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Bifurcation of limit cycles at degenerate singular point and infinity in a septic system

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Full list of author information is available at the end of the article**Abstract**

In this paper, the problem of bifurcation of limit cycles from degenerate singular point and infinity in a class of septic polynomial differential systems is investigated. Using the computer algebra system Mathematica, the limit cycle configurations of $\{(8), 3\}$ and $\{(3), 6\}$ are obtained under synchronous perturbation at degenerate singular point and infinity. To our knowledge, up to now, this is the first time that the problem of limit cycles bifurcated from degenerate singular point and infinity under synchronous perturbed conditions in a septic system has been investigated.

MSC: 34C05; 34C07**Keywords:** degenerate singular point; infinity; bifurcation of limit cycles; septic system

1 Introduction

In the qualitative theory of planar dynamical systems, bifurcation of limit cycles for planar polynomial differential system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad (1.1)$$

which belongs to the second part of the Hilbert 16th problem, is known as a hot but intractable issue. In the case of limit cycles bifurcated from the origin, a lot of work has been done (see [1, 2]). Let I_n be the maximum possible number of limit cycles in the neighborhood of the infinity (large amplitude limit cycles) of system (1.1) when $P(x, y)$, $Q(x, y)$ are of degree at most n . As far as the number of limit cycles bifurcated from infinity is concerned, representative results are as follows: cubic systems, [3] got $I_3 \geq 4$, [4] got $I_3 \geq 5$, [5] got $I_3 \geq 6$, [6, 7] got $I_3 \geq 7$; quintic systems, [8] got $I_5 \geq 4$, [9] got $I_5 \geq 6$, [10] got $I_5 \geq 8$, [11] got $I_5 \geq 11$; septic systems, [12] got $I_7 \geq 8$, [13] got $I_7 \geq 9$, [14] got $I_7 \geq 13$.

Definition 1.1 [4] We call the configuration $\{(k_1), k_2\}$ the configuration of a vector field with k_1 'small' limit cycles and k_2 'large' ones.

Few papers are concerned with bifurcation of limit cycles from the origin and infinity under synchronous perturbation: [4] obtained the limit cycle configurations of $\{(5), 2\}$ and $\{(2), 4\}$ in a cubic polynomial differential system; [15] and [16] respectively got the limit cycle configurations of $\{(5), 2\}$, $\{(2), 4\}$ and $\{(7), 2\}$, $\{(2), 6\}$ in two certain quintic systems. However, for a higher degree system, there is not any result.

In this paper, we deal with limit cycles bifurcated from degenerate singular point and infinity under synchronous perturbation in the following real septic polynomial differential system:

$$\begin{aligned} \frac{dx}{dt} &= \lambda(\delta_1 x - y)(x^2 + y^2) + X_5(x, y) + (\delta_2 x - y)(x^2 + y^2)^3, \\ \frac{dy}{dt} &= \lambda(x + \delta_1 y)(x^2 + y^2) + Y_5(x, y) + (x + \delta_2 y)(x^2 + y^2)^3, \end{aligned} \tag{1.2}$$

where $\lambda \neq 0$ and

$$X_5(x, y) = \sum_{k+j=5} A_{kj} x^k y^j, \quad Y_5(x, y) = \sum_{k+j=5} B_{kj} x^k y^j, \tag{1.3}$$

$$\begin{aligned} A_{50} &= \beta_{03} + \beta_{12} + \beta_{21} + \beta_{30}, & B_{50} &= \alpha_{03} + \alpha_{12} + \alpha_{21} + \alpha_{30}, \\ A_{41} &= -5\alpha_{03} - 3\alpha_{12} - \alpha_{21} + \alpha_{30}, & B_{41} &= 5\beta_{03} + 3\beta_{12} + \beta_{21} - \beta_{30}, \\ A_{32} &= -2(5\beta_{03} + \beta_{12} - \beta_{21} - \beta_{30}), & B_{32} &= -2(5\alpha_{03} + \alpha_{12} - \alpha_{21} - \alpha_{30}), \\ A_{23} &= 2(5\alpha_{03} - \alpha_{12} - \alpha_{21} + \alpha_{30}), & B_{23} &= -2(5\beta_{03} - \beta_{12} - \beta_{21} + \beta_{30}), \\ A_{14} &= 5\beta_{03} - 3\beta_{12} + \beta_{21} + \beta_{30}, & B_{14} &= 5\alpha_{03} - 3\alpha_{12} + \alpha_{21} + \alpha_{30}, \\ A_{05} &= -\alpha_{03} + \alpha_{12} - \alpha_{21} + \alpha_{30}, & B_{05} &= \beta_{03} - \beta_{12} + \beta_{21} - \beta_{30}. \end{aligned} \tag{1.4}$$

This paper is organized as follows. In Section 2, some preliminaries are given and system (1.2) is transferred to its complex concomitant system. In Sections 3 and 4, the first nine singular point quantities at degenerate singular point and the first seven singular point quantities at infinity are deduced. At the same time, their center conditions are obtained. In Section 5, the sufficient conditions of limit cycle configurations $\{(8), 3\}$ and $\{(3), 6\}$ are presented.

2 Conversion of the system

Obviously, the origin of system (1.2) is either a center or a focus, and it is also a degenerate singular point. System (1.2) contains no real singular point on the equator Γ_∞ of the Poincaré compactification on the sphere. Γ_∞ is called infinity on the Gauss sphere or the equator of system (1.2).

By means of polar coordinate transformation

$$x = r \cos \theta, \quad y = r \sin \theta, \tag{2.1}$$

system (1.2) takes the form

$$\frac{dr}{d\theta} = r \frac{\delta_1 \lambda + \delta_2 r^4 + r^2 [\cos \theta X_5(\cos \theta, \sin \theta) + \sin \theta Y_5(\cos \theta, \sin \theta)]}{\lambda + r^4 + r^2 [\cos \theta Y_5(\cos \theta, \sin \theta) - \sin \theta X_5(\cos \theta, \sin \theta)]}. \tag{2.2}$$

For a sufficiently small h , let

$$r = r(\theta, h) = \sum_{m=1}^{\infty} v_m(\theta) h^m \tag{2.3}$$

be the solution of Eq. (2.2) satisfying the initial value condition $r|_{\theta=0} = h$, where

$$v_1(\theta) = e^{\delta_1\theta}, \quad v_m(0) = 0, \quad m = 2, 3, \dots \tag{2.4}$$

The 0th focal value at the origin of system (1.2) is $v_1(2\pi) - 1$, i.e., $e^{2\pi\delta_1} - 1$, the k th one is $v_{2k+1}(2\pi)$, $k = 1, 2, \dots$

By means of generalized polar coordinate transformation

$$x = r^{-1} \cos \theta, \quad y = r^{-1} \sin \theta, \tag{2.5}$$

system (1.2) takes the form

$$\frac{dr}{d\theta} = -r \frac{\delta_2 + \delta_1 \lambda r^4 + r^2 [\cos \theta X_5(\cos \theta, \sin \theta) + \sin \theta Y_5(\cos \theta, \sin \theta)]}{1 + \lambda r^4 + r^2 [\cos \theta Y_5(\cos \theta, \sin \theta) - \sin \theta X_5(\cos \theta, \sin \theta)]}. \tag{2.6}$$

For a sufficiently small h , let

$$r = r(\theta, h) = \sum_{m=1}^{\infty} v_m(\theta) h^m \tag{2.7}$$

be the solution of Eq. (2.6) satisfying the initial value condition $r|_{\theta=0} = h$, where

$$v_1(\theta) = e^{-\delta_2\theta}, \quad v_m(0) = 0, \quad m = 2, 3, \dots \tag{2.8}$$

The 0th focal value at infinity of system (1.2) is $v_1(2\pi) - 1$, i.e., $e^{-2\pi\delta_2} - 1$, the k th one is $v_{2k+1}(2\pi)$, $k = 1, 2, \dots$

By means of transformation

$$z = x + iy, \quad w = x - iy, \quad T = it, \quad i = \sqrt{-1}, \tag{2.9}$$

system (1.2) can be transformed into the following complex system:

$$\begin{aligned} \frac{dz}{dT} &= \lambda(1 - i\delta_1)z^2w + b_{03}z^5 + b_{12}z^4w + b_{21}z^3w^2 + b_{30}z^2w^3 + (1 - i\delta_2)z^4w^3, \\ \frac{dw}{dT} &= -[\lambda(1 + i\delta_1)w^2z + a_{03}w^5 + a_{12}w^4z + a_{21}w^3z^2 + a_{30}w^2z^3 + (1 + i\delta_2)w^4z^3], \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} a_{30} &= \alpha_{30} + i\beta_{30}, & a_{21} &= \alpha_{21} + i\beta_{21}, \\ a_{12} &= \alpha_{12} + i\beta_{12}, & a_{03} &= \alpha_{03} + i\beta_{03}, \\ b_{30} &= \alpha_{30} - i\beta_{30}, & b_{21} &= \alpha_{21} - i\beta_{21}, \\ b_{12} &= \alpha_{12} - i\beta_{12}, & b_{03} &= \alpha_{03} - i\beta_{03}. \end{aligned} \tag{2.11}$$

It is obvious that the coefficients of system (2.10) satisfy the conjugate condition, i.e.,

$$b_{ij} = \overline{a_{ij}}, \quad (i, j) \in \{(3, 0), (2, 1), (1, 2), (0, 3)\}. \tag{2.12}$$

We call that systems (1.2) and (2.10) are associated.

3 Singular point quantities and center conditions at degenerate singular point

From Theorem 3.2 in [3], we have

Lemma 3.1 *For system (2.10)_{δ₁=δ₂=0}, we can determine uniquely an extended formal power series*

$$F(z, w) = zw \sum_{k=0}^{\infty} \frac{f_{5k}(z, w)}{(zw)^{2k}}, \tag{3.1}$$

such that

$$\left. \frac{dF}{dT} \right|_{(2.10)_{\delta_1=\delta_2=0}} = \sum_{m=1}^{\infty} \mu_m^{(0)} (zw)^{m+2}, \tag{3.2}$$

where $c_{00} = 1$, $c_{kk} = 0$, $k = 1, 2, \dots$, $f_{5k}(z, w) = \sum_{\alpha+\beta=5k} c_{\alpha\beta} z^\alpha w^\beta$, and for any positive integer m , the m th singular point quantity at the origin $\mu_m^{(0)}$ can be determined by the following recursion formulas:

$$c_{00} = 1,$$

when $(\alpha = \beta > 0)$ or $\alpha < 0$ or $\beta < 0$, $c_{\alpha\beta} = 0$;

else

$$\begin{aligned} c_{\alpha\beta} = & \frac{1}{5\lambda(\beta - \alpha)} \left((5\alpha - 5\beta)c_{-10+\alpha, -10+\beta} + b_{03}(-10 + 3\alpha - 2\beta)c_{-7+\alpha, -3+\beta} \right. \\ & + (b_{12}(-5 + 3\alpha - 2\beta) - a_{30}(5 - 2\alpha + 3\beta))c_{-6+\alpha, -4+\beta} + (b_{21}(3\alpha - 2\beta) \\ & - a_{21}(-2\alpha + 3\beta))c_{-5+\alpha, -5+\beta} + (b_{30}(5 + 3\alpha - 2\beta) - a_{12}(-5 - 2\alpha + 3\beta)) \\ & \left. \times c_{-4+\alpha, -6+\beta} - a_{03}(-10 - 2\alpha + 3\beta)c_{-3+\alpha, -7+\beta} \right), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mu_m^{(0)} = & b_{03}(-2 + m)c_{-17+5(2+m), -13+5(2+m)} + (b_{12}(-1 + m) - a_{30}(1 + m)) \\ & \times c_{-16+5(2+m), -14+5(2+m)} + (-a_{21}m + b_{21}m)c_{-15+5(2+m), -15+5(2+m)} \\ & + (-a_{12}(-1 + m) + b_{30}(1 + m))c_{-14+5(2+m), -16+5(2+m)} - a_{03}(-2 + m) \\ & \times c_{-13+5(2+m), -17+5(2+m)}. \end{aligned} \tag{3.4}$$

Using the recursion formulas to do symbolic computation, we have

Theorem 3.1 *The first nine singular point quantities at the origin of system (2.10)_{δ₁=δ₂=0} are as follows:*

$$\begin{aligned} \mu_1^{(0)} &= -a_{21} + b_{21}, \\ \mu_2^{(0)} &= \frac{a_{12}a_{30} - b_{12}b_{30}}{\lambda}, \\ \mu_3^{(0)} &= -\frac{9a_{03}a_{30}^2 + a_{12}^2b_{03} - a_{03}b_{12}^2 - 9b_{03}b_{30}^2}{8\lambda^2}, \\ \mu_4^{(0)} &= -\frac{(a_{21} + b_{21})(a_{12}^2b_{03} + 3a_{03}a_{30}b_{12} - a_{03}b_{12}^2 - 3a_{12}b_{03}b_{30})}{12\lambda^3}, \\ \mu_5^{(0)} &= -\frac{(a_{12}^2b_{03} + 3a_{03}a_{30}b_{12} - a_{03}b_{12}^2 - 3a_{12}b_{03}b_{30})(-27a_{03}b_{03} + 16a_{12}b_{12} + 216\lambda)}{648\lambda^4}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \mu_6^{(0)} &= 0, \\ \mu_7^{(0)} &= -\frac{(405a_{03}^2b_{03}^2 - 8,928a_{03}a_{12}b_{03}b_{12} + 6,400a_{12}^2b_{12}^2)(-a_{12}^2b_{03} - 3a_{03}a_{30}b_{12} + a_{03}b_{12}^2 + 3a_{12}b_{03}b_{30})}{466,560\lambda^6}, \\ \mu_8^{(0)} &= \frac{7(45a_{03}b_{03} - 32a_{12}b_{12})(a_{12}^2b_{03} + a_{03}b_{12}^2)(-a_{12}^2b_{03} - 3a_{03}a_{30}b_{12} + a_{03}b_{12}^2 + 3a_{12}b_{03}b_{30})}{69,984\lambda^7}, \\ \mu_9^{(0)} &= \frac{11a_{12}^2b_{12}^2(-224,181a_{03}b_{03} + 164,000a_{12}b_{12})(a_{12}^2b_{03} + 3a_{03}a_{30}b_{12} - a_{03}b_{12}^2 - 3a_{12}b_{03}b_{30})}{13,778,100\lambda^8}. \end{aligned}$$

In the above expression of $\mu_k^{(0)}$, we have already let $\mu_1^{(0)} = \dots = \mu_{k-1}^{(0)} = 0, k = 2, 3, \dots, 9$.

From Theorem 3.1, it is easy to get

Theorem 3.2 For system (2.10) $_{\delta_1=\delta_2=0}$, the first nine singular point quantities at the origin are zero if and only if one of the following conditions holds:

- (I) $a_{21} = b_{21}, \quad b_{30} = \frac{1}{3}a_{12}, \quad a_{30} = \frac{1}{3}b_{12}, \quad a_{12}b_{12} \neq 0;$
- (II) $a_{21} = b_{21}, \quad a_{12}a_{30} = b_{12}b_{30}, \quad a_{12}^2b_{03} = b_{12}^2a_{03},$
 $a_{12}b_{30}b_{03} = b_{12}a_{30}a_{03}, \quad b_{30}^2b_{03} = a_{30}^2a_{03}.$

Proof Putting condition (I) or (II) into expression (3.5) can easily complete the proof of sufficiency.

Now, let us prove the necessity. If $a_{12} = b_{12} = 0$, then $\mu_2^{(0)} = \mu_4^{(0)} = \mu_5^{(0)} = \mu_6^{(0)} = \mu_7^{(0)} = \mu_8^{(0)} = \mu_9^{(0)} = 0, \mu_3^{(0)} = -\frac{9(a_{03}a_{30}^2 - b_{03}b_{30}^2)}{8\lambda^2}$. Therefore, $\mu_k^{(0)} = 0 (k = 1, 2, \dots, 9)$ yields

$$a_{21} = b_{21}, \quad a_{30}^2a_{03} = b_{30}^2b_{03}, \quad a_{12} = b_{12} = 0. \tag{3.6}$$

If $a_{12}b_{12} \neq 0$, notice that $\mu_1^{(0)} = -a_{21} + b_{21} = 0, \mu_2^{(0)} = \frac{a_{12}a_{30} - b_{12}b_{30}}{\lambda} = 0$, there exist r_{32} and p such that

$$a_{21} = b_{21} = r_{21}, \quad b_{30} = pa_{12}, \quad a_{30} = pb_{12}. \tag{3.7}$$

Substituting (3.7) into expression (3.5), we get

$$\begin{aligned} \mu_3^{(0)} &= \frac{(a_{12}^2b_{03} - a_{03}b_{12}^2)(-1 + 3p)(1 + 3p)}{8\lambda^2}, \\ \mu_4^{(0)} &= \frac{(a_{12}^2b_{03} - a_{03}b_{12}^2)(-1 + 3p)r_{21}}{6\lambda^3}, \\ \mu_5^{(0)} &= \frac{(a_{12}^2b_{03} - a_{03}b_{12}^2)(-1 + 3p)(-27a_{03}b_{03} + 16a_{12}b_{12} + 216\lambda)}{648\lambda^4}, \\ \mu_7^{(0)} &= -\frac{(a_{12}^2b_{03} - a_{03}b_{12}^2)(405a_{03}^2b_{03}^2 - 8,928a_{03}a_{12}b_{03}b_{12} + 6,400a_{12}^2b_{12}^2)(-1 + 3p)}{466,560\lambda^6}, \\ \mu_8^{(0)} &= -\frac{7(-45a_{03}b_{03} + 32a_{12}b_{12})(a_{12}^2b_{03} - a_{03}b_{12}^2)(a_{12}^2b_{03} + a_{03}b_{12}^2)(-1 + 3p)}{69,984\lambda^7}, \\ \mu_9^{(0)} &= -\frac{11a_{12}^2b_{12}^2(-224,181a_{03}b_{03} + 164,000a_{12}b_{12})(a_{12}^2b_{03} - a_{03}b_{12}^2)(-1 + 3p)}{13,778,100\lambda^8}. \end{aligned} \tag{3.8}$$

$405a_{03}^2b_{03}^2 - 8,928a_{03}a_{12}b_{03}b_{12} + 6,400a_{12}^2b_{12}^2 = 0$ and $-224,181a_{03}b_{03} + 164,000a_{12}b_{12} = 0$ do not simultaneously hold, thus from $\mu_2^{(0)} = \mu_7^{(0)} = \mu_9^{(0)} = 0$, we have (i) $p = \frac{1}{3}$ or (ii) $a_{12}a_{30} -$

$b_{12}b_{30} = 0, a_{12}^2b_{03} - a_{03}b_{12}^2 = 0$. At this moment, $\mu_3^{(0)} = \mu_4^{(0)} = \mu_5^{(0)} = \mu_8^{(0)} = 0$. Condition (i) or (ii) combining with $\mu_1^{(0)} = 0$ implies the following conditions, respectively:

$$a_{21} = b_{21}, \quad b_{30} = \frac{1}{3}a_{12}, \quad a_{30} = \frac{1}{3}b_{12}, \quad a_{12}b_{12} \neq 0; \tag{3.9}$$

$$a_{21} = b_{21}, \quad a_{12}a_{30} = b_{12}b_{30}, \quad a_{12}^2b_{03} = b_{12}^2a_{03}, \quad a_{12}b_{12} \neq 0. \tag{3.10}$$

Condition (3.6) plus (3.10) is equivalent to condition (II). The proof is completed. \square

From the definition of elementary Lie invariants in [17], we can obtain

Lemma 3.2 *All the elementary Lie invariants of system (2.10)_{δ₁=δ₂=0} are as follows:*

$$\begin{aligned} &\lambda, \quad a_{21}, \quad b_{21}, \quad a_{30}b_{30}, \quad a_{12}b_{12}, \\ &a_{03}b_{03}, \quad a_{12}a_{30}, \quad b_{12}b_{30}, \\ &a_{12}^2b_{03}, \quad a_{12}b_{30}b_{03}, \quad b_{30}^2b_{03}, \\ &b_{12}^2a_{03}, \quad b_{12}a_{30}a_{03}, \quad a_{30}^2a_{03}. \end{aligned} \tag{3.11}$$

Theorem 3.3 *For system (2.10)_{δ₁=δ₂=0}, all the singular point quantities at the origin are zero if and only if the first nine singular point quantities are zero, i.e., one of the two conditions in Theorem 3.2 holds. Correspondingly, the two conditions in Theorem 3.2 are the center conditions of the origin.*

Proof When condition (I) holds, system (2.10)_{δ₁=δ₂=0} has the integrating factor $M(z, w) = (zw)^{-5}$; when condition (II) holds, system (2.10)_{δ₁=δ₂=0} satisfies the conditions of the extended symmetric principle (Theorem 2.6 in [17]). \square

4 Singular point quantities and center conditions at infinity

Lemma 4.1 [18] *For the system*

$$\begin{aligned} \frac{dz}{dT} &= z^{n+1}w^n + \sum_{k=0}^{2n} Z_k(z, w), \\ \frac{dw}{dT} &= -w^{n+1}z^n - \sum_{k=0}^{2n} W_k(z, w), \end{aligned} \tag{4.1}$$

where

$$Z_k(z, w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^\alpha w^\beta, \quad W_k(z, w) = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^\alpha z^\beta, \tag{4.2}$$

to any integer $s \neq 0, \gamma \neq 0$, we can derive successively the terms of the following formal series:

$$F(z, w) = (zw)^s \left[\sum_{k=0}^{\infty} \frac{f_{(2n+1)k}(z, w)}{(zw)^{k(n+1)}} \right]^{\frac{1}{\gamma}} \tag{4.3}$$

such that

$$\left. \frac{dF}{dT} \right|_{(4.1)} = \frac{1}{\gamma} (zw)^{n+s} \left[\sum_{k=0}^{\infty} \frac{f_{(2n+1)k}(z, w)}{(zw)^{k(n+1)}} \right]^{\frac{1}{\gamma}-1} \sum_{m=1}^{\infty} \frac{\lambda_m}{(zw)^m}, \tag{4.4}$$

and for any integer m ,

$$\lambda_m \sim -s\gamma \mu_m, \tag{4.5}$$

where μ_m is the m th singular point quantity at infinity, ' \sim ' is the symbol of algebraic equivalence, $c_{00} = 1$, $c_{(2n+1)k, (2n+1)k}$ can be taken arbitrarily, $k = 1, 2, \dots$. When $\alpha \neq \beta$, λ_m is determined by the following recursion formulas:

$$\begin{aligned} c_{\alpha\beta} &= \frac{1}{(\beta - \alpha)(2n + 1)} \sum_{k,j} \{ [n\alpha - (n + 1)\beta + (\gamma s + n + 1 - k)(2n + 1)] a_{k,j-1} \\ &\quad - [n\beta - (n + 1)\alpha + (\gamma s + n + 1 - j)] b_{j,k-1} \} \\ &\quad \times c_{\alpha+nk+(n+1)j-(n+1)(2n+1), \beta+nj+(n+1)k-(n+1)(2n+1)}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \lambda_m &= \sum_{k,j} [(\gamma s + n + 1 - k - m) a_{k,j-1} - (\gamma s + n + 1 - j - m) b_{j,k-1}] \\ &\quad \times c_{nk+(n+1)j+(m-n-1)(2n+1), nj+(n+1)k+(m-n-1)(2n+1)}. \end{aligned} \tag{4.7}$$

Take $s = -1$, $\gamma = 1$, $n = 3$, applying Lemma 4.1 to system (2.10) _{$\delta_1=\delta_2=0$} and executing calculation in Mathematica, we have

Theorem 4.1 *The first seven singular point quantities at infinity of system (2.10) _{$\delta_1=\delta_2=0$} are as follows:*

$$\begin{aligned} \mu_1^{(\infty)} &= a_{21} - b_{21}, \\ \mu_2^{(\infty)} &= -a_{12}a_{30} + b_{12}b_{30}, \\ \mu_3^{(\infty)} &= \frac{3a_{03}a_{30}^2 + 3a_{12}^2b_{03} + 8a_{03}a_{30}b_{12} - 3a_{03}b_{12}^2 - 8a_{12}b_{03}b_{30} - 3b_{03}b_{30}^2}{8}, \\ \mu_4^{(\infty)} &= -\frac{(a_{21} + b_{21})(a_{12}^2b_{03} + 3a_{03}a_{30}b_{12} - a_{03}b_{12}^2 - 3a_{12}b_{03}b_{30})}{4}, \\ \mu_5^{(\infty)} &= -\frac{(a_{12}^2b_{03} + 3a_{03}a_{30}b_{12} - a_{03}b_{12}^2 - 3a_{12}b_{03}b_{30})(-a_{03}b_{03} + 4a_{12}b_{12} + 6\lambda)}{6}, \\ \mu_6^{(\infty)} &= 0, \\ \mu_7^{(\infty)} &= \frac{5a_{12}b_{12}(-a_{03}b_{03} + 4a_{12}b_{12})(a_{12}^2b_{03} + 3a_{03}a_{30}b_{12} - a_{03}b_{12}^2 - 3a_{12}b_{03}b_{30})}{6}. \end{aligned} \tag{4.8}$$

In the above expression of $\mu_k^{(\infty)}$, we have already let $\mu_1^{(\infty)} = \dots = \mu_{k-1}^{(\infty)} = 0$, $k = 2, 3, \dots, 7$.

By the similar method used in the proof of Theorem 3.1, we can obtain the following result.

Theorem 4.2 *For system (2.10) _{$\delta_1=\delta_2=0$} , the first seven singular point quantities at infinity are zero if and only if condition (I) or (II) is satisfied.*

The following result is immediate from Theorems 3.3 and 4.2.

Theorem 4.3 *The degenerate singular point and infinity of system (2.10)_{δ₁=δ₂=0} have the same center conditions.*

5 Bifurcation of limit cycles under synchronous perturbation

From Theorems 3.1 and 4.1, we have

Theorem 5.1 *The degenerate singular point of system (1.2)_{δ₁=δ₂=0} is a fine focus of order nine if and only if the coefficients of its associated complex system (2.10)_{δ₁=δ₂=0} satisfy*

$$\begin{aligned} a_{21} = b_{21} = 0, \quad a_{30} = -\frac{1}{3}b_{12}, \quad b_{30} = -\frac{1}{3}a_{12}, \\ -27a_{03}b_{03} + 16a_{12}b_{12} + 216\lambda = 0, \\ 405a_{03}^2b_{03}^2 - 8,928a_{03}a_{12}b_{03}b_{12} + 6,400a_{12}^2b_{12}^2 = 0, \\ a_{12}^2b_{03} + a_{03}b_{12}^2 = 0, \quad a_{12}b_{12}(a_{12}^2b_{03} - a_{03}b_{12}^2) \neq 0. \end{aligned} \tag{5.1}$$

Theorem 5.2 *The infinity of system (1.2)_{δ₁=δ₂=0} is a fine focus of order seven if and only if the coefficients of its associated complex system (2.10)_{δ₁=δ₂=0} satisfy*

$$\begin{aligned} a_{21} = b_{21} = 0, \quad a_{30} = -3b_{12}, \quad b_{30} = -3a_{12}, \\ -a_{03}b_{03} + 4a_{12}b_{12} + 6\lambda = 0, \quad a_{12}b_{12}(a_{12}^2b_{03} - a_{03}b_{12}^2) \neq 0. \end{aligned} \tag{5.2}$$

Actually, when the degenerate singular point of system (1.2)_{δ₁=δ₂=0} is a fine focus of order nine, its infinity is at most a fine focus of order three; when the infinity of system (1.2)_{δ₁=δ₂=0} is a fine focus of order seven, its degenerate singular point is at most a fine focus of order three. Therefore, we have

Theorem 5.3 *By synchronous perturbation at degenerate singular point and infinity, system (1.2) has the limit cycle configurations of {(8), 3} and {(3), 6}.*

For the sake of simplicity, we will prove the first case of Theorem 5.3. The other case can be proved with the same principle.

Theorem 5.4 *Let the coefficients in (2.10) satisfy*

$$\begin{aligned} \delta_1 = \delta_2 = \epsilon_8, \quad a_{30} = -\frac{1}{3}, \quad a_{21} = \epsilon_4 - i\epsilon_7, \quad a_{12} = 1 - \epsilon_2 - \epsilon_3 - \epsilon_5 - i\epsilon_6, \\ a_{03} = -\epsilon_1 - \frac{4}{3}i\sqrt{\frac{1}{5}(31 - 2\sqrt{209})}, \quad \lambda = -\frac{4}{135}(-44 + 3\sqrt{209}), \\ b_{30} = \overline{a_{30}}, \quad b_{21} = \overline{a_{21}}, \quad b_{12} = \overline{a_{12}}, \quad b_{03} = \overline{a_{03}} \end{aligned} \tag{5.3}$$

(accordingly, the coefficients of system (1.2) are determined by (1.4), (2.11) and (5.3)), where $\epsilon_i, i = 1, \dots, 8$ are small parameters with

$$0 < \epsilon_8 \ll \epsilon_7 \ll \epsilon_6 \ll \epsilon_5 \ll \epsilon_4 \ll \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll 1. \tag{5.4}$$

Then for system (1.2), there are eight limit cycles bifurcating from the degenerate singular point, and at the same time, there are three limit cycles bifurcating from infinity.

Proof For the degenerate singular point of system (1.2), from $v_1(2\pi) = e^{2\pi\delta_1}$, $v_{2m+1}(2\pi, \delta_1) = v_{2m+1}(2\pi, 0) + \delta_1\phi_{2m+1}(a_{\alpha\beta}, b_{\alpha\beta}, \delta_1)$, Theorem 3.1 and $v_{2m+1}(2\pi) \sim i\pi\mu_m$, after computing carefully, we have the following:

$$\begin{aligned} v_1(2\pi, \delta_1) - 1 &= 2\pi\epsilon_8 + o(\epsilon_8), \\ v_3(2\pi, \delta_1) &= [-2\pi + \omega_1(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7)]\epsilon_7 + o(\epsilon_7), \\ v_5(2\pi, \delta_1) &= \left[-\frac{45}{2(-44 + 3\sqrt{209})}\pi + \omega_2(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7) \right]\epsilon_6 + o(\epsilon_6), \\ v_7(2\pi, \delta_1) &= \left[-\frac{1,215\sqrt{5(31 - 2\sqrt{209})}}{8(-44 + 3\sqrt{209})^2}\pi + \omega_3(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7) \right]\epsilon_5 + o(\epsilon_5), \\ v_9(2\pi, \delta_1) &= \left[-\frac{54,675\sqrt{5(31 - 2\sqrt{209})}}{8(-44 + 3\sqrt{209})^3}\pi + \omega_4(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7) \right]\epsilon_4 + o(\epsilon_4), \\ v_{11}(2\pi, \delta_1) &= \left[-\frac{273,375\sqrt{5(31 - 2\sqrt{209})}}{4(-44 + 3\sqrt{209})^4}\pi + \omega_5(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7) \right]\epsilon_3 + o(\epsilon_3), \\ v_{15}(2\pi, \delta_1) &= \left[\frac{22,143,375\sqrt{5(31 - 2\sqrt{209})}(-418 + 31\sqrt{209})}{16(-44 + 3\sqrt{209})^6}\pi \right. \\ &\quad \left. + \omega_6(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7) \right]\epsilon_2 + o(\epsilon_2), \\ v_{17}(2\pi, \delta_1) &= \left[-\frac{174,379,078,125\sqrt{5(31 - 2\sqrt{209})}(-29 + 2\sqrt{209})}{1,024(-44 + 3\sqrt{209})^7}\pi \right. \\ &\quad \left. + \omega_7(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7) \right]\epsilon_1 + o(\epsilon_1), \\ v_{19}(2\pi, \delta_1) &= -\frac{32,882,911,875\sqrt{5(31 - 2\sqrt{209})}(-720,929 + 49,818\sqrt{209})}{7,168(-44 + 3\sqrt{209})^8}\pi + o(1), \end{aligned}$$

where $\omega_i(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7)$ are analytic at $(0, 0, 0, 0, 0, 0, 0)$ and $\omega_i(0, 0, 0, 0, 0, 0, 0) = 0$, $i = 1, \dots, 7$.

Simultaneously, for the infinity of system (1.2), we have

$$\begin{aligned} v_1(2\pi, \delta_2) - 1 &= -2\pi\epsilon_8 + o(\epsilon_8), \\ v_3(2\pi, \delta_2) &= 2\pi\epsilon_7 + o(\epsilon_7), \\ v_5(2\pi, \delta_2) &= -\frac{2}{3}\pi\epsilon_6 + o(\epsilon_6), \\ v_7(2\pi, \delta_2) &= \frac{16}{9}\sqrt{\frac{1}{5}(31 - 2\sqrt{209})}\pi + o(1). \end{aligned}$$

According to the classical theory of Bautin, there exist eight limit cycles in a small enough neighborhood of the degenerate singular point and three limit cycles in a small enough neighborhood of the infinity. \square

6 Concluding remarks

We have presented the solution to the problem of bifurcation of limit cycles at degenerate singular point and infinity for a class of seventh degree polynomial vector field. There are two important points to make here. Firstly, by computing the first several singular point quantities with Mathematica and vanishing them, the same center conditions at degenerate singular point (Theorem 3.2) and at infinity (Theorem 4.2) are obtained. Secondly, by synchronous perturbation of degenerate singular point and infinity, the limit cycle configurations of $\{(8), 3\}$ and $\{(3), 6\}$ are determined. We hope that our work could make some contributions in this direction.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The study was carried out in collaboration between both of the authors. YW completed the main part of this paper, FL corrected the main theorems and polished the manuscript. All authors read and approved the final manuscript.

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Acknowledgements

Acknowledgments are due to the referees for their useful suggestions and valuable comments. This work is supported in part by the National Nature Science Foundation of China (No. 11101126, 11126114 and 11201211), the Fund of Henan Education Commission (No. 2011B110014), and the Scientific Research Foundation for Doctoral Scholars of Haust (09001524).

Received: 5 October 2012 Accepted: 18 December 2012 Published: 9 January 2013

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doi:10.1186/1687-1847-2013-5

Cite this article as: Wu and Li: Bifurcation of limit cycles at degenerate singular point and infinity in a septic system. *Advances in Difference Equations* 2013 2013:5.