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Van Der Waals black holes in d dimensions

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ABSTRACT: We generalize the recent solution proposed by Rajagopal et al. to arbitrary number of dimensions and horizon topologies. We comment on the regime of validity of these solution. Among our main results, we argue that the Van Der Waals (VdW) black hole (BH) metric is to be interpreted as a near horizon metric. This is supported by inspecting the energy conditions. We analyze the limiting cases of a perfect fluid, interacting points and non interacting balls gas equation of state and map them to known black holes. Finally, we provide a case study by comparing the Reissner-Nordstr om and VdW BH close to the horizon and show that they are qualitatively similar for some range of the horizon radius.

KEYWORDS: Classical Theories of Gravity, Black Holes, AdS-CFT Correspondence, Holography and condensed matter physics (AdS/CMT)

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Contents

1	Introduction	1
2	AdS Black Hole in d dimensions	3
2.1	Metric ansatz and matter source	3
2.2	Thermodynamic properties	4
3	Van der Waals Black Holes	5
3.1	Construction of the solution	5
3.2	Interpretation of the constants and domain of existence	6
3.3	Source fluid and energy conditions	8
4	Consistent VdW black hole	9
5	Limiting cases: perfect gas and interacting point gas, non interacting ball gas	12
6	Case study: why do Reissner-Nordström black holes approximatively follow the VdW EOS?	12
7	Discussion and conclusion	13
A	Deriving the d-dimensional solution	15

1 Introduction

The thermodynamic properties of black holes provide important clues to the behaviour of quantum gravity. The physics of asymptotically AdS black holes has been of considerable interest for quite some time [1] because of the AdS/CFT correspondence, which led to an effort to understand strongly coupled thermal field theories living on the AdS boundary. Recently the thermodynamic properties of these kinds of black holes have been a subject of intense interest because of their qualitatively similar behaviour to a Van der Waals fluid once rotation and/or charge are added [2–4]. However the analogy is complete [5, 6] only when the the cosmological constant $\Lambda < 0$ is interpreted as thermodynamic pressure $P = P_\Lambda$, [7, 8]

$$P_\Lambda = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}, \quad (1.1)$$

and is allowed to vary in the first law of black hole thermodynamics,

$$\delta M = T\delta S + V\delta P + \dots, \quad (1.2)$$

where the quantity V

$$V = \left(\frac{\partial M}{\partial P} \right)_{S, \dots} \tag{1.3}$$

is the thermodynamic volume and is conjugate to P [9, 10]. From this, an equation of state $P = P(V, T)$ can be written down for a given black hole. By identifying the black hole and fluid temperatures $T \sim T_f$, volumes $V \sim V_f$, and pressures $P \sim P_f$ a proper mapping to the corresponding fluid equation of state can be made [5, 6]. Charged and/or rotating AdS black hole thermodynamics has been shown to qualitatively mimic the behaviour of a standard Van der Waals (VdW) fluid. The VdW liquid/gas phase transition corresponds to a small/large black hole first-order phase transition, which terminates at a critical point characterized by the standard mean field theory critical exponents. The Gibbs free energy exhibits a swallowtail catastrophe in both cases [5, 6]. This analogy extends to a broad class of black holes in higher dimensions [11].

A VdW fluid is described by the VdW equation of state (EOS)

$$T = \left(P + \frac{a}{v^2} \right) (v - b) \tag{1.4}$$

which is a closed form 2-parameter equation of state, valid in any spatial dimension [6]. Here $v = V/N$ denotes the specific volume of the fluid, with N counting the degrees of freedom of the fluid. The parameter $a > 0$ is a measure of the intermolecular attraction between the fluid constituents and the parameter b is a measure of their volume. However the corresponding equations of state for the black hole and the VdW fluid, though qualitatively mathematically similar, are not identical. Consequently neither charge Q nor rotation J (or other parameters for more complicated black holes [11]) can be directly identified with the fluid parameters a and b . In the following, we will denote the pressure associated with the cosmological constant P_Λ to avoid confusion with the pressure of the fluid sourcing the Einstein equations.

Recently a proposal was put forward to construct an exact black hole analog of the VdW fluid in 4 space-time dimensions [12], inspired by this qualitative analogy. There is a fluid yielding a metric from the Einstein equations that can indeed give rise to the equation of state (1.4). However it can only obey the standard energy conditions for sufficiently small pressures and in a region sufficiently close to the event horizon. Here we extend this result to arbitrary number of dimension and arbitrary horizon topologies. The higher-dimensional cases may be of some physical relevance; in particular, in 5 dimensions AdS black holes have a well understood 4 dimensional *CFT* counterpart. We further examine in more detail properties of these black holes such as their domain of existence, and interpret the free constants appearing in the solutions. We also discuss the limiting cases of non interacting gas with finite size, interacting points and perfect gas, and map them to the corresponding black hole solutions.

This paper is organized as follows: In section 2, we review the basic thermodynamic properties of AdS black holes in d dimensions and extend them to a parametrized metric suited for our construction [12]. In section 3, we derive the d -dimensional solution describing a VdW black hole, and discuss the domain of existence of the latter. Next, we analyze

the source of the Einstein equation yielding the VdW black hole and discuss their energy conditions. In section 4 we provide an explicit realization of a VdW black hole satisfying the energy conditions in a range close to the horizon. In section 5, we discuss the limiting cases where $a = 0$ or $b = 0$. Finally, section 6 is devoted to providing an explicit comparison of the Reissner-Nordström black hole with the VdW black hole, in order to understand the qualitative similarities from a metric point of view. We summarize our finding in the concluding section 7.

2 AdS Black Hole in d dimensions

2.1 Metric ansatz and matter source

Let us first review the basic properties of the AdS vacuum black hole solution in d dimensions, described by the AdS-Tangherlini black hole [13]:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{k,d-2}^2, \tag{2.1}$$

where $d\Omega_{k,d-2}$ is the line element on the unit $d - 2$ constant curvature surface, with k the sign of the curvature. The vacuum solution is given by

$$f(r) = \frac{r^2}{\ell^2} + k - \frac{\mu}{r^{d-3}}, \tag{2.2}$$

where ℓ is the cosmological radius defined by $\Lambda = -\frac{(d-1)(d-2)}{2\ell^2}$ and μ is related to the ADM mass M of the black hole:

$$M = \frac{(d-2)\Omega_{k,d-2}}{16\pi}\mu. \tag{2.3}$$

This solution satisfies the AdS vacuum equation $G_{ab} + \Lambda g_{ab} = 0$. In the following, we will follow the construction of [12] and build $f(r)$ such that a desired equation of state on the black hole thermodynamics is obtained. Generically, the solution of this procedure will no longer be a vacuum solution but will contain sources. A satisfying solution will be obtained if some basic properties remain valid, such as e.g. preservation of energy conditions.

Generically, given a function f , the stress tensor is obtained using the field equations. Here, we assume the stress tensor to be given by an anisotropic fluid of the form

$$T^{ab} = \rho e_0^a e_0^b + \sum_i P_i e_i^a e_i^b = G^{ab} + \Lambda g^{ab}, \tag{2.4}$$

where e_μ^a are the components of the vielbein and $i = 1, \dots, d$ and where

$$\begin{aligned} \rho &= -P_r = -\frac{(d-2)f'(r)}{2r} - \frac{(d-3)(d-2)f(r)}{2r^2} + \frac{(d-3)(d-2)k}{2r^2} + 8\pi P_\Lambda, \\ P_\alpha &= \frac{(d-3)f'(r)}{r} + \frac{(d-4)(d-3)f(r)}{2r^2} - \frac{(d-4)(d-3)k}{2r^2} + \frac{f''(r)}{2} - 8\pi P_\Lambda, \end{aligned} \tag{2.5}$$

where α labels the coordinates in the angular sector. Note that we use the convention $G = 1/(8\pi)$, while [12] uses $G = 1$, G being Newton's constant.

2.2 Thermodynamic properties

In d space-time dimensions the cosmological constant will be interpreted as thermodynamic pressure, given by [7–10, 14]

$$P_\Lambda = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi\ell^2} \quad (2.6)$$

and without loss of generality, we write the function f as

$$f = \frac{r^2}{\ell^2} - \frac{\mu}{r^{d-3}} - h(r, P_\Lambda), \quad (2.7)$$

where h is a function to be determined and μ is related to the mass of the black hole as in (2.3).

The temperature T of a black hole in this class of solutions is obtained by requiring regularity of the Euclidean section, yielding

$$T = \frac{f'(r_h)}{4\pi}, \quad (2.8)$$

where r_h is the location of the horizon radius. The entropy is given by the Bekenstein-Hawking law

$$S = \frac{\mathcal{A}}{4} = \frac{\Omega_{d-2}}{4} r_h^{d-2}, \quad (2.9)$$

where Ω_n is the surface of the n -dimensional unit sphere and the mass is given by M provided h doesn't contain terms linear in $1/r^{d-3}$, and can be expressed in terms of other quantities by solving $f(r_h) = 0$ for M .

The first law of thermodynamics then reads

$$dM = TdS + V_\Lambda dP_\Lambda, \quad (2.10)$$

where the momentum V_Λ conjugated to P_Λ is interpreted as a volume, given by

$$V_\Lambda = \left. \frac{\partial M}{\partial P} \right|_S, \quad (2.11)$$

and M is the enthalpy,

$$M = \frac{(d-2)}{16\pi} \Omega_{k,d-2} r_h^{d-3} \left(\frac{16\pi P_\Lambda r_h^2}{(d-1)(d-2)} - h(r_h, P_\Lambda) \right). \quad (2.12)$$

The specific volume is then given by

$$v_\Lambda = \kappa \frac{V_\Lambda}{\mathcal{A}}, \quad (2.13)$$

where $\kappa = 4 \frac{d-1}{d-2}$. We are assuming one degree of freedom per Planck horizon area [12].

3 Van der Waals Black Holes

3.1 Construction of the solution

Using (2.7), the thermodynamic quantities defined in the previous sections reduce to

$$T = -\frac{(d-3)h(r_h, P_\Lambda)}{4\pi r_h} - \frac{h^{(1,0)}(r_h, P_\Lambda)}{4\pi} + \frac{4P_\Lambda r_h}{d-2} \tag{3.1}$$

$$M = \frac{\Omega_{d-2}}{d-1} P_\Lambda r_h^{d-1} - \frac{\Omega_{d-2}}{16\pi} (d-2) r_h^{d-3} h(r_h, P_\Lambda), \tag{3.2}$$

$$v_\Lambda = \frac{4r_h}{d-2} - \frac{(d-1)h^{(0,1)}(r_h, P_\Lambda)}{4\pi r_h} \tag{3.3}$$

and the entropy S is given by (2.9). Specifying an equation of state yields a set of partial differential equations for the function h . Here we impose the equation of state (1.4)

$$T = \left(P_\Lambda + \frac{a}{v_\Lambda^2} \right) (v_\Lambda - b), \tag{3.4}$$

where we set $v = v_\Lambda$, $P = P_\Lambda$ and will regard T as being given by (3.1). To solve the resultant partial differential equations that emerge from (3.4) requires an ansatz. Here we employ $h = A(r) - P_\Lambda B(r)$; inclusion of higher powers of P_Λ does not yield solutions consistent with the asymptotic AdS structure [12]. Note that strictly speaking, the equation of state provides a boundary condition on the event horizon. However, if this condition is satisfied everywhere, obviously it is also satisfied on the horizon. We shall first follow this strategy and then comment on the limitations of this approach.

Our ansatz for h yields two ordinary differential equations for A and B . The equation for B is independent of the number of the number of dimensions and is given by

$$rB' - 2B + 4\pi r b = 0, \tag{3.5}$$

which is solved by

$$B = 4\pi r(C_\ell r + b), \tag{3.6}$$

where C_ℓ is an integration constant whose meaning will be discussed later.

The equation for A is

$$r_h A'(r_h) + (d-3)A(r_h) + \frac{16\pi^2 a (d-2) r_h^2 ((d-2)(d-1)B(r_h) + 16\pi r_h^2 - 4\pi r_h b (d-2))}{((d-2)(d-1)B(r_h) + 16\pi r_h^2)^2} = 0 \tag{3.7}$$

Using (3.6), the general solution of this equation is given by (see appendix)

$$A(r; d) = \frac{a\pi x^4 (d-2)}{(1 + \tilde{C}_\ell)(d^2 - 1)} \left((d-4) {}_2F_1(1, d+1; d+2; -x) - {}_2F_1(2, d+1; d+2; -x) \right) - \frac{\pi a (d-2) x ((d-4)(d-1)x^2 - (d-3)dx + (d-1)d)}{(\tilde{C}_\ell + 1)(d-1)^2 d} + \frac{\tilde{C}_M}{r^{d-3}}, \tag{3.8}$$

The constant a can be set so that the constant term in r in the asymptotic form of f is k , similar to the $d = 4$ case. This leads to $a = \frac{d-3}{\pi(d-2)}k$. This means that the intermolecular attraction of the fluid is governed by the curvature of event horizon. A VdW black hole with a flat horizon section has $a = 0$, and one with constant negative curvature (corresponding to topological black holes [15–17]) has an effective intermolecular repulsion.

We note that the Van der Waals black holes (3.10) do not possess horizons for all generic values of the parameters. Indeed, for some values of the parameters, the horizon radius crosses 0 and becomes negative. Such solutions correspond to solitons (or bubbles); we shall not consider them here.

The allowed region in parameter space for black holes is given by $r_h > 0$. The boundary of this region can be obtained by solving for the mass in terms of other parameters upon imposing $r_h = 0$. This leads to

$$\begin{aligned} \frac{16\pi M}{2\Omega_2} &\geq -\frac{4\pi ab(1 + 4\log(12\pi b))}{(2 + 3C_\ell)^2}, \quad \text{for } d = 4, \\ \frac{16\pi M}{3\Omega_3} &\geq \frac{27\pi ab^2(1 + 6\log(12\pi b))}{4(3C_\ell + 1)^3}, \quad \text{for } d = 5, \\ \frac{16\pi M}{4\Omega_4} &\geq -\frac{100\pi ab^3(1 + 8\log(20\pi b))}{(5C_\ell + 1)^4}, \quad \text{for } d = 6, \\ \frac{16\pi M}{5\Omega_4} &\geq \frac{84375\pi ab^4(1 + 10\log(60\pi b))}{(15C_\ell + 2)^5}, \quad \text{for } d = 7, \end{aligned} \quad (3.11)$$

and other numbers of dimensions are straightforwardly computed.

Note that the specific volume (3.3) becomes

$$v = \left(\frac{4}{d-2} + (d-1)C_\ell \right) r_h + (d-1)b \quad (3.12)$$

yielding

$$\mathcal{R} = \left(\frac{(d-1)V}{\Omega_{d-2}} \right)^{\frac{1}{d-1}} \left(\frac{\Omega_{d-2}}{\mathcal{A}} \right)^{\frac{1}{d-2}} = \left(1 + \frac{(d-1)(d-2)}{4} \left(C_\ell + \frac{b}{r_h} \right) \right)^{\frac{1}{d-1}} \quad (3.13)$$

for the isoperimetric ratio \mathcal{R} . This will obey the reverse isoperimetric inequality [18] provided $C_\ell r_h + b > 0$, or in other words if C_ℓ is not too negative.

Finally, we stress that, for a given set of parameters (and horizon radius), the solution (3.10) derived here need only hold in a neighbourhood of the horizon, since the relation (3.4) involves quantities defined at the horizon. The highest derivative order appearing in this relation is 1. Therefore, the same thermodynamic properties are satisfied in a neighborhood of the horizon by black holes with

$$f(r) = \frac{r^2}{\ell^2} - \frac{\mu}{r^{d-3}} - h(r, P_\Lambda)\mathcal{H}(r, P_\Lambda), \quad (3.14)$$

where \mathcal{H} is an arbitrary function with the following properties at the horizon

$$\mathcal{H}(r_h, P_\Lambda) = 1, \quad \partial_r \mathcal{H}(r_h, P_\Lambda) = 0 = \partial_{P_\Lambda} \mathcal{H}(r_h, P_\Lambda). \quad (3.15)$$

Indeed, in this case,

$$\begin{aligned} \partial_r(\mathcal{H}(r, P_\Lambda)h(r, P_\Lambda))|_{r=r_h} &= \partial_r h(r, P_\Lambda)|_{r=r_h}, \\ \partial_{P_\Lambda}(\mathcal{H}(r, P_\Lambda)h(r, P_\Lambda))|_{r=r_h} &= \partial_{P_\Lambda}(h(r, P_\Lambda))|_{r=r_h}, \end{aligned} \tag{3.16}$$

$$\mathcal{H}(r, P_\Lambda)h(r, P_\Lambda)|_{r=r_h} = h, \tag{3.17}$$

leading to the exact same equation for h as the one obtained with $\mathcal{H} = 1$.

The function \mathcal{H} can serve as a modulating function for the fluid. If it falls off sufficiently fast for large radii, μ remains unambiguously related to the mass (or enthalpy). This also ameliorates the consequences of the linear and logarithmic terms in [12]. If the falloff rate of \mathcal{H} is sufficiently rapid, the conditions on the constant of integration C_ℓ , C_M can be relaxed (as noted above), and the near horizon spacetime has an ‘effective cosmological radius’ different from the one given by Λ .

However, this remark is true only in the case where we assume a given horizon radius, or alternatively a given mass. If the purpose is to build a family of black holes (parametrized, say, by horizon radius) where all solutions satisfy the VdW EOS, then $\mathcal{H} = 1$. Furthermore, if $\mathcal{H} \neq 1$ it is no longer clear that M be regarded as enthalpy, and that the volume v will remain the same. This is because in v is not a near horizon property but is instead a global property of the spacetime.

3.3 Source fluid and energy conditions

Recall that the metric (2.7) with the solution (3.9) does not satisfy the AdS vacuum Einstein equations. Instead the VdW black hole is sourced by a non isotropic fluid along with (2.5). The source stress tensor is conserved by construction, since it is identically the Einstein tensor, which satisfies the Bianchi identities.

A crucial remark is that close to the horizon, while ρ, P_r do not depend on \mathcal{H} , P_α does, since the f'' term yields a term of the form $-\frac{1}{2}\mathcal{H}''h$.

The energy conditions for a non isotropic fluid are given by

- Weak: $\rho \geq 0, \rho + P_i \geq 0$,
- Strong: $\rho + \sum_i P_i \geq 0, \rho + P_i \geq 0$,
- Dominant: $\rho - |P_i| \geq 0$.

In the case of van der Waals black holes, the density and pressures are given by (2.5) together with the solution for h .

In the rest of this subsection, we will consider the VdW black hole generalizing the 4 dimensional solution of [12] by setting $\mathcal{H} = 1$.

We first emphasize that since we have a vacuum solution when $h + k = 0$, and since eq. (2.5) is linear in f (and its derivatives), the density and pressure depend solely on $h + k$. As an important consequence, the properties of the sourcing fluid do not depend on the black hole mass, nor on the parameter C_M since it accounts for a mass term.

We also note that since C_M can always be reabsorbed in the definition of the mass, we set it to 0 for definiteness. Let us analyze whether it is possible to build a VdW black hole that satisfies the energy conditions close to the horizon.

First, it is instructive to inspect the small and large r behaviour of the fluid density and pressure. For large r , ρ , P_α behave as

$$\rho = -2\pi C_\ell d_1 d_2 P_\Lambda - \frac{2\pi b d_2^2 P_\Lambda}{r} + \frac{d_3 d_2 k - \frac{4\pi a d_2^2}{C_\ell d_2 d_1 + 4}}{2r^2} \quad (3.18)$$

$$P_\alpha = 2\pi C_\ell d_2 d_1 P_\Lambda + \frac{2\pi b d_3 d_2 P_\Lambda}{r} + \frac{\frac{4\pi a d_4 d_2}{C_\ell d_2 d_1 + 4} - d_4 d_3 k}{2r^2} \quad (3.19)$$

where $d_j \equiv (d - j)$. For small radii, it can easily be checked that

$$\rho = \frac{d_2 d_3}{2r^2} + \mathcal{O}(r)^{-1}, \quad P_\alpha = -\frac{d_3 d_4}{2r^2} + \mathcal{O}(r)^{-1}. \quad (3.20)$$

As a consequence, the VdW BH with $C_\ell = 0$ *always* violates the weak and strong energy conditions at large radii, and are satisfied at small radii. Although the physical relevance of the small- r is dubious, mathematically it follows that there always exists a region where these energy conditions are satisfied. This is illustrated on figure 1, where we show the weak and strong energy conditions for $b = r_h$, $a = 1/(2\pi)$ and $d = 4$. Other values of d leads to the same pattern.

Since the source of the stress tensor does not depend on the mass of the black hole, we conclude that there always exists a parameter set for which the energy conditions can be satisfied *close* to the horizon. However not all horizon radii lead to positive M , see section 3.2. The question then reduces to finding parameter sets so that the region where the energy conditions are satisfied is *outside* the horizon.

Note that for negative values of C_ℓ it is always possible to satisfy $\rho \geq 0$ everywhere. However the second part of the weak energy condition, namely $\rho + P_\alpha$, always yields a negative falloff for sufficiently large r

$$\rho + P = -\frac{2\pi b d_2 P_\Lambda}{r} + \mathcal{O}(r)^{-2}, \quad \text{as } r \rightarrow \infty \quad (3.21)$$

contrary to what was originally reported [12] for the $d = 4$ case.

This further supports the idea that the VdW black hole metric should be thought as a near horizon solution. In the next section, we will provide evidence that it is possible to build a VdW black hole that satisfies the energy conditions at least in a region close to the horizon by explicitly constructing it in some number of dimensions.

4 Consistent VdW black hole

In this section, we propose a systematic prescription for constructing VdW black holes that satisfy the energy conditions close to the horizon. For definiteness, we set $\mathcal{H} = 1$ and $C_M = 0$ in the rest of this section.

One possible way of investigating the question of consistent VdW black holes, in the sense that they satisfy the energy conditions, is to look at the near horizon solution for

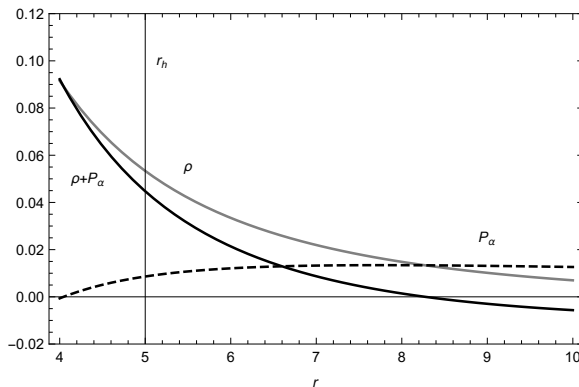


Figure 1. The weak ($\rho, \rho + P_\alpha > 0$) and strong ($\rho + P_\alpha, P_\alpha > 0$) energy conditions for $b = r_h, a = 2/(3\pi), C_\ell = 0, C_M = 0, P = 0.0009$ and $d = 5$. There exists a region where the three functions are positive outside the horizon (located at $r_h = 5$). This choice of parameters leads to a positive mass.

these. However this doesn't teach us much because the way we built the solution is such that we express the mass in terms of the horizon radius quantities. In fact, in a near horizon expansion, the free parameters are $A(r_h), B(r_h)$, which in turn are related to C_M, C_ℓ . The procedure is then to use the equations provided by the EOS, and derive them iteratively, since they are the only equations from which we've built the solution.

Instead we present an algorithm in order to build a consistent solution near the horizon. We require that mass remain positive, while the energy conditions (EC) are satisfied near the horizon for some $r > r_h$. Due to equations (2.5), (2.12) and the fact that $h(r, P_\Lambda)$ is linear in P_Λ , all these quantities can be written in the form $X = X_0 + P_\Lambda X_P$:

$$\text{EC} : \mathcal{E}_0(r) - P_\Lambda \mathcal{E}_P(r) \geq 0 \quad \text{Mass} : \mathcal{M}_0(r_h) + P_\Lambda \mathcal{M}_P(r_h) \geq 0, \quad (4.1)$$

where $\mathcal{E}_0(r), \mathcal{E}_P(r)$ are both positive and increasing with decreasing r provided $a = \frac{d-3}{\pi(d-2)}, C_\ell = 0$, and $k = 1$. Since the energy conditions are linear in density and pressure, (4.1) is generic and the form of $\mathcal{E}_0, \mathcal{E}_P$ depends on which particular energy condition is considered. This leads to a maximum pressure that saturates the energy condition just outside the horizon:

$$P_{Max}(r_h) = \mathcal{E}_0(r_h)/\mathcal{E}_P(r_h) \quad (4.2)$$

and so for a given a set of parameters (a, b, C_ℓ), it is possible to choose $P_\Lambda < P_{Max}(r_h)$ such that $M > 0$ for some finite range of $r > r_h$. This guarantees that all requisite conditions are satisfied at least in a neighbourhood of the horizon. We expect this procedure to be valid over a wide range of the parameter space and we leave a more systematic investigation for future considerations.

Setting the pressure a little bit below the maximal value, say $P_\Lambda = qP_{Max}, q < 1$, we can write M as a q independent term and a term linear in $q, M = m_0 + qm_1$. The critical value $q^* = -m_0/m_1$ is a function of r_h and corresponds to the fraction of the maximal pressure where the mass changes sign. In particular, when $q^* < 1$ the mass is positive and the energy condition considered can be satisfied in a neighborhood of the horizon.

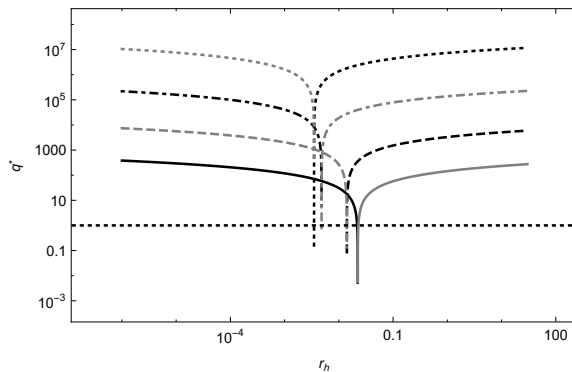


Figure 2. The value of q^* where the mass crosses 0, and where $P_\Lambda = q^*P_{Max}$. When $q^* \leq 1$, the weak energy conditions are satisfied in a neighbourhood of the horizon. If $q^* < 0$, then the mass is positive for any choice of r_h and $q^* \leq q \leq 1$. We chose $b = r_h$, $a = (d-3)/((d-2)\pi)$, $C_\ell = C_M = 0$. The black parts of the curve are q^* while the gray are $-q^*$ and d ranges from 5 to 9, from bottom to top.

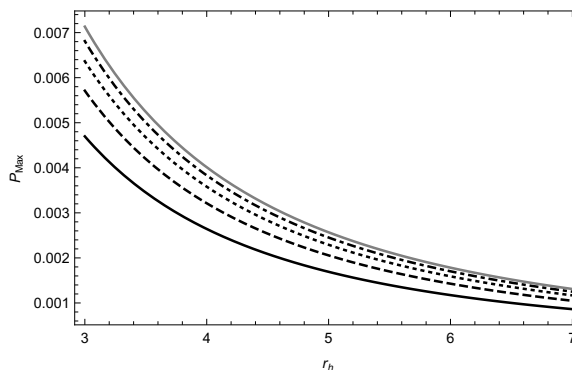


Figure 3. The maximal pressure allowing to satisfy the WEC as a function of the horizon radius, for $b = r_h$, $a = (d-3)/((d-2)\pi)$, $C_\ell = C_M = 0$ and $d = 5, 6, 7, 8, 9$ from bottom to top.

We summarize the behaviour of q^* for a number of dimensions and parameter values in figure 2.

We computed P_{Max} as the maximal pressure allowing the weak energy conditions to be fulfilled, as a function of the horizon radius and systematically found possible to build consistent VdW black holes for the case we considered. We expect that this is possible for all dimensions.

Note that b is a dimensionful parameter, so we defined $\tilde{b} = b/r_h$. We plot the maximum pressure as a function of r_h for a fixed value of \tilde{b} , and the value q^* such that the mass just vanishes in figures 2 and 3 for $\tilde{b} = 1$.

Focusing on the WEC (and fixing $k = 1$ for definiteness), we find that $P_{Max} = \alpha/r_h^2$, where α depends on d and \tilde{b} . We found (for $d = 5, 6, 7, 8, 9$) that the ratio $0 < r_h/\ell_{Min} < 1$, where ℓ_{Min} is the value of the cosmological length corresponding to P_{Max} . For the case we considered, this means that the intermediate or small black hole phase leads to consistent VdW black holes.

5 Limiting cases: perfect gas and interacting point gas, non interacting ball gas

It is instructive to consider the 3 limits of the VdW equation of state, namely, $a = 0, b = 0, a = b = 0$. These correspond respectively to a gas of free particles with finite size (the ball gas), a gas of interacting point particles, and a perfect gas.

The solutions are given by

$$f_{perfect} = \frac{16\pi(c_\ell + 1)Pr^2}{(d-2)(d-1)} - \frac{16\pi(c_M + M)r^{3-d}}{(d-2)\Omega_{d-2}} \quad (5.1)$$

$$f_{point} = 4\pi br + f_{perfect}, \quad (5.2)$$

$$f_{ball} = \frac{(d-2)a\pi}{(d-3)(1+c_\ell)} + f_{perfect}, \quad (5.3)$$

where $f_{perfect}$, f_{point} , f_{ball} are the solution for the perfect gas, the interacting point particle and the free ball gas respectively, and c_M , c_ℓ are constants of integration, rescaled from C_M , C_ℓ .

We note in passing that the Lemos string [19] has the thermodynamic properties of a perfect gas, and the Schwarzschild AdS solution has the thermodynamics of a non interacting ball with size parameter $a = (1+c_\ell)(d-3)/(\pi(d-2))$. The interaction between the gas particles is accommodated by a linear term in the metric function.

The mass M corresponds to the enthalpy in extended phase space and can be arbitrarily shifted, which was manifest from the C_M term, commensurate with the fact that we can set $C_M = 0$ without loss of generality.

Interestingly, the perfect gas black hole satisfies the Weak energy conditions everywhere as long as $c_\ell \leq 0$ and $k \neq -1$:

$$\rho_{perfect} = \frac{(d-2)(d-3)}{2r^2} - 8\pi c_\ell P, \quad (5.4)$$

$$(\rho + P)_{perfect} = \frac{d-3}{r^2}. \quad (5.5)$$

In the other cases, there is no generic structure where the WEC are satisfied everywhere.

6 Case study: why do Reissner-Nordström black holes approximatively follow the VdW EOS?

In this section, we will try to understand the qualitative VdW behaviour of the Reissner Nordström black hole, by comparing its near horizon metric with that of the VdW black hole.

The AdS Reissner-Nordström black hole solution is given by

$$f(r) = \frac{r^2}{\ell^2} + 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad (6.1)$$

where Q is the charge and M the mass of the black hole. In terms of A, B in (2.7), it is given by

$$A = \frac{Q^2}{r^2} - 1, \quad B = 0. \quad (6.2)$$

It was shown in [5] that the combination $\frac{P_\Lambda v}{T}$ at the critical point, defined as $P'_\Lambda(v_c) = 0 = P''_\Lambda(v_c)$ is the same for the VdW equation of state and the Reissner-Nordström black hole with the following identification:

$$a = \frac{3}{4\pi}, \quad b = 2\sqrt{\frac{2}{3}}Q^2. \tag{6.3}$$

For a given black hole to follow a VdW EOS, its metric components and first derivatives of the metric components should be comparable to (3.10). Let us compare explicitly the VdW metric to the Reissner-Nordström one. First, we fix the constant of integration C_ℓ by demanding that the horizon radii are the same, or equivalently that the volume of the black hole is the same:

$$V_4^Q = V_4^{VdW} \Rightarrow C_\ell = -\frac{1}{r_h}, \tag{6.4}$$

where r_h is the horizon radius of the Reissner-Nordström black hole.

Next, we compare the first derivative of the metric coefficient close to the horizon. In the static case, this is equivalent to comparing the temperature of the black holes. We find

$$T_Q = \frac{8\pi P_\Lambda r_h^4 - Q^2 + r_h^2}{4\pi r_h^3}, \quad T_{VdW} = -\frac{(b - 2r_h)(a + 4P_\Lambda r_h^2)}{4r_h^2}, \tag{6.5}$$

where T_Q , T_{VdW} are the temperature of the Reissner-Nordström and Van der Waals black holes respectively. For large horizon radius, the ratio of the temperatures is given by

$$\frac{T_Q}{T_{VdW}} = 1 + \frac{b}{2r_h} + \mathcal{O}(r_h)^{-2}, \tag{6.6}$$

while for small horizons,

$$\frac{T_Q}{T_{VdW}} = \frac{Q^2}{ab\pi r_h} + \frac{2Q^2}{ab^2\pi} + \mathcal{O}(r_h) = 1 + \frac{b}{2r_h} + \mathcal{O}(r_h) \tag{6.7}$$

where the latter expression follows upon using (6.3).

Matching the two expansions in the intermediate r_h region suggests that the ratio goes like $1 + \frac{b}{2r_h}$ over a wide range of r_h . As a consequence, the derivatives are qualitatively similar everywhere, except in the limit $r_h \rightarrow 0$ (more precisely they deviate in $1/r_h$), for the particular choice of a, b (6.3).

In figure 4, we show this comparison, and see that it qualitatively agrees for different values of P_Λ with a deviation typically for some small horizon radii.

7 Discussion and conclusion

In this paper, we constructed d -dimensional AdS black hole metrics whose thermodynamic quantities satisfy the Van der Waals equation of state. These black holes are the exact dual of a Van der Waals fluid in the conformal boundary of the AdS space. Our construction generalizes the solutions presented in [12] and is valid for arbitrary numbers of dimensions and spherical, planar and hyperbolic horizons. We discussed in some detail the properties

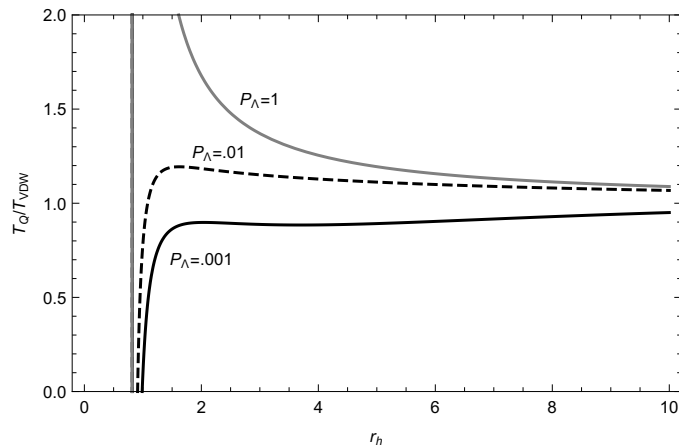


Figure 4. Comparison of the temperature of the Reissner-Nordström and VdW black holes for different values of P_Λ .

of these black holes and found that they are of positive mass for a certain portion of the parameter space, which we described as a domain of existence.

These black holes are non vacuum solutions. The source to the Einstein equations yielding the Van der Waals black hole has the form of an anisotropic fluid. The energy associated with the sourcing fluid respects the energy conditions for a certain range of parameters and in a region sufficiently close to the horizon. We found systematic deviation and violation of the energy conditions far from the horizon.

However, this situation can be improved by using a modulating function \mathcal{H} that is such that $\mathcal{H}(r_h) = 1$, $\mathcal{H}'(r_h) = 0$ and that falls off sufficiently fast. This modulating function can be chosen such that its defining property is valid over a range of r_h where the energy conditions remains satisfied. This leads in principle to a non vacuum black hole solution that is an exact Van der Waals black hole in some portion of the phase space but otherwise deviates from it outside this portion, yet satisfies the energy conditions everywhere. Indeed, it is impossible in Einstein gravity to devise an exact Van der Waals black hole valid over the whole (extended) phase space that satisfies the energy conditions everywhere. Note however that a small deviation from the weak energy condition can be physically acceptable in AdS spacetime — for instance the case of negative mass-squared scalars still leads to a positive *ADM* mass in AdS [20]. Despite the systematic violation of the Weak Energy Condition reported here, a stability analysis would be desirable to assess the potential use of the exact VdW black holes in the *AdS/CFT* picture.

We also analyzed the perfect fluid, interacting point-particle, and non-interacting ball limits of Van der Waals black holes. In particular, we noted that planar black holes are perfect gas black holes and Tangherlini black holes are non-interacting ball gas black holes. The effect of interaction in the equation of state on the metric is to add a term linear in r . These observations may provide useful motivations for a phenomenological description of CFT with the corresponding properties.

Finally, we analyzed the qualitative agreement between the thermodynamics of the

Reissner-Nordström black hole and the Van der Waals black hole by explicitly comparing both metrics in the same coordinate system and horizon location. We found that, as expected, both metrics lead to the same thermodynamic properties for large enough values of the horizon radius.

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A Deriving the d -dimensional solution

In this appendix, we give the details of the derivation of the analytic d -dimensional VdW black hole solution. First, as already pointed out in the text, the solution for the function B is given by

$$B = 4\pi r(C_\ell r + b). \tag{A.1}$$

Plugging this equation in the equation for A leads to an equation of the form

$$(r^{d-3}A)' = -\frac{N_0 r^{d-3}(N_1 + D_1 r)}{(D_0 + D_1 r)}, \tag{A.2}$$

where $N_1 = (d-2)bN_0/(4a\pi)$, $D_0 = (d-1)bN_0/(4a\pi)$, $D_1 = (4+C_\ell(d-2)(d-1))N_0/(4a(d-2)\pi)$.

Denoting $d_0 = D_0/N_0$, $d_1 = D_1/N_0$, $n_1 = N_1/N_0$, the formal solution to this equation is given by

$$\begin{aligned} r^{d-3}A = & C - \frac{r^d}{d_0^4} \left(\frac{d_1^2}{d}(3n_1 - 2d_0) + \frac{1}{r^2} \left(\frac{d_0 n_1}{d-2} + \frac{d_1(d_0 - 2n_1)r}{d-1} \right) \right) \\ & - \frac{d_1^3 r^{d+1}}{d_0^5(d+1)} \left((2d_0 - 3n_1) {}_2F_1 \left(1, d+1, d+2; -\frac{d_1}{d_0} r \right) + (d_0 - n_1) {}_2F_1 \left(2, d+1, d+2; -\frac{d_1}{d_0} r \right) \right), \end{aligned} \tag{A.3}$$

where ${}_2F_1(a, b, c; x)$ is the hypergeometric function. For integer values of d , the solution reduces to a finite polynomial with logarithmic terms, and C is an integration constant.

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