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Proof of a limited version of Mao's partition rank inequality using a theta function identity

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Abstract

Ramanujan's congruence $p(5k+4) \equiv 0 \pmod 5$ led Dyson (Eureka 8:10–15, 1944) to define a measure "rank", and then conjectured that $p(5k+4)$ partitions of $5k+4$ could be divided into subclasses with equal cardinality to give a direct proof of Ramanujan's congruence. The notion of rank was extended to rank differences by Atkin and Swinnerton-Dyer (Some properties of partitions 4:84–106, 1954), who proved Dyson's conjecture. More recently, Mao (Number Theory 133:3678–3702, 2013) proved several equalities and inequalities, leaving some as conjectures, for rank differences for partitions modulo 10 and for M_2-rank differences for partitions with no repeated odd parts modulo 6 and 10 (Mao in Ramanujan J 37:391–419, 2015). Alwaise et al. (Ramanujan J. doi:10.1007/s11139-016-9789-x, 2016) proved four of Mao's conjectured inequalities, while leaving three open. Here, we prove a limited version of one of the inequalities conjectured by Mao.

Keywords: Partitions, Ranks, Rank differences, Theta functions

Mathematics Subject Classification: Primary 11P83

1 Introduction and results

A *partition* of a positive integer n is a way of writing n as a sum of positive integers, usually written in non-increasing order of the summands or parts of the partition. The number of partitions of n is denoted by $p(n)$. For a partition λ, we denote the number of parts in the partition as $n(\lambda)$ and the largest part as $l(\lambda)$.

The celebrated Ramanujan congruences for the partition function begged for a combinatorial interpretation:

$$\begin{aligned} p(5k+4) &\equiv 0 \pmod{5}, \\ p(7k+5) &\equiv 0 \pmod{7}, \\ p(11k+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Dyson [4] defined the rank of a partition λ to be $l(\lambda) - n(\lambda)$ and conjectured that partitions for $5k+4$ and $7k+5$ can be divided into five and seven equal sub-classes respectively based on their rank. Specifically, he claimed that for s in each residue class modulo 5 or 7, respectively

$$N(s, 5, 5n+4) = \frac{p(5n+4)}{5},$$
$$N(t, 7, 7n+4) = \frac{p(7n+6)}{7},$$

where $N(s, m, n)$ denotes the number of partitions of n with rank s modulo m. Atkin and Swinnerton-Dyer [2] proved Dyson's conjecture by finding the generating functions for the rank differences $N(s, m, mk+d) - N(s, m, mk+d)$ for $k = 5, 7$. They obtained several other interesting identities apart from Ramanujan's congruences.

Lovejoy and Osburn [5] expanded on the work by Atkin and Swinnerton-Dyer to find rank differences for overpartitions and M_2-rank differences for partitions without repeated odd parts, which is defined for such a partition λ by $\left\lceil \frac{l(\lambda)}{2} \right\rceil - n(\lambda)$. The corresponding count for number of partitions of n with no repeated odd parts having its M_2-rank congruent to s modulo m is given by $N_2(s, m, n)$. They obtained all the rank difference formulas corresponding to $m = 3, 5$.

Continuing on their work, Mao [6,7] extended the results for Dyson rank differences modulo 10 and M_2 rank differences modulo 6 and 10. He obtained several interesting inequalities based on his results such as

$$N(1, 10, 5n+1) > N(5, 10, 5n+1),$$
$$N_2(0, 6, 3n+1) + N_2(1, 6, 3n+1) > N_2(2, 6, 3n+1) + N_2(3, 6, 3n+1).$$

Mao also gave some conjectures in [6,7] based on computational evidence, both for the Dyson rank and M_2-rank for partitions with unique odd parts.

Conjecture 1.1 *Computational evidence suggests that*

$$N(0, 10, 5n) + N(1, 10, 5n) > N(4, 10, 5n) + N(5, 10, 5n), \tag{1}$$
$$N(1, 10, 5n) + N(2, 10, 5n) \geq N(3, 10, 5n) + N(4, 10, 5n), \tag{2}$$
$$N_2(0, 10, 5n) + N_2(1, 10, 5n) > N_2(4, 10, 5n) + N_2(5, 10, 5n), \tag{3}$$
$$N_2(0, 10, 5n+4) + N_2(1, 10, 5n+4) > N_2(4, 10, 5n+4) + N_2(5, 10, 5n+4), \tag{4}$$
$$N_2(1, 10, 5n) + N_2(2, 10, 5n) > N_2(3, 10, 5n) + N_2(4, 10, 5n), \tag{5}$$
$$N_2(1, 10, 5n+2) + N_2(2, 10, 5n+2) > N_2(3, 10, 5n+2) + N_2(4, 10, 5n+2), \tag{6}$$
$$N_2(0, 6, 3n+2) + N_2(1, 6, 3n+2) > N_2(2, 6, 3n+2) + N_2(3, 6, 3n+2). \tag{7}$$

In (2), (5), *and* (6), $n \geq 1$, *whilst in the rest* $n \geq 0$.

Alwaise et al. [1, Theorem 1.3] proved four of these seven inequalities conjectured by Mao, namely (1), (2), (3), and (4) by using elementary methods based on the number of solutions of Diophantine equations solving for the exponents in the generating functions in the corresponding rank differences. They also observed that in (2), the strict inequality holds. However, their methods weren't strong enough to prove the remaining three conjectures, which are still open. Here, we prove a limited version of (7).

Theorem 1.2 *Mao's conjecture* (7) *is true when* $3 \nmid n+1$. *Specifically, we have that the following inequalities are true for all* $n \geq 0$*:*

$$N_2(0, 6, 9n+2) + N_2(1, 6, 9n+2) > N_2(2, 6, 9n+2) + N_2(3, 6, 9n+2), \tag{8}$$
$$N_2(0, 6, 9n+5) + N_2(1, 6, 9n+5) > N_2(2, 6, 9n+5) + N_2(3, 6, 9n+5). \tag{9}$$

2 Preliminaries

The standard q-series notation is employed which is defined as

$$(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i),$$
$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i),$$

where $n \in \mathbb{N}$ and $a \in \mathbb{C}$. The empty product $(a;q)_0$ is defined to be 1.

The following elementary identities are used in manipulation of q-series to prove equalities between expressions. For $a, b \in \mathbb{Z}$, $c \in \mathbb{C}$, and for $k \in \mathbb{N}$, we have

$$(-q;q)_\infty \cdot (q;q^2)_\infty = 1, \tag{10}$$
$$(q^a;q^b)_\infty(-q^a;q^b)_\infty = (q^{2a};q^{2b})_\infty, \tag{11}$$
$$(cq^a;q^{2b})_\infty(cq^{a+b};q^{2b})_\infty = (cq^a;q^b)_\infty, \tag{12}$$
$$(cq^a;q^{kb})_\infty \cdots (cq^{a+(k-1)b};q^{kb})_\infty = (cq^a;q^b)_\infty. \tag{13}$$

Further, we make use of the shorthand notation as employed by both Mao [6,7] and Alwaise et al. [1].

$$(a_1,\ldots,a_k;q)_n := (a_1;q)_n \cdots (a_k;q)_n,$$
$$(a_1,\ldots,a_k;q)_\infty := (a_1;q)_\infty \cdots (a_k;q)_\infty,$$
$$J_b := (q^b;q^b)_\infty,$$
$$J_{a,b} := (q^a,q^{b-a},q^b;q^b)_\infty.$$

We will also use Mao's M_2-rank difference generating function to prove our result Theorem 1.2. Mao proved the following theorem which encapsulates the pertinent rank differences.

Theorem 2.1 (Mao [7]) *We have*

$$\sum_{n\geq 0}(N_2(0,6,n)+N_2(1,6,n)-N_2(2,6,n)-N_2(3,6,n))\,q^n$$
$$= \frac{1}{J_{9,36}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+9n}}{1-q^{18n+3}} + q\frac{J_{6,36}^2 J_{18,36} J_{36}^3}{J_{3,36}^2 J_{9,36} J_{15,36}^2}$$
$$+\frac{J_{6,36}J_{18,36}^2 J_{36}^3}{2qJ_{3,36}^2 J_{9,36} J_{15,36}^2} - \frac{1}{J_{9,36}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+9n-1}}{1+q^{18n}}.$$

Apart from this, an identity of Ramanujan theta function is also used. The Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a,-b,ab;ab)_\infty$$

with $|ab| < 1$ where the equality follows from (and is equivalent to) the Jacobi triple product identity. We will use the following two special cases of the theta function and the function $\chi(q)$ which are defined as

$$\varphi(q) := f(q,q) = (-q,-q,q^2;q^2)_\infty, \tag{14}$$
$$\psi(q) := f(q,q^3) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{15}$$

$$\chi(q) := (-q;q^2)_\infty. \tag{16}$$

The following theta function identity is used in the proof of our main result.

Theorem 2.2 (Baruah and Barman [3]) *We have*

$$\varphi^2(q) + \varphi^2(q^3) = 2\varphi^2(-q^6)\frac{\chi(q)\psi(-q^3)}{\chi(-q)\psi(q^3)}.$$

3 Proof of Theorem 1.2

We denote $d(n) := N_2(0,6,n) + N_2(1,6,n) - N_2(2,6,n) - N_2(3,6,n)$ for simplicity. We will show that the generating function $\sum_{n\geq 0} d(3n+2)q^n$ has strictly positive coefficients for all $n \not\equiv 2 \pmod 3$. We first compute the generating function $\sum_{n\geq 0} d(3n+2)q^n$ using Theorem 2.1.

Proposition 3.1 *We have*

$$\sum_{n\geq 0} d(3n+2)q^n = \frac{1}{qJ_{3,12}}\left(\frac{J_{2,12}J_{6,12}^2J_{12}^3}{2J_{1,12}^2J_{5,12}^2} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{6n^2+3n}}{1+q^{6n}}\right). \tag{17}$$

Proof From the expression in Theorem 2.1, one can see that the first summand is a series in q^{3n}, the second only has q-powers which are 1 modulo 3, and the third and fourth only have q-powers which are 2 modulo 3. Now, including only exponents congruent to 2 modulo 3 in the original generating function, and then letting $q \mapsto q^{\frac{1}{3}}$, we deduce that

$$\begin{aligned}
\sum_{n\geq 0} d(3n+2)q^{3n+2} &= \frac{J_{6,36}J_{18,36}^2J_{36}^3}{2qJ_{3,36}^2J_{9,36}J_{15,36}^2} - \frac{1}{J_{9,36}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+9n-1}}{1+q^{18n}}\\
\implies \sum_{n\geq 0} d(3n+2)q^{3n} &= \frac{J_{6,36}J_{18,36}^2J_{36}^3}{2q^3J_{3,36}^2J_{9,36}J_{15,36}^2} - \frac{1}{J_{9,36}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+9n-3}}{1+q^{18n}}\\
\implies \sum_{n\geq 0} d(3n+2)q^{n} &= \frac{J_{2,12}J_{6,12}^2J_{12}^3}{2qJ_{1,12}^2J_{3,12}J_{5,12}^2} - \frac{1}{J_{3,12}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{6n^2+3n-1}}{1+q^{6n}}.
\end{aligned}$$

This completes the proof of the proposition. □

Remark 3.2 Note that the while there is a q in the denominator of the common factor above, it is canceled because the constant term of the expression inside the parentheses in (17) is zero.

We will also need the following lemma which will tie together the proof.

Lemma 3.3 *We have*

$$\frac{J_{2,12}J_{6,12}^2J_{12}^3}{J_{1,12}^2J_{5,12}^2} = \frac{\varphi^2(q)+\varphi^2(q^3)}{2}.$$

Proof We first write the expression in its constituent q-series and then use (11) to cancel common factors in both numerator and denominator. We find that

$$\begin{aligned}
\frac{J_{2,12}J_{6,12}^2J_{12}^3}{J_{1,12}^2J_{5,12}^2} &= \frac{(q^2,q^{10},q^{12};q^{12})_\infty(q^6,q^6,q^{12};q^{12})_\infty^2(q^{12};q^{12})_\infty^3}{(q,q^{11},q^{12};q^{12})_\infty^2(q^5,q^7,q^{12};q^{12})_\infty^2}\\
&= \frac{(q^2,q^{10};q^{12})_\infty(q^6,q^6,q^{12};q^{12})_\infty^2}{(q,q^7;q^{12})_\infty^2(q^5,q^{11};q^{12})_\infty^2}
\end{aligned}$$

$$= \varphi^2(-q^6)\frac{(q,q^5;q^6)_\infty(-q,-q^5;q^6)_\infty}{(q;q^6)^2_\infty(q^5;q^6)^2_\infty}$$
$$= \varphi^2(-q^6)\frac{(-q,-q^5;q^6)_\infty}{(q,q^5;q^6)_\infty}.$$

We next use (13) to reduce the q-series by multiplying the missing factors in both numerator and denominator, and simplify the expression using (15) and Theorem 2.2 as follows:

$$\begin{aligned}\frac{J_{2,12}J^2_{6,12}J^3_{12}}{J^2_{1,12}J^2_{5,12}} &= \varphi^2(-q^6)\frac{(-q,-q^5;q^6)_\infty}{(q,q^5;q^6)_\infty}\\
&= \varphi^2(-q^6)\frac{(-q,-q^5;q^6)_\infty(-q^3;q^6)_\infty(q^3;q^6)_\infty}{(q,q^5;q^6)_\infty(q^3;q^6)_\infty(-q^3;q^6)_\infty}\\
&= \varphi^2(-q^6)\frac{(-q;q^2)_\infty(q^6;q^6)_\infty(q^3;q^6)_\infty}{(q;q^2)_\infty(-q^3;q^6)_\infty(q^6;q^6)_\infty}\\
&= \varphi^2(-q^6)\frac{\chi(q)\psi(-q^3)}{\chi(-q)\psi(q^3)}\\
&= \frac{\varphi^2(q)+\varphi^2(q^3)}{2}.\end{aligned}$$

□

We now prove our result Theorem 1.2.

Proof of Theorem 1.2 We use Lemma 3.3 and Proposition 3.1, and note that all the exponents of the q-series inside the parentheses in (17) are $0 \pmod 3$. Hence,

$$\begin{aligned}\sum_{n\geq 0} d(3n+2)q^n &= \frac{1}{qJ_{3,12}}\left(\frac{J_{2,12}J^2_{6,12}J^3_{12}}{2J^2_{1,12}J^2_{5,12}} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{6n^2+3n}}{1+q^{6n}}\right)\\
&= \frac{1}{qJ_{3,12}}\left(\frac{\varphi^2(q)+\varphi^2(q^3)}{4} - \frac{1}{2} + \sum_{n\geq 1} a_{3n}q^{3n}\right),\end{aligned}$$

where $a_{3n} \in \mathbb{Z}$.

Now let $3 \nmid n+1$, then

$$\begin{aligned}d(3n+2) &= [q^n]\frac{1}{qJ_{3,12}}\left(\frac{\varphi^2(q)+\varphi^2(q^3)}{4} - \frac{1}{2} + \sum_{n\geq 1} a_{3n}q^{3n}\right)\\
&= [q^{n+1}]\left(\frac{\varphi^2(q)+\varphi^2(q^3)}{4J_{3,12}} - \frac{1}{2J_{3,12}} + \frac{1}{J_{3,12}}\sum_{n\geq 1} a_{3n}q^{3n}\right)\\
&= [q^{n+1}]\frac{\varphi^2(q)+\varphi^2(q^3)}{4J_{3,12}} - [q^{n+1}]\frac{1}{2J_{3,12}} + [q^{n+1}]\frac{1}{J_{3,12}}\sum_{n\geq 1} a_{3n}q^{3n}\\
&= [q^{n+1}]\frac{\varphi^2(q)+\varphi^2(q^3)}{4J_{3,12}}\end{aligned}$$

where $[x^k]f(x)$ denotes the coefficient of x^k in the generating function $f(x)$. It now suffices to show that all coefficients of $\frac{\varphi^2(q)+\varphi^2(q^3)}{J_{3,12}}$ are positive. This follows as

$$\begin{aligned}\frac{\varphi^2(q)+\varphi^2(q^3)}{J_{3,12}} &= \frac{2+4q+4q^2+\sum_{n\geq 3} b_n q^n}{(1-q^3)(q^9,q^{12},q^{15};q^{12})_\infty}\\
&= \left(2+4q+4q^2+\sum_{n\geq 3} b_n q^n\right)\left(\sum_{n\geq 0} q^{3n}\right)\left(1+\sum_{n\geq 1} c_n q^n\right)\end{aligned}$$

where b_i and c_i are non-negative. We can generate q^{3n+k} using the above factors by q^k from first, q^{3n} from second, and 1 from the last, where $k = 0, 1, 2$. Due to the structure of the product, each q power generated in the way described must have a positive coefficient, and so additional terms that arise with the same power would only add to the size of the coefficient. This completes our proof for Theorem 1.2 □

4 Conclusion and remarks

The method employed by Alwaise et al. [1] doesn't work for this inequality because the expression inside the parentheses in Proposition 3.1 does seem to have negative coefficients for an infinite number of coefficients.

This result is limited to $3n + 2$ when $3 \nmid n + 1$, but computational evidence suggests that $\frac{1}{1-q^{12}}\left(\frac{J_{2,12}J_{6,12}^2J_{12}^3}{2J_{1,12}^2J_{5,12}^2} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{6n^2+3n}}{1+q^{6n}}\right)$ has non-negative coefficients, and given the simplification with help of Lemma 3.3, a stronger version of the method used in along with using properties of $\varphi^2(q)$, in which the coefficient of q^n counts number of Diophantine solutions to $a^2 + b^2 = n$ might aid in proving the inequality when $3 \mid n + 1$.

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