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Research Article

Fixed Point Results in Quasimetric Spaces

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In the setting of quasimetric spaces, we prove some new results on the existence of fixed points for contractive type maps with respect to Q-function. Our results either improve or generalize many known results in the literature.

1. Introduction and Preliminaries

Let X be a metric space with metric d. We use $S(X)$ to denote the collection of all nonempty subsets of $X, Cl(X)$ for the collection of all nonempty closed subsets of $X, CB(X)$ for the collection of all nonempty closed bounded subsets of X, and H for the Hausdorff metric on $CB(X)$, that is,

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}, \quad A,B \in CB(X), \tag{1.1}$$

where $d(a,B) = \inf\{d(a,b) : b \in B\}$ is the distance from the point a to the subset B.

For a multivalued map $T : X \to CB(X)$, we say

$(a)T$ is *contraction* [1] if there exists a constant $\lambda \in (0,1)$, such that for all $x, y \in X$,

$$H(T(x),T(y)) \le \lambda d(x,y), \tag{1.2}$$

(b)T is *weakly contractive* [2] if there exist constants $h, b \in (0,1)$, $h < b$, such that for any $x \in X$, there is $y \in I_b^x$ satisfying

$$d(y,T(y)) \le hd(x,y), \tag{1.3}$$

where $I_b^x = \{y \in T(x) : bd(x,y) \le d(x,T(x))\}$.

A point $x \in X$ is called a *fixed point* of a multivalued map $T : X \to S(X)$ if $x \in T(x)$. We denote $\text{Fix}(T) = \{x \in X : x \in T(x)\}$.

A sequence $\{x_n\}$ in X is called an *orbit* of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all integer $n \geq 1$. A real valued function f on X is called *lower semicontinuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ implies that $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

Using the Hausdorff metric, Nadler Jr. [1] has established a multivalued version of the well-known Banach contraction principle in the setting of metric spaces as follows.

Theorem 1.1. *Let (X, d) be a complete metric space, then each contraction map $T : X \to CB(X)$ has a fixed point.*

Without using the Hausdorff metric, Feng and Liu [2] generalized Nadler's contraction principle as follows.

Theorem 1.2. *Let (X, d) be a complete metric space and let $T : X \to Cl(X)$ be a weakly contractive map, then T has a fixed point in X provided the real valued function $f(x) = d(x, T(x))$ on X is a lower semicontinuous.*

In [3], Kada et al. introduced the concept of w-distance in the setting of metric spaces as follows.

A function $\omega : X \times X \to [0, \infty)$ is called a w-distance on X if it satisfies the following:

(w1) $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$, for all $x, y, z \in X$;

(w2) ω is lower semicontinuous in its second variable;

(w3) for any $\varepsilon > 0$, there exists $\delta > 0$, such that $\omega(z, x) \leq \delta$ and $\omega(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Note that in general for $x, y \in X$, $\omega(x, y) \neq \omega(y, x)$ and not either of the implications $\omega(x, y) = 0 \Leftrightarrow x = y$ necessarily holds. Clearly, the metric d is a w-distance on X. Many other examples and properties of w-distances are given in [3].

In [4], Suzuki and Takahashi improved Nadler contraction principle (Theorem 1.1) as follows.

Theorem 1.3. *Let (X, d) be a complete metric space and let $T : X \to Cl(X)$. If there exist a w-distance ω on X and a constant $\lambda \in (0, 1)$, such that for each $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying*

$$\omega(u, v) \leq \lambda\omega(x, y), \tag{1.4}$$

then T has a fixed point.

Recently, Latif and Albar [5] generalized Theorem 1.2 with respect to w-distance (see, Theorem 3.3 in [5]), and Latif [6] proved a fixed point result with respect to w-distance (see, Theorem 2.2 in [6]) which contains Theorem 1.3 as a special case.

A nonempty set X together with a quasimetric d (i.e., not necessarily symmetric) is called a quasimetric space. In the setting of a quasimetric spaces, Al-Homidan et al. [7] introduced the concept of a Q-function on quasimetric spaces which generalizes the notion of a w-distance.

A function $q : X \times X \to [0,\infty)$ is called a *Q-function* on X if it satisfies the following conditions:

(Q1) $q(x,z) \leq q(x,y) + q(y,z)$, for all $x,y,z \in X$;

(Q2) If $\{y_n\}$ is a sequence in X such that $y_n \to y \in X$ and for $x \in X$, $q(x,y_n) \leq M$ for some $M = M(x) > 0$, then $q(x,y) \leq M$,

(Q3) for any $\varepsilon > 0$, there exists $\delta > 0$, such that $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ imply $d(y,z) \leq \varepsilon$.

Note that every w-distance is a Q-function, but the converse is not true in general [7]. Now, we state some useful properties of Q-function as given in [7].

Lemma 1.4. *Let (X,d) be a complete quasimetric space and let q be a Q-function on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0, then the following hold for any $x,y,z \in X$:*

(i) *if $q(x_n,y) \leq \alpha_n$ and $q(x_n,z) \leq \beta_n$ for all $n \geq 1$, then $y = z$; in particular, if $q(x,y) = 0$ and $q(x,z) = 0$, then $y = z$;*

(ii) *if $q(x_n,y_n) \leq \alpha_n$ and $q(x_n,z) \leq \beta_n$ for all $n \geq 1$, then $\{y_n\}$ converges to z;*

(iii) *if $q(x_n,x_m) \leq \alpha_n$ for any $n,m \geq 1$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*

(iv) *if $q(y,x_n) \leq \alpha_n$ for any $n \geq 1$, then $\{x_n\}$ is a Cauchy sequence.*

Using the concept Q-function, Al-Homidan et al. [7] recently studied an equilibrium version of the Ekeland-type variational principle. They also generalized Nadler's fixed point theorem (Theorem 1.1) in the setting of quasimetric spaces as follows.

Theorem 1.5. *Let (X,d) be a complete quasimetric space and let $T : X \to Cl(X)$. If there exist Q-function q on X and a constant $\lambda \in (0,1)$, such that for each $x,y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying*

$$q(u,v) \leq \lambda q(x,y), \tag{1.5}$$

then T has a fixed point.

In the sequel, we consider X as a quasimetric space with quasimetric d.

Considering a multivalued map $T : X \to S(X)$, we say

(c) T is *weakly q-contractive* if there exist Q-function q *on* X and constants $h,b \in (0,1)$, $h < b$, such that for any $x \in X$, there is $y \in J_b^x$ satisfying

$$q(y,T(y)) \leq hq(x,y), \tag{1.6}$$

where $J_b^x = \{y \in T(x) : bq(x,y) \leq q(x,T(x))\}$ and $q(x,T(x)) = \inf\{q(x,y) : y \in T(x)\}$;

(d) T is *generalized q-contractive* if there exists a Q-function q on X, such that for each $x,y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying

$$q(u,v) \leq k(q(x,y))q(x,y), \tag{1.7}$$

where k is a function of $[0,\infty)$ to $[0,1)$, such that $\limsup_{r\to t^+} k(r) < 1$ for all $t \geq 0$.

Clearly, the class of *weakly* q-contractive maps contains the class of weakly contractive maps, and the class of generalized q-contractive maps contains the classes of generalized ω-contraction maps [6], ω-contractive maps [4], and q-contractive maps [7].

In this paper, we prove some new fixed point results in the setting of quasimetric spaces for weakly *q-contractive and* generalized q-contractive multivalued maps. Consequently, our results either improve or generalize many known results including the above stated fixed point results.

2. The Results

First, we prove a fixed point theorem for weakly q-contractive maps in the setting of quasimetric spaces.

Theorem 2.1. *Let X be a complete quasimetric space and let $T : X \to Cl(X)$ be a weakly q-contractive map. If a real valued function $f(x) = q(x, T(x))$ on X is lower semicontinuous, then there exists $v_o \in X$, such that $q(v_o, T(v_o)) = 0$. Further, if $q(v_o, v_o) = 0$, then v_0 is a fixed point of T.*

Proof. Let $x_o \in X$. Since T is weakly contractive, there is $x_1 \in J_b^{x_o} \subseteq T(x_o)$, such that

$$q(x_1, T(x_1)) \leq hq(x_o, x_1), \tag{2.1}$$

where $h < b$. Continuing this process, we can get an orbit $\{x_n\}$ of T at x_o satisfying $x_{n+1} \in J_b^{x_n}$ and

$$q(x_{n+1}, T(x_{n+1})) \leq h(x_n, x_{n+1}), \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

Since $bq(x_n, x_{n+1}) \leq q(x_n, T(x_n))$ and $h < b < 1$, thus we get

$$q(x_{n+1}, T(x_{n+1})) \leq q(x_n, T(x_n)). \tag{2.3}$$

If we put $a = h/b$, then also we have

$$q(x_{n+1}, T(x_{n+1})) \leq aq(x_n, T(x_n)). \tag{2.4}$$

Thus, we obtain

$$q(x_n, T(x_n)) \leq a^n q(x_o, T(x_0)), \quad n = 0, 1, 2, \ldots, \tag{2.5}$$

and since $0 < a < 1$, hence the sequence $\{f(x_n)\} = \{q(x_n, T(x_n))\}$, which is decreasing, converges to 0. Now, we show that $\{x_n\}$ is a Cauchy sequence. Note that

$$q(x_n, x_{n+1}) \leq a^n q(x_o, x_1), \quad n = 0, 1, 2, \ldots. \tag{2.6}$$

Now, for any integer $n, m \geq 1$ with $m > n$, we have

$$q(x_n, x_m) \leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m)$$
$$\leq a^n q(x_o, x_1) + a^{n+1} q(x_o, x_1) + \cdots + a^{m-1} q(x_o, x_1) \tag{2.7}$$
$$\leq \frac{a^n}{1-a} q(x_o, x_1),$$

and thus by Lemma 1.4, $\{x_n\}$ is a Cauchy sequence. Due to the completeness of X, there exists some $v_0 \in X$, such that $\lim_{n\to\infty} x_n = v_o$. Now, since f is lower semicontinuous, we have

$$0 \leq f(v_o) \leq \liminf_{n\to\infty} f(x_n) = 0, \tag{2.8}$$

and thus, $f(v_o) = q(v_o, T(v_o)) = 0$. It follows that there exists a sequence $\{v_n\}$ in $T(v_0)$, such that $q(v_0, v_n) \to 0$. Now, if $q(v_o, v_o) = 0$, then by Lemma 1.4, $v_n \to v_0$. Since $T(v_0)$ is closed, we get $v_0 \in T(v_0)$. □

Now, we prove the following useful lemma.

Lemma 2.2. *Let (X, d) be a complete quasimetric space and let $T : X \to Cl(X)$ be a generalized q-contractive map, then there exists an orbit $\{x_n\}$ of T at x_0, such that the sequence of nonnegative numbers $\{q(x_n, x_{n+1})\}$ is decreasing to zero and $\{x_n\}$ is a Cauchy sequence.*

Proof. Let x_o be an arbitrary but fixed element of X and let $x_1 \in T(x_0)$. Since T is generalized as a q-contractive, there is $x_2 \in T(x_1)$, such that

$$q(x_1, x_2) \leq k(q(x_o, x_1))q(x_o, x_1). \tag{2.9}$$

Continuing this process, we get a sequence $\{x_n\}$ in X, such that $x_{n+1} \in T(x_n)$ and

$$q(x_n, x_{n+1}) \leq k(q(x_{n-1}, x_n))q(x_{n-1}, x_n). \tag{2.10}$$

Thus, for all $n \geq 1$, we have

$$q(x_n, x_{n+1}) < q(x_{n-1}, x_n). \tag{2.11}$$

Write $t_n = q(x_n, x_{n+1})$. Suppose that $\lim_{n\to\infty} t_n = \lambda > 0$, then we have

$$t_n \leq k(t_{n-1})t_{n-1}. \tag{2.12}$$

Now, taking limits as $n \to \infty$ on both sides, we get

$$\lambda \leq \limsup_{n\to\infty} k(t_{n-1})\lambda < \lambda, \tag{2.13}$$

which is not possible, and hence the sequence of nonnegative numbers $\{t_n\}$, which is decreasing, converges to 0. Finally, we show that $\{x_n\}$ is a Cauchy sequence. Let $\alpha = \limsup_{r\to 0^+} k(r) < 1$. There exists real number β such that $\alpha < \beta < 1$. Then for sufficiently large n, $k(t_n) < \beta$, and thus for sufficiently large n, we have $t_n < \beta t_{n-1}$. Consequently, we obtain $t_n < \beta^n t_0$, that is,

$$q(x_n, x_{n+1}) < \beta^n q(x_o, x_1), \quad n = 0, 1, 2, \ldots. \tag{2.14}$$

Now, for any integers $n, m \geq 1, m > n$,

$$\begin{aligned} q(x_n, x_m) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &< \beta^n q(x_o, x_1) + \beta^{n+1} q(x_o, x_1) + \cdots + \beta^{m-1} q(x_o, x_1) \\ &< \frac{\beta^n}{1-\beta} q(x_o, x_1), \end{aligned} \tag{2.15}$$

and thus by Lemma 1.4, $\{x_n\}$ is a Cauchy sequence. □

Applying Lemma 2.2, we prove a fixed point result for generalized q-contractive maps.

Theorem 2.3. *Let (X, d) be a complete quasimetric space then each generalized q -contractive map $T : X \to Cl(X)$ has a fixed point.*

Proof. It follows from Lemma 2.2 that there exists a Cauchy sequence $\{x_n\}$ in X such that the decreasing sequence $\{q(x_n, x_{n+1})\}$ converges to 0. Due to the completeness of X, there exists some $v_0 \in X$ such that $\lim_{n\to\infty} x_n = v_o$. Let n be arbitrary fixed positive integer then for all positive integers m with $m > n$, we have

$$q(x_n, x_m) \leq \frac{\beta^n}{1-\beta} q(x_o, x_1). \tag{2.16}$$

Let $M = (\beta^n/(1-\beta)) q(x_0, x_1)$, then $M \geq 0$. Now, note that

$$q(x_n, x_m) \leq M \Longrightarrow q(x_n, v_0) \leq M. \tag{2.17}$$

Since n was arbitrary fixed, we have

$$q(x_n, v_0) \leq \frac{\beta^n}{1-\beta} q(x_o, x_1), \quad \text{for all positive integer } n. \tag{2.18}$$

Note that $q(x_n, v_o)$ converges to 0. Now, since $x_n \in T(x_{n-1})$ and T is a generalized q-contractive map, then there is $u_n \in T(v_0)$, such that

$$q(x_n, u_n) \leq k(q(x_{n-1}, v_0)) q(x_{n-1}, v_0). \tag{2.19}$$

And for large n, we obtain

$$q(x_n, u_n) \leq k(q(x_{n-1}, v_0))q(x_{n-1}, v_0) < \beta q(x_{n-1}, v_0), \tag{2.20}$$

thus, we get

$$q(x_n, u_n) < \beta q(x_{n-1}, v_0) \leq \frac{\beta^n}{1-\beta} q(x_o, x_1). \tag{2.21}$$

Thus, it follows from Lemma 1.4 that $u_n \to v_0$. Since $T(v_0)$ is closed, we get $v_0 \in T(v_0)$. □

Corollary 2.4. *Let (X, d) be a complete quasimetric space and q a Q-function on X. Let $T : X \to Cl(X)$ be a multivalued map, such that for any $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ with*

$$q(u, v) \leq k(q(x, y))q(x, y), \tag{2.22}$$

where k is a monotonic increasing function from $(0, \infty)$ to $[0, 1)$, then T has a fixed point.

Finally, we conclude with the following remarks concerning our results related to the known fixed point results.

Remark 2.5. (1)Theorem 2.1 generalizes Theorem 1.2 according to Feng and Liu [2] and Latif and Albar [5, Theorem 3.3].

(2)Theorem 2.3 generalizes Theorem 1.3 according to Suzuki and Takahashi [4] and Theorem 1.5 according to Al-Homidan et al. [7] and contains Latif's Theorem 2.2 in [6].

(3)Theorem 2.3 also generalizes Theorem 2.1 in [8] in several ways.

(4)Corollary 2.4 improves and generalizes Theorem 1 in [9].

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