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Independent families in Boolean algebras with some separation properties

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Abstract. We prove that any Boolean algebra with the subsequential completeness property contains an independent family of size $\mathfrak{c}$, the size of the continuum. This improves a result of Argyros from the 1980s, which asserted the existence of an uncountable independent family. In fact, we prove it for a bigger class of Boolean algebras satisfying much weaker properties. It follows that the Stone space $K_{\mathcal{A}}$ of all such Boolean algebras $\mathcal{A}$ contains a copy of the Čech–Stone compactification of the integers $\beta\mathbb{N}$ and the Banach space $C(K_{\mathcal{A}})$ has l_∞ as a quotient. Connections with the Grothendieck property in Banach spaces are discussed.

1. Independent families

By an antichain in a Boolean algebra $\mathcal{A}$, we will mean a pairwise disjoint subset of $\mathcal{A}$, i.e., a $\mathcal{B} \subseteq \mathcal{A}$ such that $A \wedge B = 0$ whenever A and B are two distinct elements of $\mathcal{B}$. An independent family in a Boolean algebra $\mathcal{A}$ is a family $\{A_i : i \in I\} \subseteq \mathcal{A}$ such that

$$\bigwedge_{i\in F} A_i \wedge \bigwedge_{i\in G} A_i^{-1} \neq 0$$

for any two disjoint finite subsets $F, G \subseteq I$. For more information on Boolean algebras, see [8]. $K_{\mathcal{A}}$ will stand for the Stone space of $\mathcal{A}$ and $C(K)$ for the Banach space of real valued continuous functions on K. $[A]$ will stand for the clopen subset $\{x \in K_{\mathcal{A}} : A \in x\}$ for any A in a Boolean algebra $\mathcal{A}$, and 1_X for the characteristic function of $X \subseteq K_{\mathcal{A}}$. Unexplained notions concerning Banach spaces can be found in [2]. We will consider several separation properties in Boolean algebras.

Definition 1.1. A Boolean algebra $\mathcal{A}$ is said to have the *weak subsequential separation property* if for any countably infinite antichain in $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$, there exists $A \in \mathcal{A}$ such that both of the sets

$$\{n \in \mathbb{N} : A_n \leq A\} \quad \text{and} \quad \{n \in \mathbb{N} : A_n \wedge A = 0\}$$

are infinite.

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This is a natural generalization of the following well-known concept ([5]).

Definition 1.2. A Boolean algebra $\mathcal{A}$ is said to have the *subsequential completeness property* if given any countably infinite antichain $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$, there exists an infinite $M \subseteq \mathbb{N}$ such that in $\mathcal{A}$, the supremum of $\{A_n : n \in M\}$ exists.

One more separation property introduced in [6] is the following.

Definition 1.3. A Boolean algebra $\mathcal{A}$ is said to have the *subsequential separation property* if given any countably infinite antichain $\{A_n \ : \ n \in \mathbb{N}\} \subseteq \mathcal{A}$, there is an $A \in \mathcal{A}$ such that the set

$$\{n \in \mathbb{N} : A_{2n} \leq A \text{ and } A_{2n+1} \wedge A = 0\}$$

is infinite.

The subsequential completeness property was introduced in [5]. That paper included applications of the subsequential completeness property in the theory of Banach spaces as well as a result of Argyros (Proposition 1G) that every Boolean algebra with the subsequential completeness property has an uncountable independent family. Here we strengthen this result.

Theorem 1.4. *Suppose that $\mathcal{A}$ is an infinite Boolean algebra that has a weak subsequential separation property. Then $\mathcal{A}$ contains an independent family of cardinality $\mathfrak{c}$.*

Proof. Clearly, the Stone space of $\mathcal{A}$ cannot have a nontrivial convergent sequence and so cannot be a dispersed space; consequently, $\mathcal{A}$ is not superatomic and so contains a countable free infinite Boolean algebra $\mathcal{B} \subseteq \mathcal{A}$.

Let $\{0,1\}^{<\mathbb{N}}$ stand for finite sequences of 0s and 1s. A countable free Boolean algebra contains all countable Boolean algebras as subalgebras. We will identify a particular subalgebra $\mathcal{C} \subseteq \mathcal{A}$ with generators $\{A_s : s \in \{0,1\}^{<\mathbb{N}}\}$ such that whenever $\mathcal{D}$ is a countable free Boolean algebra generated by free generators $\{D_s : s \in \{0,1\}^{<\mathbb{N}}\}$, there is a surjective homomorphism $h\colon \mathcal{D} \to \mathcal{C}$ such that $h(D_s) = A_s$ and the kernel of h is generated by the elements of the form $A_s \wedge A_t$ such that $s \subsetneq t$.

In particular, it follows that whenever $s \subsetneq t$, then

$$A_s \wedge A_t = 0. \tag{$*$}$$

Now we note that if n is an integer and $u_i \in \{0,1\}^{<\mathbb{N}}$ for $1 \leq i \leq n$, then

$$A_{u_1} \wedge \cdots \wedge A_{u_n} \neq 0, \tag{$**$}$$

unless $u_i \subseteq u_j$ for some distinct $1 \leq i, j \leq n$. Indeed, suppose that $u_i \not\subseteq u_j$ for all distinct $1 \leq i, j \leq n$. Consider an arbitrary element of the kernel of h; it may be assumed to be of the form

$$\bigvee_{1 \leq i \leq k} (D_{s_i} \wedge D_{t_i})$$

for some $k \in \mathbb{N}$ and $s_i, t_i \in \{0,1\}^{<\mathbb{N}}$ satisfying $s_i \subsetneq t_i$ for all $1 \leq i \leq k$. Note that $\{u_1, \ldots, u_n\} \cap \{s_j, t_j\}$ has at most one element for each $1 \leq j \leq k$. Let $v_j \in \{s_j, t_j\}$ be the other element for $1 \leq j \leq k$. So $\{v_1, \ldots, v_k\}$ and $\{u_1, \ldots, u_n\}$ are disjoint, and by the independence of the generators we have

$$0 \neq \bigwedge_{1 \leq i \leq k} D_{v_i}^{-1} \wedge \bigwedge_{1 \leq i \leq n} D_{u_i}.$$

So

$$0 \neq \bigwedge_{1 \leq i \leq k} (D_{s_i}^{-1} \vee D_{t_i}^{-1}) \wedge \bigwedge_{1 \leq i \leq n} D_{u_i}$$

and hence

$$\bigwedge_{1 \leq i \leq n} D_{u_i} \not\leq \bigvee_{1 \leq i \leq k} (D_{s_i} \wedge D_{t_i}),$$

as required for $(**)$. Now by $(*)$ for every $x \in 2^{\mathbb{N}}$, let us consider an antichain $\{A_s : s \subseteq x\}$. By the subsequential separation property for each $x \in 2^{\mathbb{N}}$, there are $A_x \in \mathcal{A}$ as in Definition 1.1. We will show that $\{A_x : x \in 2^{\mathbb{N}}\}$ is the required independent family. Let $x_1, \ldots, x_n$ be distinct elements of $2^{\mathbb{N}}$ and let $F \subseteq \{1, \ldots, n\}$ and $G = \{1, \ldots, n\} \setminus F$. Let $m \in \mathbb{N}$ be such that $u_i = x_i|m$ are all distinct (and so none is included in the other). For $i \leq n$, let $m_i \geq m$ be such that $A_{x_i|m_i} \leq A_{x_i}$ if $i \in F$ and $A_{x_i|m_i} \wedge A_{x_i} = 0$ if $i \in G$. The sequences $x_1|m_1, \ldots, x_n|m_n$ satisfy the hypothesis of $(**)$ for $u_1, \ldots, u_n$, so

$$0 \neq \bigwedge_{1 \leq i \leq n} A_{x_i|m_i} \leq \bigwedge_{i \in F} A_{x_i} \wedge \bigwedge_{i \in G} A_{x_i}^{-1}$$

as required for the independence of $\{A_x : x \in 2^{\mathbb{N}}\}$. □

In the literature, there are several more weakenings of the subsequential completeness property that are stronger than our weak subsequential separation property, most notably the subsequential interpolation property introduced in [3] and applied in several other papers and the subsequential separation property of [6]. Hence, the above theorem applies to the algebras with these properties as well.

Corollary 1.5. *If $\mathcal{A}$ is a Boolean algebra having a subsequential separation property and $K_{\mathcal{A}}$ is its Stone space, then $\beta\mathbb{N}$ is a subspace of $K_{\mathcal{A}}$ and l_∞ is a quotient of $C(K_{\mathcal{A}})$.*

Proof. It is well known that a free Boolean algebra with $\mathfrak{c}$ generators maps homeomorphically onto any Boolean algebra of cardinality $\mathfrak{c}$, and so, for example, onto $\wp(\mathbb{N})$. Use the Sikorski extension theorem to obtain a homeomorphism of $\mathcal{A}$ onto $\wp(\mathbb{N})$. By the Stone duality, it follows that the Stone space of $\wp(\mathbb{N})$, which is homeomorphic to $\beta\mathbb{N}$, is a subspace of $K_{\mathcal{A}}$. Restricting continuous functions on $K_{\mathcal{A}}$ to a copy of $\beta\mathbb{N}$ gives, by the Tietze extension theorem, a norm one linear operator onto $C(\beta\mathbb{N})$, which is known to be isometric to l_∞. □

It follows that many Banach spaces present in the literature have l_∞ as a quotient, in particular the spaces of [5] or [9]. In [9], besides Boolean separations a lattice version of the subsequential completeness property is considered for connected compact K. Namely (see [9, 5.1]) we consider spaces K such that given any pairwise disjoint ($f_n \cdot f_m = 0$) sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions $f_n\colon K \to [0,1]$, there is an infinite $M \subseteq \mathbb{N}$ such that in $C(K)$ the supremum of $(f_n)_{n\in M}$ exists. It is not difficult to generalize the proof of Theorem 1.4 to conclude that such K always contain $\beta\mathbb{N}$ as well.

2. The Grothendieck property of Banach spaces

In this section, we would like to direct the attention of the reader to some links between the weak subsequential separation property and the Grothendieck property, which originated in the theory of Banach spaces.

Definition 2.1. Let X be a Banach space. We say that X has the *Grothendieck property* if and only if weak* convergence of sequences in X^* is equivalent to weak convergence. A Boolean algebra $\mathcal{A}$ has the Grothendieck property if and only if the Banach space $C(K_\mathcal{A})$ has the Grothendieck property.

The Grothendieck property for Boolean algebras was introduced and first investigated by Schachermayer in [10]. It can be quite nicely characterized using finitely additive signed measures on Boolean algebras. Recall that the Riesz representation theorem says that the dual to a $C(K)$ space is isometric to the space of Radon measures on K with the variation norm, i.e., all continuous functionals on $C(K)$ are of the form $\int f d\mu$ for μ a Radon measure on K. A Radon measure means a signed, countably additive, Borel, regular measure. If K is totally disconnected (has a basis of clopen sets), then any finitely additive, bounded, signed measure on Clop($\mathcal{A}$) extends uniquely to a Radon measure on K. So there is a one to one correspondence between such measures and elements of the dual Banach space to $C(K_\mathcal{A})$. For more information on this, see [11].

Lemma 2.2. *Suppose $\mathcal{A}$ is a Boolean algebra, $K_\mathcal{A}$ its Stone space, and $C(K_\mathcal{A})$ the Banach space of all real-valued continuous functions on $K_\mathcal{A}$ with the supremum norm. $C(K_\mathcal{A})$ has the Grothendieck property if and only if whenever*

- $\{A_n : n \in \mathbb{N}\}$ *is an antichain of $\mathcal{A}$,*
- $\varepsilon > 0$,
- μ_n *is a bounded sequence of bounded, finitely additive signed measures on $\mathcal{A}$ such that $|\mu_n(A_n)| > \varepsilon$,*

there is $A \in \mathcal{A}$ such that

$$(\mu_n(A) : n \in \mathbb{N})$$

is not a convergent sequence of the reals.

Proof. Suppose that $C(K_{\mathcal{A}})$ has the Grothendieck property and A_n, ε, and μ_n are as above. Let x_n^* be elements of the dual to $C(K_{\mathcal{A}})$ that are uniquely determined by the condition $x_n^*(1_{[A]}) = \mu_n(A)$ for $A \in \mathcal{A}$. Using the Rosenthal lemma (see, for example, [2]) going to a subsequence, we may assume that

$$\sum_{n \in \mathbb{N} \setminus \{k\}} |\mu_k(A_n)| < \varepsilon/3$$

holds for every $k \in \mathbb{N}$. It follows that $|\int 1_{\bigcup_{n\in\mathbb{N}}[A_{2n}]} dx_k^*| < \varepsilon/3$ if k is an odd integer and $|\int 1_{\bigcup_{n\in\mathbb{N}}[A_{2n}]} dx_k^*| > 2\varepsilon/3$ if k is an even integer. In other words, the element of the bidual to $C(K_{\mathcal{A}})$ corresponding to the Borel set $\bigcup_{n\in\mathbb{N}}[A_{2n}]$ of $K_{\mathcal{A}}$ witnesses the fact that $(x_n^*)_{n\in\mathbb{N}}$ is not weakly convergent. By the Grothendieck property, it is not weakly* convergent. As $(x_n^*)_{n\in\mathbb{N}}$ is bounded, it means that $(\mu_n(A) : n \in \mathbb{N})$ is not a convergent sequence of the reals for some $A \in \mathcal{A}$ since the span of characteristic functions of clopen sets is dense in $C(K_{\mathcal{A}})$ by the Stone–Weierstrass theorem.

For the converse implication, suppose that $(x_n^*)_{n\in\mathbb{N}}$ is a bounded sequence in the dual to $C(K_{\mathcal{A}})$ that is not weakly convergent; assume the condition of the lemma, and conclude that $(x_n^*)_{n\in\mathbb{N}}$ is not weakly* convergent. $(x_n^*)_{n\in\mathbb{N}}$, as a nonconvergent sequence in a compact dual ball, has at least two distinct accumulation points. If every sequence of $(x_n^*)_{n\in\mathbb{N}}$ contains a weakly convergent subsequence (and so weakly* convergent), then we conclude that $(x_n^*)_{n\in\mathbb{N}}$ is not weakly* convergent, as required. Otherwise, by choosing a subsequence of $(x_n^*)_{n\in\mathbb{N}}$, we may assume that it does not have any weakly convergent subsequence. So by the Eberlein–Smulian theorem, $\{x_n^* : n \in \mathbb{N}\}$ is not relatively weakly compact, and hence by the Grothendieck–Dieudonne characterization of weakly compact subsets of the duals to Banach spaces $C(K)$, we obtain an antichain $\{A_n : n \in \mathbb{N}\}$ of $\mathcal{A}$, an $\varepsilon > 0$, and an infinite $M \subseteq \mathbb{N}$ such that such that $|\nu_n([A_n])| > \varepsilon$ where ν_n is the Radon measure on $K_{\mathcal{A}}$ corresponding to x_n^*. Note that the restriction μ_n of ν_n to the family of all characteristic functions of clopen subsets of $K_{\mathcal{A}}$ satisfies the condition of the lemma; hence, there is an $A \in \mathcal{A}$ as stated there. The function $1_{[A]}$ witnesses the fact that $(x_n^*)_{n\in\mathbb{N}}$ is not weak* convergent, as required. □

It is proved by Haydon in [6, 6.3] that Boolean algebras with the subsequential separation property have the Grothendieck property.

Definition 2.3. We say that a Boolean algebra $\mathcal{A}$ a *positive Grothendieck property* if and only if whenever

- $\{A_n : n \in \mathbb{N}\}$ is an antichain of $\mathcal{A}$,
- $\varepsilon > 0$,
- μ_n is a bounded sequence of bounded, finitely additive non-negative measures on $\mathcal{A}$ such that $\mu_n(A_n) > \varepsilon$,

there is $A \in \mathcal{A}$ such that $(\mu_n(A) : n \in \mathbb{N})$ is not a convergent sequence of the reals.

Proposition 2.4. *If a Boolean algebra $\mathcal{A}$ has a weakly subsequential separation property, then $C(K_{\mathcal{A}})$ has the positive Grothendieck property.*

Proof. Let $\{A_n : n \in \mathbb{N}\}$, $\varepsilon > 0$ and μ_n, $n \in \mathbb{N}$, be as in Definition 2.3. Applying [12, Lemmas 1 and 2] (see also [6, 6.2 and 6.3]), we may assume that there is an antichain $\{B_n : n \in \mathbb{N}\}$ and finitely additive bounded measures λ_n and ν_n such that $\mu_n = \lambda_n + \nu_n$, where ν_n weakly converges, $\lambda_n(K) = \eta$, and $\lambda_n(B_n) > 3\eta/4$. By the Dieudonne–Grothendieck theorem applied to the μ_n, we conclude that they do not form a relatively weakly compact set, and hence are not a weakly convergent sequence; hence, $\eta \neq 0$.

Now use the weak subsequential separation property to obtain $A \in \mathcal{A}$ such that both of the sets $M_1 = \{n \in \mathbb{N} : B_n \leq A\}$ and $M_0 = \{n \in \mathbb{N} : B_n \wedge A = 0\}$ are infinite. For each $n \in M_1$, we have

$$\mu_n(A) = \lambda_n(A \cap B_n) + \lambda_n(A \setminus B_n) + \nu_n(A) > 3\eta/4 - \eta/4 + \nu_n(A) = \eta/2 + \nu_n(A),$$

and for each $n \in M_0$, we have

$$\mu_n(A) = \lambda_n(A \cap B_n) + \lambda_n(A \setminus B_n) + \nu_n(A) < \eta/4 + \nu_n(A).$$

As $\nu_n(A)$ converges, $\eta \neq 0$; since weakly convergent sequences are weakly* convergent, we conclude that $\mu_n(A)$ does not converge, as required. □

Proposition 2.5. *There is a Boolean algebra with the weak subsequential separation property that does not have the Grothendieck property and so does not have the subsequential separation property.*

Proof. This is a classical example $\mathcal{A}$ (see, for example, [10]) of the Boolean algebra of all subsets M of $\mathbb{N}$ such that $2k \in M$ if and only if $2k+1 \in M$ for all but finitely many $k \in \mathbb{N}$. It is well known that $\mu_n = \delta_{2n} - \delta_{2n+1}$ form a weakly* convergent sequence in $C(K_{\mathcal{A}})$ that is not weakly convergent and so $\mathcal{A}$ does not have the Grothendieck property. On the other hand, given an antichain $\{A_n : n \in \mathbb{N}\}$ in $\mathcal{A}$, there is an infinite $M \subseteq \mathbb{N}$ such that there are pairwise disjoint $B_n \in \mathcal{A}$ with $A_n \subseteq B_n$ for $n \in M$ such that $2k \in B_n$ if and only if $2k+1 \in B_n$ for all $k \in \mathbb{N}$ and al $n \in M$. Infinite unions of such B_n provide elements witnessing the separation. □

The Grothendieck property of $C(K_{\mathcal{A}})$ does not imply in ZFC the existence of an independent family of cardinality $\mathfrak{c}$ in $\mathcal{A}$. Namely, assuming the continuum hypothesis, Talagrand proved in [12] that there is a Boolean algebra $\mathcal{A}$ such that $C(K_{\mathcal{A}})$ has the Grothendieck property but l_∞ is not a quotient of $C(K_{\mathcal{A}})$. In particular, $\beta\mathbb{N}$ is not a subset of $K_{\mathcal{A}}$ and so $\mathcal{A}$ has no uncountable independent family. Moreover it is proved in [1] that it is consistent that there is a Boolean algebra $\mathcal{A}$ that has the Grothendieck property but has cardinality strictly smaller than 2^ω (ground model after adding Sacks reals). On the other hand, assuming $\mathfrak{p} = 2^\omega$ Haydon, Levy, and Odell proved in [7] that each nonreflexive Banach space with the Grothendieck property (in particular, each of the form $C(K_{\mathcal{A}})$ for $\mathcal{A}$ infinite) has l_∞ as a quotient. However, we do not know the answer to the following.

Question 2.6. *Is it consistent that each Boolean algebra with the Grothendieck property has an independent family of cardinality* $\mathfrak{c}$*?*

3. Efimov's problem

The affirmative answer to the question above could be considered a weak solution to the Efimov problem (see [4]) whether it is consistent that any compact K without a nontrivial convergent sequence has a subspace homeomorphic to $\beta\mathbb{N}$ or there is in ZFC a compact space without a nontrivial convergent sequence and without a copy of $\beta\mathbb{N}$. Indeed, subsets of a compact space K can be considered as subsets of the dual ball to the Banach space $C(K)$ or Radon measures (points of K correspond to pointwise measures) with the weak* topology. And so, instead of nontrivial convergent sequences or copies of $\beta\mathbb{N}$ in K, one can ask for the same subspaces in the dual ball or the Radon measures. The Grothendieck property of $C(K)$, in a sense, asserts that there are no nontrivial convergent sequences among Radon measures (not just pointwise measures, here nontrivial means those which are not convergent in the weak topology, or those which can be separated by a Borel subset of the compact space, the notion changes as the dual ball always contains copies of intervals) and it easily implies the nonexistence of nontrivial (in the sense of having distinct terms) sequences of points of K. Of course, the negative answer to the above question would solve the original Efimov problem.

Note that Efimov's problem is equivalent to asking if an analogous property to our weak subsequential separation property for points of the Stone space instead of elements of the Boolean algebra implies the existence of an independent family of cardinality $\mathfrak{c}$.

Also, Talagrand proved (see [12]) that the dual ball to $C(K)$ contains a copy of $\beta\mathbb{N}$ if and only if ℓ_∞ is a quotient of $C(K)$. So another weak version of the Efimov problem would be to ask if it is consistent that whenever K has no convergent sequence, ℓ_∞ is a quotient of $C(K)$. One should note here that consistently it is not the case (again the example of [12]) and that the result of [7] gives the consistency of the statement that for every compact K, the dual ball to $C(K)$ with the weak* topology either contains a copy of $\beta\mathbb{N}$ or a convergent sequence that is not weakly convergent.

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