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Linear differential equations for families of polynomials

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Abstract
In this paper, we present linear differential equations for the generating functions of the Poisson-Charlier, actuarial, and Meixner polynomials. Also, we give an application for each case.

Keywords: actuarial polynomials; Meixner polynomials; Poisson-Charlier polynomials

1 Introduction

As is well known, the *Poisson-Charlier polynomials* $C_k(x;a)$ are *Sheffer sequences* (see [1–4]) with $g(t) = e^{a(e^t-1)}$ and $f(t) = a(e^t - 1)$, which are given by the generating function

$$C(x,t) = e^{-t}(1 + t/a)^x = \sum_{n\geq 0} C_n(x;a)\frac{t^n}{n!} \quad (a \neq 0). \tag{1}$$

They satisfy the Sheffer identity

$$C_n(x + y;a) = \sum_{k=0}^{n} \binom{n}{k} a^{k-n} C_k(y;a)(x)_{n-k},$$

where $(x)_n$ is the *falling factorial* (see [5]). Moreover, these polynomials satisfy the recurrence relation

$$C_{n+1}(x;a) = a^{-1}xC_n(x - 1;a) - C_n(x;a) \quad \bigl(\text{see [5]}\bigr).$$

The first few polynomials are $C_0(x;a) = 1$, $C_1(x;a) = -\frac{(a-x)}{a}$, $C_2(x;a) = \frac{(a^2-x-2ax+x^2)}{a^2}$.

The actuarial polynomials $a_n^{(\beta)}(x)$ are given by the generating function of Sheffer sequence

$$F(x,t) = e^{\beta t+x(1-e^t)} = \sum_{n\geq 0} a_n^{(\beta)}(x)\frac{t^n}{n!} \quad \bigl(\text{see [5]}\bigr), \tag{2}$$

and the Meixner polynomials of the first kind $m_n(x;\beta,c)$ are also introduced in [5] as follows:

$$M(x,t)=\sum_{n\geq 0} m_n(x;\beta,c)\frac{t^n}{n!}=(1-t/c)^x(1-x)^{-x-\beta}. \tag{3}$$

In mathematics, Meixner polynomials of the first kind (also called discrete Laguerre polynomials) are a family of discrete orthogonal polynomials introduced by Josef Meixner (see [6–10]). They are given in terms of binomial coefficients and the (rising) Pochhammer symbol by

$$m_n(x,\beta,c)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k}k!(x-\beta)_{n-k}c^{-k} \quad \bigl(\text{see [5]}\bigr).$$

Some interesting identities and properties of the Poisson-Charlier, actuarial, and Meixner polynomials can be derived from umbral calculus (see [11–13]). Kim and Kim [12] introduced nonlinear Changhee differential equations for giving special functions and polynomials. Many researchers have studied the Poisson-Charlier, actuarial and Meixner polynomials in the mathematical physics, combinatorics, and other applied mathematics (for example, see [14, 15]).

In this paper, we study linear differential equations arising from the Poisson-Charlier, actuarial, and Meixner polynomials and derive new recurrence relations for those polynomials from our differential equations.

2 Poisson-Charlier polynomials

Recall that the falling polynomials $(x)_N$ are defined by $(x)_N=(x-1)\cdots(x-N+1)$ for $N\geq 1$ with $(x)_0=1$. For brevity, we denote the generating functions $C(x,t)$ and $\frac{d^j}{dt^j}C(x;t)$ by C and $C^{(j)}$ for $j\geq 0$.

Lemma 1 *The generating function $C^{(N)}$ is given by $(\sum_{i=0}^{N}a_i(N,x)(t+a)^{-i})C$, where $a_0(N,x)=(-1)^N$, $a_N(N,x)=(x)_N$, and*

$$a_i(N,x)=(x-i+1)a_{i-1}(N-1,x)-a_i(N-1,x)\quad (1\leq i\leq N-1).$$

Proof Clearly, $a_0(0,x)=1$. For $N=1$, by (1) we have $C^{(1)}=(-1+x(t+a)^{-1})C$, which proves the lemma for $N=1$ (here $a_0(1,x)=-1$ and $a_1(1,x)=x$). Assume that $C^{(N)}$ is given by $(\sum_{i=0}^{N}a_i(N,x)(t+a)^{-i})C$. Then

$$\begin{aligned}C^{(N+1)}&=\left(-\sum_{i=0}^{N}a_i(N,x)i(t+a)^{-i-1}\right)C+\left(\sum_{i=0}^{N}a_i(N,x)(t+a)^{-i}\right)\bigl(-1+x(t+a)^{-1}\bigr)C\\&=\left(\sum_{i=1}^{N+1}(x-i+1)a_{i-1}(N,x)(t+a)^{-i}-\sum_{i=0}^{N}a_i(N,x)(t+a)^{-i}\right)C.\end{aligned}$$

This shows that the generating function $C^{(N+1)}$ is given by

$$\begin{aligned}&\left(-a_0(N,x)+\sum_{i=1}^{N}\bigl((x-i+1)a_{i-1}(N,x)-a_i(N,x)\bigr)(t+a)^{-i}\right.\\&\quad\left.+(x-N)a_N(N,x)(t+a)^{-N-1}\right)C.\end{aligned}$$

Comparing with $C^{(N+1)}=(\sum_{i=0}^{N+1}a_i(N+1,x)(t+a)^{-i})C$, we complete the proof. □

In order to obtain an explicit formula for the generating function $C^{(N)}$, we need the following lemma.

Lemma 2 *For all* $0 \le i \le N$, *the coefficient's* $a_i(N,x)$ *in Lemma* 1 *are given by*

$$a_i(N,x) = (x)_i \binom{N}{i}(-1)^{N-i}.$$

Proof By Lemma 1 we have that

$$a_i(N+1,x) = (x-i+1)a_{i-1}(N,x) - a_i(N,x), \quad 0 \le i \le N+1,$$

with $a_0(0,x) = 1$ and $a_i(N,x) = 0$ whenever $i > N$ or $i < 0$. Define $A_i(x;t) = \sum_{N\ge i} a_i(N,x)t^N$. Then we have

$$A_i(x;t) = \frac{(x+1-i)t}{1+t}A_{i-1}(x)$$

with $A_0(x;t) = \frac{1}{1+t}$. By induction on i we derive that $A_i(x,t) = \frac{(x)_i t^i}{(1+t)^{i+1}}$. Hence, by the fact that $\frac{1}{(1+t)^{i+1}} = \sum_{j\ge 0}\binom{i+j}{i}(-1)^j t^j$ we obtain that $a_i(N,x) = (x)_i\binom{N}{i}(-1)^{N-i}$, as required. □

Thus, by Lemmas 1 and 2 we can state the following result.

Theorem 3 *The linear differential equations*

$$C^{(N)} = \left(\sum_{i=0}^{N}(x)_i\binom{N}{i}(-1)^{N-i}(t+a)^{-i}\right)C \quad (n = 0,1,\ldots)$$

have a solution $C(x,t) = e^{-t}(1+t/a)^x$, *where* $(x)_i = x(x-1)\cdots(x+1-i)$ *with* $(x)_0 = 1$.

As an application of Theorem 3, we obtain the following corollary.

Corollary 4 *For all* $k, N \ge 0$,

$$C_{k+N}(x;a) = \sum_{i=0}^{N}\sum_{m=0}^{k}(x)_i\binom{N}{i}\binom{k}{m}(-1)^{N-i+m}(i+m-1)_m a^{-i-m}C_{k-m}(x;a).$$

Proof By (1) and Theorem 3 we have

$$C^{(N)} = \left(\sum_{i=0}^{N}(x)_i\binom{N}{i}(-1)^{N-i}(t+a)^{-i}\right)\sum_{\ell\ge 0}C_\ell(x;a)\frac{t^\ell}{\ell!}.$$

Since $\frac{1}{(1+t)^{i+1}} = \sum_{j\ge 0}\binom{i+j}{i}(-1)^j t^j$, we obtain

$$C^{(N)} = \sum_{k\ge 0}\sum_{i=0}^{N}\sum_{m=0}^{k}(x)_i\binom{N}{i}\binom{k}{m}(-1)^{N-i+m}(i+m-1)_m a^{-i-m}C_{k-m}(x;a)\frac{t^k}{k!}.$$

By comparing coefficients of t^k we complete the proof. □

3 Actuarial polynomials

For brevity, we denote the generating functions $F(x,t) = e^{\beta t + x(1-e^t)}$ and $\frac{d^j}{dt^j}F(x;t)$ by F and $F^{(j)}$ for $j \geq 0$.

Lemma 5 *The generating function $F^{(N)}$ is given by $(\sum_{i=0}^N b_i(N,x)e^{it})F$, where $b_0(N,x) = \beta^N$, $b_N(N,x) = (-x)^N$, and $b_i(N,x) = -xb_{i-1}(N-1,x) + (\beta+i)b_i(N-1,x)$ $(1 \leq i \leq N-1)$.*

Proof Clearly, $b_0(0,x) = 1$. For $N = 1$, by (2) we have $F^{(1)} = (\beta - xe^t)F$, which proves the lemma for $N = 1$ (here $b_0(1,x) = \beta$ and $b_1(1,x) = -x$). Assume that $F^{(N)}$ is given by $(\sum_{i=0}^N b_i(N,x)e^{it})F$. Then

$$\begin{aligned} F^{(N+1)} &= \left(\sum_{i=0}^N b_i(N,x)ie^{it}\right)F + \left(\sum_{i=0}^N b_i(N,x)e^{it}\right)(\beta - xe^t)F \\ &= \left(\sum_{i=0}^N (\beta+i)a_i(N,x)e^{it} - x\sum_{i=1}^{N+1} b_{i-1}(N,x)e^{it}\right)F, \end{aligned}$$

which shows that the generating function $F^{(N+1)}$ is given by

$$\left(\beta b_0(N,x) + \sum_{i=1}^N \bigl(-xa_{i-1}(N,x) + (\beta+i)b_i(N,x)\bigr)e^{it} - xb_N(N,x)e^{(N+1)t}\right)F.$$

Comparing with $F^{(N+1)} = (\sum_{i=0}^{N+1} b_i(N+1,x)e^{it})C$, we complete the proof. □

Lemma 6 *For all $0 \leq i \leq N$, the coefficients $b_i(N,x)$ in Lemma 5 are given by*

$$b_i(N,x) = (-x)^i \sum_{j=i}^N \binom{N}{j} \beta^{N-j} S(j,i),$$

where $S(n,k)$ are the Stirling numbers (for example, see [16]) of the second kind.

Proof By Lemma 5 we have that

$$b_i(N+1,x) = -xb_{i-1}(N,x) + (\beta+i)b_i(N,x), \quad 0 \leq i \leq N+1,$$

with $b_0(0,x) = 1$ and $b_i(N,x) = 0$ whenever $i > N$ or $i < 0$. Define $B_i(x;t) = \sum_{N \geq i} b_i(N,x)t^N$. Then we have

$$B_i(x;t) = \frac{-xt}{1-(\beta+i)t}B_{i-1}(x)$$

with $B_0(x;t) = \frac{1}{1-\beta t}$. By induction on i we derive that

$$B_i(x,t) = \frac{(-xt)^i}{(1-\beta t)(1-(\beta+1)t)\cdots(1-(\beta+i)t)} = \frac{(-xt)^i}{(1-\beta t)^{i+1}} \prod_{j=0}^i \frac{1}{1-jt/(1-\beta t)}.$$

Hence, since $\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{n\geq k} S(n,k)x^n$ (for example, see [16]), where $S(n,k)$ are the Stirling numbers of the second kind, we obtain that

$$B_i(x,t) = (-x)^i \sum_{j\geq i} S(j,i) \frac{t^j}{(1-\beta t)^{j+1}}.$$

Since $\frac{1}{(1+t)^{i+1}} = \sum_{j\geq 0} \binom{i+j}{i}(-1)^j t^j$, we obtain that

$$B_i(x,t) = (-x)^i \sum_{j\geq i} \sum_{\ell\geq 0} \binom{j+\ell}{j} \beta^\ell S(j,i) t^{J+\ell}.$$

Thus, by finding the coefficients of t^N we complete the proof. □

Thus, by Lemmas 5 and 6 we can state the following result.

Theorem 7 *The linear differential equations*

$$F^{(N)} = \sum_{i=0}^{N} \left((-x)^i e^{it} \sum_{j=i}^{N} \binom{N-1}{j-1} \beta^{N-j} S(j,i) \right) F \quad (N = 0, 1, \ldots)$$

have a solution $F(x,t) = e^{\beta t + x(1-e^t)}$.

Recall that $F(x,t) = e^{\beta t+x(1-e^t)} = \sum_{n\geq 0} a_n^{(\beta)}(x) \frac{t^n}{n!}$, which is the generating function for the actuarial polynomials $a_n^{(\beta)}(x)$ (see (2)). As an application of Theorem 7, we obtain the following corollary.

Corollary 8 *For all* $k, N \geq 0$,

$$a_{N+k}^{(\beta)}(x) = \sum_{i=0}^{N} \sum_{m=0}^{k} b_i(N;x) \binom{k}{m} i^{k-m} a_m^{(\beta)}(x),$$

where $b_i(N,x) = (-x)^i \sum_{j=i}^{N} \binom{N-1}{j-1} \beta^{N-j} S(j,i)$.

Proof By (2) and Theorem 7 we have $F^{(N)} = (\sum_{i=0}^{N} b_i(N,x)e^{it}) \sum_{\ell\geq 0} a_\ell^{(\beta)}(x) \frac{t^\ell}{\ell!}$. Thus,

$$F^{(N)} = \sum_{k\geq 0} \sum_{i=0}^{N} \sum_{m=0}^{k} b_i(N,x) \binom{k}{m} i^{k-m} a_m^{(\beta)}(x) \frac{t^k}{k!}.$$

By comparing the coefficients of t^{N+k} we complete the proof. □

4 Meixner polynomials of the first kind

Recall that the *rising polynomials* $\langle x\rangle_N$ are defined by $\langle x\rangle_N = x(x+1)\cdots(x+N-1)$ with $\langle x\rangle_0 = 1$. For brevity, we denote the generating functions $M(x,t) = (1-t/c)^x(1-x)^{-x-\beta}$ and $\frac{d^j}{dt^j} M(x;t)$ by M and $M^{(j)}$ for $j \geq 0$, respectively.

Theorem 9 *The linear differential equations*

$$M^{(N)} = \left(\sum_{i=0}^{N}(-1)^i\binom{N}{i}(x)_{N-i}\langle x+\beta\rangle_i(t-1)^{-i}(t-c)^{-(N-i)}\right)M \quad (N = 0,1,\ldots)$$

have a solution $M = M(x,t) = (1-t/c)^x(1-x)^{-x-\beta}$.

Proof We proceed the proof by induction on N. Clearly, the theorem holds for $N = 0$. By (3) we have $M^{(1)} = (x(t-c)^{-1} - (x+\beta)(t-1)^{-1})M$, which proves the theorem for $N = 1$. Assume that the theorem holds for $N \geq 1$. Then by the induction hypothesis we have

$$\begin{aligned}
&M^{(N+1)}\\
&= \frac{d}{dt}\left(\sum_{i=0}^{N}(-1)^i\binom{N}{i}(x)_{N-i}\langle x+\beta\rangle_i(t-1)^{-i}(t-c)^{-(N-i)}\right)M\\
&= \left\{\left(\sum_{i=0}^{N}(-1)^{i+1}i\binom{N}{i}(x)_{N-i}\langle x+\beta\rangle_i(t-1)^{-i-1}(t-c)^{-(N-i)}\right)M\right.\\
&\quad+\left(\sum_{i=0}^{N}(-1)^{i+1}(N-i)\binom{N}{i}(x)_{N-i}\langle x+\beta\rangle_i(t-1)^{-i}(t-c)^{-(N+1-i)}\right)M\\
&\quad+\left(\sum_{i=0}^{N}(-1)^i\binom{N}{i}(x)_{N-i}\langle x+\beta\rangle_i(t-1)^{-i}(t-c)^{-(N-i)}\right)\\
&\quad\left.\times\left(x(t-c)^{-1}-(x+\beta)(t-1)^{-1}\right)M\right\}.
\end{aligned}$$

After rearranging the indices of the sums, we obtain

$$\begin{aligned}
&M^{(N+1)}\\
&= \left(\sum_{i=1}^{N+1}(-1)^i(i-1)\binom{N}{i-1}(x)_{N+1-i}\langle x+\beta\rangle_{i-1}(t-1)^{-i}(t-c)^{-(N+1-i)}\right)M\\
&\quad+\left(\sum_{i=0}^{N}(-1)^{i+1}(N-i)\binom{N}{i}(x)_{N-i}\langle x+\beta\rangle_i(t-1)^{-i}(t-c)^{-(N+1-i)}\right)M\\
&\quad+\left(\sum_{i=0}^{N}(-1)^i\binom{N}{i}x(x)_{N-i}\langle x+\beta\rangle_i(t-1)^{-i}(t-c)^{-(N+1-i)}\right)M\\
&\quad+\left(\sum_{i=1}^{N+1}(-1)^i\binom{N}{i-1}(x)_{N+1-i}(x+\beta)\langle x+\beta\rangle_{i-1}(t-1)^{-i}(t-c)^{-(N+1-i)}\right)M.
\end{aligned}$$

This implies

$$M^{(N+1)} = \left(\sum_{i=0}^{N+1}(-1)^i\binom{N+1}{i}(x)_{N+1-i}\langle x+\beta\rangle_i(t-1)^{-i}(t-c)^{-(N+1-i)}\right)M,$$

and the induction step is completed. □

From (3) we have $M^{(N)} = \sum_{k\geq 0} m_{k+N}(x;\beta,c)\frac{t^k}{k!}$ for all $N \geq 0$. Similarly to the previous section, we have a recurrence relation for the coefficients of $m_n(x;\beta,c)$.

Corollary 10 *For all $k, N \geq 0$,*

$$m_{k+N}(x;\beta,c) = (-1)^N \sum_{i=0}^{N} (-1)^i \binom{N}{i} (x)_{N-i} \langle x+\beta\rangle_i \sum_{\ell+m+n=k} \frac{k!\binom{i+\ell-1}{\ell}\binom{N+m-i-1}{m}}{n! c^{N-i+m}} m_n(x;\beta,c).$$

Proof By Theorem 9 we have

$$M^{(N)} = \left(\sum_{i=0}^{N} (-1)^i \binom{N}{i} (x)_{N-i} \langle x+\beta\rangle_i (t-1)^{-i} (t-c)^{-(N-i)}\right) \sum_{\ell\geq 0} m_\ell(x;\beta,c)\frac{t^\ell}{\ell!}.$$

Thus, since $(t-c)^{-s} = (-1)^s \sum_{\ell\geq 0}\binom{s+\ell-1}{\ell} c^{-s-\ell} t^\ell$, we obtain

$$M^{(N)} = (-1)^N \sum_{i=0}^{N} (-1)^i \binom{N}{i} (x)_{N-i} \langle x+\beta\rangle_i \times \sum_{\ell\geq 0}\sum_{m\geq 0}\sum_{n\geq 0} \binom{i+\ell-1}{\ell}\binom{N+m-i-1}{m} m_n(x;\beta,c)\frac{c^{-N-m+i} t^{\ell+m+n}}{n!}.$$

Hence, by finding the coefficients of t^k in the generating function $M^{(N)}$ we complete the proof. □

5 Results and discussion

In this paper, the Poisson-Charlier polynomials, actuarial, and Meixner polynomial are introduced. We study linear differential equations arising from the Poisson-Charlier, actuarial, and Meixner polynomials and present some their recurrence relations. Linear differential equations for various families of polynomials are derived. Furthermore, some particular cases of the results are presented.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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