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Baxter Operator and Archimedean Hecke Algebra

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Abstract: In this paper we introduce Baxter integral $\mathcal{Q}$-operators for finite-dimensional Lie algebras $\mathfrak{gl}_{\ell+1}$ and $\mathfrak{so}_{2\ell+1}$. Whittaker functions corresponding to these algebras are eigenfunctions of the $\mathcal{Q}$-operators with the eigenvalues expressed in terms of Gamma-functions. The appearance of the Gamma-functions is one of the manifestations of an interesting connection between Mellin-Barnes and Givental integral representations of Whittaker functions, which are in a sense dual to each other. We define a dual Baxter operator and derive a family of mixed Mellin-Barnes-Givental integral representations. Givental and Mellin-Barnes integral representations are used to provide a short proof of the Friedberg-Bump and Bump conjectures for $G = GL(\ell+1)$ proved earlier by Stade. We also identify eigenvalues of the Baxter $\mathcal{Q}$-operator acting on Whittaker functions with local Archimedean L-factors. The Baxter $\mathcal{Q}$-operator introduced in this paper is then described as a particular realization of the explicitly defined universal Baxter operator in the spherical Hecke algebra $\mathcal{H}(G(\mathbb{R}), K)$, K being a maximal compact subgroup of G. Finally we stress an analogy between $\mathcal{Q}$-operators and certain elements of the non-Archimedean Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p))$.

1. Introduction

The notion of the $\mathcal{Q}$-operator was introduced by Baxter as an important tool to solve quantum integrable systems [Ba]. These operators were initially constructed for a particular class of quantum integrable systems associated with affine Lie algebras $\widehat{\mathfrak{gl}}_{\ell+1}$ and its quantum/elliptic generalizations. A new class of integral $\mathcal{Q}$-operators corresponding to the $\widehat{\mathfrak{gl}}_{\ell+1}$-Toda chain was later proposed by Pasquier and Gaudin [PG]. Its generalization to Toda chains for other classical affine Lie algebras was proposed recently in [GLO1,GLO2,GLO3].

In this paper we introduce integral Baxter $\mathcal{Q}$-operators for Toda chains corresponding to the finite-dimensional classical Lie algebras $\mathfrak{gl}_{\ell+1}$ and $\mathfrak{so}_{2\ell+1}$. These integral operators are closely related with the recursion operators in the Givental integral representation of

Whittaker functions (see [Gi,JK] for $\mathfrak{gl}_{\ell+1}$ and [GLO3] for other classical Lie algebras). It is well known that $\mathfrak{g}$-Whittaker functions are common eigenfunctions of the complete set of mutually commuting $\mathfrak{g}$-Toda chain quantum Hamiltonians. The quantum Hamiltonians arise as projections of the generators of the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. One of the characteristic properties of the introduced Baxter integral operators for a finite-dimensional classical Lie algebra $\mathfrak{g}$ is that the corresponding $\mathfrak{g}$-Whittaker functions are their eigenfunctions. Moreover, integral $\mathcal{Q}$-operators provide a complete set of integral equations defining $\mathfrak{g}$-Whittaker functions. Similarly to the relation of the Hamiltonians with the generators of the center $\mathcal{Z}$, we construct universal Baxter operators in a spherical Hecke algebra whose projection gives the Baxter operator for Toda chains. Other projections provide Baxter operators for other quantum integrable systems (e.g. Sutherland models).

The eigenvalues of the Baxter operators acting on Whittaker functions are expressed in terms of a product of Gamma-functions. The appearance of the Gamma-functions implies a close connection between Givental and Mellin-Barnes integral representations [KL1] for $\mathfrak{gl}_{\ell+1}$-Whittaker functions. We discuss this relation in some detail. Note that the representation theory interpretation [GKL] of the Mellin-Barnes integral representation uses the Gelfand-Zetlin construction of the maximal commutative subalgebra of $\mathcal{U}(\mathfrak{gl}_{\ell+1})$. One can guess a connection between Mellin-Barnes and Givental representations on a general ground by noticing that Givental diagrams for classical Lie algebras [GLO3] are identical to Gelfand-Zetlin patterns [BZ]. Moreover both constructions are most natural for classical Lie algebras. In this note we discuss a duality relation between recursive structures of Givental and Mellin-Barnes integral representations. We construct a dual version of the Baxter $\mathcal{Q}$-operator and derive a set of relations between recursive/Baxter operators and their duals. We also propose a family of mixed Mellin-Barnes-Givental integral representations interpolating between Mellin-Barnes and Givental integral representations of Whittaker functions.

We use the Mellin-Barnes integral representation to give simple proofs of Bump-Friedberg and Bump conjectures on Archimedean factors arising in the application of the Rankin-Selberg method to analytic continuations of $GL(\ell+1)\times GL(\ell+1)$ and $GL(\ell+1)\times GL(\ell)$ automorphic L-functions. We also discuss a relation with the proofs given by Stade [St1,St2]. The proof in [St1,St2] is based on a recursive construction of $\mathfrak{gl}_{\ell+1}$-Whittaker functions generalizing the construction due to Vinogradov and Takhtajan [VT]. As it was noticed in [GKLO] and is explicitly demonstrated below, the Stade recursion basically coincides with the Givental recursion (see also the recent detailed discussion in [St3]). We also show that the Bump-Friedberg and Bump conjectures are simple consequences of the Mellin-Barnes integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker function.

The Rankin-Selberg method is a powerful tool of studying analytic properties of automorphic L-functions. The application of the Baxter $\mathcal{Q}$-operators and closely related recursive operators to a derivation of analytic properties of L-functions using the Rankin-Selberg method is not accidental. We remark that the eigenvalues of the $\mathcal{Q}$-operators acting on $\mathfrak{g}$-Whittaker functions are given by Archimedean local L-factors and the integral $\mathcal{Q}$-operators should be naturally considered as elements of the Archimedean Hecke algebra $\mathcal{H}(G(\mathbb{R}),K)$, K being a maximal compact subgroup of G. We construct the corresponding universal Baxter operator as an element of the spherical Hecke algebra $\mathcal{H}(G(\mathbb{R}),K)$. We also describe non-Archimedean counterparts of the universal Baxter operators as elements of non-Archimedean Hecke algebras $\mathcal{H}(GL(\ell+1,\mathbb{Q}_p),GL(\ell+1,\mathbb{Z}_p))$. The consideration of Archimedean and non-Archimedean universal $\mathcal{Q}$-operators on an equal footing provides a uniform description of the automorphic forms as their common eigenfunctions (replacing the more traditional approach based on the algebra of the invariant differential operators as a substitute of $\mathcal{H}(G(\mathbb{R}), K)$).

Let us note that the connection of the Baxter operators with Archimedean L-factors implies in particular that there is a hidden parameter in the $\mathcal{Q}$-operator corresponding to a choice of a finite-dimensional representation of the Langlands dual Lie algebra. In this sense, $\mathcal{Q}$-operators considered in this paper correspond to standard representations of the classical Lie algebras. We are going to consider the $\mathcal{Q}$-operators corresponding to more general representations in a separate publication.

One should stress that there are various Hecke algebras relevant to the study of the quantum Toda chains. For example for $B \subset G$ being a Borel subgroup, the Hecke algebra $\mathcal{H}(G, B)$ of B-biinvariant functions is closely related to the scattering data of quantum Toda chains [STS]. The Hecke algebra $\mathcal{H}(G, N)$, N being the unipotent radical of B also deserves consideration. Note that the representations of $\mathcal{H}(G, N)$ contain certain information about the scattering data of the theory and its center is isomorphic to $\mathcal{H}(G, K)$.

Finally let us remark that the constructions of affine integral $\mathcal{Q}$-operators and their eigenvalues for the action on Whittaker functions [PG] together with the considerations of this paper imply an intriguing possibility to interpret the eigenvalues of affine $\mathcal{Q}$-operators as a kind of local Archimedean L-factors. It is natural to expect that these L-factors should be connected with 2-dimensional local fields in the sense of Parshin [Pa]. We are going to discuss this fascinating possibility elsewhere.

The plan of this paper is as follows. In Sect. 2 we recall the Givental integral representation for $\mathfrak{gl}_{\ell+1}$ and introduce the Baxter $\mathcal{Q}$-operator for $\mathfrak{gl}_{\ell+1}$. In Sect. 3 we consider the relation between Givental and Mellin-Barnes integral representations of $\mathfrak{gl}_{\ell+1}$-Whittaker functions and introduce the dual Baxter operator. In Sect. 4 we use the Mellin-Barnes integral representation to prove the Bump-Friedberg and Bump conjecture and discuss the relation with [St1,St2]. In Sect. 5 we identify eigenvalues of the Baxter $\mathcal{Q}$-operator with local Archimedean L-factors and construct universal Baxter operators as elements of the spherical Hecke algebra $\mathcal{H}(G(\mathbb{R}), K)$. The main result of this paper is given in Theorem 5.1. We also discuss an analogy between $\mathcal{Q}$-operators and certain elements of the non-Archimedean Hecke algebra $\mathcal{H}(GL(\ell+1, \mathbb{Q}_p), GL(\ell+1, \mathbb{Z}_p))$. Finally in Sect. 6 a generalization to $\mathfrak{so}(2\ell+1)$ is given.

2. Baxter Operator for $\mathfrak{gl}_{\ell+1}$

2.1. Whittaker functions as matrix elements. Let us recall two constructions of $\mathfrak{g}$-Whittaker functions as matrix elements of infinite-dimensional representations of $\mathcal{U}(\mathfrak{g})$ and a relation of $\mathfrak{g}$-Whittaker functions with eigenfunctions of $\mathfrak{g}$-Toda quantum chains.

Let us first describe the construction based on the Gauss decomposition. According to Kostant [Ko1,Ko2], $\mathfrak{gl}_{\ell+1}$-Whittaker function can be defined as a certain matrix element in a principal series representation of $G = GL(\ell+1, \mathbb{R})$. Let $\mathcal{U}(\mathfrak{g})$ be a universal enveloping algebra of $\mathfrak{g} = \mathfrak{gl}_{\ell+1}$ and V, V' be $\mathcal{U}(\mathfrak{g})$-modules, dual with respect to a non-degenerate invariant pairing $\langle . , . \rangle : V' \times V \to \mathbb{C}$, $\langle v', Xv \rangle = -\langle Xv', v \rangle$ for all $v \in V$, $v' \in V'$ and $X \in \mathfrak{g}$. Let $B_- = N_- AM$ and $B_+ = AMN_+$ be Langlands decompositions of opposite Borel subgroups. Here $N_\pm$ are unipotent radicals of $B_\pm$, A is the identity component of the vector Cartan subgroup and M is the intersection of the centralizer of the vector Cartan subalgebra with the maximal compact subgroup $K \subset G$. We will assume that the actions of the Borel subalgebras $\mathfrak{b}_+ = \mathrm{Lie}(B_+)$ on V and $\mathfrak{b}_- = \mathrm{Lie}(B_-)$

on V' are integrated to the actions of the corresponding subgroups. Let $\chi_{\pm}:\mathfrak{n}_{\pm}\to\mathbb{C}$ be the characters of $\mathfrak{n}_{\pm}$ defined by $\chi_+(e_i):=-1$ and $\chi_-(f_i):=-1$ for all $i=1,\ldots,\ell$, where e_i, f_i are generators of $\mathfrak{n}_+$ and $\mathfrak{n}_-$ which correspond to the simple roots. A vector $\psi_R\in V$ is called a Whittaker vector with respect to χ_+ if

$$e_i\psi_R=-\psi_R,\quad i=1,\ldots,\ell, \tag{2.1}$$

and a vector $\psi_L\in V'$ is called a Whittaker vector with respect to χ_- if

$$f_i\psi_L=-\psi_L\,,\quad i=1,\ldots,\ell. \tag{2.2}$$

One defines a Whittaker model $\mathcal{V}$ as a space of functions on G such that $f(ng)=\chi_{N_+}(n)f(g), n\in N_+, \chi_{N_+}(n)=\chi_+(\log n)$. The $\mathcal{U}(\mathfrak{g})$-module admits a Whittaker model with respect to the character χ if it is equivalent to a sub-representation of $\mathcal{V}$.

Let $\mathcal{V}_{\underline{\lambda}}=\mathrm{Ind}_B^G\chi_{\underline{\lambda}}$ be a principal series representation of G induced from the generic character $\chi_{\underline{\lambda}}$ of B trivial on $N\subset B$ with $\underline{\lambda}=(\lambda_1,\ldots,\lambda_{\ell+1})$. It is realized in the space of functions $f\in C^\infty(G)$ satisfying equation

$$f(bg)=\chi_{\underline{\lambda}}(b)f(g),$$

where $b\in B$. The action of G is given by the right action $\pi_{\underline{\lambda}}(g)f(x)=f(xg^{-1})$. We will be interested in the infinitesimal form $\mathrm{Ind}_{\mathrm{U}(\mathfrak{b})}^{\mathrm{U}(\mathfrak{g})}\chi_{\underline{\lambda}}$ of this representation given by

$$(Xf)(g)=\frac{d}{dt}f(ge^{-tX})|_{t\to0}.$$

Define the $(\mathfrak{g},B)$-module as a $\mathfrak{g}$-module such that the action of the Borel subalgebra $\mathfrak{b}\subset\mathfrak{g}$ is integrated to the action of the Borel subgroup B, $\mathfrak{b}=\mathrm{Lie}(B)$. Consider an irreducible $(\mathfrak{g},B)$-submodule $\mathcal{V}_{\underline{\lambda}}^{(0)}$ of $\mathcal{V}_{\underline{\lambda}}$ given by the Schwartz space $\mathcal{S}(N_-)$ of functions on N_- exponentially decreasing at infinity with all their derivatives. This $(\mathfrak{g},B)$-module always admits a Whittaker model. Below we will denote by ψ_L,ψ_R the Whittaker vectors in $\mathcal{V}_{\underline{\lambda}}^{(0)}$ and its dual. Following Kostant [Ko1,Ko2] (see also [Et] for a recent discussion) we define a $\mathfrak{g}$-Whittaker function in terms of the invariant pairing of Whittaker modules as follows:

$$\Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(x)=e^{-\langle\rho,x\rangle}\langle\psi_L\,,\pi_{\underline{\lambda}}(e^{h_x})\,\psi_R\rangle,\quad x\in\mathfrak{h}, \tag{2.3}$$

where $h_x:=\sum_{i=1}^{\ell+1}\langle\omega_i,x\rangle\,h_i,\omega_i$ is a basis of fundamental weights of $\mathfrak{g}$, $\rho=1/2\sum_{\alpha>0}\alpha$ and $\pi_{\underline{\lambda}}(e^{h_x})$ is an action of e^{h_x} in the representation $\mathcal{V}_{\underline{\lambda}}$. It was shown in [Ko1] that $\mathfrak{g}$-Whittaker function is a common eigenfunction of a complete set of commuting Hamiltonians of the $\mathfrak{g}$-Toda chain. A complete set of commuting Hamiltonians of the $\mathfrak{g}$-Toda chain is generated by the differential operators $\mathcal{H}_k\in\mathrm{Diff}(\mathfrak{h}), k=1,\ldots,\ell+1$ on the Cartan subalgebra $\mathfrak{h}$ defined in terms of the generators $\{c_k\}$ of the center $\mathcal{Z}(\mathfrak{g})\subset\mathcal{U}(\mathfrak{g})$ as follows:

$$\mathcal{H}_k\Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(x)=e^{-\langle\rho,x\rangle}\langle\psi_L\,,\pi_{\underline{\lambda}}(e^{h_x})\,c_k\,\psi_R\rangle. \tag{2.4}$$

More explicitly one has

$$\Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x})=e^{-\sum_{i=1}^{\ell+1}x_i\rho_i}\langle\psi_L\,,\pi_{\underline{\lambda}}(e^{\sum_{i=1}^{\ell+1}x_iE_{i,i}})\,\psi_R\rangle, \tag{2.5}$$

where $\rho_j = \frac{\ell}{2} + 1 - j$, $j = 1, \ldots, \ell+1$ are the components of ρ in the standard basis of $\mathbb{R}^{\ell+1}$, $\underline{x} = (x_1, \ldots, x_{\ell+1})$ and $E_{i,j}$ are the standard generators of $\mathcal{U}(\mathfrak{gl}_{\ell+1})$. The linear and quadratic Hamiltonians in this case are given by

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}} = -\imath \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_i}, \tag{2.6}$$

$$\tilde{\mathcal{H}}_2^{\mathfrak{gl}_{\ell+1}} = -\frac{1}{2} \sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial {x_i}^2} + \sum_{i=1}^{\ell} e^{x_i - x_{i+1}}. \tag{2.7}$$

Let us introduce a generating function for $\mathfrak{gl}_{\ell+1}$-Toda chain Hamiltonians as

$$t^{\mathfrak{gl}_{\ell+1}}(\lambda) = \sum_{j=1}^{\ell+1} (-1)^j \lambda^{\ell+1-j} \mathcal{H}_j^{\mathfrak{gl}_{\ell+1}}(x, \partial_x), \tag{2.8}$$

where $\tilde{\mathcal{H}}_2^{\mathfrak{gl}_{\ell+1}} = \frac{1}{2}(\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}})^2 - \mathcal{H}_2^{\mathfrak{gl}_{\ell+1}}$. Then the $\mathfrak{gl}_{\ell+1}$-Whittaker function satisfies the following equation

$$t^{\mathfrak{gl}_{\ell+1}}(\lambda) \;\; \Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}) = \prod_{j=1}^{\ell+1} (\lambda - \lambda_j) \;\; \Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}), \tag{2.9}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{\ell+1})$ and $\underline{x} = (x_1, \ldots, x_{\ell+1})$.

The appropriately normalized $\mathfrak{gl}_{\ell+1}$-Whittaker function (2.3) is a solution of Eqs. (2.9) invariant with respect to the actions of the Weyl group $W = S_{\ell+1}$ given by $s : \lambda_i \to \lambda_{s(i)}$, $s \in W$. The W-invariant $\mathfrak{gl}_{\ell+1}$- Whittaker functions provide a basis of W-invariant functions in $\mathbb{R}^{\ell+1}$ (see e.g. [STS, KL2]).

Theorem 2.1. *For the properly normalized W-invariant $\mathfrak{gl}_{\ell+1}$-Whittaker functions the following orthogonality and completeness relations hold*

$$\int_{\mathbb{R}^{\ell+1}} \overline{\Psi}_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}) \, \Psi_{\underline{\nu}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}) \prod_{j=1}^{\ell+1} dx_j = \frac{1}{(\ell+1)! \; \mu^{(\ell+1)}(\underline{\lambda})} \sum_{w \in W} \delta^{(\ell+1)}(\underline{\lambda} - w(\underline{\nu})), \tag{2.10}$$

$$\int_{\mathbb{R}^{\ell+1}} \overline{\Psi}_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}) \, \Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{y}) \mu^{(\ell+1)}(\underline{\lambda}) \prod_{j=1}^{\ell+1} d\lambda_j = \delta^{(\ell+1)}(\underline{x} - \underline{y}), \tag{2.11}$$

where

$$\mu^{(\ell+1)}(\underline{\lambda}) = \frac{1}{(2\pi)^{\ell+1}(\ell+1)!} \prod_{j \neq k} \frac{1}{\Gamma(\imath\lambda_k - \imath\lambda_j)}. \tag{2.12}$$

There exists another construction of $\mathfrak{gl}_{\ell+1}$-Whittaker functions that uses a pairing of the spherical vector (i.e. a vector invariant with respect to the maximal compact subgroup $K = SO(\ell+1,\mathbb{R})$ of $GL(\ell+1,\mathbb{R})$) and a Whittaker vector (see e.g. [J,Ha]). Consider the following function:

$$\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g) = e^{-\rho(g)}\langle\phi_K, \pi_{\underline{\lambda}}(g)\,\psi_R\rangle, \tag{2.13}$$

where $\rho(g)$ is given by $\rho(kan) = \langle\rho, \log a\rangle$, ϕ_K is a spherical vector in $\mathcal{V}_{\underline{\lambda}}$,

$$\phi_K(bgk) = \chi_{\underline{\lambda}}(b)\phi_K(g), \quad k \in K, \quad b \in B_+. \tag{2.14}$$

The function $\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g)$ defined by (2.13) satisfies the functional equation

$$\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(kgn) = \tilde{\chi}_{N_-}(n)\,\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(g), \quad k \in K, \quad n \in N_-, \tag{2.15}$$

where $\tilde{\chi}_{N_-}(n) = \exp(2\sum_{j=1}^{\ell} n_{j+1,j})$. Thus (2.13) descends to a function on the space A of the diagonal matrices $a = \mathrm{diag}(e^{\tilde{x}_1}, \ldots, e^{\tilde{x}_{\ell+1}})$ entering the Iwasawa decomposition $KAN_- \to GL(\ell+1,\mathbb{R})$. We fix a normalization of the matrix element so that the function (2.13) is W-invariant. The resulting function on A is related to the $\mathfrak{gl}_{\ell+1}$-Whittaker function (2.3) by a simple redefinition of the variables.

Lemma 2.1. *The following relation between $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{x})$ and $\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{\underline{\tilde{\lambda}}}(\underline{\tilde{x}})$ holds*:

$$\widetilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{\underline{\tilde{\lambda}}}(\underline{\tilde{x}}) = \Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{x}), \tag{2.16}$$

where $\underline{\tilde{x}} = (\tilde{x}_1, \ldots, \tilde{x}_{\ell+1})$, $\underline{\tilde{\lambda}} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{\ell+1})$ are expressed through $\underline{x} = (x_1, \ldots, x_{\ell+1})$, $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{\ell+1})$ as follows

$$\tilde{x}_j = \frac{1}{2}x_j, \qquad \tilde{\lambda}_j = 2\lambda_j.$$

2.2. Recursive and Baxter operators. The following integral representation for $\mathfrak{gl}_{\ell+1}$-Whittaker function was introduced by Givental [Gi] (see also [JK]).

Theorem 2.2. *$\mathfrak{gl}_{\ell+1}$-Whittaker functions (2.5) admit an integral representation*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(x_1, \ldots, x_{\ell+1}) = \int_{\mathbb{R}^{\frac{\ell(\ell+1)}{2}}} \prod_{k=1}^{\ell}\prod_{i=1}^{k} dx_{k,i}\; e^{\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x)}, \tag{2.17}$$

where

$$\begin{aligned}\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x) \;&= \imath\sum_{k=1}^{\ell+1}\lambda_k\left(\sum_{i=1}^{k}x_{k,i} - \sum_{i=1}^{k-1}x_{k-1,i}\right)\\ &\quad - \sum_{k=1}^{\ell}\sum_{i=1}^{k}\left(e^{x_{k+1,i}-x_{k,i}} + e^{x_{k,i}-x_{k+1,i+1}}\right),\end{aligned} \tag{2.18}$$

and $x_i := x_{\ell+1,i}$, $i = 1, \ldots, \ell+1$.

The interpretation of the Givental integral formula as a matrix element (2.5) was first obtained in [GKLO], where it was also noted that the integral representation (2.17) of the $\mathfrak{gl}_{\ell+1}$-Whittaker function has a recursive structure over the rank of the Lie algebra $\mathfrak{gl}_{\ell+1}$.

Corollary 2.1. *The following integral operators $Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k}$ provide a recursive construction of $\mathfrak{gl}_{\ell+1}$-Whittaker functions:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(\underline{x}_{\,\ell+1}) = \int_{\mathbb{R}^\ell} \prod_{i=1}^{\ell} dx_{\ell,i}\;\; Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\,\ell+1}, \underline{x}_{\,\ell}|\lambda_{\ell+1})\Psi^{\mathfrak{gl}_\ell}_{\lambda_1,\ldots,\lambda_\ell}(\underline{x}_{\,\ell}), \tag{2.19}$$

$$\begin{aligned} &Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\,\ell+1}, \underline{x}_{\,\ell}|\lambda_{\ell+1}) \\ &\quad = \exp\left\{\imath\lambda_{\ell+1}\Big(\sum_{i=1}^{\ell+1} x_{\ell+1,i} - \sum_{i=1}^{\ell} x_{\ell,i}\Big) - \sum_{i=1}^{\ell}\Big(e^{x_{\ell+1,i}-x_{\ell,i}} + e^{x_{\ell,i}-x_{\ell+1,i+1}}\Big)\right\}, \end{aligned} \tag{2.20}$$

where $\underline{x}_{\,k} = (x_{k,1}, \ldots, x_{k,k})$ and we assume that $Q^{\mathfrak{gl}_1}_{\mathfrak{gl}_0}(x_{11}|\lambda_1) = e^{\imath\lambda_1 x_{1,1}}$.

Definition 2.1. *Baxter operator $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda)$ for $\mathfrak{gl}_{\ell+1}$ is an integral operator with the kernel*

$$\begin{aligned} &\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{x},\ \underline{y}|\,\lambda) \\ &\quad = \exp\left\{\imath\lambda\sum_{i=1}^{\ell+1}(x_i - y_i) - \sum_{i=1}^{\ell}\Big(e^{x_i-y_i} + e^{y_i-x_{i+1}}\Big) - e^{x_{\ell+1}-y_{\ell+1}}\right\}, \end{aligned} \tag{2.21}$$

where we assume $x_i := x_{\ell+1,i}$ and $y_i := y_{\ell+1,i}$.

Note that the Baxter operator defined above is non-trivial even for $\mathfrak{gl}_1$.

Theorem 2.3. *The Baxter operator $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda)$ satisfies the following identities:*

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda)\cdot\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda') = \mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda')\cdot\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda), \tag{2.22}$$

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\gamma)Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\lambda) = \Gamma(\imath\gamma - \imath\lambda)\;\; Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\lambda)\mathcal{Q}^{\mathfrak{gl}_\ell}(\gamma), \tag{2.23}$$

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda)\cdot T^{\mathfrak{gl}_{\ell+1}}(\lambda') = T^{\mathfrak{gl}_{\ell+1}}(\lambda')\cdot\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda), \tag{2.24}$$

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda - \imath) = \imath^{\,\ell+1}\;\; T^{\mathfrak{gl}_{\ell+1}}(\lambda)\;\; \mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda), \tag{2.25}$$

where

$$T^{\mathfrak{gl}_{\ell+1}}(\underline{x}, \underline{y}|\lambda) = t^{\mathfrak{gl}_{\ell+1}}(\underline{x}, \partial_{\underline{x}}|\lambda)\delta^{\ell+1}(x - y), \tag{2.26}$$

$$t^{\mathfrak{gl}_{\ell+1}}(\underline{x}, \partial_{\underline{x}}|\lambda) = \sum_{j=1}^{\ell+1}(-1)^j\lambda^{\ell+1-j}\,\mathcal{H}^{\mathfrak{gl}_{\ell+1}}_j(\underline{x}, \partial_{\underline{x}}). \tag{2.27}$$

Proof. The commutativity of $\mathcal{Q}$-operators

$$\int_{\mathbb{R}^{\ell+1}} \mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{y},\underline{x}|\lambda)\;\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{x},\underline{z}|\lambda')\prod_{j=1}^{\ell+1}dx_j \tag{2.28}$$

$$=\int_{\mathbb{R}^{\ell+1}} \mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{y},\underline{x}|\lambda')\;\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{x},\underline{z}|\lambda)\prod_{j=1}^{\ell+1}dx_j, \tag{2.29}$$

is proved using the following change of variables x_i:

$$\begin{aligned} x_1 &\longmapsto -x_1+z_1+\ln\left(e^{y_1}+e^{z_2}\right),\\ x_i &\longmapsto -x_i-\ln\left(e^{-y_{i-1}}+e^{-z_i}\right)+\ln\left(e^{y_i}+e^{z_{i+1}}\right), \quad 1<i\le\ell,\\ x_{\ell+1} &\longmapsto -x_{\ell+1}+y_{\ell+1}-\ln\left(e^{-y_\ell}+e^{-z_{\ell+1}}\right). \end{aligned}$$

The proof of (2.23) is similar to the proof of the commutativity (2.22). The commutation relations (2.24) and the difference equation (2.25) then easily follow from (2.23) and (2.10), (2.11). □

Corollary 2.2. *The following relation holds:*

$$\int_{\mathbb{R}^{\ell+1}}\prod_{i=1}^{\ell+1}dx_i\;\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{y},\,\underline{x}|\,\gamma)\,\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{x})\;=\;\prod_{i=1}^{\ell+1}\Gamma(\imath\gamma-\imath\lambda_i)\;\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{y}), \tag{2.30}$$

where $\underline{x}=(x_1,\dots,x_{\ell+1})$, $\underline{y}=(y_1,\dots,y_{\ell+1})$ *and* $\underline{\lambda}=(\lambda_1,\dots,\lambda_{\ell+1})$.

Finally let us provide an expression for the kernel of the Baxter $\mathcal{Q}$-operator in the parametrization naturally arising in the construction of $\mathfrak{gl}_{\ell+1}$-Whittaker functions using Iwasawa decomposition (see (2.13) and Lemma 2.1). Let $\tilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(\underline{\tilde{x}},\,\underline{\tilde{y}}|\tilde{\lambda})$ be defined by

$$\begin{aligned}&\tilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(\underline{\tilde{x}},\,\underline{\tilde{y}}|\tilde{\lambda})\\ &=2^{\ell+1}\exp\left\{\imath\tilde{\lambda}\sum_{i=1}^{\ell+1}(\tilde{x}_i-\tilde{y}_i)-\sum_{k=1}^{\ell}\left(e^{2(\tilde{x}_k-\tilde{y}_k)}+e^{2(\tilde{y}_k-\tilde{x}_{k+1})}\right)-e^{2(\tilde{x}_{\ell+1}-\tilde{y}_{\ell+1})}\right\}.\end{aligned}$$

Proposition 2.1. *The following relation holds:*

$$\int_{\mathbb{R}^{\ell+1}}\prod_{i=1}^{\ell+1}d\tilde{x}_i\;\tilde{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(\underline{\tilde{y}},\,\underline{\tilde{x}}|\,\tilde{\gamma})\,\tilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{\underline{\tilde{\lambda}}}(\underline{\tilde{x}})\;=\;\prod_{i=1}^{\ell+1}\Gamma\Big(\frac{\imath\tilde{\gamma}-\imath\tilde{\lambda}_i}{2}\Big)\;\tilde{\Psi}^{\mathfrak{gl}_{\ell+1}}_{\underline{\tilde{\lambda}}}(\underline{\tilde{y}}). \tag{2.31}$$

3. Givental versus Mellin-Barnes Integral Representations

An important property of the Givental integral representation is its recursive structure with respect to the rank of the Lie algebra. There is another integral representation [KL1] for $\mathfrak{gl}_{\ell+1}$-Whittaker functions generalizing the Mellin-Barnes integral representation for low ranks. This representation also has a recursive structure. Its interpretation in terms of representation theory uses the Gelfand-Zetlin construction of a maximal commutative subalgebra in $\mathcal{U}(\mathfrak{gl}_{\ell+1})$ [GKL]. In this section we compare recursive structures of Givental and Mellin-Barnes representations and demonstrate that these two integral representations should be considered as dual to each other. We propose the construction of the dual Baxter operator based on Mellin-Barnes integral representations. We also construct a family of new integral representations interpolating between Givental and Mellin-Barnes representations. Finally we introduce a symmetric recursive construction of $\mathfrak{gl}_{\ell+1}$-Whittaker functions such that the corresponding recursive operator is expressed through the Baxter and dual Baxter operators. The Givental and Mellin-Barnes integral representations are then obtained from the symmetric integral representations by simple manipulations.

Let us first recall the Mellin-Barnes integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker functions [KL1].

Theorem 3.1. *The following integral representation of* $\mathfrak{gl}_{\ell+1}$*-Whittaker function holds:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}}(\underline{x}) = \int_{\mathcal{S}} \prod_{n=1}^{\ell} \frac{\prod\limits_{k=1}^{n}\prod\limits_{m=1}^{n+1} \Gamma(\imath\gamma_{n+1,m} - \imath\gamma_{n})}{(2\pi)^n n! \prod\limits_{s\neq p} \Gamma(\imath\gamma_{ns} - \imath\gamma_{np})} e^{\imath \sum\limits_{n=1}^{\ell+1}\sum\limits_{j=1}^{n}(\gamma_{nj}-\gamma_{n-1,j})x_n} \prod_{\substack{n=1\\ j\leq n}}^{\ell} d\gamma_{nj}, \qquad (3.1)$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{\ell+1}) := (\gamma_{\ell+1,1}, \ldots, \gamma_{\ell+1,\ell+1})$, $\underline{x} = (x_1, \ldots, x_{\ell+1})$ *and the domain of integration* $\mathcal{S}$ *is defined by the conditions* $\min_j\{\mathrm{Im}\,\gamma_{kj}\} > \max_m\{\mathrm{Im}\,\gamma_{k+1,m}\}$ *for all* $k = 1, \ldots, \ell$. *Recall that we assume* $\gamma_{nj} = 0$ *for* $j > n$.

Corollary 3.1. *The following recursive relation holds:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}_{\ell+1}}(x_1, \ldots, x_{\ell+1}) = \int_{\mathcal{S}_\ell} \widehat{Q}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{\gamma}_{\ell+1}, \underline{\gamma}_{\ell}|x_{\ell+1}) \Psi^{\mathfrak{gl}_\ell}_{\underline{\gamma}_\ell}(x_1, \ldots, x_\ell)\, \mu^{(\ell)}(\underline{\gamma}_\ell) \prod_{j=1}^{\ell} d\gamma_{\ell,j}\,, \qquad (3.2)$$

where

$$\widehat{Q}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{\gamma}_{\ell+1}, \underline{\gamma}_{\ell}|x_{\ell+1}) = e^{\imath(\sum\limits_{j=1}^{\ell+1}\gamma_{\ell+1,j} - \sum\limits_{k=1}^{\ell}\gamma_{\ell,k})x_{\ell+1}} \prod_{k=1}^{\ell}\prod_{m=1}^{\ell+1} \Gamma(\imath\gamma_{\ell+1,m} - \imath\gamma_{\ell,k}), \qquad (3.3)$$

the measure $\mu^{(\ell)}(\underline{\gamma}_\ell)$ *is defined by (2.12) and* $\underline{\gamma}_k = (\gamma_{k,1}, \ldots, \gamma_{k,k})$. *We imply* $\Psi^{\mathfrak{gl}_1}_{\gamma_{1,1}}(x_1) = e^{\imath\gamma_{1,1}x_1}$. *The domain of integration* $\mathcal{S}_\ell$ *is defined by the conditions* $\min_j\{\mathrm{Im}\gamma_{\ell,j}\} > \max_m\{\mathrm{Im}\gamma_{\ell+1,m}\}$.

We call the integral operator Corollary 3.1 the Mellin-Barnes recursive operator.

Let us stress that the recursive structure of the Mellin-Barnes integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker functions is dual to that of the Givental integral representation. Indeed, the Givental recursive operator $Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}$ depends on an additional "spectral" variable $\lambda_{\ell+1}$ and acts in the space of functions of the "coordinate" variables $\underline{x}$, while the dual Mellin-Barnes recursive operator $\widehat{Q}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}$ depends on the additional "coordinate" variable $x_{\ell+1}$ and acts in the space of functions of the "spectral" variables $\underline{\gamma}$. Using the orthogonal and completeness relations (2.10), (2.11) one can show that these two operators are related by a conjugation by the integral operator with the kernel $\Psi^{\mathfrak{gl}_\ell}_{\underline{\gamma}_\ell}(\underline{x}_\ell)$.

Proposition 3.1. *The following integral representation for the kernel of the recursive operator $\widehat{Q}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}$ holds:*

$$\widehat{Q}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{\gamma}_{\ell+1},\underline{\gamma}_\ell|x_{\ell+1,\ell+1}) = \int_{\mathbb{R}^\ell}\prod_{j=1}^{\ell}dx_{\ell+1,j}\;\overline{\Psi}^{\mathfrak{gl}_\ell}_{\underline{\gamma}_\ell}(\underline{x}'_{\ell+1})\;\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}_{\ell+1}}(\underline{x}_{\ell+1})$$

$$=\int_{\mathbb{R}^\ell}\prod_{j=1}^{\ell}dx_{\ell+1,j}\prod_{k=1}^{\ell}dx_{\ell,k}\,\overline{\Psi}^{\mathfrak{gl}_\ell}_{\underline{\gamma}_\ell}(\underline{x}'_{\ell+1})\;Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1},\underline{x}_\ell\gamma_{\ell+1,\ell+1})\Psi^{\mathfrak{gl}_\ell}_{\underline{\gamma}'_{\ell+1}}(\underline{x}_\ell),\quad(3.4)$$

where $\underline{x}_k=(x_{k,1},\ldots,x_{k,k})$, $\underline{x}'_k=(x_{k,1},\ldots,x_{k,k-1})$.

In view of the above duality for the recursive operators it is natural to introduce an operator dual to the Baxter $\mathcal{Q}$-operator.

Definition 3.1. *The dual Baxter operator $\widehat{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(z)$ is an integral operator with the kernel*

$$\widehat{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(\underline{\gamma}_{\ell+1},\underline{\beta}_{\ell+1}|z)=\prod_{i=1}^{\ell+1}\prod_{j=1}^{\ell+1}\Gamma(\gamma_{\ell+1,i}-\imath\beta_{\ell+1,j})e^{\imath z(\sum_{i=1}^{\ell+1}\gamma_{\ell+1,i}-\sum_{j=1}^{\ell+1}\beta_{\ell+1,j})},\qquad(3.5)$$

acting on the space of functions of $\underline{\gamma}=(\gamma_1,\ldots,\gamma_{\ell+1})$ as

$$\widehat{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(z)\cdot F(\underline{\gamma})=\int_{\mathcal{S}_{\ell+1}}\widehat{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(\underline{\gamma},\underline{\tilde{\gamma}}|z)\;F(\underline{\tilde{\gamma}})\;\mu^{(\ell+1)}(\underline{\tilde{\gamma}})\prod_{j=1}^{\ell+1}d\tilde{\gamma}_j\,.\qquad(3.6)$$

Proposition 3.2. *The $\mathfrak{gl}_{\ell+1}$-Whittaker function satisfies the following relation:*

$$\widehat{\mathcal{Q}}^{\mathfrak{gl}_{\ell+1}}(z)\cdot\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}_{\ell+1}}(\underline{x}_{\ell+1})=e^{-e^{(x_{\ell+1,\ell+1}-z)}}\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}_{\ell+1}}(\underline{x}_{\ell+1}).\qquad(3.7)$$

Proof. We should prove that

$$\int_{\mathcal{S}_{\ell+1}}e^{\imath z\,(\sum_{i=1}^{\ell+1}\lambda_{\ell+1,i}-\sum\gamma_{\ell+1,i})}\frac{\prod\limits_{i,j=1}^{\ell+1}\Gamma(\imath\lambda_{\ell+1,i}-\imath\gamma_{\ell+1,j})}{\prod\limits_{i\neq j}\Gamma(\imath\gamma_{\ell+1,j}-\imath\gamma_{\ell+1,i})}\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}_{\ell+1}}(\underline{x}_{\ell+1})\prod_{j=1}^{\ell+1}d\gamma_{\ell+1,j}$$

$$=(2\pi)^{\ell+1}(\ell+1)!\,e^{-e^{(x_{\ell+1,\ell+1}-z)}}\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}_{\ell+1}}(\underline{x}_{\ell+1}).\qquad(3.8)$$

Due to the orthogonality condition (2.10) this is equivalent to the following:

$$\int_{\mathbb{R}^{\ell+1}}e^{-e^{(x_{\ell+1,\ell+1}-z)}}\overline{\Psi}^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}_{\ell+1}}(\underline{x}_{\ell+1})\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}_{\ell+1}}(\underline{x}_{\ell+1})\prod_{j=1}^{\ell+1}dx_{\ell+1,j}$$

$$=e^{\imath z\sum_{i=1}^{\ell+1}(\lambda_{\ell+1,i}-\gamma_{\ell+1,i})}\prod_{i=1}^{\ell+1}\prod_{j=1}^{\ell+1}\Gamma(\imath\lambda_{\ell+1,i}-\imath\gamma_{\ell+1,j}).\qquad(3.9)$$

Using the recursive relation (3.2) one can rewrite this as

$$\int_{\mathbb{R}^{\ell+1}\times\mathcal{S}_\ell\times\mathcal{S}_\ell}\prod_{j=1}^{\ell+1}dx_{\ell+1,j}\,\prod_{j=1}^{\ell}d\lambda_{\ell,j}\,\prod_{j=1}^{\ell}d\gamma_{\ell,j}$$

$$\times e^{\imath x_{\ell+1,\ell+1}(\sum_{i=1}^{\ell+1}(\lambda_{\ell+1,i}-\gamma_{\ell+1,i})-\sum_{k=1}^{\ell}(\lambda_{\ell,k}-\gamma_{\ell,k}))-e^{(x_{\ell+1,\ell+1}-z)}} \tag{3.10}$$

$$\times\frac{\prod\limits_{i=1}^{\ell+1}\prod\limits_{k=1}^{\ell}\Gamma(\lambda_{\ell+1,i}-\imath\lambda_{\ell,k})\Gamma(\imath\gamma_{\ell,k}-\imath\gamma_{\ell+1,i})}{(2\pi)^{2\ell}(\ell!)^2\prod\limits_{k\neq l}\Gamma(\imath\lambda_{\ell,l}-\imath\lambda_{\ell,k})\Gamma(\imath\gamma_{\ell,l}-\imath\gamma_{\ell,k})}\overline{\Psi}^{\mathfrak{gl}_\ell}_{\underline{\gamma}_\ell}(\underline{x}'_{\ell+1})\Psi^{\mathfrak{gl}_\ell}_{\underline{\lambda}_\ell}(\underline{x}'_{\ell+1})$$

$$= e^{-\imath z\sum_{i=1}^{\ell+1}(\lambda_{\ell+1,i}-\gamma_{\ell+1,i})}\prod_{i=1}^{\ell+1}\prod_{j=1}^{\ell+1}\Gamma(\imath\lambda_{\ell+1,i}-\imath\gamma_{\ell+1,j}), \tag{3.11}$$

where $\underline{x}'_{\ell+1}=(x_{\ell+1,1},\dots,x_{\ell+1,\ell})$. Using the orthogonality condition (2.10) with respect to the $\underline{x}'_{\ell+1}$ and integrating over $\underline{\gamma}_\ell$ we see that (3.7) is equivalent to the following:

$$\frac{1}{(2\pi)^\ell\,\ell\,!}\int_{-\infty}^{\infty}dx_{\ell+1,\ell+1}\;e^{-\imath(x_{\ell+1,\ell+1}-z)\sum_{i=1}^{\ell+1}(\lambda_{\ell+1,i}-\gamma_{\ell+1,i})-e^{x_{\ell+1,\ell+1}-z}}$$

$$\times\int_{\mathcal{S}'_\ell}\prod_{j=1}^{\ell}d\lambda_{\ell,j}\;\frac{\prod\limits_{i=1}^{\ell+1}\prod\limits_{k=1}^{\ell}\Gamma(\lambda_{\ell+1,i}-\imath\lambda_{\ell,k})\Gamma(\imath\lambda_{\ell,k}-\imath\gamma_{\ell+1,i})}{\prod\limits_{k\neq l}\Gamma(\imath\lambda_{\ell,l}-\imath\lambda_{\ell,k})} \tag{3.12}$$

$$=\prod_{i=1}^{\ell+1}\prod_{j=1}^{\ell+1}\Gamma(\imath\lambda_{\ell+1,i}-\imath\gamma_{\ell+1,j}).$$

where the contour of integration $\mathcal{S}'_\ell$ in above formulas is deformed so as to separate the sequences of poles going down $\{\gamma_{\ell+1,j}-\imath k,\ j=1,\dots,\ell+1, k=0,\dots,\infty\}$ from the sequences of poles going up $\{\lambda_{\ell+1,j}+\imath k,\ j=1,\dots,\ell+1, k=0,\dots,\infty\}$. We assume also that $\gamma_{\ell+1,j}\neq\lambda_{\ell+1,k}$ for any j,k. The last identity is a simple consequence of the following integral formula due to Gustafson (see [Gu], Theorem 5.1, p. 81):

$$\frac{1}{(2\pi)^\ell}\int_{\mathcal{S}'_\ell}\prod_{j=1}^{\ell}d\lambda_{\ell,j}\;\frac{\prod\limits_{i=1}^{\ell+1}\prod\limits_{k=1}^{\ell}\Gamma(\lambda_{\ell+1,i}-\imath\lambda_{\ell,k})\Gamma(\imath\lambda_{\ell,k}-\imath\gamma_{\ell+1,i})}{\prod\limits_{k\neq l}\Gamma(\imath\lambda_{\ell,l}-\imath\lambda_{\ell,k})}$$

$$=\ell\,!\,\frac{\prod\limits_{i=1}^{\ell+1}\prod\limits_{j=1}^{\ell+1}\Gamma(\imath\lambda_{\ell+1,i}-\imath\gamma_{\ell+1,j})}{\Gamma\Big(\sum\limits_{i=1}^{\ell+1}\imath\lambda_{\ell+1,i}-\sum\limits_{i=1}^{\ell+1}\imath\gamma_{\ell+1,i}\Big)}.\quad\square$$

Proposition 3.3. *The following symmetric recursive relation for* $\mathfrak{gl}_{\ell+1}$*-Whittaker functions holds:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}_{\ell+1}}(\underline{x}_{\ell+1})=e^{\imath\gamma_{\ell+1,\ell+1}x_{\ell+1,\ell+1}}\;\widehat{\mathcal{Q}}^{\mathfrak{gl}_\ell}(x_{\ell+1,\ell+1})\cdot\mathcal{Q}^{\mathfrak{gl}_\ell}(\gamma_{\ell+1,\ell+1})\Psi^{\mathfrak{gl}_\ell}, \tag{3.13}$$

where $\underline{x}'_{\ell+1} = (x_{\ell+1,1},\dots,x_{\ell+1,\ell})$, $\underline{\gamma}'_{\ell+1} = (\gamma_{\ell+1,1},\dots,\gamma_{\ell+1,\ell})$ *and the action of the Baxter operator* $\mathcal{Q}^{\mathfrak{gl}_\ell}$ *and its dual* $\widehat{\mathcal{Q}}^{\mathfrak{gl}_\ell}$ *is given by*

$$\Big(\widehat{\mathcal{Q}}^{\mathfrak{gl}_\ell}(x_{\ell+1,\ell+1})\cdot\mathcal{Q}^{\mathfrak{gl}_\ell}(\gamma_{\ell+1,\ell+1})\Psi^{\mathfrak{gl}_\ell}\Big)_{\underline{\gamma}_{\ell+1}}(\underline{x}_{\ell+1}) \tag{3.14}$$
$$=\int\prod_{j=1}^{\ell}d\gamma_{\ell,j}\prod_{j=1}^{\ell}dx_{\ell,j}\;\;\mu^{(\ell)}(\underline{\gamma}_\ell)\,\widehat{\mathcal{Q}}^{\mathfrak{gl}_\ell}(\underline{\gamma}'_{\ell+1},\underline{\gamma}_\ell|\,x_{\ell+1,\ell+1})$$
$$\times\,\mathcal{Q}^{\mathfrak{gl}_\ell}(\underline{x}'_{\ell+1},\underline{x}_\ell|\,\gamma_{\ell+1,\ell+1})\Psi^{\mathfrak{gl}_\ell}_{\underline{\gamma}_\ell}(\underline{x}_\ell).$$

Proof. Let us start with the Mellin-Barnes recursive relation

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}_{\ell+1}}(\underline{x}_{\ell+1})$$
$$=\int\prod_{j=1}^{\ell}d\gamma_{\ell,j}\;\;\mu^{(\ell)}(\underline{\gamma}_\ell)\prod_{i=1}^{\ell+1}\prod_{j=1}^{\ell}\Gamma(\imath\gamma_{\ell+1,i}-\imath\gamma_{\ell,j})$$
$$\times\,e^{\imath x_{\ell+1,\ell+1}(\sum_{i=1}^{\ell+1}\gamma_{\ell+1,i}-\sum_{i=1}^{\ell}\gamma_{\ell,i})}\Psi^{\mathfrak{gl}_\ell}_{\underline{\gamma}_\ell}(\underline{x}'_{\ell+1}).$$

Using the properties of the Baxter operator we have

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}_{\ell+1}}(\underline{x}_{\ell+1})=e^{\imath\gamma_{\ell+1,\ell+1}x_{\ell+1,\ell+1}}\int\prod_{j=1}^{\ell}dx_{\ell,j}\;\;\mathcal{Q}^{\mathfrak{gl}_\ell}(\underline{x}'_{\ell+1},\underline{x}_\ell|\gamma_{\ell+1,\ell+1})$$
$$\times\Big(\int\prod_{j=1}^{\ell}d\gamma_{\ell,j}\;\;\mu^{(\ell)}(\underline{\gamma}_\ell)\prod_{i=1}^{\ell}\prod_{j=1}^{\ell}\Gamma(\imath\gamma_{\ell+1,i}-\imath\gamma_{\ell,j})$$
$$\times e^{\imath x_{\ell+1,\ell+1}(\sum_{i=1}^{\ell}\gamma_{\ell+1,i}-\sum_{i=1}^{\ell}\gamma_{\ell,i})}\;\Psi^{\mathfrak{gl}_\ell}_{\underline{\gamma}_\ell}(\underline{x}_\ell)\Big)$$
$$=e^{\imath\gamma_{\ell+1,\ell+1}x_{\ell+1,\ell+1}}\;\;\widehat{\mathcal{Q}}^{\mathfrak{gl}_\ell}(x_{\ell+1,\ell+1})\cdot\mathcal{Q}^{\mathfrak{gl}_\ell}(\gamma_{\ell+1,\ell+1})\Psi^{\mathfrak{gl}_\ell}.$$

Note that one can equally start with a Givental recursive relation and use the eigenvalue property (3.7) of the dual Baxter operator. □

The Givental and Mellin-Barnes recursions are easily obtained from the symmetric recursion (3.13). This provides a direct and inverse transformation of the Givental representation into the Mellin-Barnes one. Moreover, this leads to a family of the intermediate Givental-Mellin-Barnes representations. Indeed, to obtain $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}_{\ell+1}}(\underline{x}_{\ell+1})$ from $\Psi^{\mathfrak{gl}_\ell}_{\underline{\gamma}_\ell}(\underline{x}_\ell)$ one can either use the integral operator $Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{x}_{\ell+1},\underline{x}_\ell|\gamma_{\ell+1,\ell+1})$ or the integral operator $\widehat{Q}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\underline{\gamma}_{\ell+1},\underline{\gamma}_\ell|x_{\ell+1,\ell+1})$. This leads to the following family of mixed Mellin-Barnes-Givental integral representations of $\mathfrak{gl}_{\ell+1}$-Whittaker function

$$\Psi^{\mathfrak{gl}_{\ell+1}}=Q^{(\epsilon_1)}\cdot Q^{(\epsilon_2)}\cdots Q^{(\epsilon_\ell)}\;\;\Psi^{\mathfrak{gl}_1},\quad\epsilon=L,R\,, \tag{3.15}$$

where $Q^{(L)}$ is the integral operator with the integral kernel $Q^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k}$, $Q^{(R)}$ is the integral operator with the integral kernel $\widehat{Q}^{\mathfrak{gl}_{k+1}}_{\mathfrak{gl}_k}$ and the integral operators act on $\underline{\gamma}$- or $\underline{x}$-variables depending on ϵ_i. Various choices of $\{\epsilon_i\}$ in (3.15) provide various integral representations of the $\mathfrak{gl}_{\ell+1}$-Whittaker function.

4. Archimedean Factors in Rankin-Selberg Method

In this section we apply the dual recursion operator and Baxter operators discussed in the previous section to simplify calculations of the correction factors arising in the Rankin-Selberg method applied to $GL(\ell+1)\times GL(\ell+1)$ and $GL(\ell+1)\times GL(\ell)$. Note that these calculations are an important step in the proof of the functional equations for the corresponding automorphic L-functions using the Rankin-Selberg approach. Explicit expressions for these correction factors in terms of Gamma-functions were conjectured by Friedberg-Bump and Bump and proved later by Stade [St1,St2]. The proofs in [St1,St2] are based on a recursive generalization of the integral representation of $\mathfrak{gl}_{\ell+1}$-Whittaker functions, $\ell = 2$ first derived by Vinogradov and Takhtajan [VT]. The recursion in [St1,St2] changes the rank by two $\ell-1 \to \ell+1$. It was noted in [GKLO] that this recursion is basically the Givental recursion applied twice.

In this section we will demonstrate that using the recursive properties of the Mellin-Barnes representation and the dual Baxter operator one can give a one-line proof of the Friedberg-Bump and Bump conjectures. We start with a brief description of the relevant facts about automorphic L-functions, the Rankin-Selberg method and the Bump-Freidberg and Bump conjectures. For more details see e.g. [Bu,Go].

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and G be a reductive Lie group. An automorphic representation π of $G(\mathbb{A})$ can be characterized by an automorphic form ϕ_π such that it is an eigenfunction of any element of the global Hecke algebra $\mathcal{H}(G(\mathbb{A}))$. The global Hecke algebra can be represented as a product $\mathcal{H}(G(\mathbb{A})) = (\otimes_p \mathcal{H}_p)\otimes\mathcal{H}_\infty$ of the local non-Archimedean Hecke algebras $\mathcal{H}_p = \mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p))$ for each prime p and an Archimedean Hecke algebra $\mathcal{H}_\infty = \mathcal{H}(G(\mathbb{R}), K)$, where K is a maximal compact subgroup in $G(\mathbb{R})$. The local Hecke algebra $\mathcal{H}_p$ is isomorphic to a representation ring of a simply connected complex Lie group ${}^LG_0$, Langlands dual to G (e.g. $A_\ell, B_\ell, C_\ell, D_\ell$ are dual to A_ℓ, C_ℓ, B_ℓ, D_ℓ respectively). For each unramified representation of $G(\mathbb{Q}_p)$ one can define an action of $\mathcal{H}_p$ such that an automorphic form ϕ_π is a common eigenfunction of all elements of $\mathcal{H}_p$ for all primes p and thus defines a set of homomorphisms $\mathcal{H}_p \to \mathbb{C}$. Identifying local Hecke algebras with the representation ring of ${}^LG_0$ one can describe this set of homomorphisms as a set of conjugacy classes g_p in ${}^LG_0$.

Given a finite-dimensional representation $\rho_V : {}^LG_0 \to GL(V,\mathbb{C})$ one can construct an L-function corresponding to an automorphic form ϕ in the form of the Euler product as follows:

$$L(s,\phi,\rho_V) = \prod_p{}' L_p(s,\phi,\rho_V) = \prod_p{}' \det_V(1-\rho_V(g_p)\,p^{-s})^{-1}, \tag{4.1}$$

where $\prod'_p$ is a product over primes p such that the corresponding representation of $G(\mathbb{Q}_p)$ is not ramified. It is natural to complete the product by including local L-factors corresponding to Archimedean and ramified places. L-factors for ramified representations can be taken trivial. For the Archimedean place the Hecke eigenfunction property is usually replaced by the eigenfunction property with respect to the ring of invariant differential operators on $G(\mathbb{R})$. The corresponding eigenvalues are described by a conjugacy class t_∞ in the Lie algebra ${}^L\mathfrak{g}_0 = \mathrm{Lie}({}^LG_0)$. The Archimedean L-factor is given by [Se]

$$L_\infty(s,\phi,\rho_V) = \prod_{j=1}^{\ell+1}\Big(\pi^{-\frac{s-\alpha_j}{2}}\,\Gamma\Big(\frac{s-\alpha_j}{2}\Big)\Big) = \det_V\Big(\pi^{-\frac{s-\rho_V(t_\infty)}{2}}\,\Gamma\Big(\frac{s-\rho_V(t_\infty)}{2}\Big)\Big), \tag{4.2}$$

where $\rho_V(t_\infty)=\mathrm{diag}(\alpha_1,\dots\alpha_{\ell+1})$. The complete L-function

$$\Lambda(s,\phi,\rho)=L(s,\phi,\rho)L_\infty(s,\phi,\rho), \tag{4.3}$$

should satisfy the functional equation of the form

$$\Lambda(1-s,\phi,\rho)=\epsilon(s,\phi,\rho)\Lambda(s,\phi_{\pi^\vee},\rho^\vee),$$

where the ϵ-factor is of the exponential form $\epsilon(s,\phi,\rho)=A\,B^s$ and $\pi^\vee$, $\rho^\vee$ are dual to π,ρ.

In the Rankin-Selberg method one considers automorphic L-functions associated with automorphic representations of the products $G\times\tilde{G}$ of reductive groups. Let $\rho_V: {}^LG_0\to\mathrm{End}(V)$, $\tilde{\rho}_{\tilde{V}}: {}^L\tilde{G}_0\to\mathrm{End}(\tilde{V})$ be finite-dimensional representations of dual groups and let $g_p\in {}^LG_0$, $\tilde{g}_p\in {}^L\tilde{G}_0$ be representatives of the conjugacy classes corresponding to automorphic forms ϕ and $\tilde{\phi}$. One defines the L-function $L(s,\pi\times\tilde{\pi},\rho\times\tilde{\rho})$ as follows:

$$L(s,\phi\times\tilde{\phi},\rho\times\tilde{\rho})=\prod_p{}' \det_{V\otimes\tilde{V}}(1-\rho_V(g_p)\otimes\tilde{\rho}_V(\tilde{g}_p)\,p^{-s})^{-1}. \tag{4.4}$$

The L-function (4.4) up to a correction factor can be naturally written as an integral of the product of automorphic forms ϕ and $\tilde{\phi}$ with a simple kernel function. Given an explicit expression for the correction factor, this integral representation can be an important tool for studying analytic properties of $L(s,\phi\times\tilde{\phi})$ as a function of s.

In the following we consider the Rankin-Selberg method in the case of $G\times\widetilde{G}$ being either $GL(\ell+1)\times GL(\ell+1)$ or $GL(\ell+1)\times GL(\ell)$ with ρ and $\tilde{\rho}$ being standard representations. We start with the case of $GL(\ell+1)\times GL(\ell+1)$. Consider the following zeta-integral:

$$Z(s,\phi\times\tilde{\phi})=\int_{GL(\ell+1,\mathbb{Q})Z_{\mathbb{A}}^{(\ell+1)}\backslash GL(\ell+1,\mathbb{A})}\phi(g)\tilde{\phi}(g)\mathcal{E}(g,s)\,dg, \tag{4.5}$$

where the Eisenstein series is

$$\mathcal{E}(g,s)=\zeta((\ell+1)s)\sum_{\gamma\in P(\ell+1,\ell,\mathbb{Z})\backslash GL(\ell+1,\mathbb{Z})} f_s(\gamma g). \tag{4.6}$$

Here $Z_{\mathbb{A}}^{(\ell+1)}$ is the center of $GL(\ell+1,\mathbb{A})$, $\zeta(s)=\sum_{n=1}^\infty n^{-s}$ is the Riemann zeta-function and

$$f_s\in\mathrm{Ind}_{P(\ell+1,\ell,\mathbb{A})}^{GL(\ell+1,\mathbb{A})}\,\delta_P^s,$$

where δ_P denotes the modular function of the parabolic subgroup $P(\ell+1,\ell,\mathbb{A})$ of $GL(\ell+1,\mathbb{A})$ with the Levi factor $GL(\ell,\mathbb{A})\times GL(1,\mathbb{A})$.

Using the Rankin-Selberg unfolding technique (4.5) can be represented in the form

$$Z(s,\phi\times\tilde{\phi})=L(s,\phi\times\tilde{\phi})\Psi(s,\phi\times\tilde{\phi}),$$

where the correction factor $\Psi(s,\phi\times\tilde{\phi})$ is a convolution of two $\mathfrak{gl}_{\ell+1}$-Whittaker functions. The Bump-Freidberg conjecture proved in [St1] claims that $\Psi(s,\phi\times\tilde{\phi})$ is equal to the Archimedean local L-factor.

Theorem 4.1 (Bump-Freidberg-Stade).

$$\Psi(s,\phi\times\tilde{\phi}) = L_\infty(s,\phi\times\tilde{\phi}) = \prod_{j=1}^{\ell+1}\prod_{k=1}^{\ell+1}\pi^{-\frac{s-\alpha_j-\tilde{\alpha}_k}{2}}\Gamma\Big(\frac{s-\alpha_j-\tilde{\alpha}_k}{2}\Big), \tag{4.7}$$

where $\rho_V(t_\infty)=\mathrm{diag}(\alpha_1,\ldots\alpha_{\ell+1})$ *and* $\tilde{\rho}_V(\tilde{t}_\infty)=\mathrm{diag}(\tilde{\alpha}_1,\ldots\tilde{\alpha}_{\ell+1})$ *correspond to the automorphic representations* ϕ *and* $\tilde{\phi}$ *as in (4.2).*

The proof of the theorem can be reduced to the following identity proved by Stade (we rewrite Theorem 1.1, [St2] in our notations).

Lemma 4.1. *The following integral relation holds:*

$$\int_{\mathbb{R}^{\ell+1}}\prod_{j=1}^{\ell+1}dx_{\ell+1,j}\; e^{-e^{x_{\ell+1,\ell+1}}}\,\overline{\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}_{\ell+1}}}(\underline{x}_{\ell+1})\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}_{\ell+1}+\underline{t}}(\underline{x}_{\ell+1})$$
$$=\prod_{k=1}^{\ell+1}\prod_{j=1}^{\ell+1}\Gamma(\imath t+\imath\lambda_{\ell+1,k}-\imath\gamma_{\ell+1,j}), \tag{4.8}$$

where $\underline{t}=(t,\ldots,t)\in\mathbb{R}^{\ell+1}$.

Proof. The proof readily follows from the proof of Proposition 3.2. □

Next we consider the Rankin-Selberg method for $GL(\ell+1)\times GL(\ell)$, ρ and $\tilde{\rho}$ being standard representations of $GL(\ell+1)$ and $GL(\ell)$. In this case one has to study the following integral:

$$Z(s,\phi\times\tilde{\phi}) = \int_{GL(\ell,\mathbb{Z})Z^{(\ell)}_{\mathbb{A}}\backslash GL(\ell,\mathbb{A})}\phi(\begin{pmatrix} g & \\ & 1\end{pmatrix})\,\tilde{\phi}(g)\,|\det(g)|^{s-1/2}dg, \tag{4.9}$$

where $Z^{(\ell)}_{\mathbb{A}}$ is the center of $GL(\ell,\mathbb{A})$. Using the Rankin-Selberg unfolding technique, the integral (4.9) can be represented in the form

$$Z(s,\phi\times\tilde{\phi}) = L(s,\phi\times\tilde{\phi})\Psi(s,\phi\times\tilde{\phi}),$$

where the correction factor $\Psi(s,\phi\times\tilde{\phi})$ is a convolution of $\mathfrak{gl}_{\ell+1}$- and $\mathfrak{gl}_\ell$-Whittaker functions. The Bump conjecture proved in [St1] claims that $\Psi(s,\phi\times\tilde{\phi})$ is equal to the Archimedean local L-factor.

Theorem 4.2 (Bump-Stade).

$$\Psi(s,\phi\times\tilde{\phi}) = L_\infty(s,\phi\times\tilde{\phi}) = \prod_{j=1}^{\ell+1}\prod_{k=1}^{\ell}\pi^{-\frac{s-\alpha_j-\tilde{\alpha}_k}{2}}\Gamma\Big(\frac{s-\alpha_j-\tilde{\alpha}_k}{2}\Big), \tag{4.10}$$

where $\rho_V(t_\infty)=\mathrm{diag}(\alpha_1,\ldots\alpha_{\ell+1})$ *and* $\tilde{\rho}_V(\tilde{t}_\infty)=\mathrm{diag}(\tilde{\alpha}_1,\ldots\tilde{\alpha}_{\ell})$ *correspond to the automorphic representations* ϕ *and* $\tilde{\phi}$ *as in (4.2).*

The proof of the theorem is equivalent to the proof of the following integral identity (we rewrite Theorem 3.4, [St2] using our notations):

Lemma 4.2.

$$\int_{\mathbb{R}^{\ell+1}}\prod_{j=1}^{\ell+1}dx_{\ell+1,j}\;\overline{\Psi}^{\mathfrak{gl}_{\ell}}_{\underline{\gamma}_{\ell}}(\underline{x}'_{\ell+1})\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}_{\ell+1}+\underline{t}}(\underline{x}_{\ell+1})$$
$$=\delta\Big(\imath(\ell+1)t+\imath\sum_{i=1}^{\ell+1}\lambda_{\ell+1,i}-\imath\sum_{k=1}^{\ell}\gamma_{\ell,k}\Big)\prod_{i=1}^{\ell+1}\prod_{k=1}^{\ell}\Gamma(\imath t+\imath\lambda_{\ell+1,i}-\imath\gamma_{\ell,k}),\quad (4.11)$$

where $\underline{t}=(t,\dots,t)\in\mathbb{R}^{\ell+1}$, $\underline{x}'_{\ell+1}=(x_{\ell+1,1},\dots,x_{\ell+1,\ell})$ *and* $\delta(x)$ *is the Dirac* δ-*function.*

Proof. To verify this statement we substitute into the l.h.s. of (4.11) the following recursive relation:

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\lambda}_{\ell+1}+\underline{t}}(\underline{x}_{\ell+1})=\widehat{Q}(x_{\ell+1,\,\ell+1})\cdot\Psi^{\mathfrak{gl}_{\ell}}_{\underline{\lambda}_{\ell}}(\underline{x}'_{\ell+1}).$$

Then applying the orthogonality relation from Theorem 2.1 and integrating over $x_{\ell+1,\,\ell+1}$ we obtain the r.h.s. (4.11). □

Let us stress that one should not expect to have expressions for $\Psi(s,\phi\times\tilde{\phi})$ as products of Gamma-functions for more general cases $GL(\ell+n)\times GL(\ell)$, $n>1$. From the point of view of Mellin-Barnes recursive construction, $\Psi(s,\phi\times\tilde{\phi})$ are the kernels of recursive operators corresponding to the change of rank $\ell\to\ell+n$ and thus are given by compositions of elementary recursive operators. This leads to general expressions for $\Psi(s,\phi\times\tilde{\phi})$ in terms of the integrals of the products of Gamma-functions. Let us remark that in this paper we consider Rankin-Selberg method as a method for studying properties of matrix elements of the natural (recursive) operators acting in the space of automorphic forms. One can expect that this point of view might be useful in the investigation of other properties of automorphic L-functions.

Let us comment on Stade's proof of Theorems 4.1, 4.2. The proof in [St1,St2] is based on the recursive relation connecting $\mathfrak{gl}_{\ell+1}$- and $\mathfrak{gl}_{\ell-1}$-Whittaker functions. Below we derive this recursion from the following form of the Givental recursion.

Proposition 4.1. *The following recursive relations for* $\mathfrak{gl}_{\ell+1}$-*Whittaker functions holds:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\dots,\lambda_{\ell+1}}(\underline{x}_{\ell+1})=\int_{\mathbb{R}^{\ell-1}}\prod_{i=1}^{\ell-1}dx_{\ell-1,i}\;Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_{\ell-1}}(\underline{x}_{\ell+1},\underline{x}_{\ell-1}|\lambda_{\ell+1},\lambda_{\ell})$$
$$\times\Psi^{\mathfrak{gl}_{\ell-1}}_{\lambda_1,\dots,\lambda_{\ell-1}}(\underline{x}_{\ell-1}),\quad (4.12)$$

$$Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_{\ell-1}}(\underline{x}_{\ell+1},\underline{x}_{\ell-1}|\lambda_{\ell+1},\lambda_{\ell})$$
$$=\int_{\mathbb{R}^{\ell}}\prod_{j=1}^{\ell}dx_{\ell,j}\;\exp\Big\{\imath\lambda_{\ell+1}\Big(\sum_{i=1}^{\ell+1}x_{\ell+1,i}-\sum_{k=1}^{\ell}x_{\ell,k}\Big)$$
$$-\sum_{k=1}^{\ell}\Big(e^{x_{\ell+1,k}-x_{\ell,k}}+e^{x_{\ell,k}-x_{\ell+1,k+1}}\Big)$$
$$+\imath\lambda_{\ell}\Big(\sum_{k=1}^{\ell}x_{\ell,k}-\sum_{j=1}^{\ell-1}x_{\ell-1,j}\Big)-\sum_{k=1}^{\ell-1}\Big(e^{x_{\ell,k}-x_{\ell-1,k}}+e^{x_{\ell-1,k}-x_{\ell,k+1}}\Big)\Big\}.\quad (4.13)$$

Proof. The recursive relation (4.12) is the Givental recursive relation (2.19) applied twice. □

Theorem 4.3 (Stade). *The following recursion relation for $\mathfrak{gl}_{\ell+1}$-Whittaker functions holds:*

$$\Psi^{\mathfrak{gl}_{\ell+1}}_{\lambda_1,\ldots,\lambda_{\ell+1}}(\underline{x}_{\,\ell+1}) = \int_{\mathbb{R}^{\ell-1}} \prod_{j=1}^{\ell-1} dx_{\ell-1,j}\;\; K_{\ell+1,\,\ell-1}(\underline{x}_{\,\ell+1},\,\underline{x}_{\,\ell-1}|\,\lambda_{\ell+1},\lambda_\ell) \times \Psi^{\mathfrak{gl}_{\ell-1}}_{\lambda_1,\ldots\lambda_{\ell-1}}(\underline{x}_{\,\ell-1}), \tag{4.14}$$

where $K_{\ell+1,\,\ell-1}(\underline{x}_{\,\ell+1},\,\underline{x}_{\,\ell-1}|\,\underline{\lambda})$ is given by the following explicit formula:

$$\begin{aligned} &K_{\ell+1,\,\ell-1}(\underline{x}_{\,\ell+1},\,\underline{x}_{\,\ell-1}|\,\lambda)\\ &= 2^{1-\ell}\exp\left\{\frac{\imath(\lambda_\ell+\lambda_{\ell+1})}{2}\Big(\sum_{i=1}^{\ell+1}x_{\ell+1,i}-\sum_{j=1}^{\ell-1}x_{\ell-1,j}\Big)\right\}\\ &\quad\times\prod_{i=1}^{\ell}K_{\imath(\lambda_\ell-\lambda_{\ell+1})}\left(2\sqrt{\left(e^{x_{\ell+1,i}}+e^{x_{\ell-1,i-1}}\right)\left(e^{-x_{\ell+1,i+1}}+e^{-x_{\ell-1,i}}\right)}\right). \end{aligned}\tag{4.15}$$

Here we use the following integral representation for the Macdonald function:

$$K_\nu(y)=\int_0^\infty \frac{dt}{t}\; t^\nu e^{-y(t+t^{-1})/2}.$$

Proof. At first we substitute into the expression for $K_{\ell+1,\,\ell-1}$ the integral representation with integration variables t_i for Macdonald functions $K_{\imath(\lambda_\ell-\lambda_{\ell+1})}$. Then we make the following change of variables t_i:

$$\begin{aligned} t_1 &= e^{x_{\ell,1}}\sqrt{\frac{e^{-x_{\ell+1,2}}+e^{-x_{\ell-1,1}}}{e^{x_{\ell+1,1}}}},\\ t_k &= e^{x_{\ell,k}}\sqrt{\frac{e^{-x_{\ell+1,k+1}}+e^{-x_{\ell-1,k}}}{e^{x_{\ell+1,k}}+e^{x_{\ell-1,k-1}}}},\quad t_\ell = e^{x_{\ell,\ell}}\sqrt{\frac{e^{-x_{\ell+1,\ell+1}}}{e^{x_{\ell+1,\ell}}+e^{x_{\ell-1,\ell-1}}}}, \end{aligned}\tag{4.16}$$

for $k=1,\ldots,\ell$ and $j=1,\ldots,\ell-1$. Thus we obtain the following identity between the kernels:

$$K_{\ell+1,\,\ell-1}(\underline{x}_{\,\ell+1},\,\underline{x}_{\,\ell-1}|\,\lambda) = Q^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_{\ell-1}}(\underline{x}_{\,\ell+1},\,\underline{x}_{\,\ell-1}|\,\lambda). \tag{4.17}$$

This reduces the Stade recursion to the Givental recursive procedure. □

The appearance of the Gamma-functions both in the Mellin-Barnes integral representation of the $\mathfrak{gl}_{\ell+1}$-Whittaker functions and in the expressions for the Archimedean L-factors is not accidental. In the next section we explain this connection by relating the constructed Baxter operator with a universal Baxter operator considered as an element of the Archimedean Hecke algebras $\mathcal{H}(G(\mathbb{R}),K)$, where K is a maximal compact subgroup of $G(\mathbb{R})$.

5. Universal Baxter Operator

5.1. Universal Baxter operator in $\mathcal{H}(G(\mathbb{R}), K)$. In this section we will argue the Baxter $\mathcal{Q}$-operator for the $\mathfrak{gl}_{\ell+1}$-Toda chain can (and should) be considered as a realization of the universal Baxter operator considered as elements of the spherical Hecke algebra $\mathcal{H}(GL(\ell+1,\mathbb{R}), K)$, K being a maximal compact subgroup of $GL(\ell+1,\mathbb{R})$. We also consider non-Archimedean analogs of the universal Baxter operator as an element of a local Hecke algebra $\mathcal{H}(GL(\ell+1,\mathbb{Q}_p), GL(\ell+1,\mathbb{Z}_p))$. Both in Archimedean and non-Archimedean cases the eigenvalues of the Baxter $\mathcal{Q}$-operators acting on $\mathfrak{gl}_{\ell+1}$-Whittaker functions are given by the corresponding local L-factors.

Let us start with the definition of the spherical Hecke algebra $\mathcal{H}_\infty = \mathcal{H}(G(\mathbb{R}), K)$, where K is a maximal compact subgroup of $G(\mathbb{R})$. Algebra $\mathcal{H}_\infty$ is defined as an algebra of K-biinvariant functions on G, $\phi(g) = \phi(k_1 g k_2)$, $k_1, k_2 \in K$ acting by a convolution

$$\phi * f(g) = \int_G \phi(g\tilde{g}^{-1})\, f(\tilde{g}) d\tilde{g}. \tag{5.1}$$

To ensure the convergence of the integrals one usually imposes the condition of compact support on K-biinvariant functions. We will consider slightly more general class of exponentially decaying functions.[1]

By the multiplicity one theorem [Sha], there is a unique smooth spherical vector $\langle\phi_K|$ in a principal series irreducible representation $\mathcal{V}_{\underline{\gamma}} = \mathrm{Ind}_{B_-}^G\, \chi_{\underline{\gamma}}$. The action of a K-biinvariant function ϕ on the spherical vector $\langle\phi_K|$ in $\mathcal{V}_{\underline{\gamma}}$ is given by the multiplication by a character Λ_ϕ of the Hecke algebra:

$$\phi * \langle\phi_K| \equiv \int_G dg \phi(g^{-1})\, \langle\phi_K|\pi_\gamma(g) = \Lambda_\phi(\gamma)\langle\phi_K|. \tag{5.2}$$

In particular, the elements ϕ of the Hecke algebra should act by convolution on the Whittaker function as follows:

$$\phi * \Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(g) = \Lambda_\phi(\underline{\gamma})\Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(g), \quad \phi \in \mathcal{H}_\infty. \tag{5.3}$$

Here the Whittaker function $\Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}$ is considered as a function on G such that

$$\Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(kan) = \chi_{N_-}(n)\, \Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(a), \tag{5.4}$$

where $kan \in KAN_- \to G$ is the Iwasawa decomposition.

In the previous section we construct the Baxter integral operator acting on the $\mathfrak{gl}_{\ell+1}$-Whittaker function (considered as a function on the subspace A of the diagonal matrices) as

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda)\ \cdot\ \Psi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}) = \prod_{j=1}^{\ell+1} \pi^{-\frac{\imath\lambda-\imath\gamma_j}{2}}\, \Gamma\Big(\frac{\imath\lambda-\imath\gamma_j}{2}\Big)\ \Psi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}), \tag{5.5}$$

[1] This should be compared with the use of exponentially decreasing functions instead of functions with compact support in the Mathai-Quillen construction of the representative of the Thom class.

where the kernel of the operator $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda)$ is given by

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\underline{x},\underline{y}|\lambda) = 2^{\ell+1}\exp\Big\{\imath\lambda\sum_{i=1}^{\ell+1}(x_i-y_i)-\pi\sum_{k=1}^{\ell}\Big(e^{2(x_k-y_k)}+e^{2(y_k-x_{k+1})}\Big)-\pi e^{2(x_{\ell+1}-y_{\ell+1})}\Big\}.$$

Note that here we use a parametrization of Baxter operator naturally arising in the description of Whittaker functions in terms of Iwasawa decomposition. In this section we will use only this type of the parametrization and drop the tildes in the corresponding notations (see (2.13) and Lemma 2.1). We also take coupling constants in the Toda chain $g_i=\pi^2$ to agree with the standard normalizations in Representation theory.

Let us recall that we introduce the $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}}(\underline{x})$ as a matrix element multiplied by the factor $\exp(-\langle\rho,x\rangle)$ (see (2.5), (2.13)). In the construction of the universal Baxter operator it is more natural to consider a modified Whittaker function $\Phi^{\mathfrak{gl}_{\ell+1}}$ equal to the matrix elements itself

$$\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}}(\underline{x})=e^{\langle\rho,x\rangle}\,\Psi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}}(\underline{x}). \tag{5.6}$$

Define a modified Baxter $\mathcal{Q}$-operator:

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}_0(\lambda)=e^{\langle\rho,x\rangle}\,\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda)e^{-\langle\rho,x\rangle}.$$

It has the kernel

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}_0(\underline{x},\underline{y}|\lambda)=2^{\ell+1}\exp\Big\{\sum_{j=1}^{\ell+1}(\imath\lambda+\rho_j)(x_j-y_j)$$
$$-\pi\sum_{k=1}^{\ell}\Big(e^{2(x_k-y_k)}+e^{2(y_k-x_{k+1})}\Big)-\pi e^{2(x_{\ell+1}-y_{\ell+1})}\Big\},$$

where $\rho\in\mathbb{R}^{\ell+1}$, with $\rho_j=\frac{\ell}{2}+1-j,\quad j=1,\dots,\ell+1$, and it acts on the modified Whittaker functions as follows:

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}_0(\lambda)\,\cdot\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}}(\underline{x})=\prod_{j=1}^{\ell+1}\pi^{-\frac{\imath\lambda-\imath\gamma_j}{2}}\,\Gamma\Big(\frac{\imath\lambda-\imath\gamma_j}{2}\Big)\,\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}}(\underline{x}). \tag{5.7}$$

We would like to find an element $\phi_{\mathcal{Q}_0(\lambda)}$ in $\mathcal{H}_\infty$ such that the following relation holds:

$$\phi_{\mathcal{Q}_0(\lambda)}\,*\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}}(g)=\prod_{j=1}^{\ell+1}\pi^{-\frac{\imath\lambda-\imath\gamma_j}{2}}\,\Gamma\Big(\frac{\imath\lambda-\imath\gamma_j}{2}\Big)\,\Phi^{\mathfrak{gl}_{\ell+1}}_{\underline{\gamma}}(g), \tag{5.8}$$

and the restriction of $\phi_{\mathcal{Q}_0(\lambda)}$ to the subspace of functions satisfying (5.4) coincides with the operator $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}_0(\lambda)$. We shall call such $\phi_{\mathcal{Q}_0(\lambda)}$ a universal Baxter operator.

Theorem 5.1. *Let $\phi_{\mathcal{Q}_0(\lambda)}(g)$ be a K-biinvariant function on $G=GL(\ell+1,\mathbb{R})$ given by*

$$\phi_{\mathcal{Q}_0(\lambda)}(g)=2^{\ell+1}|\det g|^{\imath\lambda+\frac{\ell}{2}}e^{-\pi\mathrm{Tr}g^tg}. \tag{5.9}$$

i) *Then, the action of* $\phi_{\mathcal{Q}_0(\lambda)}$ *on the functions satisfying (5.4) descends to the action of* $\mathcal{Q}_0^{\mathfrak{gl}_{\ell+1}}(\lambda)$ *defined by (5.7);*

ii) *The action of* $\phi_{\mathcal{Q}_0(\lambda)}$ *on the modified Whittaker function* $\Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(g)$ *by a convolution is given by*

$$\big(\phi_{\mathcal{Q}_0(\lambda)} * \Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}\big)(g) = L_\infty(\lambda)\ \Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(g), \tag{5.10}$$

where $L_\infty(\lambda)$ *is the local Archimedean L-factor,*

$$L_\infty(\lambda) = \prod_{j=1}^{\ell+1} \pi^{-\frac{\imath\lambda-\imath\gamma_j}{2}}\,\Gamma\Big(\frac{\imath\lambda-\imath\gamma_j}{2}\Big). \tag{5.11}$$

Proof. i) The action of the K-biinvariant function on $\mathfrak{gl}_{\ell+1}$-Whittaker functions is given by

$$\begin{aligned}\big(\phi * \Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}\big)(g) &= \int_G d\tilde{g}\,\phi(g\tilde{g}^{-1})\,\Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(\tilde{g})\\ &= \int_G d\tilde{g}\,\phi(g\tilde{g}^{-1})\,\langle k|\pi_{\underline{\gamma}}(\tilde{g})|\psi_R\rangle.\end{aligned} \tag{5.12}$$

Fix the Iwasawa decomposition $\tilde{g} = \tilde{k}\tilde{a}\tilde{n}$, $\tilde{k} \in K$, $\tilde{a} \in A$, $\tilde{n} \in N_-$ of a generic element $\tilde{g} \in G$ and let $\delta_{B_-}(\tilde{a}) = \det_{\mathfrak{n}_-} \mathrm{Ad}_{\tilde{a}}$. We shall use the notation $d^\times a = da \cdot \det(a)^{-1}$ for $a \in A$. We have for $a \in A$,

$$\begin{aligned}\big(\phi * \Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}\big)(a) &= \int_{AN_-} d^\times\tilde{a}d\tilde{n}\,\delta_{B_-}(\tilde{a})\,\phi(a\tilde{n}^{-1}\tilde{a}^{-1})\,\chi_{N_-}(\tilde{n})\,\Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(\tilde{a})\\ &= \int_A d^\times\tilde{a}\ K_\phi(a,\tilde{a})\,\Phi_{\underline{\gamma}}^{\mathfrak{gl}_{\ell+1}}(\tilde{a})\end{aligned} \tag{5.13}$$

with

$$K_\phi(a,\tilde{a}) = \int_{N_-} d\tilde{n}\ \delta_{B_-}(\tilde{a})\ \phi(a\tilde{n}^{-1}\tilde{a}^{-1})\,\chi_{N_-}(\tilde{n}),$$

$$\chi_{N_-}(\tilde{n}) = \exp\Big\{2\pi\imath\sum_{i=1}^{\ell}\tilde{n}_{i+1,i}\Big\}.$$

Thus to prove the first statement of the theorem we should prove the following;

$$\mathcal{Q}_0^{\mathfrak{gl}_{\ell+1}}(\underline{x},\underline{y}|\lambda) = \int_{N_-} d\tilde{n}\ \delta_{B_-}(\tilde{a})\,\phi_{\mathcal{Q}_0(\lambda)}(a\tilde{n}^{-1}\tilde{a}^{-1}|\lambda)\,\chi_{N_-}(\tilde{n}), \tag{5.14}$$

where

$$\begin{gathered} a = \mathrm{diag}\,(e^{x_1},\ldots,e^{x_{\ell+1}}), \qquad \tilde{a} = \mathrm{diag}(e^{y_1},\ldots e^{y_{\ell+1}}),\\ \delta_{B_-}(\tilde{a}) = e^{-2\langle\rho,\log\tilde{a}\rangle} = e^{\sum_{i>j}(y_i-y_j)}.\end{gathered} \tag{5.15}$$

For $g = a\tilde{n}^{-1}\tilde{a}^{-1}$ we have

$$\det\ g = e^{\sum_{i=1}^{\ell+1}(x_i-y_i)}, \quad \mathrm{Tr}\ g^t g = \sum_{i=1}^{\ell+1} e^{2(x_i-y_i)} + \sum_{i>j} u_{ij}^2 e^{2(x_i-y_j)}, \tag{5.16}$$

where $u = \tilde{n}^{-1} \in N_-$. Taking into account that $\chi_{N_-}(\tilde{n}) = \chi_{N_-}(u^{-1}) = \exp(-2\pi\imath \sum_{i=1}^{\ell} u_{i+1,i})$, we obtain

$$\begin{aligned}
&\mathcal{Q}_0^{\mathfrak{gl}_{\ell+1}}(\underline{x}, \underline{y}|\lambda) \\
&= 2^{\ell+1}\int_{N_-} du\; e^{\sum_{i>j}(y_i-y_j)}\, e^{-2\pi\imath\sum_{i=1}^{\ell}u_{i+1,i}} \\
&\quad\times \exp\Big\{\sum_{i=1}^{\ell+1}(\imath\lambda+\frac{\ell}{2})(x_i-y_i) - \pi\sum_{i=1}^{\ell+1}e^{2(x_i-y_i)} - \pi\sum_{i>j}u_{ij}^2 e^{2(x_i-y_j)}\Big\} \qquad (5.17)\\
&= 2^{\ell+1}\exp\Big\{(\imath\lambda+\frac{\ell}{2})\sum_{i=1}^{\ell+1}(x_i-y_i) - \pi\sum_{i=1}^{\ell+1}e^{2(x_i-y_i)}\Big\}\, e^{\sum_{i>j}(y_i-y_j)} \\
&\quad\times\int_{\mathbb{R}^\ell}\prod_{i=1}^{\ell}du_{i+1,i}\;\exp\Big\{-2\pi\imath\sum_{i=1}^{\ell}u_{i+1,i} - \pi\sum_{i=1}^{\ell}u_{i+1,i}^2 e^{2(x_{i+1}-y_i)}\Big\} \\
&\quad\times\prod_{i>j+1}\int du_{ij}\;\exp\Big\{-\pi u_{ij}^2 e^{2(x_i-y_j)}\Big\}. \qquad (5.18)
\end{aligned}$$

Computing the integrals by using the formula

$$\int_{-\infty}^{\infty} e^{-\imath\omega x - px^2}dx = \sqrt{\frac{\pi}{p}}\;\; e^{\frac{-\omega^2}{4p}} \qquad (5.19)$$

we readily obtain that

$$\begin{aligned}
\mathcal{Q}_0^{\mathfrak{gl}_{\ell+1}}(\underline{x}, \underline{y}|\lambda) = 2^{\ell+1}\exp\Big\{&\sum_{i=1}^{\ell+1}(\imath\lambda+\rho_i)(x_i-y_i) \\
-\pi\sum_{i=1}^{\ell}\Big(&e^{2(x_i-y_i)} + e^{2(y_i-x_{i+1})}\Big) - \pi e^{2(x_{\ell+1}-y_{\ell+1})}\Big\}, \qquad (5.20)
\end{aligned}$$

where $\rho_j = \frac{\ell}{2}+1-j,\quad j=1,\ldots,\ell+1$. This completes the proof of the first statement of the theorem.
ii) The proof of (5.10) follows from the results of Sect. 2. □

It is instructive to provide a direct proof of (5.10). To do so let us first recall standard facts in the theory of spherical functions (see [HC] for details).

There is a general integral expression for the K-biinvariant function in terms of eigenvalues $\Lambda_\phi(\underline{\gamma})$ (5.3). Consider the action on the spherical functions

$$\varphi_{\underline{\gamma}}(g) = \langle\phi_K|\pi_{\underline{\gamma}}(g)|\phi_K\rangle, \qquad (5.21)$$

normalized by the condition $\varphi_{\underline{\gamma}}(e) = 1$. The explicit integral representation for $\varphi_{\underline{\gamma}}(g)$ is

$$\varphi_{\underline{\gamma}}(g) = \int_K dk\; e^{\langle h(gk),\imath\underline{\gamma}+\underline{\rho}\rangle}, \qquad (5.22)$$

where $\int_K dk = 1$ and $h(g) = \log a$, where $g = kan \in KAN_- \to G$ is the Iwasawa decomposition. Then we have

$$\begin{gathered} \phi * \varphi_{\underline{\gamma}}(g) = \Lambda_\phi(\underline{\gamma})\varphi_{\underline{\gamma}}(g), \\ \Lambda_\phi(\underline{\gamma}) = \phi * \varphi_{\underline{\gamma}}(e). \end{gathered} \tag{5.23}$$

Thus the eigenvalues can be written in terms of the spherical transform as follows:

$$\Lambda_\phi(\underline{\gamma}) = \int_G dg\ \phi(g^{-1})\varphi_{\underline{\gamma}}(g) = 2^{-(\ell+1)} \int_{A^+} d^\times a\ \phi(a^{-1})\varphi_{\underline{\gamma}}(a), \tag{5.24}$$

where we have used the Cartan decomposition $G = KA^+(M \setminus K)$ to represent the first integral as an integral over diagonal matrices. Here we define $A^+ = \exp \mathfrak{a}^+$, where $\mathfrak{a}^+$ consists of the diagonal matrices of the form $\mathrm{diag}\,(e^{x_1}, \dots, e^{x_{\ell+1}}),\ \ x_1 \le x_2 \le \dots \le x_{\ell+1}$ and M is the normalizer of $\mathfrak{a}$ in K. Notice that $2^{\ell+1} = |M|$.

Proposition 5.1. *The following integral relation holds:*

$$\begin{aligned} \Lambda_{\phi_{\mathcal{Q}_0(\lambda)}}(\underline{\gamma}) &= 2^{-(\ell+1)} \int_{A^+} d^\times a\ \phi_{\mathcal{Q}_0(\lambda)}(a^{-1})\, \varphi_{\underline{\gamma}}(a) \\ &= \prod_{j=1}^{\ell+1} \pi^{-\frac{\imath\lambda - \gamma_j}{2}}\, \Gamma\Big(\frac{\imath\lambda - \imath\gamma_j}{2}\Big), \end{aligned} \tag{5.25}$$

where $\rho_j = \frac{\ell}{2} + 1 - j,\quad j = 1, \dots, \ell+1.$

Proof. Using the integral representation (5.24), the l.h.s. of (5.25) is given by

$$\int_{K\times A^+} dk d^\times a\ \ |\det a|^{-\imath\lambda - \frac{\ell}{2}} e^{-\pi \mathrm{Tr}\,(a^t a)^{-1}}\ e^{<h(ak), \imath\underline{\gamma}+\underline{\rho}>}. \tag{5.26}$$

Using Cartan and Iwasawa decompositions we have

$$\begin{aligned} &\int_{K\times A^+} dk\ d^\times a\ \ |\det a|^{-\imath\lambda - \frac{\ell}{2}} e^{-\pi \mathrm{Tr}\,(a^t a)^{-1}}\ e^{<h(ak), \imath\underline{\gamma}+\underline{\rho}>} \\ &= \int_{K\times A^+\times K} dk'\ d^\times a\, dk\ \ |\det k'ak|^{-\imath\lambda - \frac{\ell}{2}} e^{-\pi \mathrm{Tr}\,((k'ak)^t (kak'))^{-1}} e^{<h(k'ak), \imath\underline{\gamma}+\underline{\rho}>} \\ &= 2^{\ell+1} \int_{K\times A^+\times M\backslash K} dk'\ d^\times a\, dk\ \ |\det k'ak|^{-\imath\lambda - \frac{\ell}{2}} e^{-\pi \mathrm{Tr}\,((k'ak)^t (kak'))^{-1}} e^{<h(k'ak), \imath\underline{\gamma}+\underline{\rho}>} \\ &= 2^{\ell+1} \int_G dg\, |\det g|^{-\imath\lambda - \frac{\ell}{2}} e^{-\pi \mathrm{Tr}\,(g^t g)^{-1}}\ e^{<h(g), \imath\underline{\gamma}+\underline{\rho}>} \\ &= 2^{\ell+1} \int_{K\times A\times N_-} dn\, d^\times a\, dk\, \delta_{B_-}(a) |\det a|^{-\imath\lambda - \frac{\ell}{2}} e^{-\pi \mathrm{Tr}\,(n^t a^2 n)^{-1}}\ e^{\langle \log(a), \imath\underline{\gamma}+\underline{\rho}\rangle} \\ &= 2^{\ell+1} \int_{A\times N_-} dn\, d^\times a\, \delta_{B_-}(a) |\det a|^{-\imath\lambda - \frac{\ell}{2}} e^{-\pi \mathrm{Tr}\,(n^t a^2 n)^{-1}}\ e^{<\log(a), \imath\underline{\gamma}+\underline{\rho}>} \\ &= \prod_{j=1}^{\ell+1} \pi^{-\frac{\imath\lambda - \gamma_j}{2}}\, \Gamma\Big(\frac{\imath\lambda - \imath\gamma_j}{2}\Big), \end{aligned} \tag{5.27}$$

where the formula

$$\int_{-\infty}^{+\infty} dx\; e^{\nu x} e^{-ae^{-2x}} = \frac{1}{2}\, a^{\frac{\nu}{2}}\, \Gamma(-\frac{\nu}{2})$$

was used. □

The integral operator constructed above can be considered as a universal Baxter operator on matrix elements between the spherical vector and any other vector in the representation space. In particular it is easy to describe explicitly an action of the Baxter operators on the space of zonal spherical functions. In this case one obtains the Baxter operator for the Sutherland model at a particular value of the coupling constant.

5.2. Non-Archimedean analog of Baxter operator. Let us construct a non-Archimedean analog of the universal Baxter $\mathcal{Q}$-operator introduced above. In the non-Archimedean case the local Hecke algebra $\mathcal{H}_p = \mathcal{H}(GL(\ell+1,\mathbb{Q}_p), K_p)$, $K_p = GL(\ell+1,\mathbb{Z}_p)$ is defined as an algebra of the compactly supported K_p-biinvariant functions on $GL(\ell+1,\mathbb{Q}_p)$. Note that K_p is a maximal compact subgroup of $GL(\ell+1,\mathbb{Q}_p)$. Consider a set $\{T_p^{(i)}\}$, $i = 1,\dots,(\ell+1)$ of generators of $\mathcal{H}(GL(\ell+1,\mathbb{Q}_p), K_p)$ given by the characteristic functions of the following subsets:

$$\mathcal{O}_i = K_p \cdot \mathrm{diag}(\underbrace{p,\dots,p}_{i}, 1\dots,1) \cdot K_p \subset GL(\ell+1,\mathbb{Q}_p). \tag{5.28}$$

The action of $T_p^{(i)}$ on functions $f \in C(G/K)$ is then given by the following integral formula:

$$(T_p^{(i)} f)(g) = \int_{\mathcal{O}_i} f(gh)dh. \tag{5.29}$$

This can be considered as a convolution with characteristic function $T_p^{(i)}$ of $\mathcal{O}_i$. For an appropriately defined non-Archimedean $\mathfrak{gl}_{\ell+1}$-Whittaker function W_σ [Sh,CS] one has

$$T_p^{(i)}\, W_\sigma = \mathrm{Tr}_{V_{\omega_i}}\, \rho_i(\sigma)\; W_\sigma, \tag{5.30}$$

where $\rho_i : GL(\ell+1,\mathbb{C}) \to \mathrm{End}(V_{\omega_i},\mathbb{C})$, $V_{\omega_i} = \wedge^i \mathbb{C}^{\ell+1}$ is a representation of $GL(\ell+1,\mathbb{C})$ corresponding to the fundamental weight ω_i and σ is a conjugacy class in $GL(\ell+1,\mathbb{C})$ corresponding to a non-Archimedean Whittaker function W_σ. Note that, in contrast with (5.30), the standard normalization of $T_p^{(i)}$ includes an additional factor $p^{-i(i-1)/2}$. More generally, one considers Hecke operators $T_p^{(V)}$ associated to arbitrary finite dimensional representations $\rho_V : GL(\ell+1,\mathbb{C}) \to \mathrm{End}(V,\mathbb{C})$ satisfying

$$T_p^{(V)}\, W_\sigma = \mathrm{Tr}_V\, \rho_V(\sigma)\; W_\sigma. \tag{5.31}$$

It is natural to arrange the generators of $\mathcal{H}_p$ into the following generating function:

$$T_p(\lambda) = \sum_{j=1}^{\ell+1} (-1)^j\, p^{-(\ell+1-j)\lambda}\, T_p^{(j)}. \tag{5.32}$$

We introduce another generating function

$$\mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda) = \sum_{n=0}^{\infty} p^{-n\lambda}\; T_p^{(S^n V)}, \tag{5.33}$$

where $V = \mathbb{C}^{\ell+1}$ is the standard representation of $\mathfrak{gl}_{\ell+1}(\mathbb{C})$. The generating functions (5.32), (5.33) satisfy the following relations:

$$\mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda) \cdot \mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda') = \mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda') \cdot \mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda), \tag{5.34}$$

$$\mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda) \cdot T_p(\lambda') = T_p(\lambda') \cdot \mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda), \tag{5.35}$$

$$1 = T_p(\lambda) \cdot \mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda), \tag{5.36}$$

and the operators $T_p(\lambda)$ and $\mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda)$ act on the non-Archimedean analog of the Whittaker function as

$$T_p(\lambda)\; W_\sigma = \det_V(1 - p^{-\lambda}\rho_V(\sigma))\, W_\sigma, \tag{5.37}$$

$$\mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda)\; W_\sigma = \det_V(1 - p^{-\lambda}\rho_V(\sigma))^{-1}\, W_\sigma. \tag{5.38}$$

Thus the eigenvalues of $\mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda)$ are given by the local non-Archimedean L-factors

$$L_p(s) = \det_V(1 - p^{-s}\rho_V(\sigma))^{-1}, \tag{5.39}$$

where we use a more traditional notation $s := \lambda$.

Comparing (5.34), (5.35), (5.36) with (2.22), (2.24), (2.25) one can see that the $\mathfrak{gl}_{\ell+1}$ Baxter $\mathcal{Q}$-operator appears quite similar to the generating function $\mathcal{Q}_p^{\mathfrak{gl}_{\ell+1}}(\lambda)$ in the Hecke algebra $\mathcal{H}(GL(\ell+1,\mathbb{Q}_p), K_p)$ and the analog of $T_p(\lambda)$ is given by (2.26). In particular both operators share the property that their eigenvalues are given by local L-factors.

One can represent Archimedean and non-Archimedean Baxter operators in a unified form. Let us rewrite (5.33) as

$$\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}(\lambda)(g) = \sum_{(n_1,,\ldots n_{\ell+1})\in\mathbb{Z}_+^{\ell+1}} (p^{n_1}\cdots p^{n_{\ell+1}})^{\imath\lambda}\delta_{\underline{n}}(g), \tag{5.40}$$

where $\underline{n} = (n_1,\ldots,n_{\ell+1})$, $\delta_{\underline{n}}(g)$ is a characteristic function of $\mathcal{O}_{\underline{n}} \subset GL(\ell+1,\mathbb{Q}_p)$,

$$\mathcal{O}_{\underline{n}} = K_p \cdot \mathrm{diag}(p^{n_1},\ldots,p^{n_{\ell+1}}) \cdot K_p. \tag{5.41}$$

On the other hand the (universal) Archimedean Baxter $\mathcal{Q}$-operator (5.9) can be written in the following form:

$$\phi_{\mathcal{Q}_0(\lambda)}(g) = \int dt_1\cdots dt_{\ell+1}\,(t_1\cdots t_{\ell+1})^{\imath\lambda} e^{-\pi\sum_{j=1}^{\ell+1} t_i^2}\delta_{\underline{t}}(g), \tag{5.42}$$

where $\delta_{\underline{t}}(g)$, $\underline{t} = (t_1,\ldots,t_{\ell+1})$ is an appropriately defined function with the support at $\mathcal{O}_{\underline{t}} \subset GL(\ell+1,\mathbb{R})$,

$$\mathcal{O}_{\underline{t}} = K \cdot \mathrm{diag}(t_1,\ldots,t_{\ell+1}) \cdot K. \tag{5.43}$$

The integral formulas (5.40) and (5.42) are compatible in the sense of the standard correspondence between Archimedean and non-Archimedean integrals (see e.g. [W]).

6. Baxter Operator for $\mathfrak{so}_{2\ell+1}$

In the next section we define a Baxter $\mathcal{Q}$-operator for $\mathfrak{g} = \mathfrak{so}_{2\ell+1}$ and demonstrate that the relation between local L-factors and eigenvalues of $\mathcal{Q}$-operators holds in this case. A more systematic discussion of the general case will be given elsewhere.

According to [Ko1], the $\mathfrak{so}_{2\ell+1}$-Whittaker function can be written in terms of the invariant pairing of Whittaker modules as follows

$$\Psi_\lambda^{\mathfrak{so}_{2\ell+1}}(x) = e^{-\langle\rho,x\rangle}\langle\psi_L\,,\pi_\lambda(e^{h_x})\,\psi_R\rangle, \qquad x \in \mathfrak{h}, \tag{6.1}$$

where $h_x := \sum_{i=1}^{\ell}\langle\omega_i, x\rangle\, h_i$, ω_i is a basis of the fundamental weights of $\mathfrak{so}_{2\ell+1}$. Note that $\mathfrak{so}_{2\ell+1}$-Whittaker functions are common eigenfunctions of the complete set of the commuting $\mathfrak{so}_{2\ell+1}$-Toda chain Hamiltonians $\mathcal{H}_{2k} \in \mathrm{Diff}(\mathfrak{h})$, $k = 1, \ldots, \ell$ defined by

$$\mathcal{H}_{2k}^{\mathfrak{so}_{2\ell+1}}\Psi_\lambda^{\mathfrak{so}_{2\ell+1}}(x) = e^{-\langle\rho,x\rangle}\langle\psi_L\,,\pi_\lambda(e^{h_x})\,c_{2k}\,\psi_R\rangle, \tag{6.2}$$

where $\{c_{2k}\}$ are generators of the center $\mathcal{Z}(\mathfrak{so}_{2\ell+1}) \subset \mathcal{U}(\mathfrak{so}_{2\ell+1})$. For the quadratic Hamiltonian we have

$$\mathcal{H}_2^{\mathfrak{so}_{2\ell+1}} = -\frac{1}{2}\sum_{i=1}^{\ell}\frac{\partial^2}{\partial {x_i}^2} + \frac{1}{2}e^{x_1} + \sum_{i=1}^{\ell-1}e^{x_{i+1}-x_i}. \tag{6.3}$$

Let us introduce a generating function for the $\mathfrak{so}_{2\ell+1}$-Toda chain Hamiltonians as

$$t^{\mathfrak{so}_{2\ell+1}}(\lambda) = \sum_{j=1}^{\ell}(-1)^j\lambda^{2\ell+1-2j}\mathcal{H}_{2j}^{\mathfrak{so}_{2\ell+1}}(x). \tag{6.4}$$

Then the $\mathfrak{so}_{2\ell+1}$-Whittaker function satisfies the following equation:

$$t^{\mathfrak{so}_{2\ell+1}}(\lambda)\;\;\Psi_{\underline{\lambda}}^{\mathfrak{so}_{2\ell+1}}(\underline{x}) = \lambda\prod_{j=1}^{\ell}(\lambda^2-\lambda_j^2)\;\;\Psi_{\underline{\lambda}}^{\mathfrak{so}_{2\ell+1}}(\underline{x}), \tag{6.5}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{\ell+1})$ and $\underline{x} = (x_1, \ldots, x_{\ell+1})$.

Theorem 6.1. *Eigenfunctions of the $\mathfrak{so}_{2\ell+1}$-Toda chain admit the integral representation:*

$$\Psi_{\lambda_1,\ldots,\lambda_\ell}^{\mathfrak{so}_{2\ell+1}}(x_{\ell,1}, \ldots, x_{\ell,\ell}) \;=\; \int_{\mathbb{R}^{\ell^2}}\prod_{k=1}^{\ell-1}\prod_{i=1}^{k}dx_{k,i}\prod_{k=1}^{\ell}\prod_{i=1}^{k}dz_{k,i}\;e^{\mathcal{F}^{\mathfrak{so}_{2\ell+1}}(x,z)},$$

where

$$\begin{aligned}\mathcal{F}^{\mathfrak{so}_{2\ell+1}}(x,z) \;=\; & -\imath\lambda_1(x_{1,1}-2z_{1,1}) - \imath\sum_{n=2}^{\ell}\lambda_n\Big(\sum_{i=1}^{n}x_{n,i} - 2\sum_{i=1}^{n}z_{n,i} + \sum_{i=1}^{n-1}x_{n-1,i}\Big)\\ & -\Big\{\sum_{n=1}^{\ell}e^{z_{n,1}} + \sum_{k=2}^{\ell}\sum_{n=k+1}^{\ell}\Big(e^{x_{n-1,k}-z_{n,k}} + e^{x_{n,k}-z_{n,k}}\Big)\\ & + \sum_{n=k}^{\ell}\Big(e^{z_{n,k}-x_{n-1,k-1}} + e^{z_{n,k}-x_{n,k-1}}\Big) + \sum_{n=1}^{\ell}e^{x_{n,n}-z_{n,n}}\Big\},\end{aligned} \tag{6.6}$$

where we set $x_i := x_{\ell,i}$, $1 \le i \le \ell$.

This integral representation was proposed in [GLO3] (we made an additional change of variables $z_{\ell,1}\longmapsto -z_{\ell,1}+\ln\Big(e^{x_{\ell,1}}+e^{x_{\ell-1,1}}\Big)$ in the integral representation given in [GLO3]).

Corollary 6.1. *The following integral operators $Q^{\mathfrak{so}_{2\ell+1}}_{\mathfrak{so}_{2\ell-1}}$ provide a recursive construction of the $\mathfrak{so}_{2\ell+1}$-Whittaker function:*

$$\Psi^{\mathfrak{so}_{2\ell+1}}_{\lambda_1,\ldots,\lambda_\ell}(\underline{x}_\ell)=\int_{\mathbb{R}^{\ell-1}}\prod_{i=1}^{\ell-1}dx_{\ell-1,i}\;\; Q^{\mathfrak{so}_{2\ell+1}}_{\mathfrak{so}_{2\ell-1}}(\underline{x}_\ell,\underline{x}_{\,\ell-1}|\lambda_\ell)\Psi^{\mathfrak{so}_{2\ell-1}}_{\lambda_1,\ldots,\lambda_{\ell-1}}(\underline{x}_{\,\ell-1}),\tag{6.7}$$

where

$$\begin{aligned}Q^{\mathfrak{so}_{2\ell+1}}_{\mathfrak{so}_{2\ell-1}}(\underline{x}_\ell,\underline{x}_{\,\ell-1}|\lambda_\ell)&=\int_{\mathbb{R}^\ell}\prod_{i=1}^{\ell}dz_{\ell,i}\\ &\times\exp\Big\{-\imath\lambda_\ell\Big(\sum_{i=1}^{\ell}x_{\ell,i}-2\sum_{i=1}^{\ell}z_{\ell,i}+\sum_{i=1}^{\ell-1}x_{\ell-1,i}\Big)\Big\}\\ &\times\exp\Big\{-\Big(e^{z_{\ell,1}}+\sum_{i=1}^{\ell-1}\Big(e^{x_{\ell-1,i}-z_{\ell,i}}+e^{z_{\ell,i+1}-x_{\ell-1,i}}\Big)\\ &\qquad+\sum_{i=1}^{\ell-1}\Big(e^{x_{\ell,i}-z_{\ell,i}}+e^{z_{\ell,i+1}-x_{\ell,i}}\Big)+e^{x_{\ell,\ell}-z_{\ell,\ell}}\Big)\Big\}.\end{aligned}\tag{6.8}$$

For $\ell=1$ we set

$$Q^{\mathfrak{so}_3}_{\mathfrak{so}_1}(x_{1,1}|\lambda_1)=\int_{\mathbb{R}}dz_{1,1}e^{\imath\lambda_1x_{1,1}-2\imath\lambda_1z_{1,1}}\exp\Big\{-\Big(e^{z_{1,1}}+e^{x_{1,1}-z_{1,1}}\Big)\Big\}.\tag{6.9}$$

Below $\Psi^{\mathfrak{so}_{2\ell+1}}_{\lambda}(x)$ will always denote the unique W-invariant solution of (6.5) (class one principal series Whittaker function). Note that the space of W-invariant Whittaker functions $\Psi^{\mathfrak{so}_{2\ell+1}}_{\lambda}(x)$ provides a basis in the space of W-invariant functions on $\mathbb{R}^\ell$.

Definition 6.1. *The Baxter $\mathcal{Q}$-operator for $\mathfrak{so}_{2\ell+1}$ is given by*

$$\begin{aligned}\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\underline{y},\underline{x}|\lambda)\;&=\int_{\mathbb{R}^{\ell+1}}\prod_{i=1}^{\ell+1}dz_i\;\exp\Big\{-\imath\lambda\Big(\sum_{i=1}^{\ell}y_i-2\sum_{i=1}^{\ell+1}z_i+\sum_{i=1}^{\ell}x_i\Big)\Big\}\\ &\times\exp\Big\{-e^{z_1}-\sum_{i=1}^{\ell}\Big(e^{y_i-z_i}+e^{z_{i+1}-y_i}+e^{x_i-z_i}+e^{z_{i+1}-x_i}\Big)\Big\},\end{aligned}\tag{6.10}$$

where $\underline{y}=(y_1,\ldots,y_\ell)$ and $\underline{x}=(x_1,\ldots,x_\ell)$.

Theorem 6.2. *The operator $\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\lambda)$ satisfies the following identities:*

$$\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\lambda)\,\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\lambda')=\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\lambda')\,\mathcal{Q}^{\mathfrak{so}_{\ell+1}}(\lambda),\tag{6.11}$$

$$\mathcal{Q}^{\mathfrak{so}_{\ell+1}}(\lambda)\cdot\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}_{\mathfrak{so}_{2\ell-1}}(\lambda')=\Gamma\Big(\imath\lambda'-\imath\lambda\Big)\Gamma\Big(-\imath\lambda'-\imath\lambda\Big)\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}_{\mathfrak{so}_{2\ell-1}}(\lambda')\cdot\mathcal{Q}^{\mathfrak{so}_{\ell-1}}(\lambda),\tag{6.12}$$

$$\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\lambda)\,T^{\mathfrak{so}_{2\ell+1}}(\lambda')=T^{\mathfrak{so}_{2\ell+1}}(\lambda')\,\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\lambda),\tag{6.13}$$

$$\lambda\mathcal{Q}^{\mathfrak{so}_{\ell+1}}(\lambda+\imath)=\imath^{2\ell}\,\mathcal{Q}^{\mathfrak{so}_{\ell+1}}(\lambda)\;T^{\mathfrak{so}_{2\ell+1}}(\lambda),\tag{6.14}$$

where

$$T^{\mathfrak{so}_{2\ell+1}}(\underline{x},\underline{y}|\lambda)=t^{\mathfrak{so}_{2\ell+1}}(\underline{x},\partial_{\underline{x}}|\lambda)\delta^{(\ell)}(\underline{x}-\underline{y}), \tag{6.15}$$

$$t^{\mathfrak{so}_{2\ell+1}}(\underline{x},\partial_{\underline{x}}|\lambda)=\sum_{j=1}^{\ell+1}(-1)^j\lambda^{2\ell+1-2j}\,\mathcal{H}_{2j}^{\mathfrak{so}_{2\ell+1}}(\underline{x},\partial_{\underline{x}}). \tag{6.16}$$

Proof. We will prove the commutativity of $\mathcal{Q}$-operators (6.11). The relation (6.12) can be proved using a similar approach. The other identities then easily follow.

To prove (6.11) we should verify the following identity between the kernels:

$$\int_{\mathbb{R}^{\ell+1}}\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\underline{y},\underline{x}|\lambda)\;\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\underline{x},\underline{z}|\lambda')\;\prod_{j=1}^{\ell+1}dx_j \tag{6.17}$$

$$=\int_{\mathbb{R}^{\ell+1}}\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\underline{y},\underline{x}|\lambda')\;\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\underline{x},\underline{z}|\lambda)\;\prod_{j=1}^{\ell+1}dx_j, \tag{6.18}$$

where

$$\begin{aligned}&\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\underline{y},\underline{x}|\lambda)\\&=\int_{\mathbb{R}^{\ell+1}}\prod_{i=1}^{\ell+1}du_i\;\exp\Big\{-\imath\lambda\Big(\sum_{i=1}^{\ell}y_i-2\sum_{i=1}^{\ell+1}u_i+\sum_{i=1}^{\ell}x_i\Big)\Big\}\\&\quad\times\exp\Big\{-e^{u_1}-\sum_{i=1}^{\ell}\Big(e^{y_i-u_i}+e^{u_{i+1}-y_i}+e^{x_i-u_i}+e^{u_{i+1}-x_i}\Big)\Big\},\end{aligned} \tag{6.19}$$

$$\begin{aligned}&\mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\underline{x},\underline{z}|\lambda')\\&=\int_{\mathbb{R}^{\ell+1}}\prod_{i=1}^{\ell+1}dv_i\;\exp\Big\{-\imath\lambda'\Big(\sum_{i=1}^{\ell}x_i-2\sum_{i=1}^{\ell+1}v_i+\sum_{i=1}^{\ell}z_i\Big)\Big\}\\&\quad\times\exp\Big\{-e^{v_1}-\sum_{i=1}^{\ell}\Big(e^{x_i-v_i}+e^{v_{i+1}-x_i}+e^{z_i-v_i}+e^{v_{i+1}-z_i}\Big)\Big\}.\end{aligned} \tag{6.20}$$

The proof is given by the following sequence of elementary transformations. Let us first make a change of variables u_i and v_i in (6.17):

$$\begin{aligned}&u_1\longmapsto -u_1+\ln\Big(e^{y_1}+e^{x_1}\Big),\\&u_i\longmapsto -u_i-\ln\Big(e^{y_{i-1}}+e^{x_{i-1}}\Big)+\ln\Big(e^{y_i}+e^{x_i}\Big),\quad 1<i\leq\ell,\\&v_1\longmapsto -v_1+\ln\Big(e^{x_1}+e^{z_1}\Big),\\&v_i\longmapsto -v_i-\ln\Big(e^{x_{i-1}}+e^{z_{i-1}}\Big)+\ln\Big(e^{x_i}+e^{z_i}\Big),\quad 1<i\leq\ell.\end{aligned}$$

We introduce additional integration variables $u_{\ell+1}$ and $v_{\ell+1}$ in (6.17) using integral formulas:

$$\begin{aligned}\Big(e^{-y_\ell}+e^{-x_\ell}\Big)^{-2\imath\lambda'} &=\Gamma(2\imath\lambda')^{-1}\int_{\mathbb{R}} du_{\ell+1}\exp\Big\{2\imath\lambda' u_{\ell+1}-e^{u_{\ell+1}-y_\ell}-e^{u_{\ell+1}-x_\ell}\Big\},\\ \Big(e^{-x_\ell}+e^{-z_\ell}\Big)^{-2\imath\lambda} &=\Gamma(2\imath\lambda)^{-1}\int_{\mathbb{R}} dv_{\ell+1}\exp\Big\{2\imath\lambda' v_{\ell+1}-e^{v_{\ell+1}-x_\ell}-e^{v_{\ell+1}-z_\ell}\Big\}.\end{aligned} \tag{6.21}$$

Then let us modify the variables $x_i,\ i=1,\dots,\ell$ as:

$$x_i \longmapsto -x_i-\ln\Big(e^{-u_i}+e^{-z_i}\Big)+\ln\Big(e^{u_{i+1}}+e^{z_{i+1}}\Big), \tag{6.22}$$

and use the following integral representations to introduce the additional variables x_0 and $x_{\ell+1}$:

$$\begin{aligned}\Big(e^{u_1}+e^{v_1}\Big)^{-\imath(\lambda+\lambda')} &=\Gamma(\imath(\lambda+\lambda'))^{-1}\int_{\mathbb{R}} dx_0\\ &\quad\times\exp\Big\{-\imath(\lambda+\lambda')x_0-e^{u_1-x_0}-e^{z_1-x_0}\Big\},\\ \Big(e^{-u_{\ell+1}}+e^{-v_{\ell+1}}\Big)^{\imath(\lambda+\lambda')} &=\Gamma(-\imath(\lambda+\lambda'))^{-1}\\ &\quad\times\int_{\mathbb{R}} dx_{\ell+1}\exp\Big\{-\imath(\lambda+\lambda')x_{\ell+1}-e^{x_{\ell+1}-u_{\ell+1}}-e^{x_{\ell+1}-v_{\ell+1}}\Big\}.\end{aligned}$$

Now we make the following sequence of changes of the variables:

$$\begin{aligned}u_1 &\longmapsto -u_1-\ln\Big(1+e^{-x_0}\Big)+\ln\Big(e^{y_1}+e^{x_1}\Big),\\ u_i &\longmapsto -u_i-\ln\Big(e^{y_{i-1}}+e^{x_{i-1}}\Big)+\ln\Big(e^{y_i}+e^{x_i}\Big),\quad 1<i\le\ell,\\ u_{\ell+1} &\longmapsto -u_{\ell+1}+x_{\ell+1}-\ln\Big(e^{-y_\ell}+e^{-x_\ell}\Big),\end{aligned}$$

$$\begin{aligned}v_1 &\longmapsto -v_1-\ln\Big(1+e^{-x_0}\Big)+\ln\Big(e^{x_1}+e^{z_1}\Big),\\ v_i &\longmapsto -v_i-\ln\Big(e^{x_{i-1}}+e^{z_{i-1}}\Big)+\ln\Big(e^{x_i}+e^{z_i}\Big),\quad 1<i\le\ell,\\ v_{\ell+1} &\longmapsto -v_{\ell+1}+x_{\ell+1}-\ln\Big(e^{-x_\ell}+e^{-z_\ell}\Big),\end{aligned}$$

$$\begin{aligned}x_0 &\longmapsto -x_0+\ln\Big(e^{u_1}+e^{z_1}\Big),\\ x_i &\longmapsto -x_i-\ln\Big(e^{-u_i}+e^{-z_i}\Big)+\ln\Big(e^{u_{i+1}}+e^{z_{i+1}}\Big),\quad 1\le i\le\ell,\\ x_{\ell+1} &\longmapsto -x_{\ell+1}-\ln\Big(e^{-u_{\ell+1}}+e^{-z_{\ell+1}}\Big).\end{aligned} \tag{6.23}$$

One integrates out the variables x_0 and $x_{\ell+1}$ and modifies the variables u_i and v_i as follows

$$\begin{aligned}u_1 &\longmapsto -u_1+\ln\Big(e^{y_1}+e^{x_1}\Big),\\ u_i &\longmapsto -u_i-\ln\Big(e^{y_{i-1}}+e^{x_{i-1}}\Big)+\ln\Big(e^{y_i}+e^{x_i}\Big),\quad 1<i<\ell,\\ u_\ell &\longmapsto -u_\ell-\ln\Big(e^{-y_{\ell-1}}+e^{-x_{\ell-1}}\Big),\end{aligned}$$

$$\begin{aligned} v_1 &\longmapsto -v_1 + \ln\Big(e^{x_1} + e^{z_1}\Big), \\ v_i &\longmapsto -v_i - \ln\Big(e^{x_{i-1}} + e^{z_{i-1}}\Big) + \ln\Big(e^{x_i} + e^{z_i}\Big), \quad 1 < i < \ell, \\ v_\ell &\longmapsto -v_\ell - \ln\Big(e^{-x_{\ell-1}} + e^{-z_{\ell-1}}\Big). \end{aligned}$$

Integrating out $u_{\ell+1}$ and $v_{\ell+1}$, one completes the proof of (6.11). □

Corollary 6.2. *The following identity holds:*

$$\begin{aligned} &\int_{\mathbb{R}^\ell} \prod_{i=1}^{\ell} dx_{\ell,i}\, \mathcal{Q}^{\mathfrak{so}_{2\ell+1}}(\underline{y}, \underline{x}|\gamma)\, \Psi^{\mathfrak{so}_{2\ell+1}}_{\underline{\lambda}}(\underline{x}) \\ &= \prod_{i=1}^{\ell} \Gamma\Big(\imath\lambda_i - \imath\gamma\Big) \prod_{i=1}^{\ell} \Gamma\Big(-\imath\lambda_i - \imath\gamma\Big)\, \Psi^{\mathfrak{so}_{2\ell+1}}_{\underline{\lambda}}(\underline{y}). \end{aligned} \tag{6.24}$$

Finally let us note that this result is in agreement with the interpretation of the eigenvalues of $\mathcal{Q}$-operators as local Archimedean L-functions corresponding to automorphic representations of reductive Lie groups discussed above.

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