



An alternative transformation in ranking using l_1 -norm in data envelopment analysis

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Abstract Jahanshahloo et al. (Appl Math Comput 153:215–224, 2004) propose a method for ranking extremely efficient decision making units (DMUs) in data envelopment analysis (DEA) using super-efficiency technique and l_1 -norm and they show that the presented method is able to eliminate the existing difficulties in some methods. This paper suggests an alternative transformation to convert the nonlinear model proposed by Jahanshahloo et al. (Appl Math Comput 153:215–224, 2004) into a linear programming form. The present paper shows that model with this transformation is equivalent to the above-mentioned nonlinear model. The motivation of this work is to linearize the proposed nonlinear model by Jahanshahloo et al. (Appl Math Comput 153:215–224, 2004) which has the higher order of complexity.

Keywords Data envelopment analysis (DEA) · Ranking · Efficiency · Extremely efficient · Chinese cities

Introduction

For many applications, ranking DMUs is an important and essential procedure to decision makers in DEA, especially when there are extremely efficient DMUs. In these regards, several methods have been proposed for ranking of the extreme efficient DMUs, see Andersen and Petersen (1993), Mehrabian et al. (1999) and Jahanshahloo et al. (2004). A DMU is called extremely efficient if it cannot be

represented as a linear combination (with nonnegative coefficients) of the remaining DMUs (Charnes et al. 1991). Andersen and Petersen (AP) (Andersen and Petersen 1993) proposed a new procedure to rank efficient DMUs. The AP model determines the rank of a given DMU by removing it from the reference set and by computing its super-efficiency score. However, the AP model may be infeasible in some cases. It is proved that super-efficient DEA models are infeasible (see Thrall 1996; Seiford and Zhu 1999). Mehrabian et al. (1999) suggested the MAJ model for complete ranking of efficient DMUs, but their approach lacks feasibility in some cases, too. To overcome the drawbacks of the AP (Andersen and Petersen 1993) and MAJ (Mehrabian et al. 1999) models, Jahanshahloo et al. (2004) presented a method to rank the extremely efficient DMUs in DEA models with constant and variable returns to scale using L_1 -norm. Their proposed model is a nonlinear programming form which has the higher order of complexity in solving. In addition, a complex procedure was applied in Jahanshahloo et al. (2004) to provide the nonlinear model into a linear one which obtains an approximately optimal solution. Wu and Yan (2010) have also suggested an effective transformation to convert the nonlinear model in Jahanshahloo et al. (2004) into a linear model. Recently, Ziari and Raissi (2016) proposed an approach to rank the efficient DMUs in DEA based on minimizing distance of the under evaluation DMU to the frontier of efficiency. Following this trend, the present paper is an attempt to provide an alternative transformation to convert the nonlinear model in Jahanshahloo et al. (2004) into a linear model. The proposed model in this paper linearizes model in Jahanshahloo et al. (2004) in a way which is different from the presented method in Wu and Yan (2010). To linearize model in Jahanshahloo et al. (2004), the presented model uses the number of fewer

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auxiliary variables with respect to proposed model in Wu and Yan (2010). Furthermore, it is shown that the model with this transformation is equivalent to the nonlinear model and is easier to be solved. The rest of the paper is organized as follows. Section 2 reviews some ranking methods, especially the model by Jahanshahloo et al. (2004). The paper proposes an alternative transformation in Sect. 3. Section 4 re-executes the empirical example by Jahanshahloo et al. (2004) for illustration. The last Section concludes the study.

Review of some ranking models

In this section, we review some ranking models in data envelopment analysis. In the following subsections, it is assumed that there are n DMUs and for each DMU $_j$ ($j = 1, \dots, n$) a vector of inputs (X_j) is considered to produce a vector of outputs (Y_j), where $X_j = (x_{1j}, x_{2j}, \dots, x_{mj})$ and $Y_j = (y_{1j}, y_{2j}, \dots, y_{sj})$. It is also assumed that $X_j \geq 0, Y_j \geq 0, X_j \neq 0$, and $Y_j \neq 0$ for every $j = 1, \dots, n$.

AP model

The ranking method proposed by Andersen and Petersen (1993) is a super efficiency model. In the AP model, DMU under evaluation is excluded from reference set and using other units, the rank of given DMU is obtained.

The input-oriented AP model using the CRS super-efficiency model is as follows:

$$\begin{aligned} \min \theta \\ \text{s.t.} \quad & \sum_{j=1, j \neq k}^n \lambda_j x_{ij} \leq \theta x_{ik}, \quad i = 1, \dots, m \\ & \sum_{j=1, j \neq k}^n \lambda_j y_{rj} \geq y_{rk}, \quad r = 1, \dots, s \\ & \lambda_j \geq 0, \quad j = 1, \dots, n, j \neq k, \end{aligned} \quad (1)$$

where $\lambda_j, j = j = 1, \dots, n, j \neq k$ and θ are the variables of model.

In addition, the output-oriented AP model using the CRS super-efficiency model is as follows:

$$\begin{aligned} \max \phi \\ \text{s.t.} \quad & \sum_{j=1, j \neq k}^n \lambda_j x_{ij} \leq x_{ik}, \quad i = 1, \dots, m \\ & \sum_{j=1, j \neq k}^n \lambda_j y_{rj} \geq \phi y_{rk}, \quad r = 1, \dots, s \\ & \lambda_j \geq 0, \quad j = 1, \dots, n, j \neq k, \end{aligned} \quad (2)$$

where $\lambda_j, j = j = 1, \dots, n, j \neq k$ and ϕ are the variables of model.

The main drawbacks of this model are infeasibility and instability for some DMUs. It can be said that a model is stable if the DMU under evaluation which is efficient remains efficient after perturbation on data.

MAJ model

This ranking model is proposed by Mehrabian et al. (1999) to solve infeasibility of AP models in some cases. The MAJ model can be expressed as follows:

$$\begin{aligned} \min 1 + w \\ \text{s.t.} \quad & \sum_{j=1, j \neq k}^n \lambda_j x_{ij} \leq x_{ik} + w, \quad i = 1, \dots, m \\ & \sum_{j=1, j \neq k}^n \lambda_j y_{rj} \geq y_{rk}, \quad r = 1, \dots, s \\ & \lambda_j \geq 0, \quad j = 1, \dots, n, j \neq k, \end{aligned} \quad (3)$$

where $\lambda_j, j = j = 1, \dots, n, j \neq k$ and w are the variables of model.

L_1 -norm model

In this subsection, the model of Jahanshahloo et al. (2004) is explained as follows. Then the production possibility set (PPS) with constant returns to scale T_c and the PPS with variable returns to scale T_v are defined as:

$$T_c = \left\{ (X, Y) \mid X \geq \sum_{j=1}^n \lambda_j X_j, Y \leq \sum_{j=1}^n \lambda_j Y_j, \lambda_j \geq 0, j = 1, \dots, n \right\}, \quad (4)$$

and

$$T_v = \left\{ (X, Y) \mid X \geq \sum_{j=1}^n \lambda_j X_j, Y \leq \sum_{j=1}^n \lambda_j Y_j, \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n \right\}, \quad (5)$$

respectively.

DMU $_k$ is assumed to be extremely efficient. By removing (X_k, Y_k) from T_c , the production possibility set T'_c is defined as:

$$T'_c = \left\{ (X, Y) \mid X \geq \sum_{j=1, j \neq k}^n \lambda_j X_j, Y \leq \sum_{j=1, j \neq k}^n \lambda_j Y_j, \lambda_j \geq 0, j = 1, \dots, n \right\}. \quad (6)$$

To obtain the ranking score of DMU $_k$ the model in Jahanshahloo et al. (2004) is considered as follows:

$$\begin{aligned} \min \Gamma_k^c(X, Y) &= \sum_{i=1}^m |x_i - x_{ik}| + \sum_{r=1}^s |y_r - y_{rk}| \\ \text{s.t.} \quad &\sum_{j=1, j \neq k}^n \lambda_j x_{ij} \leq x_i, \quad i = 1, \dots, m \\ &\sum_{j=1, j \neq k}^n \lambda_j y_{rj} \geq y_r, \quad r = 1, \dots, s \\ &x_i \geq 0, \quad y_r \geq 0 \quad i = 1, \dots, m, \quad r = 1, \dots, s \\ &\lambda_j \geq 0, \quad j = 1, \dots, n, \quad j \neq k, \end{aligned} \tag{7}$$

where $X = (x_1, \dots, x_m)$, $Y = (y_1, \dots, y_s)$ and $\lambda = (\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_n)$ are the variables of the model (7) and $\Gamma_k^c(X, Y)$ is the distance between (X_k, Y_k) and (X, Y) by l_1 -norm.

To convert model (4) into a linear model, [1] defines the set T_c'' as follows:

$$T_c'' = T_c' \cap \left\{ (X, Y) \mid X \geq X_k \text{ and } Y \leq Y_k \right\} \tag{8}$$

and they apply the scaling input and output data by normalization. After these changes, an approximately optimal solution of model (7) is obtained by solving a linear programming model related to it.

In next section, an alternative transformation to model (7) is considered and to obtain the optimal solution, an equivalent linear programming model is solved.

An alternative transformation

For converting model (7) into a linear model, the following transformation is utilized: $|x_i - x_{ik}| \leq a_i \quad i = 1, \dots, m$ and $|y_r - y_{rk}| \leq b_r \quad r = 1, \dots, s$. Thus, we have:

$$\begin{cases} x_i - x_{ik} \leq a_i, & i = 1, \dots, m, \\ x_i - x_{ik} \geq -a_i, & i = 1, \dots, m, \end{cases} \text{ and } \begin{cases} y_r - y_{rk} \leq b_r, & r = 1, \dots, s, \\ y_r - y_{rk} \geq -b_r, & r = 1, \dots, s. \end{cases}$$

Then Model (7) can be converted into the following linear programming problem:

$$\begin{aligned} \min \Gamma_k^c(X, Y) &= \sum_{i=1}^m a_i + \sum_{r=1}^s b_r \\ \text{s.t.} \quad &\sum_{j=1, j \neq k}^n \lambda_j x_{ij} \leq x_i, \quad i = 1, \dots, m \\ &\sum_{j=1, j \neq k}^n \lambda_j y_{rj} \geq y_r, \quad r = 1, \dots, s \\ &x_i - x_{ik} \leq a_i, \quad i = 1, \dots, m, \\ &-x_i + x_{ik} \leq a_i, \quad i = 1, \dots, m, \\ &y_r - y_{rk} \leq b_r, \quad r = 1, \dots, s, \\ &-y_r + y_{rk} \leq b_r, \quad r = 1, \dots, s, \\ &x_i \geq 0, \quad y_r \geq 0, \quad a_i \geq 0, \quad b_r \geq 0, \quad i = 1, \dots, m, \quad r = 1, \dots, s, \\ &\lambda_j \geq 0, \quad j = 1, \dots, n, \quad j \neq k, \end{aligned} \tag{9}$$

where $X = (x_1, \dots, x_m)$, $Y = (y_1, \dots, y_s)$, $a = (a_1, \dots, a_m)$, $b = (b_1, \dots, b_s)$ and $\lambda = (\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_n)$ are the variables of model (9).

It is obvious that model (9) is equivalent with model (7). Furthermore, model (9) is a linear programming problem which can easily provide the optimal solution of model (7). The above-proposed model includes $3(m + n)$ constraints whereas Wu and Yan (2010) model has $2(m + n)$ constraints that obviously the Wu and Yan (2010) model is more efficient than the above model. To overcome this problem, we proposed the following model which is more efficient with respect to Wu and Yan (2010) model. The efficient proposed model is as follows:

$$\begin{aligned} \min \Gamma_k^c(X, Y) &= \sum_{i=1}^m (x_i - x_{ik}) + \sum_{r=1}^s (-y_r + y_{rk}) \\ \text{s.t.} \quad &\sum_{j=1, j \neq k}^n \lambda_j x_{ij} \leq x_i, \quad i = 1, \dots, m \\ &\sum_{j=1, j \neq k}^n \lambda_j y_{rj} \geq y_r, \quad r = 1, \dots, s \\ &x_i \geq x_{ik}, \quad i = 1, \dots, m \\ &y_r \leq y_{rk}, \quad r = 1, \dots, s \\ &x_i \geq 0, \quad y_r \geq 0 \quad i = 1, \dots, m, \quad r = 1, \dots, s \\ &\lambda_j \geq 0, \quad j = 1, \dots, n, \quad j \neq k, \end{aligned} \tag{10}$$

where $X = (x_1, \dots, x_m)$, $Y = (y_1, \dots, y_s)$ and $\lambda = (\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_n)$ are the variables of model (10).

Next, model (10) is reformulated by imposing the convexity constraint $\sum_{j=1, j \neq k}^n \lambda_j = 1$ on (10), to extend the model (10) from the constant returns to scale to the variable returns to scale case. Then, to obtain the ranking score of DMU under evaluation, the following linear programming problem is solved:

$$\begin{aligned} \min \Gamma_k^c(X, Y) &= \sum_{i=1}^m (x_i - x_{ik}) + \sum_{r=1}^s (-y_r + y_{rk}) \\ \text{s.t.} \quad &\sum_{j=1, j \neq k}^n \lambda_j x_{ij} \leq x_i, \quad i = 1, \dots, m \\ &\sum_{j=1, j \neq k}^n \lambda_j y_{rj} \geq y_r, \quad r = 1, \dots, s \\ &\sum_{j=1, j \neq k}^n \lambda_j = 1 \\ &x_i \geq x_{ik}, \quad i = 1, \dots, m \\ &y_r \leq y_{rk}, \quad r = 1, \dots, s \\ &\lambda_j \geq 0, \quad j = 1, \dots, n, \quad j \neq k \\ &x_i \geq 0, \quad y_r \geq 0 \quad i = 1, \dots, m, \quad r = 1, \dots, s \\ &\lambda_j \geq 0, \quad j = 1, \dots, n, \quad j \neq k, \end{aligned} \tag{11}$$

Table 1 The data for 28 Chinese cities

DMU1	Input 1	Input 2	Input 3	Output 1	Output 2	Output 3
1	483.01	1,397,736	616,961	6,785,798	1,594,957	1,088,699
2	371.95	855,509	385,453	2,505,984	545,140	835,745
3	268.23	685,584	341,941	2,292,025	406,947	473,600
4	202.02	452,713	117,424	1,158,016	135,939	336,165
5	197.93	471,650	112,634	1,244,124	204,909	317,709
6	178.96	423,124	189,743	1,187,130	190,178	605,037
7	148.04	367,012	97,004	658,910	86,514	239,760
8	184.93	408,311	111,904	993,238	1,411,954	353,896
9	123.33	245,542	91,861	854,188	135,327	239,360
10	116.91	305,316	91,710	606,743	78,357	208,188
11	129.62	295,812	92,409	736,545	114,365	298,112
12	106.26	198,703	53,499	454,684	67,154	233,733
13	89.70	210,891	95,642	494,196	78,992	118,553
14	109.26	282,209	84,202	842,854	149,186	243,361
15	85.50	184,992	49,357	776,285	116,974	234,875
16	72.17	222,327	73,907	490,998	117,854	118,924
17	76.18	161,159	47,977	482,448	67,857	158,250
18	73.21	144,163	43,312	515,237	114,883	101,231
19	86.72	190,043	55,326	625,514	173,099	130,423
20	69.09	158,436	66,640	382,880	74,126	123,968
21	77.69	135,046	46,198	867,467	65,229	262,876
22	97.42	206,926	66,120	830,142	128,279	242,773
23	54.96	79,563	43,192	521,684	37,245	184,055
24	67.00	144,092	43,350	869,973	86,859	194,416
25	46.30	100,431	31,428	604,715	55,989	127,586
26	65.12	96,873	28,112	601,299	37,088	224,855
27	20.09	50,717	54,650	145,792	11,816	24,442
28	69.81	117,790	30,976	319,218	31,726	169,051

Table 2 The results of ranking by applying model (11)

DMU	1	2	6	8	21	23	24	25	26	27
Ranking result	1	8	3	2	7	10	9	5	6	4
Value of obj. function	1.521	0.015	0.128	0.700	0.024	0.010	0.013	0.038	0.037	0.090

Model (11) is similar to model (10) which can be solved by any classical mathematical tool such as GAMS.

Empirical example

In this section, we apply DEA model (11) on the data set used by Jahanshahloo et al. (2004), with the assumption of variable returns to scale. The data set consists of 28 DMUs with 3 inputs and 3 outputs. Data are originally reported by Charnes et al. (1989) which comprised 28 Chinese cities

(DMUs) in 1983. The inputs are labor, working funds, and investment. The outputs are gross industrial output value, profit and taxes, and retail sales. The data in Table 1 should be normalized before applying model (11). Table 2 includes the ranking results for 10 extremely efficient DMUs in model (11) (DMU₁, DMU₂, DMU₆, DMU₈, DMU₂₁, DMU₂₃, DMU₂₄, DMU₂₅, DMU₂₆, DMU₂₇). Note that in the empirical study these results are the same as the studies by Jahanshahloo et al. (2004) and Wu and Yan (2010). In this study, we have presented a method for converting the nonlinear programming model by Jahanshahloo et al. (2004) into linear programming problem



using an alternative transformation, and it can easily be solved. It is worth nothing that some of the input data of Table 3 in Jahanshahloo et al. (2004) were mistakenly recorded and in this Section we have corrected them.

Conclusion

Jahanshahloo et al. (2004) propose a new ranking method using super-efficiency technique and l_1 -norm. It was shown that the proposed method is able to eliminate the existing problems in some methods. In this regard, this paper provides an alternative transformation for converting the nonlinear programming model by Jahanshahloo et al. (2004) into the linear programming model which is equivalent to the original nonlinear model. In addition, we proposed the new efficient model which gives the same solutions of original model proposed by Jahanshahloo et al. (2004). Considering the higher order of complexity of nonlinear model (7), the proposed treatment in this article is easier to be utilized.

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