, and then we give the ideas of the proof of the Yamabe problem in domains using the method of critical points at infinity. This same procedure works on manifolds, and it gives a positive answer to the Yamabe conjecture. The same idea also works for the CR Yamabe problem that will be presented later. Finally, we relax the geometric problem by including sign-changing solutions and we present different problems that share the same behaviour, and where the lack of compactness is present.

The second part of this survey deals with contact geometry. In this part the approach of A. Bahri toward finding a solution to the Weinstein conjecture is presented. We start by setting the tools and presenting the space of variations, and then we describe the bubbling phenomena that happen in this case. Note that, in this specific case, the bubbling phenomena are different in nature from the one studied in the first part of the survey. We then present another kind of difficulty that appears in these kinds of problems, namely the violation of the Fredholm condition. This difficulty makes a classical Morse theoretical approach non-applicable. We then conclude by presenting the result of Bahri [15] for the Weinstein conjecture on the three dimensional sphere.

2 Yamabe type problems

2.1 Geometric overview

The origin of the Yamabe problem dates back to 1960, initiated by Yamabe in [68] as a program for finding an Einstein metric via a min–max procedure. It consists of fixing a conformal class of a metric g, namely $[g]$ and finding a metric in that class with constant curvature. The problem is in fact variational, that is, in order to solve the problem one needs to find critical points of the energy functional

$$E(g') = \frac{\int_M R_{g'} \mathrm{d}v_{g'}}{\left(\int_M v_{g'}\right)^{\frac{n-2}{n}}}.$$

An easy computation shows that if $g' = u^{\frac{4}{n-2}} g$, the curvature transforms as follows:

$$L_g u = R_u u^{\frac{n+2}{n-2}},$$

where R_u is the scalar curvature of the new metric g' and L_g is a self-adjoint, second-order elliptic operator called the conformal Laplacian operator defined by

$$L_g u = -4\frac{n-1}{n-2}\Delta u + Ru.$$

Hence the energy functional becomes

$$E(u) = \frac{\int_M u L_g u \mathrm{d}v_g}{\left(\int_M u^{\frac{2n}{n-2}} \mathrm{d}v_g\right)^{\frac{n-2}{n}}}.$$

For the sake of notation we will write $2^* = \frac{2n}{n-2}$ and $c(n) = \frac{n-2}{4(n-1)}$. The critical points of E then satisfy

$$L_g u = \frac{\int_M u L_g u \mathrm{d}v_g}{\int_M u^{2^*} \mathrm{d}v_g} u^{2^*-1}. \tag{2.1}$$

Hence the new curvature we obtain is $R_u = \frac{\int_M u L_g u \mathrm{d}v_g}{\int_M u^{2^*} \mathrm{d}v_g}$ which is constant. One of the approaches that was used in the literature is direct minimization. This approach considers the minimization problem

$$Y(M, [g]) = \inf_{u \in H^1(M) \setminus \{0\}} E(u).$$

The constant $Y([g])$ depends only on the conformal class of g and the following theorem was proved.

Theorem 2.1 (Yamabe [68], Trudinger [62], Aubin [7]) *If $Y(M, [g]) < Y(S^n, [g_0])$, where g_0 is the standard metric on the sphere, then a minimizer exists for $Y(M, [g])$ and the Yamabe problem can be solved.*

Later on it was proved by Schoen and Yau [64] and Schoen [63], that if $Y(M, [g]) = Y(S^n, [g_0])$ then (M, g) is conformal to (S^n, g_0). For more details regarding their approaches the reader is directed to [42] and [6] and the references therein.

In our case we are interested in a different approach, one that is more variational. Indeed, since the problem is variational, one is tempted to use the classical methods, namely, Morse theory. Let us recall first one of the main ingredients of variational theory, that is, the deformation lemma. For this matter, we start by fixing some notation. Let X be a Banach–Finsler manifold and $f \in C^1(X, \mathbb{R})$, we define for $a \in \mathbb{R}$

$$f_a = \{x \in X; f(x) \leq a\}.$$

Lemma 2.2 (Deformation Lemma) *Let X be a Banach–Finsler manifold and $f \in C^1(X, \mathbb{R})$ and assume that f satisfies the (PS) condition on the interval $[a, b]$. If f has no critical values in $[a, b]$, then f_a is a deformation retract of f_b.*

In summary, the deformation lemma says that if we do not cross a critical value of f, then the topology of the level set does not change. In fact, if we add the hypothesis that f is Morse, then we have a precise description of what happens when we cross a critical value.

Lemma 2.3 (Morse Lemma) *Let $f \in C^2(X, \mathbb{R})$ where X again is a Banach–Finsler manifold and assume that f satisfies the (PS) condition on $[a, b]$. Without loss of generality we will assume that f has only one critical point x_0 with critical value $c \in (a, b)$ such that the Morse index of x_0, $i_{x_0} = k \in \mathbb{N}$. Then f_b is homotopy equivalent to $f_a \cup_f D^{i_{x_0}}$.*

The Morse lemma states that f_b has the same topology as f_a with a disk of dimension i_{x_0} attached to it. Taking these two lemmas into account, we have a precise description of the change of the topology in the space of variations and the critical points of the function defined on them. Note that the Palais–Smale condition is a major assumption in both of these lemmas.

2.2 The Palais–Smale condition

From the previous section, it is clear that the Palais–Smale condition is a compactness assumption that is, fundamental in the understanding this variational problem. We will see that this is what makes the problem that we are dealing with non-trivial. For now, we will focus our study on a model problem on an open bounded set $\Omega \subset \mathbb{R}^n$, that is, we are interested in solving a problem of the form

$$\begin{cases} -\Delta u = u^{2^*-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

Although there are many choices for the energy functional, they all turn out to be equivalent up to rescaling. For instance, if we pick the functional J_p defined on $H_0^1(\Omega)$ by

$$J_p(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{p+1} \int_\Omega u^{p+1} \mathrm{d}x,$$

where $1 < p \leq 2^* - 1$. The critical points of J_p satisfy

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

Notice that when $p = 2^* - 1$ then the critical points of J_p are solutions to our problem (2.2). The following theorem then holds:

Theorem 2.4 *The functional J_p satisfies the (PS) condition for $1 < p < 2^* - 1$.*

This is a simple exercise and it relies mainly on the compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. In particular, this theorem allows us to use diverse variational approaches such that the Mountain pass lemma or direct minimization (after a modification of the functional) to prove the existence of a solution to (2.3). Moreover, one can even use some topological index theorems to prove a multiplicity result if we allow the solution to change sign. Note however that this Sobolev injection fails to be compact when $p = 2^* - 1$.

Now, we define the standard bubble U to be the solution of problem (2.2), when $\Omega = \mathbb{R}^n$, namely

$$-\Delta U = U^{2^*-1} \quad \text{on } \mathbb{R}^n. \tag{2.4}$$

The function U can be computed explicitly, and up to a multiplicative constant

$$U(x) = \frac{1}{(1 + |x|^2)^{\frac{n-2}{2}}}. \tag{2.5}$$

Since in this case the problem is invariant under translation and rescaling, we have that

$$U_{x_0,\lambda} = \lambda^{\frac{n}{2^*}} U((\cdot - x_0)/\lambda),$$

is also a solution to (2.4). An important fact is that the standard bubble U is the unique positive solution to (2.4) up to translation and rescaling and if we set

$$S_n = \inf_{u \in D^{1,2}(\mathbb{R}^n) \setminus \{0\}} \frac{\int |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^n} |u|^{2^*} \mathrm{d}x\right)^{\frac{2}{2^*}}},$$

the best Sobolev constant of the embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, then S_n is achieved by $\tilde{U} = c_0 U$, where $c_0 = S_n^{\frac{1}{2^*-1}}$. We will also use the projection operator onto $H_0^1(\Omega)$, that is, $P : D^{1,2}(\mathbb{R}^n) \to H_0^1(\Omega)$ defined by $Pu = v$ if and only if $v \in H_0^1(\Omega)$ satisfies

$$\begin{cases} -\Delta v = u & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

Now one can characterize the Palais–Smale sequences of J_{2^*-1}.

Theorem 2.5 *Let $(u_k)_{k \geq 1}$ be a positive (PS) sequence for J_{2^*-1}. Then there exists a function $u_\infty \in H_0^1(\Omega)$ solution to (2.2) and ℓ sequences of points $x_1^k, \ldots, x_\ell^k \in \Omega$ converging, respectively, to $x_1, \ldots, x_\ell$ and sequences of real numbers $\lambda_1^k, \ldots, \lambda_\ell^k$, such that*

(a) $\|u_k - u_\infty - \sum_{i=1}^{\ell} PU_{x_i^k, \lambda_i^k}\|_{H_0^1} \to 0.$

(b) $J_{2^*-1}(u_k) = I_{2^*-1}(u_\infty) + \ell J_\infty(U) + o(1).$

(c) $\lambda_i^k d(x_i^k, \partial\Omega) \to \infty$ for $1 \leq i \leq \ell$ and $\left(\frac{\lambda_i^k}{\lambda_j^k} + \frac{\lambda_j^k}{\lambda_i^k} + \lambda_i^k \lambda_j^k |x_i^k - x_j^k|^2\right)^{-1} \to 0$, for $i \neq j$.

In particular, J_{2^*-1} does not satisfy the (PS) condition. In fact, in terms of existence of positive solutions to the problem (2.2), we have the following result that is a direct consequence of the Pohozaev identity.

Theorem 2.6 *Assume that the set Ω is star shaped. Then problem (2.2) has no positive solution. Moreover if $p > 2^* - 1$, the problem (2.3) has no solution.*

So surprisingly, the shape of the underlying set has an effect on the existence of solutions and by looking at this result, one starts doubting the existence of a positive solution.

One can also consider the approach of constructing solutions for the subcritical case $p < 2^* - 1$ and then let $p \to 2^* - 1$. In [46,47,49,54], it was shown that for similar problems we can exhibit solutions to the subcritical problem or a perturbation of the subcritical problem, which converge to a sum of bubbles when we approach the critical exponent.

In [9], A. Bahri introduced the idea of seeing these bubbles as asymptotes for the functional. As shown in Fig. 1 below, asymptotes can behave like critical points in terms of change in the topology of the level set,
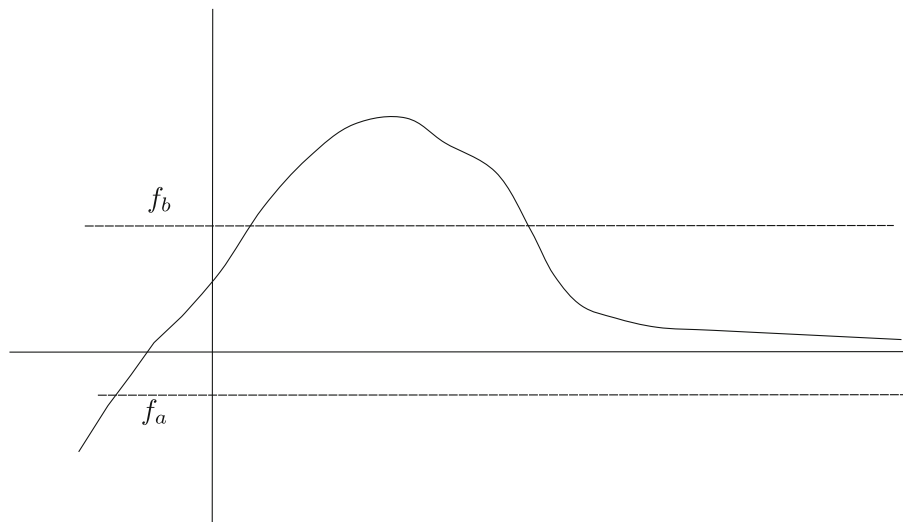
Fig. 1 Level sets in the presence of symptotes

hence the name critical points at infinity. So if we consider the critical points at infinity as critical points, the deformation lemma still holds although classical variational analysis will not give us a distinction between actual critical points and critical points at infinity.

In his work (see [8,9,16,18]), A. Bahri considered a slightly modified functional, namely

$$J(u) = \frac{1}{\int_\Omega u^{2^*} \mathrm{d}x},$$

on the space of variations $\Sigma_+ = \{u \in H_0^1(\Omega); u > 0, \int_\Omega |\nabla u|^2 \mathrm{d}x = 1\}$. It is easy to see that critical points of J on Σ_+ correspond to positive solutions to (2.2). Now in order to find actual critical points, a more in depth study of the critical points at infinity and their contribution to the topology of the level sets is needed. The main theorem regarding the existence of solutions to (2.2) can be stated as follows.

Theorem 2.7 *If there exists a positive integer d such that $H_d(\Omega, \mathbb{Z}_2) \neq 0$, then problem (2.2) has a positive solution.*

Here $H_d(\Omega, \mathbb{Z}_2)$ is the singular homology group of Ω with coefficients in $\mathbb{Z}_2$. Notice that this theorem says that if the underlying set Ω has non-trivial topology (for example an annulus or a domain with holes), then one can find a solution. In what follows we present the main ideas of the proof.

The proof proceeds by contradiction, that is, we assume that there is no positive solution to (2.2). First, we define the neighbourhoods of critical points at infinity namely

$$V(k, \varepsilon) = \left\{ \begin{array}{l} u \in \Sigma_+; \exists (x_1, \ldots, x_k) \in \Omega^k, (\lambda_1, \ldots, \lambda_k) \in (\frac{1}{\varepsilon}, +\infty)^k \text{ s.t.} \\ \|u - \frac{1}{\sqrt{k}} \sum_{i=1}^k PU_{x_i, \lambda_i}\|_{H_0^1} < \varepsilon, \lambda_i d(x_i, \partial\Omega) > \frac{1}{\varepsilon} \text{ and } \varepsilon_{ij} < \varepsilon \end{array} \right\}, \quad (2.7)$$

where for $x \in \Omega, d(x, \partial\Omega)$ is the distance of x to the boundary and

$$\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |x_i - x_j| \right)^{-\frac{n-2}{2}}, \quad \text{for } i \neq j.$$

With these newly defined sets, one has that if $(u_q)_{q \geq 0}$ is a (PS) sequence for J, then there exists $\varepsilon_q \to 0$ and $k \in \mathbb{N}$ such that $u_q \in V(k, \varepsilon_k)$. Moreover, $J(u_q) \to b_k = k^{\frac{2}{n-2}} S$. S is a constant related to the best constant of the Sobolev embedding and it is irrelevant for now. The proof relies then on the following two propositions.

Proposition 2.8 *There exists k_0 positive and $\lambda_0 > 0$ so that for every $k > k_0$ and $\lambda > \lambda_0$ we have, for any $x_1, \ldots, x_k \in \Omega$ and $\alpha_1, \ldots, \alpha_k \geq 0$ satisfying $\sum_{i=1}^k \alpha_i = 1$ the following holds:*

$$J \left(\frac{\sum_{i=1}^k \alpha_i PU_{(x_i, \lambda)}}{\| \sum_{i=1}^k \alpha_i PU_{(x_i, \lambda)} \|} \right) < (k)^{\frac{2}{n-2}} S$$

Proposition 2.9 *For every* $k \in \mathbb{N}^*$, *there exists* $\varepsilon_k > 0$ *such that for* $\varepsilon_k > \varepsilon > 0$, *if* $u \in V(k, \varepsilon)$, *the minimization problem*

$$\min \left\{ \left\| u - \sum_{i=1}^{k} \alpha_i \, PU_{x_i, \lambda i} \right\|, \lambda_i > 0, \alpha_i > 0, x_i \in \Omega \right\},$$

has a unique solution $(\alpha_1, \ldots, \alpha_k, x_1, \ldots, x_k, \lambda_1, \ldots, \lambda_k)$, *up to permutation.*

This proposition gives us a continuous parametrization of $V(k, \varepsilon)$ for ε small. And its proof is standard and similar to the one in Bahri–Coron in [18] or in the book [9]. Now if we define the set

$$W_p = \{u \in \Sigma_+; \, J(u) < b_{k+1}\},$$

then one has:

Lemma 2.10 *For a given* $k \in \mathbb{N}^*$ *and* $\varepsilon > 0$, *if the functional* J *has no critical points, then the pair* (W_k, W_{k-1}) *retracts by deformation to* $(W_{k-1} \cap A_k, W_{k-1})$ *where* $A_k \subset V(k, \varepsilon)$.

The proof of this Lemma mainly relies on the study of the flow

$$\frac{\partial u}{\partial t} = -\lambda(u)u + \lambda(u)^{\frac{n+2}{n-2}}(-\Delta)^{-1}(u^{\frac{n+2}{n-2}}),$$

where $\lambda(u) = J(u)^{\frac{n-2}{2}}$. This flow has very good properties, indeed.

Lemma 2.11 (i) *The flow defined above, is well defined for data in* $H_0^1(\Omega)$
 (ii) *It preserves* Σ_+
 (iii) *It decreases the energy* J
 (iv) *If* $\frac{\partial u}{\partial t}$ *converges to zero in* $H_0^1(\Omega)$, *then* $u(t)$ *is a (PS), sequence.*

Proof The proof of this lemma is elementary:

 (i) This is a differential equation on the Hilbert space H_0^1, and it satisfies the Cauchy–Lipschitz condition for the existence.
 (ii) The flow is positivity preserving. This can be seen by reasoning on the first time the function $u(t)$ hits a zero, and at that point we see that the derivative must be positive, hence the positive cone is preserved. Moreover, it preserves the unit ball, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t} ||u(t)||^2 = \int_\Omega \frac{\partial u}{\partial t}(-\Delta)u = \int_\Omega u(-\lambda(u)(-\Delta)u + \lambda(u)^{\frac{n+2}{n-2}} u^{\frac{n+2}{n-2}}) = 0.$$

 (iii) It follows by direct computation of $\frac{\mathrm{d}}{\mathrm{d}t} J(u)$, in fact it is the descending gradient flow of J with respect to the H_0^1 norm that we introduced.
 (vi) The energy is strictly decreasing along the flow. Hence if $\frac{\partial u}{\partial t}$ converges to zero, it defines a (PS) sequence since it is already the gradient flow of J. $\square$

Remark 2.12 Notice that the flow that is used here is a non-local flow, but what we gain from it is that the local existence is guarantied by the classical Cauchy–Lipschitz existence result of ODE, since indeed the flow is an ODE on a Banach manifold. In the literature there are other flows that were used. Of course, one automatically thinks about the L^2 flow, giving us a heat type equation. There is another flow used in the study of the Yamabe equation, hence comes the name, Yamabe flow (which is the parallel of the Ricci flow, that is, we evolve the metric by its scalar curvature). This last flow is mainly a fast diffusion flow. From the last two flows, we gain regularity coming from the heat kernel, but the problem with these flows is that they can blow up in finite time; therefore, the study of the convergence at infinity can be a bit tricky.

Now using this flow, under the assumption that there are no critical points, we can deform W_k into W_{k-1} as long as $|| - \lambda(u)u + \lambda(u)^{\frac{n+2}{n-2}}(-\Delta)^{-1}(u^{\frac{n+2}{n-2}})||$ is bounded from below by a positive constant and knowing that this fact is violated by divergent (PS) sequences, (from vi)) using their characterization in Theorem (2.5) we have that the flow will be in $V(k, \varepsilon)$ after a certain time, and that is the region that we call A_k and this finishes the proof of Lemma (2.10).

Notice that in Proposition (2.8) the function $\sum_{i=1}^{k} \alpha_i PU_{(x_i,\lambda)}$ acts like a convex combination of the points $(x_1, \ldots, x_k) \in \Omega^k$. We consider the set B_k, of formal convex combination of Dirac masses on Ω, that is,

$$B_k = \left\{ \sum_{i=1}^{k} \alpha_i \delta_{x_i}, \alpha_i \geq 0, \sum_{i=1}^{p} \alpha_i = 1, (x_1, \ldots, x_k) \in \Omega^p \right\}.$$

Let $f_k(\lambda) : B_k \longmapsto \Sigma_+$ defined by

$$f_k(\lambda)(\sum_{i=1}^{k} \alpha_i \delta_{x_i}) = \frac{\sum_{i=1}^{k} \alpha_i PU_{(x_i,\lambda)}}{|| \sum_{i=1}^{k} \alpha_i PU_{(x_i,\lambda)}||}.$$

Clearly $W_{k-1} \subset W_k$ and $B_{k-1} \subset B_k$ and we have

Lemma 2.13 *There exist $\lambda_0 > 0$ and $k_0 > 0$ such that for $\lambda > \lambda_0$ and $k > k_0$, $f_k(\lambda)$ maps B_k into W_k, moreover $f_k(\lambda)_* : H_*(B_k, B_{k-1}) \longmapsto H_*(W_k, W_{k-1})$ is trivial.*

Now we are set to find a contradiction by showing that if Ω has enough topology, the map $f_k(\lambda)$ is not trivial. By assumption, we have that $H_d(\Omega) \neq 0$, therefore due to a result of R. Thom [59], we have the existence of a d-dimensional manifold and a continuous map $h : V \to \Omega$ such that if $[V] \in H_d(V)$ is the orientation class of V, we have $h_*([V]) \neq 0$. Considering $S_k = \{x \in V^k; \text{ there exists } i \neq j \in [1, k]\}$ and T_k an open equivariant neighbourhood (under the action of the symmetric group σ_k) of S_k in V^k, (the existence of this neighbourhood can be found in [18] Appendix C)

Lemma 2.14 *There exists an isomorphism $q_k : H_*(V^k \times_{\sigma_k} \Delta_{k-1}, S_k \times \cup_{\sigma_k} V^k \times \partial \Delta_{k-1}) \longmapsto H_*(B_k(V), B_{k-1}(V))$, where Δ_k is the standard k-simplex. Moreover, this isomorphism is induced by the natural map between those two topological spaces which is a homeomorphism.*

Here, $B_k(V)$ is defined in the same way as B_k, replacing Ω by V. So if we take $\overline{V}_k = V^k/T_k$ then $\overline{V}_k \times_{\sigma_k} \Delta_{k-1}$ is a $kd + k - 1$-dimensional manifold with boundary, and as a consequence of the previous lemma we get that

$$H_*(B_k(V), B_{k-1}(V)) \cong H_*(\overline{V}_k \times_{\sigma_k} \Delta_{k-1}, \partial(\overline{V}_k \times_{\sigma_k} \Delta_{k-1})),$$

hence we take $w_k \in H_{kd+d-1}(B_k(V), B_{k-1}(V))$ the inverse image of the orientation class of the manifold $\overline{V}_k \times_{\sigma_k} \Delta_{k-1}$ with $\mathbb{Z}_2$ coefficient.

Lemma 2.15 *There exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$,*

$$f_k(\lambda)_*(w_k) \neq 0.$$

Notice that with this lemma the theorem is proved since there is a contradiction with Lemma (2.13).

Proof Using the commuting diagram below and the module structure, the proof is reduced to the case $k = 1$. So one needs to show that $f_1(\lambda)_*(w_1) \neq 0$.

$$
\begin{array}{ccc}
H_*(W_k, W_{k-1}) & \xrightarrow{\;\partial\;} & H_{*-1}(W_{k-1}, W_{k-2}) \\
\uparrow{\scriptstyle f_k(\lambda)_*} & & \uparrow{\scriptstyle f_{k-1}(\lambda)_*} \\
H_*(B_k, B_{k-1}) & \xrightarrow[\;\partial\;]{} & H_{*-1}(B_{k-1}, B_{k-2})
\end{array}
$$

First, since for $\varepsilon > 0$ small enough $J_{S+\varepsilon} \subset V(1, \eta)$ where η converges to zero as ε does. Then, it can be parametrized like $V(1, \eta)$, hence there exists a continuous map $g : J_{S+\varepsilon} \longmapsto \Omega$ that sends the Ω-part of the parametrization and using the flow defined earlier we can construct a retraction $r : W_1 \longmapsto J_{S+\varepsilon}$. Now it is easy to see that by construction $g \circ r \circ f_1(\lambda) = id_\Omega$, therefore $f_1(\lambda)_*(w_1) \neq 0$, and this finishes the proof. $\square$

Combining this result with the non-existence result for solutions in the star-shaped (hence contractible) case, one is led to think that this result is sharp. That is, that the topology of Ω is the only constraint on existence and non-existence of solutions. Unfortunately, this is not the case. Indeed, the geometry of Ω plays a role in the existence of solutions as shown in [26] and [36] where the authors construct contractible sets on which they can solve problem (2.2). In his book [9], A. Bahri pointed out another set that seems to be indicative of the existence or non-existence of critical points. We now begin introducing this set. We set G the Green's function of the operator $-\Delta$ on Ω with zero boundary condition and H its regular part so that

$$G(x, y) = \frac{1}{|x - y|^{n-2}} - H(x, y).$$

We define then for $(x_1, \ldots, x_k) \in \Omega^k$ the matrix $M(x_1, \ldots, x_k) = (M_{ij})_{1 \le i, j \le k}$ with entries $M_{ij} = -G(x_i, x_j)$ if $i \ne j$ and $M_{ii} = H(x_i, x_j)$. Since this matrix is real symmetric it is diagonalizable, so we set $\rho(x_1, \ldots, x_k)$ its smallest eigenvalue. We can then introduce the important set

$$I_k = \{x \in \Omega^k; \rho(x) \le 0\}.$$

If we assume for instance that 0 is a regular value of ρ (which is the case for instance if Ω is a thin annulus; see [1]), we have that

Theorem 2.16 [20] *Using the notations in the proof above, we have*

(i) *For $\varepsilon > 0$ small enough, the functional J does not have any critical point in $V(k, \varepsilon)$.*
(ii) *The only critical points at infinity of J correspond to $\sum_{i=1}^k PU_{x_i, \infty}$ where $k \in \mathbb{N}^*$ and $x_1, \cdot, x_k$ satisfy*

$$\rho(x_1, \ldots, x_k) > 0 \quad and \quad \rho'(x_1, \ldots, x_k) = 0.$$

(iii) *There exists $k_0 \in \mathbb{N}^*$ such that above b_{k_0}, J does not have any critical point at infinity.*
(iv) *If moreover we assume that J has no critical points in Σ_+, then*

$$H_*(W_k, W_{k-1}) = H_*(\Omega^k \times_{\sigma_k} \Delta_{k-1}, \Omega^k \times \partial \Delta_{k-1} \cup_{\sigma_k} I_k \times \Delta_{k-1}).$$

So the set I_k gives us the contribution of a critical point at infinity to $(J_{b_k+\varepsilon}, J_{b_k-\varepsilon})$. Based on this A. Bahri conjectured the following:

Conjecture 2.17 *If for $k \ge 2$, I_k is not contractible, the problem (2.2) has at least one solution.*

2.3 Extension to other situations

The method that we described works perfectly in the case of a compact oriented manifold M, since it already has a non-trivial orientation class. The reasoning is the same if we replace the operator $(-\Delta)$ by the conformal Laplacian L_g. Hence, this gives a positive solution to the Yamabe problem; see for instance [8,16]. The same procedure works for higher order conformal invariants such as the Q-curvature. It also works for the fourth-order conformally invariant operator P_g called the Paneitz–Branson operator, although an extra condition for coercivity needs to be added.

The case of CR-manifolds is very similar to the setting in Riemannian geometry. Indeed, given a $2n + 1$-dimensional CR-manifold (M, θ), with volume form $dv_\theta = \theta \wedge d\theta^n$, we want to study the functional

$$E(\theta') = \frac{\int_M W_{\theta'} dv_{\theta'}}{(\int_M dv_{\theta'})^{\frac{n}{n+1}}},$$

where W_θ is the Webster curvature relative to the contact form θ, we want to study the functional for all the contact forms defining the same contact structure as θ, that is, the set $[\theta] = \{f\theta; f > 0\}$. Similarly to the scalar curvature case, if we set $q = 2n + 2$ then the Webster curvature under conformal change $\theta' = u^{\frac{4}{q-2}}\theta$,

$$W_u = u^{-\frac{q+2}{q-2}} L_b u,$$

where L_b is a second-order sub-elliptic operator defined by

$$L_b u = \frac{2q}{q-2} (\Delta_b u + W_\theta u).$$

Therefore the problem of prescribing a constant Webster curvature is equivalent to finding critical points of the functional

$$E(u) = \frac{\int_M u L_b u \, \mathrm{d}v}{\left(\int_M |u|^{2q} q - 2 \mathrm{d}v \right)^{\frac{q-2}{q}}}$$

on the space $S_2^1(M)$, the Folland–Stein space, which is the equivalent of the Sobolev space $H^1(M)$ in the CR setting. Positive critical points of E satisfy the Euler–Lagrange equation

$$L_b u = E(u) u^{\frac{q+2}{q-2}}. \tag{2.8}$$

Therefore, $W_u = E(u)$ which is a constant.

Now if one considers the minimization problem

$$\lambda(M, [\theta]) = \inf_{u \in S_2^1(M) \setminus \{0\}} E(u),$$

we get that the constant $\lambda(M, [\theta])$ is a CR invariant, moreover,

Theorem 2.18 [40] *Let M be a smooth $2n + 1$-dimensional CR manifold with contact form θ. Then*

(i) $\lambda(M, [\theta]) \leq \lambda(S^{2n+1}, [\theta_0])$
(ii) *If $\lambda(M, [\theta]) < \lambda(S^{2n+1}, [\theta_0])$ then it is achieved by a smooth function and hence the problem of prescribing constant Webster curvature is solved.*

We also have the following theorem.

Theorem 2.19 [41] *Suppose M is a compact strictly pseudoconvex $(2n + 1)$-dimensional CR manifold. If $n \geq 2$ and M is not locally CR equivalent to S^{2n+1}, then $\lambda(M, [\theta]) < \lambda(S^{2n+1}, [\theta_0])$, and thus, the CR Yamabe problem can be solved on M.*

The case when M is locally conformal to the standard CR sphere and when $n = 1$ still remains. Again, the problem in this case is not compact. That is, the embedding $S_2^1(M) \subset L^{\frac{2q}{q-2}}$ is not compact and hence bubbling occurs when studying (PS) sequences. These bubbles can be identified similarly to the case of the Yamabe problem and one can apply the same reasoning with careful estimates to show that

Theorem 2.20 [31] *Let (M, θ) be an orientable compact real $(2n + 1)$-dimensional CR manifold, locally CR equivalent to S^{2n+1}. Then there exists a contact form conformal to θ with constant Webster curvature.*

Also for the case $n = 1$, we have

Theorem 2.21 ([32] *Let M be a 3-dimensional compact manifold not locally CR equivalent to S^3. Then the CR Yamabe problem, (i.e. (2.8)) has a solution.*

The proof of these results follows closely to what was presented above in the case of the Riemannian case. One just needs to consider a different bubble, replacing the U that was defined in (2.5). If we use the coordinates $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathbb{H}^n$, then the bubble in the Heisenberg group takes the form

$$\tilde{U} = \frac{c_0}{\left((1 + |x|^2 + |y|^2)^2 + t^2 \right)^{\frac{q-2}{4}}},$$

where c_0 is a fixed constant. The other bubbles are obtained from this one using translations and rescaling in the Heisenberg group this time. That is, if we consider the group law "·" defined for $\xi_0 = (x_0, y_0, t_0)$ and $\xi_1 = (x_1, y_1, t_1)$, by

$$\xi_0 \cdot \xi_1 = (x_0 + x_1, y_0 + y_1, t_0 + t_1 + (x_1 y_0 - x_0 y_1)),$$

then the translation operator is defined by $\tau_{\xi_0}(\xi) = \xi_0 \cdot \xi$. One also needs to consider the dilation operator $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ defined by

$$\delta_\lambda(\xi_0) = (\lambda x_0, \lambda y_0, \lambda^2 t_0).$$

Now the rest of the bubbles take the form

$$\tilde{U}_{\lambda,\xi} = \lambda^{\frac{2-q}{2}} \tilde{U} \circ \delta_{\frac{1}{\lambda}} \circ \tau_{\xi^{-1}}.$$

Using careful estimates, one can show that the method of critical points at infinity applies.

2.4 Problem in full generality

So far, the discussions above were focused on the geometric problem, namely the solution of the problem needed to be positive. If we remove this constraint and allow sign-changing solutions, the problem becomes richer. However, one drawback is that the bubbles in this case cannot be identified explicitly. Actually, there are multiple situations in which we cannot identify the bubbles explicitly, a few of them are presented below:

2.4.1 The sign-changing Yamabe problem

Now, we consider the same problem,

$$L_g u = |u|^{2^*-2} u, \tag{2.9}$$

where again L_g is the conformal Laplacian, but here the sign of the solution is not important. The problem does not lack difficulty since the embedding $H^1(M) \subset L^{2^*}(M)$ is still not compact, moreover we now have to deal with the fact that the bubbles are not explicit since we only have a classification of the standard bubble in the positive case. Nevertheless, one can show existence of solutions when there is an infinite group G of isometries that acts on M without fixed points. In this case, one can use this action to exclude bubbling phenomena by restricting the space of variations to functions invariant under the group action. The first result in this nature was proved by Ding [25] for the case of the sphere S^n under the action of the group $G_k = O(k) \times O(n+1-k)$, for $k \geq 2$, by looking at S^n as the unit sphere of $\mathbb{R}^{n+1}$. The theorem states that

Theorem 2.22 [25] *The sign-changing Yamabe problem on S^n has infinitely many solutions.*

We will give the idea of the proof in general. So we consider the functional

$$E_2(u) = \frac{1}{2} \int_M u L_g u \, dx - \frac{1}{2^*} \int_M |u|^{2^*} \, dx$$

Now if $(u_i)_{i \geq 0}$ is a (PS) sequence of E_2, then there exist $k \geq 0$ and sequences $a_i^j \longrightarrow a^j \in M$, for $1 \leq j \leq k$ and a sequence of numbers R_i^j converging to zero, a solution $u_\infty \in H^1(M)$ of the problem (2.9) and solutions $u^j \in H^1(\mathbb{R}^n)$ of $-\Delta u = |u|^{\frac{4}{n-2}} u$ on $\mathbb{R}^n$, such that up to subsequence, we have

$$u_i = u_\infty + \sum_{j=1}^{k} \omega_i^j + o(1) \text{ in } H^1(M),$$

where

$$\omega_i^j = (R_i^j)^{-\frac{n-2}{2}} \beta_j(x)(\rho_{i,j}^{-1})^*(u^j)$$

and

$$\rho_{i,j}(x) = \exp_{a^j}(R_i^j x).$$

Here, β_k is a non-negative function equals to 1 in $B_1(a^j)$ and zero outside $B_2(a^j)$. Moreover, we have

$$E_2(u_i) = E_2(u_\infty) + \sum_{j=1}^{k} E_2(u^j) + o(1).$$

Now, let G be a subgroup of $Isom(M)$ that acts on M with no fixed points. We consider the space $H_G^1(M)$ of functions invariant under G, that is,

$$H_G^1(M) = \{u \in H^1(M); u(g \cdot x) = u(x), \forall g \in G\}.$$

Also notice that the functional E_2 is invariant under G. So if one shows that E_2 satisfies (PS) on $H_G^1(M)$, the problem is solved using the min–max variational argument of Ambrozzetti and Rabinowitz [3]. Let us consider (u_i) a (PS) sequence for E_2. Then according to the characterization stated above we have

$$E_2(u_i) = E_2(u_\infty) + \sum_{i=1}^{\ell} c_i + o(1),$$

where $c_i = E_2(u^i)$, we can show that $c_i \geq c > 0$. The main point is that the number of bubbles is finite and that the energy is finite. In particular, if $(u_i)_{i \geq 0}$ is a (PS) sequence that concentrates on $x_1, \ldots, x_k$, then $u_i(g \cdot)$ concentrates at $g \cdot x_1, \ldots, g \cdot x_k$; but if $u_i \subset H_G^1(M)$, then $u_i(g \cdot) = u_i$. Hence u_i concentrates at all the orbits of $x_1, \ldots, x_\ell$ but the orbits are infinite because G is, therefore the set of concentrations needs to be empty and hence the (PS) condition holds.

Of course since we are in the Riemannian setting, one can consider any operator that is conformally invariant. For instance if we set P_k the $GJMS$ operators of even order on a manifold for $1 \leq k < n$ and k even, which are differential with leading term $(-\Delta)^{\frac{k}{2}}$, or on manifolds that appear as the conformal boundary of Einstein manifolds and in this case they can be defined for k odd and they are pseudo-differential operator of fractional order $\frac{k}{2}$ (see [33]), then the problem reads as

$$P_k u = |u|^{2_n^* - 2} u, \tag{2.10}$$

where $2_n^* = \frac{2n}{n-k}$. For instance, for $k = 2$, we have

$$P_2 u = (\Delta_g)^2 u + div_g \left(\frac{2}{3} R_g g - 2 Ric_g\right) du,$$

and on the sphere S^n they are of the form

$$P_k = \begin{cases} \prod_{j=0}^{\frac{k-2}{2}} (-\Delta_{g_0} + k(k-j-1)) & \text{if } k \text{ is even.} \\ (-\Delta_{g_0} + \frac{k-2}{2})^{\frac{1}{2}} \prod_{j=0}^{\frac{j-3}{2}} (-\Delta_{g_0} + k(k-j-1)) & \text{if } k \text{ is odd .} \end{cases} \tag{2.11}$$

It is important to mention that in the work [24], the authors exhibit a new kind of sign-changing solutions obtained by the superposition of one large bubble and a large number of negative bubble arranged along a submanifold of $\mathbb{R}^n$. These solutions are different from the ones obtained in [25] since they are not invariant under the group $O(2) \times O(n-1)$.

2.4.2 Morse theory for sign-changing Yamabe

After the study of the Yamabe problem and the understanding of the change of topology that occurs when crossing the critical values b_k, there were more extensions to the work in the sign-changing case. Notice that now we have the existence of solutions in most cases (compared to the case of the standard Yamabe problem). A. Bahri and S. Chanillo extended the study of the sign-changing Yamabe problem by finding a clear description of the critical points at infinity (under mild assumptions) and the parametrization of the neighbourhood at infinity; see [17]. This result was again improved in [19]. Recently, before his departure, A. Bahri wrote a paper [14], dedicated to the anniversary of J.M. Coron focusing on the study of the sign-changing Yamabe problem from a Morse theoretical point of view. We will proceed by giving a small idea of the content of this study.

If one considers a Morse function f on a compact manifold M under the assumption that its descendant gradient flow is Morse–Smale, then one can construct what we call the Morse homology. Namely, if we let $crit_k$ be the set of critical points of f with Morse index equal to k, then we can define the chain complex
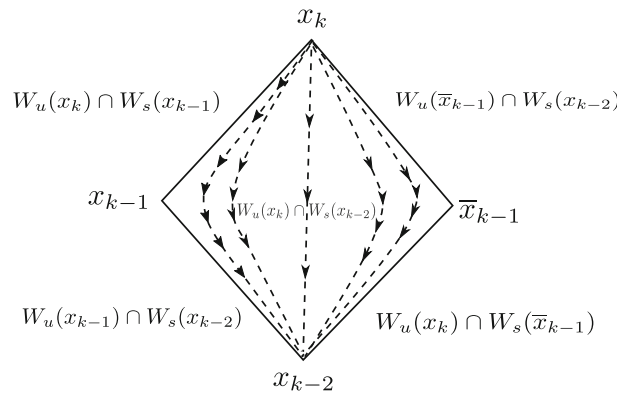
Fig. 2 Moduli spaces of flow lines

$(C_*(f), \partial)$ by taking C_k the vector space generated by the critical points of index k with coefficients in $\mathbb{Z}_2$, that is,

$$C_k(f) = span_{\mathbb{Z}_2}\{x \in crit_k\} = \bigoplus_{x \in crit_k} \mathbb{Z}_2 \otimes x.$$

The boundary operator $\partial : C_k \to C_{k-1}$ is then defined on the generators by

$$\partial x_k = \sum_{y_{k-1} \in crit_{k-1}} \langle x_k, y_{k-1} \rangle y_{k-1},$$

where $\langle x_k, y_{k-1} \rangle$ is the number of flow lines between x_k and x_{k-1} modulo 2. The main theorem in Morse theory states that $\partial \circ \partial = 0$. This leads to the construction of Morse homology which happens to coincide with the classical singular homology. To understand this relation in general we write it explicitly in the following way

$$\partial \circ \partial x_k = \sum_{x_{k-2} \in crit_{k-2}} \left(\sum_{y_{k-1} \in crit_{k-1}} \langle x_k, y_{k-1} \rangle \langle y_{k-1}, x_{k-2} \rangle \right) x_{k-2} \tag{2.12}$$

So the equation $\partial \circ \partial = 0$, is equivalent to

$$\sum_{y_{k-1} \in crit_{k-1}} \langle x_k, y_{k-1} \rangle \langle y_{k-1}, x_{k-2} \rangle = 0.$$

This means that the individual flow relations between x_k and x_{k-2} are the ones that matter. This is equivalent to the fact that the moduli space of flow lines between two critical points of difference of index 2 can be compactified. Hence the reason that the relation $\partial \circ \partial$ is often seen as a compactness relation. This can be represented by the kite in Fig. 2 below.

In general, if we are in the setting of a functional on a Banach–Finsler manifold satisfying the (PS) condition, then the same results hold. However, if the (PS) condition does not hold and we are in the presence of asymptotes and critical points at infinity then the relation is false. But, if we include the critical points at infinity as critical points, then it is like compactifying the space with additional points. The boundary operator then takes the form of

$$\partial x_k = \sum_{y_{k-1} \in crit_{k-1}} \langle x_k, y_{k-1} \rangle y_{k-1} + \sum_{y_{k-1}^\infty \in crit_{k-1}^\infty} \langle x_k, y_{k-1}^\infty \rangle y_{k-1}^\infty$$

Therefore, we can write ∂ as the sum of the classical boundary operator between genuine critical points and a boundary operator at infinity counting the flow lines linking a genuine critical point and a critical point at infinity (see Fig. 3), that is,

$$\partial = \partial_0 + \partial_\infty.$$

Now notice that one has $\partial \circ \partial = 0$ but what about $\partial_0 \circ \partial_0$ and $\partial_\infty \circ \partial_\infty$?
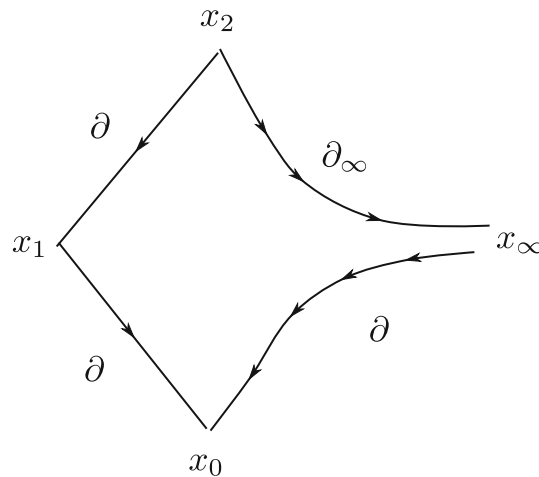
Fig. 3 Kite with asymptotes

In the paper [14], Bahri states many observations about the sign-changing Yamabe problem on S^3 and on an open set $\Omega \in \mathbb{R}^n$. By a study of the different configurations that critical points at infinity can have and by constructing another flow at infinity, he gives some conditions for the relation $\partial_\infty \circ \partial_\infty = 0$ to hold. Now if one assumes that the conditions are satisfied in a given situation, then one could define a homology $H_*(\partial_\infty)$. But it remains open to compute or identify this homology and study its significance. In [14], the reader can find a series of open problems and conjectures about this homology, and how to make sense of it and its existence.

2.4.3 The sign-changing CR Yamabe problem

Similar to the previous section, we will consider the CR-Yamabe type problem with no restriction on the sign of the solution, that is,

$$L_b u = |u|^{\frac{4}{q-2}} u \tag{2.13}$$

The same result holds for the CR case.

Theorem 2.23 [48] *Let (M, θ) be a K-contact manifold. Then the problem* (2.13) *has infinitely many sign-changing solutions. In particular, the CR Yamabe problem on the standard sphere S^{2n+1} has infinitely many sign-changing solutions.*

The proof idea of the proof is similar to that of the Riemannian case, but here we consider the space of $S_2^1(M)$ functions that are invariant under the flow of ξ, the Reeb vector field (in the sphere this flow generates the Hopf-fibration). The fact that the functions are invariant under the action of ξ, and M is K-contact implies that the sublaplacian commutes with the action of ξ and thus the sublaplacian is equal then to the laplacian. But the critical exponent for the sublaplacian $\frac{2q}{q-2}$ is less than the critical exponent for the laplacian $\frac{2(q-1)}{q-3}$. Hence we are in the setting of the subcritical Yamabe type problem and therefore we have infinitely many solutions. This result was extended to find other kinds of families of solutions on the sphere in [55], by use of the action of groups like the one used in [25]. These groups now need to preserve the CR structure. The question of whether one can construct solutions similar to the work [24] in the CR setting, still remains open.

2.4.4 The spinorial Yamabe problem

In this case (M, g, Σ) is a compact spin manifold with spin structure Σ. We define the Dirac operator $D : C^\infty(\Sigma M) \to C^\infty(\Sigma M)$ locally by

$$Du = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} u$$

This operator also satisfies a conformal invariance property and one is interested in solving the problem

$$Du = |u|^{\frac{4}{n-1}} u \tag{2.14}$$

For the details regarding spin structures and the construction of the Dirac operator, the reader is referred to [30]. Notice that in this case the equation is not scalar, which adds to its difficulty. The problem in this case is variational and the solutions to this problem correspond to the critical points of the functional $E : H^{\frac{1}{2}}(\Sigma M) \to \mathbb{R}$ defined by

$$E(u) = \frac{1}{2} \int_M \langle Du, u \rangle \mathrm{d}x - \frac{n-1}{2n} \int_M |u|^{\frac{2n}{n-1}} \mathrm{d}x$$

In contrast with the previous two problems, an extra difficulty appears, namely that the functional is strongly indefinite. That is, the linearized operator has infinitely many positive eigenvalues and infinitely many negative eigenvalues. Moreover, because of the non-compactness of the embedding $H^{\frac{1}{2}}(\Sigma M) \to L^{\frac{2n}{n-1}}(\Sigma M)$ the problem is again critical and bubbling occurs. In fact, even dealing with the sub-critical problem is challenging, that is, if we consider the problem

$$Du = |u|^{p-2} u, \tag{2.15}$$

where $2 \leq p < \frac{2n}{n-1}$, the existence of solutions is not a trivial matter. Different methods were applied to find such solutions.

Theorem 2.24 [38,43] *Let (M, g, Σ) be a compact spin manifold. Then the problem* (2.15) *has infinitely many solutions.*

The proof in [38] relies on a version of the min–max theorem adapted to strongly indefinite functionals, this method is similar in nature to the one introduced by P. Rabinowitz in [58] to find periodic orbits of Hamiltonian systems. We will discuss this part in depth in the second part of the survey. Another proof was provided in [43] using a topological method where we construct a Floer-type homology relative to the problem. After computing this homology we deduce the existence of infinitely many solutions. In a preprint that we received through private communications, T. Isobe also developed a Floer homology theory for the problem, from which one can extract existence and multiplicity results.

Now let us go back to the critical problem, that is, $p = \frac{2n}{n-1}$. As we mentioned before, in this case bubbling may occur. Indeed, in his work [39], Isobe proved the following result regarding the (PS) sequences of E.

Theorem 2.25 (Isobe) *Let u_i be a (PS) sequence for the functional E then there exist $k \geq 0$ and sequences $a_i^j \longrightarrow a^j \in M$, for $1 \leq j \leq k$ and a sequence of numbers R_i^j converging to zero, a solution $u_\infty \in H^{\frac{1}{2}}(M)$ of problem* (2.14) *and solutions $u^j \in H^{\frac{1}{2}}(\mathbb{R}^n)$ of $Du = |u|^{\frac{2}{n-1}} u$ on $\mathbb{R}^n$, such that up to subsequence, we have*

$$u_i = u_\infty + \sum_{j=1}^k \omega_i^j + o(1) \text{ in } H^{\frac{1}{2}}(\Sigma M),$$

where

$$\omega_i^j = (R_i^j)^{-\frac{n-1}{2}} \beta_j(x)(\rho_{i,j}^{-1})^*(u^j)$$

and

$$\rho_{i,j}(x) = \exp_{a^j}(R_i^j x).$$

Here, β_k is a non-negative function equals to 1 in $B_1(a^j)$ and zero outside $B_2(a^j)$. Moreover, we have

$$E(u_i) = E(u_\infty) + \sum_{j=0}^k E(u^j) + o(1).$$

Hence, if we can find an adequate group action and show that this action is compatible with the spin structure then we can get an existence and multiplicity result similar to [25,48,55]. In [44], we prove that this is the case by working on S^n and using the isometry groups $O(k) \times O(n + 1 - k)$.

Theorem 2.26 [44] *There exists infinitely many solutions to the spinorial Yamabe problem* (2.14) *on the sphere S^n.*

2.4.5 The Yang–Mills equation

Consider a compact smooth Riemannian manifold (M, g). Let G be a compact Lie group (we will stick to the cases $G = S^1$ or $SU(2)$), and (P, π) a principal G-bundle over M. The gauge group, $\mathcal{G}$, consists of maps $u : P \to G$ that are equivariant, that is,

$$u(pg) = g^{-1}u(p)g, \quad \text{for all } p \in P, g \in G.$$

If $\mathfrak{g}$ is the Lie algebra of G, we can define a connection on P as a $\mathfrak{g}$-equivariant 1-form with fixed values in the vertical direction of TP. That is, $A \in \Omega^1(P, \mathfrak{g})$ such that

$$A_{pg}(vg) = g^{-1}A_p(v)g, \quad \text{for all } v \in T_pP, g \in G,$$

and

$$A_p(p\xi) = \xi, \quad \text{for all } \xi \in \mathfrak{g}, p \in P.$$

We will denote by $\mathcal{A}(P)$ the space of smooth connections on P. Each connection gives rise to a covariant derivative in the following way: if $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$, then for $A \in \mathcal{A}(P)$ we associate the covariant derivative $\nabla_A : \Gamma(\mathfrak{g}_P \to \Gamma(T^*M \otimes \mathfrak{g}_P)$, defined by

$$\nabla_A s = \mathrm{d}s + [A, s],$$

where the symbol Γ here is used to denote smooth sections. Then, the gauge group acts on $\mathcal{A}(P)$ in the following way: if $u \in \mathcal{G}(P)$ and $A \in \mathcal{A}(P)$, then

$$u^*A = u^{-1}Au + u^{-1}du.$$

This covariant derivative can be extended as an exterior derivative $d_A : \Omega^k_{Ad}(P, \mathfrak{g}) \to \Omega^{k+1}_{Ad})(P, \mathfrak{g})$ as follows

$$d_A s = \mathrm{d}s + [A \wedge s],$$

where here $\Omega^k_{Ad}(P, \mathfrak{g})$ is the space of horizontal, equivariant k-forms and $[\cdot \wedge \cdot]$ is defined for $A, B \in \Omega^1_{Ad})(P, \mathfrak{g})$ by

$$[A \wedge B](X, Y) = [A(X), B(Y)] - [A(Y), B(X)]$$

for all $X, Y \in T_pP$. In contrast with the usual exterior derivative, d_A^2 may not vanish and we can define F_A the curvature 2-form of A by

$$d_A d_A s = [F_A \wedge s].$$

This leads to the formula

$$F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2_{Ad}(P, \mathfrak{g}).$$

The main properties of the curvature are:

Proposition 2.27 *The curvature form F_A satisfies*

(i) *F. is $\mathcal{G}(P)$ equivariant, i.e. for all $A \in \mathcal{A}(P)$, and for all $u \in \mathcal{G}(P)$,*

$$F_{u^*A} = uF_A.$$

(ii) *F_A satisfies the Bianchi identity, that is,*

$$d_A F_A = 0.$$

The main objective here is to study the space $\mathcal{A}(P)/\mathcal{G}(P)$. One way of studying its topology for instance is to consider an invariant Morse functional on $\mathcal{A}(P)$ and study its critical points. One such functional is what we call the Yang–Mills functional defined by

$$Y(A) = \frac{1}{2} \int_M |F_A|^2 \mathrm{d}x.$$

Then, the right space to work with is $\mathcal{A}_2^1(P)$ the space of Sobolev connections that are in L^2 and have their first differential in L^2. It is the analogue of the space H^1 for connections. Notice that since F_A contains a quadratic term, we need the connections to be at least in L^4 in order to have $Y(A)$ well defined. Using the Sobolev embedding, we have that if the dimension of M is less or equal than 4, then the functional is well defined on the space $\mathcal{A}_2^1(P)$. Critical points of this functional are called Yang–Mills connections and they satisfy the equation

$$d_A^* F_A = 0.$$

This equation combined with the Bianchi identity shows that the curvature of a Yang–Mills connection satisfies an elliptic equation of the form

$$\begin{cases} d_A F_A = 0, \\ d_A^* F_A = 0 \end{cases} \tag{2.16}$$

For more details on the construction of the space of connections and properties of the curvature, we refer the reader to [27,66].

As in the previous sections, the property that we investigate first is the (PS) condition and for this we have the following result.

Theorem 2.28 [65] *Let $(A_k)_{k \geq 0}$ be a sequence of connections in $\mathcal{A}_2^1(P)$ such that $Y(A_k)$ is uniformly bounded, then there exists a subsequence that we denote again by $(A_k)_{k \geq 0}$ and a sequence of gauge transformations $(u_k)_{k \geq 0}$ in $\mathcal{G}_2^2(P)$ such that $(u_k^* A_k)_{k \geq 0}$ weakly converges in $\mathcal{A}_2^1(P)$.*

This theorem allows us to show that Y satisfies the (PS) condition if $dim(M) < 4$, and this is again because of the compactness of the Sobolev embedding in L^4. For the proof of such theorem we refer to [66]. This fact was used in [4], in the case of Riemann surfaces to study the gauge group and hence the structure of vector bundles over M. In dimension 4, the situation starts to get complicated, since the embedding of the Sobolev space in L^4 is continuous but not compact. In fact, one has the following theorem.

Theorem 2.29 [60,61] *Let $P \rightarrow M$ be a principal G-bundle with characteristic classes (k, η) and (A_k) a PS sequence for Y then there exists*

(1) *a set $B = \{x_1, \ldots, x_p\} \subset M$,*
(2) *a principal G-bundle $P_0 \rightarrow M$ with Pontryagin number k_0 and characteristic class η,*
(3) *a connection $A_\infty \in \mathcal{A}(P_0)$, solution to the Yang–Mills equation,*
(4) *a sequence of p-pairs $(P_i, A^i)_{1 \leq i \leq p}$ such that $P_i \rightarrow S^4$ is a principal G-bundle with Pontryagin number k_i and $A^i \in \mathcal{A}(P_i)$ is a solution to the Yang–Mills equation on S^4 and a sequence of gauge transformations $(u_k)_{k \geq 0}$ so that $u_k^* A_k$ converges to A_∞ in $\mathcal{A}_{2,loc}^1(P_{|M \setminus B})$ and $k_0 + \sum_{i=1}^p k_i = k$ and*

$$Y(A_k) = Y(A_\infty) + \sum_{i=1}^p Y(A^i) + o(1).$$

Based on this theorem, we have a characterization of the bubbling phenomena equivalent to the one of the sign-changing Yamabe problem. In fact, studying just the minima of the functional Y, is very similar to the original Yamabe problem. The minima of Y are called ASD connections (anti-self dual) or instantons. This study as in the case of the Yamabe, is intimately tied to topological properties of the manifold M and this led K. Donaldson [27,28] to define a new invariant for 4-manifolds. One may notice that in the case of ASD connections, bubbling occurs only with ASD connections on S^4. After the stereographic projection on $\mathbb{R}^4$ one has this formula for the basic instantons

$$B_1 = \frac{-x_2 \mathfrak{i} - x_3 \mathfrak{j} - x_4 \mathfrak{k}}{\epsilon^2 + |x|^2}$$

$$B_2 = \frac{x_1 \mathfrak{i} - x_4 \mathfrak{j} + x_3 \mathfrak{k}}{\epsilon^2 + |x|^2}$$

$$B_3 = \frac{x_4 \mathfrak{i} + x_1 \mathfrak{j} - x_2 \mathfrak{k}}{\epsilon^2 + |x|^2}$$

$$B_4 = \frac{-x_3 \mathfrak{i} + x_2 \mathfrak{j} + x_1 \mathfrak{k}}{\epsilon^2 + |x|^2},$$

where $(\mathfrak{i}, \mathfrak{j}, \mathfrak{k})$ is the standard basis of $\mathfrak{su}(2)$. Notice that the decay and the form of these instantons is similar to the standard bubble of the Yamabe problem. In fact, this decay or behaviour has nothing to do with the problem itself, but rather it is coming from the conformal invariance and the stereo-graphic projection between S^4 and $\mathbb{R}^4$ which is a conformal transformation. In the general case, C.H. Taubes [61], gave a very explicit description of the change of topology when we cross a critical level of a critical point at infinity. However, we will omit the details of this statement due to its technical nature.

3 Contact geometry

Although this section differs from the previous one, it still resides in the variational realm. We start by giving the general setting of the problem. Let M be a 3-dimensional compact manifold and α a contact form on M. That is, $\alpha \wedge d\alpha$ is a volume form on M. Such a pair (M, α) is called a contact manifold. Now, for a given contact manifold there is a vector field tightly related to it. This vector field ξ is called the Reeb vector field and it is uniquely characterized by

$$\begin{cases} \alpha(\xi) = 1 \\ d\alpha(\xi, \cdot) = 0 \end{cases}$$

Now, the problem is to find periodic orbits of ξ. This problem is known as the Weinstein conjecture. We will start first by describing the origin of the problem. So we consider the standard example in $\mathbb{C}^2$ with the coordinate system $z = (z_1, z_2)$ and let H be a function that has at most quadratic growth at infinity. We propose then to solve the problem:

$$\begin{cases} \dot{z}_1 = H_{z_2} \\ \dot{z}_2 = -H_{z_1} \\ z(0) = z(1). \end{cases} \tag{3.1}$$

That is, finding a function $z : [0, 1] \longrightarrow \mathbb{C}^2$ satisfying the previous equation.

A compact way of writing this problem is to use the complex structure J, so that solutions of (3.1) satisfy $\dot{z} = J \nabla H$. This problem is variational, that is, solutions of are critical points of the functional F defined by

$$F(z) = \int_0^1 J z' \cdot z - \int_0^1 H(z(t)) dt.$$

The natural space of variations here is $H_{per}^{\frac{1}{2}}([0, 1], \mathbb{C}^2)$. It is important to notice that the functional is strongly indefinite. Let us try to write this problem in a formal way. That is, we consider the standard symplectic structure of $\mathbb{C}^2 = \mathbb{R}^4$ defined by $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and given a Hamiltonian H, one can define the Hamiltonian vector field X_H by

$$i_{X_H} w_0 = dH.$$

It can be easily checked that the flow of X_H corresponds to solutions of (3.1). If we set the map φ_t^H as the 1-parameter group of X_H then we have that

$$H(\varphi_t^H(x)) = H(x)$$

for all x, hence the level sets of H are invariant under the flow of X_H. From this property, one can ask the following question: given a hypersurface $S = H^{-1}(1)$ does it contain a Hamiltonian periodic orbit?

This question can be answered in the affirmative provided that the surface S is convex (or star shaped). This was the famous work of P. Rabinowitz in [58]. He used a type of min–max argument on sets with a given

S^1 action after introducing an approximation for the space of variations. Then, after estimating the critical values of the approximated problem, he showed that they are bounded independently of the approximation. Since (PS) holds in this case, then we have convergence when removing the approximation.

Later, this method was formulated differently in terms of a homology type construction and generalized to a setting of other symplectic manifolds having special kinds of contact hypersurfaces. This construction is called the Rabinowitz–Floer homology (see [2,21–23]). This method was then further extended to problems unrelated to symplectic or contact geometry. For instance, the case of the non-linear Dirac equation stated above in [43] and then to more general problems in [49,51].

This result initiated the Weinstein conjecture in full generality for contact manifolds. One of the very interesting results in answering this question was

Theorem 3.1 [37] *If (M, α) is an overtwisted contact structure, then ξ has at least one contractible periodic orbit.*

The approach of H. Hofer used the construction of a family of pseudo-holomorphic curves with boundaries on the overtwisted disk. Then, using monotonicity of this boundary one sees that the family will blow-up which, after rescaling, yields the existence of a periodic orbit of the Reeb vector field.

The variational problem can be formulated in full generality as follows. Let $J : H^1(S^1, M) \to \mathbb{R}$ defined by

$$J(x) = \int_0^1 \alpha_{x(t)}(\dot{x}(t)) \mathrm{d}t.$$

One can easily see that critical points of this functional are periodic orbits of ξ. But there are many issues with this setting. In fact, the functional does not satisfy the (PS) condition on H^1, this is because it only controls one component of $\dot{x}$, that is, the one along ξ. The other issue is that the functional is strongly indefinite, therefore each critical point has an infinite Morse index and co-index. Hence, even if we can handle the bubbling or the non-compactness coming from the violation of (PS), we still cannot describe the change of the topology when crossing a critical set. Actually, the topology of the level sets does not change in this case. The idea introduced by A. Bahri in his work, [10–12,19], was to first remove one of these difficulties, namely changing the space of variations to make the functional J a definite functional with finite Morse index at the critical points. Under a convexity assumption one can reduce the difficulty of the problem by restricting the functional to a smaller space of variations.

Definition 3.2 We say that the contact form α admits a Legendre transform induced by v, or is v-convex, if :

(a) there exists a C^1 never vanishing vector field v in the kernel of α.
(b) the 1-form $\beta(\cdot) = i_v \mathrm{d}\alpha = \mathrm{d}\alpha(v, \cdot)$, is a contact form with the same orientation as α.

This v-convexity condition is satisfied in many cases. For example, it is satisfied by the standard contact structure on the sphere, the first overtwisted contact structure of Gonzalo–Varela [35], all the tight contact forms of the torus. There are other cases when this holds, some others can be found in [43,44]. The substitute space of variation is now defined by

$$\mathcal{C}_\beta = \left\{ x \in \mathcal{L}_\beta; \alpha_x(\dot{x}) = c > 0 \right\},$$

where c is a non-prescribed constant. Then the following holds :

Theorem 3.3 *The space $\mathcal{C}_\beta - M$ is a Hilbert manifold and its tangent space at a loop x, such that $\dot{x} = a\xi + bv$, is given by the set of vectors $Z = \lambda\xi + \mu v + \eta[\xi, v]$ such that:*

$$\begin{cases} \dot{\lambda} = b\eta - \int_0^1 b\eta \\ \dot{\eta} = \overline{\mu} b\eta + a\mu - \lambda b, \end{cases} \tag{3.2}$$

where $\overline{\mu} = \mathrm{d}\alpha(v, [v, [\xi, v]])$.

In addition to this structural property of the space of variations, we have

Theorem 3.4 [11] *The critical points of J restricted to $\mathcal{C}_\beta$ are the periodic orbits of ξ and these critical points have finite Morse index. Moreover, the difference between the Morse indices of the periodic orbits are the same whether J is restricted to $\mathcal{C}_\beta$ or the free loop space $\Lambda(S^1, M) = H^1(S^1, M)$.*

The main benefit of working in this space is that the critical points and the asymptotes of the functional are well understood. So one can start a Morse theoretical approach if we can settle the compactness issue. However, the natural question that comes first is understanding the topology of $\mathcal{C}_\beta$ since if it has rich topology and the functional J satisfies (PS), then we can deduce right away that there is a periodic orbit of ξ and even deduce a multiplicity result. In [52], we prove under some assumptions on α that $\mathcal{C}_\beta$ does indeed have rich topology. In order to state the precise result, let us introduce the following rotation condition.

Definition 3.5 We say that $\ker \alpha$ turns well along v, if, starting from any x_0 in M, the rotation of $\ker \alpha$ along the v-orbit in a transported frame exceeding π.

This condition has a dynamical characterization, (see [34]) and using this characterization one can show that

Lemma 3.6 *If one of the following conditions is satisfied:*

$$(i) \quad |\overline{\mu}| < 2$$

$$(ii) \ \text{there exists a map } u \text{ on } M \text{ such that} \overline{\mu} = u_v$$

then α turns well along v. Moreover, if $\overline{\mu} = 0$ then α is tight.

With this in mind we have the following result.

Theorem 3.7 [52] *Let (M, α) be a contact closed manifold such that α is v-convex and it terns well along v, then the injection*

$$\mathcal{C}_\beta \hookrightarrow \Lambda(S^1, M)$$

is an S^1-equivariant homotopy equivalence.

This result shows in particular that $\mathcal{C}_\beta$ and $\Lambda(S^1, M)$ have the same topology. The proof of this result consists of three steps. First, since β is a contact form, then $\mathcal{L}_\beta$ the space of Legendrian curves of β have the same topology as $\Lambda(S^1, M)$. The second step is to construct a deformation $\Phi : [0, 1] \to \mathcal{L}_\beta$ via flow of a vector field, so that $\Phi(0) = id_{\mathcal{L}_\beta}$ and $\Phi(1)(\cdot) \in \mathcal{C}_\beta^+$, where the intermediate space $\mathcal{C}_\beta^+$ is defined by

$$\mathcal{C}_\beta^+ = \left\{ x \in \mathcal{L}_\beta; \alpha(\dot{x}) \geq 0 \right\}.$$

The last step then consists of deforming curves in $\mathcal{C}_\beta^+$ to curves in $\mathcal{C}_\beta$ again via the flow of a vector field. This last step consists of removing Dirac masses along v of different multiplicity.

Now by looking at the functional on $\mathcal{C}_\beta$, it can be seen that J is insensitive to pieces of orbits along v. Therefore, one should expect that if we construct a deformation flow the limiting set should contain curves having back and forth runs along v. First, let us find the necessary expressions in order to construct a decreasing deformation. Let $Z = \lambda \xi + \mu v + \eta w$, where w is the Reeb vector field of β and $x \in \mathcal{C}_\beta$ such that $\dot{x} = a\xi + bv$ then we have

$$Z \cdot a = -\int_0^1 b\eta \tag{3.3}$$

and

$$Z \cdot b = \frac{\overline{\lambda b + \dot{\eta}}}{a} + a\eta\tau - b\eta\overline{\mu}_\xi, \tag{3.4}$$

where τ is defined by $[\xi, [\xi, v]] = -\tau v$ and $\overline{\mu}$ as above. In fact, the flow along any vector field is determined by the data of b. So most of the evolution, will be studied on b. Now by looking at (3.3), we see that the natural decreasing flow here is given by taking $\eta = b$. This choice will give us a heat type equation in (3.4). The flow chosen this way has good geometric properties, such as decreasing the number of zeros of b and decreasing the linking between curves (see [10,12]). However, the problem here is its limiting behaviour. Indeed, in the case of blow-up the flow will converge to the expected behaviour curve, that is, b will be a summation of Dirac δ functions and a term which is absolutely continuous. That is,

$$b(t) \rightharpoonup \sum_{i=1}^n b_i \delta_{t_i} + \phi_0.$$

The term ϕ_0 is not well understood and it is not something that is expected during the evolution of the flow. Therefore, in [10], another flow was introduced. Its construction is very technical and delicate, and it is studied extensively in [67]. We can define the set of asymptotes

$$\Sigma_k = \{x \in \mathcal{C}_\beta^+ \text{ s.t. } x \text{ is made of } k \text{ pieces of } \xi \text{ and } k \text{ pieces of } v\}.$$

Notice that Σ_1 is the set of periodic orbits of ξ and an example of a curve in Σ_5 is presented in Fig. (4) below.
Now we can state

Theorem 3.8 [10] *There exists a flow Φ defined by a vector field Z on $T\mathcal{C}_\beta$ such that*

(i) *J is decreasing along the flow. That is,*

$$\frac{\mathrm{d}}{\mathrm{d}s} J(\Phi_s(x)) \leq 0$$

(ii) *if $b(s, t)$ blows-up on a finite time T then*

$$b(s, t) \rightharpoonup \sum_{k=1}^{n} c_i \delta_{t_i}$$

(iii) *The limiting curve in this case belongs to the stratified set $\bigcup_{k \geq 1} \Sigma_k$.*

The sets Σ_k are the equivalent of the sets $V(k, \varepsilon)$ in the case of the Yamabe problem. So now one has a characterization of when the (PS) condition is violated. Of course a study of the flow at infinity is required to understand the properties of the functional around the critical points at infinity. In order to properly define the critical points at infinity, we need to introduce a few definitions. First let us call ϕ the transport map along the orbits of v and ψ the transport map along the orbits of ξ.

Definition 3.9 Two points x_0 and $x_1 = \phi_{t_0}(x_0)$ along an orbit of v are said to be conjugate, if the transport of the vector ξ from x_0 is a multiple of ξ at x_1, that is,

$$D\phi_{t_0}(\xi_{x_0}) = \theta \xi_{x_1}, \qquad \theta \in \mathbb{R}.$$

A piece of an orbit of ξ between two points x_0 and x_1 is said to be characteristic, if the contact plane makes exactly $k\pi$ rotations from x_0 to x_1.

Now we are able to define the different critical points at infinity.

Definition 3.10 A curve $x \in \Gamma_k$ is said to be

(i) a false critical point at infinity if all the v pieces of the curve are between conjugate points.
(ii) a true critical point at infinity if the ξ pieces of the curve are characteristic.

This distinction of the critical points at infinity is made mainly because of their nature. For instance in the case of true critical points at infinity, each characteristic piece behaves as a separate critical point with its own index for the variational problem. In fact it can play the role of a superposition of multiple critical points which makes the problem even harder to study. These properties were investigated in [11] and [19]. The problem that remains is the stability of the homology if it does exist. In fact one can compute the homology in certain cases, but proving its stability along a perturbation of the contact structure is a real challenge. As in the case of the sign-changing Yamabe problem, we can write the boundary operator as $\partial = \partial_0 + \partial_\infty$ then one needs to show that

$$\partial_\infty \circ \partial_0 = \partial_0 \circ \partial_\infty = 0. \tag{3.5}$$

In that case ∂_0 will define a homology own its own. The problem is that relation (3.5) can be violated under deformation of the contact structure. In [53], we studied this contact homology in the case of the torus with all its contact structure. First, we recall the existence of the following sequence of tight contact structures on T^3 defined by

$$\alpha_n = \cos(nz)\mathrm{d}x + \sin(nz)\mathrm{d}y, \qquad n \in \mathbb{N},$$
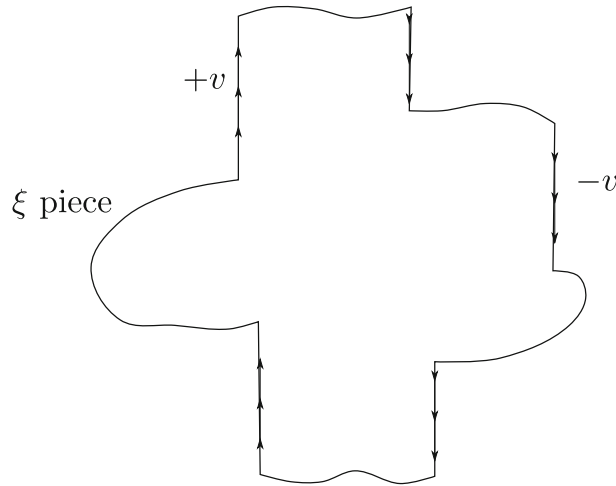
then we have

Fig. 4 Curve in Σ_5

Theorem 3.11 (MV8) *Let g be a homotopy class of the two-dimensional torus* T^2. *Then for every* $n \in \mathbb{N}$, *we have*

$$H_k(\alpha_n, g) = \begin{cases} \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \ n \ times, & if \ k = 0, 1 \\ 0, & if \ k > 1 \end{cases} \tag{3.6}$$

Moreover, in the same work, we exhibit more structures between the different homologies for the contact structures

Theorem 3.12 *Let p and k be positive integers. Then there exists a morphism*

$$f_* : H_*(\alpha_{kp}, g) \longrightarrow H_*(\alpha_p, g).$$

Moreover, this homomorphism corresponds to an equivariant homology reduction under the action of the group $\mathbb{Z}_k$, *that is,*

$$H_*(\alpha_p, g) = H_*^{\mathbb{Z}_k}(\alpha_{kp}, g).$$

This can be explained by the following commuting diagram

This result is then extended to S^1 torus bundles defined as follow. Given a matrix $A \in SL_2(\mathbb{Z})$, we define the space

$$Y_A = T^2 \times \mathbb{R}/(x, y, z) = (A(x, y), z + 2\pi).$$

We know that these spaces contain infinitely many contact structures. The construction of such structures starts by taking a strictly increasing function h and considering the contact form α_h on $\mathbb{R}^3$ defined by

$$\alpha_h = \cos(h(z))\mathrm{d}x + \sin(h(z))\mathrm{d}y.$$

This contact homology is locally stable. That is, it is well defined and stable under small perturbations although the situation becomes more complicated if we consider big perturbations.

3.1 Fredholm condition

The main issue causing this instability, is no longer the lack of compactness, but instead it is another condition that is, fundamental in the study of Morse theory and Morse relations. It is the Fredholm condition. That is the linearized operator is Fredholm. This allows the use of the implicit function theorem and hence the gluing of orbits and the construction of the homology. In our case, this violation is caused mainly by the translations along v. This space of invariance is large enough that it violates the Fredholm assumption. In order to understand this violation, let us take an explicit problem consisting of the wave equation on an interval with periodic boundary conditions:

$$\begin{cases} u_{tt} - u_{xx} = f \\ u(0) = u(1) \end{cases} \tag{3.7}$$

If we set T the operator defined by $Tu = u_{tt} - u_{xx}$ then we can decompose the space L^2 for example as,

$$L^2 = H^+ \oplus H_0 \oplus H^-,$$

where H^+ (resp. H^-) corresponds to the space spanned by the eigenfunctions with positive (resp. negative) eigenvalues and H^0 its kernel. Then, one can verify that H_0 contains the set

$$\overline{H} = \left\{ h(t - x); h \in C_{per}^{\infty}[0, 1] \right\}$$

Hence it is infinite dimensional. This is the reason that some non-linear problems involving the wave equation require extra assumptions on the non-linearity, such as convexity or monotonicity to overcome this violation. In our case, if we are given a critical point x of J, and we modify it a bit to a curve x_ε by inserting a small back and forth run along v, then if we expand the functional J at x_ε we find

$$J(x_\varepsilon) = J(x) + \varepsilon(1 - \alpha_{\phi(-s)}(D\varphi(s)(\xi)) + o(\varepsilon),$$

where ϕ is the transport map along v introduced above. Hence, if $\alpha_{\varphi(-s)}(D\varphi(s)(\xi) > 1$ we have an extra decreasing direction. A special study for this violation was done for the specific case of the exotic contact forms of Gonzalo–Varela on S^3 (see [35]), defined by

$$\alpha_n = - \left(\cos\left(\frac{\pi}{4} + n\pi(x_3^2 + x_4^2) \right) (x_2\mathrm{d}x_1 - x_1\mathrm{d}x_2) + \sin\left(\frac{\pi}{4} + n\pi(x_3^2 + x_4^2) \right) (x_4\mathrm{d}x_3 - x_3\mathrm{d}x_4) \right).$$

The Fredholm violation was first studied by A. Bahri in [13] for the first exotic contact form α_1 of J. Gonzalo and F. Varela, which is v-convex, as was proved in the work of V. Martino [56]. In another work [45], we prove that this also holds for the exotic form α_3 even though it is not v-convex, and in fact after a deeper study of the dynamics of v one can show that all the exotic forms α_n violate the Fredholm condition. The study in this case is explicit and computational and it involves identifying some characteristic surfaces (their number grows with respect to n), and then study the transport equations along v and its rotation between these surfaces. In the same work, we also find the critical points and their Morse indices and we show that for symmetry reasons they come as circles of periodic orbits as is the case of the standard sphere.

As we pointed out, the contact form α_3 is not v-convex, at least for the vector field v that comes as a natural extension of the vector field introduced in [56]. That means that at a certain time the form β, changes its orientation from positively oriented to negatively oriented. In the case where it is negatively oriented, we have that α_3 and β are rotating in opposite directions. It was conjectured by A. Bahri that if we set $\mathcal{P} = -\mathrm{d}\alpha(v, [\xi, v])$ then there exists a foliation (a distribution that does not rotate) transverse to ξ, in the region $\mathcal{P} < 0$. In the paper [45], we partially answer this conjecture

Theorem 3.13 [45] *There exist $\delta > 0$ and small and a function F defined on the set $[\mathcal{P} \leq \delta]$ such that $dF(v) = 0$ and $dF(\xi) > 0$ for every $x \in [\mathcal{P} < \delta]$.*

3.2 Existence result for the first contact form

The main issue that was pointed out in [15], is the point to circle Morse relation. That is, if we start from an S^1 set of periodic orbits and then we deform the contact form then, due to the Fredholm violation, this action can be lost during the deformation reaching a single periodic orbit. That is one point with no S^1 action to a circle of periodic orbits. To avoid this complexity, A. Bahri used a powerful topological tool, that is, the Fadell and Rabinowitz index [29]. This index can be defined as follows.

Definition 3.14 Let X be a topological space with an effective action of S^1. For every closed set $A \subset X$ invariant under the action of S^1, we define $i_{S^1}(A)$ by

$$i_{S^1}(A) = \inf\{k \geq 1; \text{ there exists an equivariant map } f : A \to S^{2k-1}\},$$

and $i_{S^1}(\emptyset) = 0$.

Now clearly if one takes $X = S^\infty$ then $i_{S^1}(S^{2n-1}) = n$ and $i_{S^1}(X) = \infty$. This definition is an alternative one to the Fadell–Rabinowitz index defined, and goes as follows: given a set K such that S^1 acts on K in an effective way. The action of S^1 on $S^\infty \times K$ is free, hence we have a principal S^1-bundle $q_K : S^\infty \times K \to (S^\infty \times K)/S^1$ and a classifying map $f : (S^\infty \times K)/S^1 \to \mathbb{C}P^\infty$, such that if $p : S^\infty \times K \to S^\infty$ is the natural projection and $q : S^\infty \to \mathbb{C}P^\infty$ is the standard principal S^1-bundle, we have $f \circ q_K = q \circ p$. Then the index is defined by

$$i_{S^1}(K) = \max\{k \in \mathbb{N}; f^*(\alpha) \neq 0\},$$

where α is the generator of the rational cohomology group $H^1(\mathbb{C}P^\infty, \mathbb{Q}) = \mathbb{Q}$. Basically, the index is a way of counting the cohomology generators of a space X transversally to the S^1 action.

Using some estimates on two subsets of $\cup_{k \geq 1} \Sigma_k$, Bahri proved the following result in [15].

Theorem 3.15 [15] *Consider a contact form α on S^3 and assume that the Reeb vector field ξ has no periodic orbits of index 1. Then the functional J has at least one periodic orbit of index $2k - 1$ for k possibly large.*

In the same work, he conjectured many results regarding the Fadell–Rabinowitz index of certain sets and also on the nature of the orbits of index one provided they exist.

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