Int. J. Disaster Risk Sci. 2011, 2 (3): 1–10
doi:10.1007/s13753-011-0011-x

ARTICLE

A Cover Technique to Verify the Reliability of a Model for Calculating Fuzzy Probabilities

Chongfu Huang*

State Key Laboratory of Earth Surface Processes and Resource Ecology, Beijing Normal University, Beijing 100875, China
Key Laboratory of Environmental Change and Natural Disasters, Ministry of Education of China, Beijing Normal University, Beijing 100875, China
Academy of Disaster Reduction and Emergency Management, Ministry of Civil Affairs and Ministry of Education of China, Beijing Normal University, Beijing 100875, China

Abstract Many models have been suggested to calculate fuzzy probabilities in risk analysis. In general, the reliability of a model is demonstrated by practical effects or proved theoretically. In this article we suggest a new approach called the cover technique to verify the model's reliability. The technique is based on a hypothesis that a statistical result can approximately confirm a fuzzy probability as a fuzzy-set-valued probability. A cover is constructed by many biprobability distributions. The consistency degree of a cover and a fuzzy probability distribution is employed to verify the reliability of a model. We present a case that shows how to construct a distribution-cover and calculate the consistency degree of the cover and a possibility-probability distribution. A series of numerical experiments with random samples from a normal distribution verify the reliability of the interior-outer-set model.

Keywords cover technique, fuzzy probability, histogram, interior-outer-set model, possibility-probability distribution

1 Introduction

In recent years the problem of estimating a fuzzy probability with a small sample has been given much attention in risk analysis. With incomplete information, it is difficult to clearly see the scenes in the future associated with some adverse incident. The scenes are fuzzy risks (Huang and Ruan 2008) and we would employ models to calculate fuzzy probabilities for representing risks. For example, the fuzzy probability of earthquake magnitude given in Karimi and Hülermeier (2007) represents the fuzzy seismic risk found in the North Anatolian Fault.

Since fuzzy theory was born, the fuzzy community started thinking of fuzzy probability. Most researchers accept the concept of the probability of a fuzzy event (Zadeh 1968) where a basic probability distribution is given.

Following an approach to model uncertainty that was pioneered by Ramsey (1931) and further developed by de Finetti (1937), Williams (1975), and Walley (1991), de Cooman (2005) has presented a sound and deep approach to vague probability.

In statistical applications, imprecise probabilities usually come from subjectively assessed prior probabilities. Fuzzy set theory is applicable to the modeling of imprecise subjective probabilities, and is suggested by many researchers, for example Freeling (1980), Watson, Weiss, and Donnell (1979), and Dubois and Prade (1989).

There is an urgent need to verify whether a model that calculates fuzzy probabilities is reliable before it can be employed in risk analysis. For example, we suppose that a group of terrorists monitored by a security department has slipped into a city. According to statistical data, the department could estimate the probability of death toll x resulting from the attack of the group, denoted as $p(x)$, and employ it to describe the risk of the terrorism attack. However, nobody believes the $p(x)$ because the available data are scarce. Thus, a fuzzy probability $\underset{\sim}{p}(x)$ would be a reasonable improvement for risk analysis of a potential terrorism attack. It is necessary to verify the reliability of the model used to calculate $\underset{\sim}{p}(x)$ before we suggest it to the security department.

Many models have been suggested to calculate fuzzy probabilities. Some have been demonstrated with practical effects (Tanaka, Fan, and Toguchi 1983; de Cooman 2005), and others would be proven by using mathematical theory (Moeller and Beer 2003).

In this article we develop the histogram-covering approach (Huang and Jia 2008) into a more general cover technique to verify the reliability of a model that is employed to calculate fuzzy probabilities. The technique is based on a hypothesis that a statistical result can approximately confirm a fuzzy probability as a fuzzy-set-valued probability. The key concept in the technique is "biprobability distribution" that is a probability of probability of event occurrence. Many biprobability distributions form a cover.

* E-mail: hchongfu@gmail.com

This article is organized as follows: Section 2 presents the cover hypothesis; Section 3 introduces the cover of probability distributions; Section 4 defines the consistency degree of a cover and a possibility-probability distribution (PPD); Section 5 presents two kinds of covers constructed with histograms; Section 6 introduces interior-outer-set model to calculate a possibility-probability distribution. In section 7, we verify that interior-outer-set model is reliable. We conclude this article with section 8.

2 Primeval Hypotheses

This study illustrates that a statistical result can approximately confirm a fuzzy probability represented as a fuzzy-set-valued probability.

First of all, let us look at the example of observing balls drawn from an urn to estimate the probability of drawing a red ball. The urn contains black balls, brown balls, red balls, orange balls, yellow balls, green balls, blue balls, purple balls, grey balls, and white balls. If we draw all balls from the urn, we can accurately estimate the probability of drawing a red ball. In case we draw a small number of balls, the probability cannot be accurately estimated in terms of statistics.

There is no loss in generality when we suppose that there are n red balls and m non-red balls in the urn. Furthermore, suppose a ball is drawn at random from the urn. By the relative frequency approach, the probability of obtaining a red ball is $P = n / (n + m)$.

The problem we study is to estimate the probability by observing s balls drawn from an urn, where $s \ll n+m$. We suppose that there are n_s red balls among s balls. $\hat{P} = n_s / s$ is used to estimate P. In terms of statistics, $\hat{P} \neq P$. In other words, we cannot determine an exact probability of obtaining a red ball until we draw all balls from the urn. The fuzzy framework suits for representing the uncertainty in the probability estimate.

Let M be a model to fuzzify $\hat{P}$ so that we can obtain a fuzzy probablity $\underset{\sim}{P}$, particularly expressed with a possibility distribution $\pi(p)$, $p \in [0,1]$ to represent the uncertainty in the probability estimate. For example, the model in Eq. 1 would be used to fuzzify $\hat{P}$ into a possibility distribution shown in Figure 1a. When all balls are drawn to estimate the probability, the fuzziness will be zero (Figure 1b).

$$\pi(p) = \begin{cases} 1, & p \in [a,b], \\ 0, & \text{otherwise}, \end{cases} \quad \text{Eq. 1}$$

where $a = \max\{0, \hat{P} - [1 - s / (n + m)]^9\}$, $b = \min\{1, \hat{P} + [1 - s / (n + m)]^9\}$.

Obviously, nobody can confirm whether the model in Eq. 1 is suitable to represent the uncertainty of the probability of obtaining a red ball with respect to an experiment where s balls are drawn from $n + m$ balls in an urn. We suggest two statistical hypotheses to verify the reliability of a model M. The hypothesis is called the cover hypothesis.

2.1 Subjective Cover Hypothesis

Consider the following case: In an experiment group there are l statisticians and one fuzzy engineer. Observing s balls drawn from an urn filled with S balls, $s \ll S$, they estimate the probability of obtaining a red ball.

The statisticians are good at estimating the probability by using their experience. The estimate given by ith statistician is denoted as $p^{(i)}$.

The fuzzy engineer is interested in mining fuzzy information carried by a small sample and good at constructing a fuzzy model M to fuzzify a probability $\hat{P}$ that is estimated by using the relative frequency into a possibility distribution $\pi(p)$.

From the point of view of statistics, it is easy to understand that sample $X_{\text{statisticians}} = \{p^{(1)}, p^{(2)}, \cdots, p^{(l)}\}$ provides confidence information about the probability of obtaining a red ball. $p^{(1)} = p^{(2)} = \cdots = p^{(l)}$ implies that the probability is $p^{(1)}$ with confidence. Regarding X as a general sample, we can obtain a probability distribution such as a histogram. Any probability distribution $P_{\text{statisticians}}(p)$ based on $X_{\text{statisticians}}$ is called a subject cover.

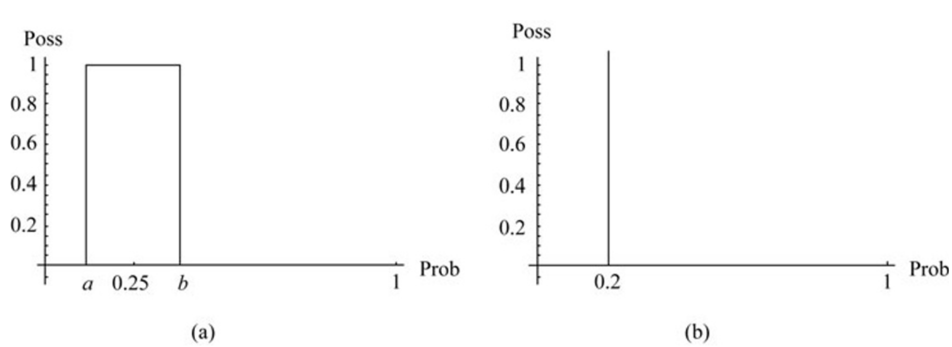

Figure 1. A possibility distribution to represent the uncertainty of the probability of obtaining a red ball with respect to an urn filled with 20 red balls and 80 non-red balls. (a) 20 balls are drawn to estimate the probability. Among them, 5 balls are red and 15 balls are non-red. $\hat{P} = 5/20 = 0.25$, $(1 - s/(n+m))^9 = (1 - 20/100)^9 = 0.1342$, $a = \max\{0, 0.25 - 0.1342\} = 0.1158$, $b = \min\{1, 0.25 + 0.1342\} = 0.3842$. (b) All balls are drawn to estimate the probability. $\hat{P} = 20/100 = 0.2$, $a = b = 0.2$.

From the point of view of possibility theory (Zadeh 1978), $\pi(p)$ implies that the probability is p in possibility $\pi(p)$ in terms of confidence restriction. Therefore, if a subject cover is similar as $\pi(p)$, the cover confirms, in some degree, that the fuzzy model M is reliable.

Our hypothesis is that there must exist a subject cover $P_{\text{statisticians}}(p)$ to verify whether a fuzzy model M is reliable.

It is not difficult to invite l statisticians to participate in the experiment, where the statisticians estimate the probability of obtaining a red ball with the s observations and their experience. Therefore, the hypothesis is accepted that the set of their estimates is a subject cover.

2.2 Random Cover Hypothesis

Consider the following case: In an experiment group there are one statistician and one fuzzy engineer. The statistician runs N experiments, and each time he draws s balls from the urn. The estimate of the probability of obtaining a red ball from the ith experiment is also denoted as $p^{(i)}$.

$X_{\text{experiments}} = \{p^{(1)}, p^{(2)}, \cdots, p^{(N)}\}$ also provides confidence information about the probability of obtaining a red ball. Any probability distribution $P_{\text{experiments}}(p)$ based on the $X_{\text{experiments}}$ is call a random cover.

Observing s balls drawn from an urn filled with S balls, $s \ll S$, the fuzzy engineer employs a fuzzy M to obtain a fuzzy distribution $\pi(p)$ for the probability of obtaining a red ball.

The random cover hypothesis is described as that there must exist a random cover $P_{\text{experiments}}(p)$ to verify whether a fuzzy model M is reliable.

It is easy to run many experiments when $s \ll S$, and the results of the experiments are different. Therefore, the hypothesis is accepted that the set of the results is a random cover.

3 Cover of Probability Distributions

The primeval hypotheses in section 2 serve as the fuzzy models that fuzzify a probability value. In risk analysis, the fuzzy risk is frequently related to a possibility-probability distribution (Huang and Moraga 2002; Karimi and Hülermeier 2007; Huang and Ruan 2008), defined in Eq. 2.

$$\Pi_{\Omega,U_p} = \{\pi_x(p) \mid x \in \Omega, p \in U_p\} \qquad \text{Eq. 2}$$

where Ω stands for the population from which we draw a sample ω, U_p is the universe of discourse of probability. Let ξ be a real function defined on Ω, then $x = \xi(\omega)$, $\omega \in \Omega$, is a random variable. x and ω are equipollent to identify an event. $\pi_x(p)$ is the possibility that an event occurs with probability p.

We extend the concept of the cover to correspond with probability distributions $p^{(i)}(x)$, $i = 1,2,\cdots,l$, instead of probability values $p^{(i)}$, $i = 1,2,\cdots,l$, shown in section 2.

Let X be a sample drawn from a population Ω with a theoretical probability distribution $p(x)$. Let γ be a statistical method, such as Maximum Likelihood, which processes X to give an estimate of the probability distribution, written as $p_X^{\gamma}(x)$.

That is, for a population Ω, the theoretical probability distribution $p(x)$ is unique, but the different samples X_1, $X_2, \cdots, X_N$ lead us to have different estimates $p_{X_1}^{\gamma}(x)$, $p_{X_2}^{\gamma}(x), \cdots, p_{X_N}^{\gamma}(x)$.

$\forall x_0 \in \Omega$, to estimate $p(x_0)$, we have N values $p_{X_1}^{\gamma}(x_0)$, $p_{X_2}^{\gamma}(x_0), \cdots, p_{X_N}^{\gamma}(x_0)$. They form a sample, called probability sample, written as W_{x_0}, that is, $W_{x_0} = \{p_{X_1}^{\gamma}(x_0), p_{X_2}^{\gamma}(x_0), \cdots, p_{X_N}^{\gamma}(x_0)\}$.

Let ϕ be a statistical model employed to process W_{x_0} and obtain a probability distribution at x_0, called biprobability distribution, written as $\mathbf{p}_{W_{x_0}}^{\phi}(p), p \in [0,1]$.

For example, Let

$$p(x) = \frac{1}{0.372\sqrt{2\pi}} \exp[-\frac{(x-6.86)^2}{2 \times 0.372^2}], -\infty < x < \infty, \qquad \text{Eq. 3}$$

x is a random variable obeying normal distribution $N(6.86, 0.372^2)$. With 10 random seed numbers, respectively, running Program 2 in Huang and Shi (2002), a generator of random numbers, with MU=6.86, the standard deviation SIGMA=0.372, and the sample size N=11, we obtain 10 samples, one of them is,

$$\begin{aligned} X_1 &= \{x_1, x_2, \cdots, x_{11}\} \\ &= \{6.91, 6.59, 6.31, 6.50, 7.03, 6.49, \\ &\quad 7.27, 7.13, 6.72, 7.42, 6.34\}. \end{aligned} \qquad \text{Eq. 4}$$

Let γ be the Maximum Likelihood, we have 10 probability distributions from the samples. For example, from X_1, we obtain,

$$p_{X_1}^{\gamma}(x) = \frac{1}{0.365\sqrt{2\pi}} \exp[-\frac{(x-6.79)^2}{2 \times 0.372^2}], -\infty < x < \infty.$$

For $x_0 = 7.3$, we have $p_{X_1}^{\gamma}(7.3) = 0.414$. Totally,

$$\begin{aligned} W_{7.3} &= \{p_{X_1}^{\gamma}(7.3), p_{X_2}^{\gamma}(7.3), \cdots, p_{X_{10}}^{\gamma}(7.3)\} \\ &= \{0.414, 0.523, \cdots, 0.947\}. \end{aligned} \qquad \text{Eq. 5}$$

When we assume that $W_{7.3}$ in Eq. 5 obeys normal distribution, with Maximum Likelihood to be ϕ, we have a biprobability distribution,

$$\mathbf{p}_{W_{7.3}}^{\phi}(p) = \frac{1}{0.273\sqrt{2\pi}} \exp[-\frac{(p-0.494)^2}{2 \times 0.273^2}], 0 \le p \le 1,$$

which is shown in Figure 2. In practice, the biprobability distribution is not the normal distribution inferred by using the central limit theorem, because the integration of the function $\mathbf{p}_{W_{7.3}}^{\phi}(p)$ in [0,1] is less than 1. That we use the normal distribution assumption is to reduce the complexity in discussing the property of a biprobability distribution.

Let

$$C(x,p) = \mathbf{p}_{W_x}^{\phi}(p), x \in \Omega, p \in [0,1]$$

be a family of biprobability distributions corresponding to a population with distribution $p(x)$. For any fixed x, $C(x, p)$ is

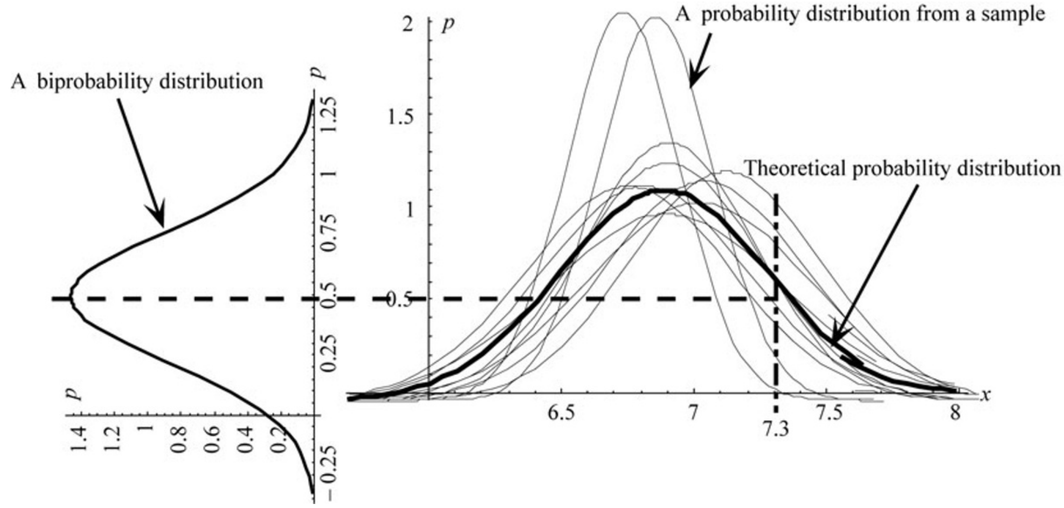

Figure 2. A biprobability distribution is a probability distribution of the probability that an event occurs at x_0. The theoretical probability distribution is the normal distribution *N*(6.86,0.372²). 10 samples lead us to have 10 probability distributions. For x_0 = 7.3, we have 10 probability values from which a biprobability distribution is produced.

a probability distribution with respect to variable p. It is a density function for a continuous distribution defined on interval [0,1], or a discrete function for a discrete distribution defined on a universe U_p of discourse of probability. The family $C(x, p)$ is called a cover of probability distributions. Obviously, $C(x, p)$ is a random cover but not a subjective cover. According to the random cover hypothesis suggested in section 2, we infer that there must exist a cover $C(x, p)$ to verify the reliability of a fuzzy model M.

4 Consistency Degree of a Cover and a PPD

The primeval hypotheses suggested in section 2 only assert that it is possible to verify the reliability of a fuzzy model with a cover. In this section, we suggest an approach to compare a cover and a PPD for verification. The reliability degree of the model is measured by using the consistency degree of the cover and the PPD.

4.1 Consistency Degree

The concept of consistency is quite rough. Strictly speaking, a PPD and a cover C are consistent if and only if they are equality. In many cases, $\forall x \in \Omega, \sup_p\{\pi_x(p)\} = 1$, but $\sup_p\{C(x, p)\} < 1$. Let

$$\theta_x(p) = C(x, p) / \sup_p\{C(x, p)\}.$$

$\Theta = \{\theta_x(p)\}$, $x \in \Omega, p \in U_p$, is called a normalized cover.

We define that $\pi_x(p)$ and $C(x, p)$ are consistent if and only if $\forall x \in \Omega$, $\pi_x(p) = \theta_x(p)$.

It is interesting to notice that from Figure 2 we know that $\forall x$, $C(x, p)$ is a probability distribution; therefore it is impossible for $\sup_p\{C(x, p)\}$ to be very small. $\theta_x(p)$ is defined as a quotient, to force $\sup_p\{\theta_x(p)\}$ to be equal to 1. Hence, C and Π may coincide well, when Θ and Π can coincide.

Obviously, in a numerical experiment, in general, a PPD is not equal to a cover because the size of a sample is always limited. Therefore, it is necessary to weaken the condition of consistency.

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and U_p be the universe of discourse of probability. Let $\Pi = \{\pi_x(p)\}$ and $\Theta = \{\theta_x(p)\}$, be a PPD and a cover, defined on $\Omega \times U_p$, respectively.

Obviously, both $\pi_x(p)$ and $\theta_x(p)$ are two-variable bounded functions. And, $0 \le \pi_x(p)$, $\theta_x(p) \le 1$, that is, they are 0–1 bounded functions. Hence, our problem can be simplified to study the consistency between two functions defined on a domain U.

Let $\mathcal{F}$ be a set of two-variable functions with domain U, denoted as $\mathcal{F} = \{f(x, y) : U\}$. In our case, $\mathcal{F}$ is a set of 0–1 bounded functions, and $U = \Omega \times U_p$.

Definition 1. *Let* $f_1(x, y), f_2(x, y) \in \mathcal{F}$. $f_2(x, y)$ *is strictly consistent with* $f_1(x, y)$ *if and only if* $f_1(x, y) = f_2(x, y)$, $\forall (x, y) \in U$.

Let $\mathcal{F} = \{f(x, y) : U\}$ and U_x, U_y be the domains of x, and y, respectively, that is, $U = U_x \times U_y$.

Definition 2. *Let* $f_1(x, y), f_2(x, y) \in \mathcal{F}$.

$$d(f_1, f_2) = \frac{1}{\int_{U_x}\int_{U_y} dx\, dy} \int_{U_x}\int_{U_y} |f_1(x, y) - f_2(x, y)|\, dx\, dy \quad \text{Eq. 6}$$

is called the naive distance between f_1 *and* f_2.

$$\eta = 1 - d(f_1, f_2)$$

is called the consistent approximation.

η is an reasonable index to measure the consistency degree between two functions. However, with it, we overlook the information that $f_1(x, y)$ and $f_2(x, y)$ would reach extreme values in different points. Particularly, when $f_1(x, y)$ and $f_2(x, y)$ are equal to zero in the main part of U, η is not a good index for consistency. η is an *upper consistency*.

Let $\mathcal{F}$ be a set of 0–1 bounded functions, $f_1(x, y), f_2(x, y) \in \mathcal{F}$. Let

$$A = \{(x, y) \mid f_1(x, y) = 1, (x, y) \in U\},$$
$$B = \{(x, y) \mid f_2(x, y) = 1, (x, y) \in U\}.$$

A, B are called the *kernels* of $f_1(x, y)$, $f_2(x, y)$, respectively. We employ $Z = A \cup B$, called the *peak set*, to show information that $f_1(x, y)$ and $f_2(x, y)$ would reach 1 in different points.

Definition 3. *Let $\mathcal{F}$ be a set of 0–1 bounded functions, $f_1(x, y), f_2(x, y) \in \mathcal{F}$ with a peak set Z.*

$$D(f_1, f_2) = \begin{cases} \dfrac{1}{\iint_Z dx\,dy} \iint_Z |f_1(x, y) - f_2(x, y)|\,dx\,dy, & Z \text{ is integrable;} \\ \dfrac{1}{|Z|} \sum_i \sum_j |f_1(x_i, y_j) - f_2(x_i, y_j)|, & Z \text{ is a discrete set} \end{cases} \quad \text{Eq. 7}$$

is called the extremal error between f_1 and f_2, where $|Z|$ is cardinal number of Z.

$$\beta = 1 - D(f_1, f_2)$$

is called the consistent kernel.

In the case that the peak set Z is not integrable, nor discrete, the expression of the consistent kernel may be complex. In practice, the Z is usually discrete.

Obviously, if $A = B$, then $\beta = 1$. Otherwise, the largest error between $f_1(x, y)$, $f_2(x, y)$ on the peak set Z determines the consistent kernel. β is a *lower consistency*.

Definition 4. *Let $f_1(x, y)$, $f_2(x, y)$ be 0–1 bounded functions with U. $f_2(x, y)$ is consistent with $f_1(x, y)$ in degree $\alpha = (\eta + \beta) / 2$, if and only if their consistent approximation is η and consistent kernel is β.*

α is also called the *consistency degree* of $f_2(x, y)$ to $f_1(x, y)$.

4.2 Consistency Degree of a Cover and a PPD

Let Ω be a population and U_p a universe of discourse of probability. Given a normalized cover $\Theta = \{\theta_x(p)\}$ and a PPD $\Pi = \{\pi_x(p)\}$ defined on $\Omega \times U_p$, we study the consistency degree of Θ and Π.

Employing formula in Eq. 6 and Eq. 7, respectively, we obtain the naive distance and the extremal error between Θ and Π, shown in Eq. 8 and Eq. 9.

$$d(\Theta, \Pi) = \frac{1}{\int_\Omega \int_{U_p} dx\,dp} \int_\Omega \int_{U_p} |\theta_x(p) - \pi_x(p)|\,dx\,dp, \quad \text{Eq. 8}$$

$$D(\Theta, \Pi) = \begin{cases} \dfrac{1}{\iint_Z dx\,dp} \iint_Z |\theta_x(p) - \pi_x(p)|\,dx\,dp, & Z \text{ is integrable;} \\ \dfrac{1}{|Z|} \sum_i \sum_j |\theta_{x_i}(p_j) - \pi_{x_i}(p_j)|, & Z \text{ is a discrete set} \end{cases} \quad \text{Eq. 9}$$

where $Z = \{(x, p) \mid \theta_x(p) = 1, \pi_x(p) = 1, x \in \Omega, p \in U_p\}$.

According to Definition 4, the consistency degree of Θ and Π is $\alpha(\Theta, \Pi)$ shown in Eq. 10:

$$\alpha(\Theta, \Pi) = \frac{\eta(\Theta, \Pi) + \beta(\Theta, \Pi)}{2} = 1 - \frac{d(\Theta, \Pi) + D(\Theta, \Pi)}{2}. \quad \text{Eq. 10}$$

$\alpha(\Theta, \Pi)$ is called a consistency degree of a cover with normalization Θ and a PPD Π. In other words, the consistency degree of a cover and a PPD is defined by the consistency degree of the cover's normalization and the PPD.

5 Two Kinds of Covers Constructed with Histograms

A histogram is a graph of grouped (binned) data in which the number of values in each bin is represented by the area of a rectangular box. A relative frequency histogram, as an estimate of the probability distribution of a continuous variable, is a bar graph constructed in such a way that the area of each bar is proportional to the fraction of observations in the category that it represents.

5.1 Relative Frequency Histogram

Let $X = \{x_i \mid i = 1, 2, \cdots, n\}$ be a sample drawn from Ω with a probability density distribution (PDF) $p(x)$, and $I_j = [x_0 + (j-1)h, x_0 + jh), j = 1, 2, \cdots, m$, be m intervals for constructing a histogram with X.

$$p_X^h(x) = \frac{1}{n}(\text{number of } x_i \text{ in the same interval as } x), \quad \text{Eq. 11}$$

is called a relative frequency histogram (RFH) with respect to X. $p_X^h(x)$ is an estimate of probability that an event occurs in the same interval as x.

Let u_j be the midpoint of interval I_j. We obtain a discrete domain of definition of $p_X^h(x)$

$$U = \{u_j \mid j = 1, 2, \cdots, m\}.$$

Hence, a RFH $p_X^h(x)$ can be represented by using a discrete distribution:

$$H = \{p_X^h(u_1), p_X^h(u_2), \cdots, p_X^h(u_m)\}.$$

5.2 Probability Sample

Let $X_1, X_2, \cdots, X_N$ be N samples drawn from Ω, and $I_1, I_2, \cdots, I_m$ be m intervals for constructing N RFHs with the given samples. In other words, we employ the same intervals to make all histograms.

For an interval I_j, from N RFHs we obtain N probability estimate values. Hence, we obtain a probability sample

$$W_{I_j} = \{p^h_{X_1}(u_j), p^h_{X_2}(u_j), \cdots, p^h_{X_N}(u_j)\}.$$

The sample W_{I_j} is a set of the probability values estimated with a set of samples drawn from a population. The cardinal number of set W_{I_j} and set $\{X_1, X_2, \cdots, X_N\}$ are the same (both are N). Probability $p^h_X(u)$, estimated by using RFH with X in u, is only a possible value to probability that an event occurs in the interval which includes u.

5.3 Natural Cover of Histograms

According to N (the size of the probability sample), we divide probability domain [0,1] into t probability intervals, $A_k = [(k-1)\delta, k\delta)$, $k = 1,2,\cdots,t$, where t can be obtained by using Eq. 12 suggested by Otness and Encysin (1972), and probability step $\delta = 1/t$.

$$t = 1.87(N-1)^{2/5}. \qquad \text{Eq. 12}$$

Then, with probability sample W_{I_j}, we can construct a biprobability distribution, denoted as $\mathbf{p}^h_{W_{I_j}}(p_k)$, $k = 1,2,\cdots,t$, where p_k is the midpoint of A_k. Mathematically,

$$\mathbf{p}^h_{W_{I_j}}(p_k) = \frac{1}{N}(\text{number of } p^h_X(u_j) \text{ in the same bin as } p_k),\ p^h_X(u_j) \in W_{I_j}.$$

Let

$$\theta_{I_j}(p_k) = \mathbf{p}^h_{W_{I_j}}(p_k) / \sup_k\{\mathbf{p}^h_{W_{I_j}}(p_k)\}.$$

$\Theta_{I_j} = \{\theta_{I_j}(p_k) \mid k = 1,2,\cdots,t\}$ is called a normalized cover with respect to I_j.

From m event intervals $I_1, I_2, \cdots, I_m$, we obtain m biprobability distributions: $\mathbf{p}^h_{W_{I_1}}(p_k), \mathbf{p}^h_{W_{I_2}}(p_k), \cdots, \mathbf{p}^h_{W_{I_m}}(p_k)$. They lead to a discrete cover that can be represented by a matrix in Eq. 13, which is called the natural cover of histograms.

$$\Theta = \begin{matrix} \\ I_1 \\ I_2 \\ \vdots \\ I_m \end{matrix} \begin{matrix} A_1 & A_2 & \cdots & A_t \\ \left(\begin{matrix} \theta_{I_1}(p_1) & \theta_{I_1}(p_2) & \cdots & \theta_{I_1}(p_t) \\ \theta_{I_2}(p_1) & \theta_{I_2}(p_2) & \cdots & \theta_{I_2}(p_t) \\ \vdots & \vdots & & \vdots \\ \theta_{I_m}(p_1) & \theta_{I_m}(p_2) & \cdots & \theta_{I_m}(p_t) \end{matrix}\right) \end{matrix} \qquad \text{Eq. 13}$$

5.4 Distribution-Cover of Histograms

According to n (the size of the sample drawn from Ω with PDF $p(x)$), we construct a discrete universe of discourse of probability shown in Eq. 14.

$$U_p = \{p_k \mid k = 0,1,2,\cdots,n\} = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}. \qquad \text{Eq. 14}$$

For an interval I_j, we employ the information distribution formula in Eq. 15 to make a soft histogram estimation (Huang 2002b) by using Eq. 16.

$$\chi_{ik} = \begin{cases} 1 - n\,|\,p^h_{X_i}(u_j) - p_k\,|, & \text{if } |\,p^h_{X_i}(u_j) - p_k\,| \le 1/n; \\ 0, & \text{if } |\,p^h_{X_i}(u_j) - p_k\,| > 1/n. \end{cases} \qquad \text{Eq. 15}$$

$$\mathbf{p}^D_{W_{I_j}}(p_k) = \frac{\sum_{i=1}^{N} \chi_{ik}}{N},\ k = 0,1,2,\cdots,n. \qquad \text{Eq. 16}$$

Let

$$\theta^D_{I_j}(p_k) = \mathbf{p}^D_{W_{I_j}}(p_k) / \sup_k\{\mathbf{p}^D_{W_{I_j}}(p_k)\}. \qquad \text{Eq. 17}$$

Then,

$$\Theta^D_{I_j} = \{\theta^D_{I_j}(p_k) \mid k = 0,1,2,\cdots,n\} \qquad \text{Eq. 18}$$

is called a normalized cover with respect to I_j.

From m event intervals $I_1, I_2, \cdots, I_m$, we obtain m biprobability distributions: $\mathbf{p}^D_{W_{I_1}}(p_k), \mathbf{p}^D_{W_{I_2}}(p_k), \cdots, \mathbf{p}^D_{W_{I_m}}(p_k)$. They lead to a discrete cover that can be represented by a matrix in Eq. 19, which is called a distribution-cover of histograms.

$$\Theta^D = \begin{matrix} \\ I_1 \\ I_2 \\ \vdots \\ I_m \end{matrix} \begin{matrix} p_1 & p_2 & \cdots & p_t \\ \left(\begin{matrix} \theta^D_{I_1}(p_1) & \theta^D_{I_1}(p_2) & \cdots & \theta^D_{I_1}(p_t) \\ \theta^D_{I_2}(p_1) & \theta^D_{I_2}(p_2) & \cdots & \theta^D_{I_2}(p_t) \\ \vdots & \vdots & & \vdots \\ \theta^D_{I_m}(p_1) & \theta^D_{I_m}(p_2) & \cdots & \theta^D_{I_m}(p_t) \end{matrix}\right) \end{matrix} \qquad \text{Eq. 19}$$

Huang (2000) proved that, in the case where we only have a small sample to estimate a probability distribution, a soft histogram estimation is better than a histogram estimation in a higher work efficiency about 28 percent. In other words, if we need a sample including 30 observations for the histogram method, then less 28 percent is 30–30×28% = 30–8 = 22, a sample with 22 observations can give a soft histogram estimation in a similarly accurate way. Therefore, we employ the distribution-cover of histograms to verify the reliability of a model for calculating fuzzy probabilities.

6 PPD Calculated by Interior-Outer-Set Model

Interior-outer-set model (IOSM) (Huang 2002a) is suggested to calculate, with a sample $X = \{x_i \mid i = 1,2,\cdots,n\}$, a PPD in Eq. 2 defined on $\mathbf{I} \times U_p$, where,

$$\mathbf{I} = \{I_j \mid j = 1,2,\cdots,m\},$$

and U_p is shown in Eq. 14.

Let u_j be the midpoint of interval I_j, $\Delta \equiv u_{j+1} - u_j$, $j = 1,2,\cdots,j-1$. Let

$$q_{ij} = \begin{cases} 1 - |\,x_i - u_j\,|/\Delta, & \text{if } |\,x_i - u_j\,| \le \Delta; \\ 0, & \text{if } |\,x_i - u_j\,| > \Delta. \end{cases}$$

q_{ij} is called the information gain of that observation x_i distributed to controlling point u_j.

Definition 5. $X_{in-j} = X \cap I_j$ *is called an interior set of interval I_j. The elements of X_{in-j} are called the interior points of I_j.*

Let A and B be two sets. $A \setminus B = \{x | x \in A, x \notin B\}$ is called their set difference.

Definition 6. *Let $X_{\text{in}-j}$ be the interior set of interval I_j. $X_{\text{out}-j} = X \setminus X_{\text{in}-j}$ is called an outer set of interval I_j. The elements of X_{out-j} are called the outer points of I_j.*

$\forall x_i \in X$, if $x_i \in X_{\text{in}-j}$ we say that it loses information, by gain at $1 - q_{ij}$, to another interval, we use $q_{ij}^- = 1 - q_{ij}$ to represent the loss; if $x_i \in X_{\text{out}-j}$ we say that it gives information, by gain at q_{ij}, to I_j, we use q_{ij}^+ to represent the addition. q_{ij} means that x_i may leave I_j in possibility q_{ij}^- if $x_i \in X_{\text{in}-j}$, or x_i may join I_j in possibility q_{ij}^+ if $x_i \in X_{\text{out}-j}$.

q_{ij}^- is called the *leaving possibility*, and q_{ij}^+ called the *joining possibility*. The leaving possibility of an outer point is defined as 0 (it has gone). The joining possibility of an interior point is defined as 0 (it has been in the interval).

Any model based q_{ij}^+ and q_{ij}^- to calculate a PPD on $\mathbf{I} \times U_p$ is called an IOSM. The first IOSM was suggested in Huang (1998) and applied to study the risk of crop flood giving a better result to support risk management in crops avoiding flood than the traditional probability method (Huang 2002a). In Huang and Moraga (2002), the model was transformed into a matrix algorithm. The second IOSM was suggested in Moraga and Huang (2003) with complexity in the $O(n \log n)$ class instead of complexity $O(n^2)$ and applied to make soft risk maps (Zhang 2005). The third IOSM was introduced in Zong (2004) to smooth the abrupt slopes in a PPD where the membership is less than or equal to 0.5 if it is not 1. This paper focuses on exploring a new approach to verify reliability, not to improve IOSM. Therefore, we employ the second IOSM in Eq. 20 to calculate a PPD.

$$\pi_{I_j}(p) = \begin{cases} \text{1st (smallest) element of } Q_j^-, & \text{if } p = 0; \\ \text{2nd element of } Q_j^-, & \text{if } p = \dfrac{1}{n}; \\ \vdots & \vdots \\ \text{Last (largest) element of } Q_j^-, & \text{if } p = \dfrac{n_j - 1}{n}; \\ 1, & \text{if } p = \dfrac{n_j}{n}; \\ \text{1st (largest) element of } Q_j^+, & \text{if } p = \dfrac{n_j + 1}{n}; \\ \text{2nd element of } Q_j^+, & \text{if } p = \dfrac{n_j + 2}{n}; \\ \vdots & \vdots \\ \text{Last element of } Q_j^+, & \text{if } p = 1. \end{cases} \qquad \text{Eq. 20}$$

where Q_j^- is the list of q_{ij}^- according to ascending magnitude, Q_j^+ is the list of q_{ij}^+ according to descending magnitude, and $|Q_j^-| = n_j$, that is, we suppose that there are n_j observations falling in interval I_j.

Then, from a given sample X we can obtain a PPD on $\mathbf{I} \times U_p$, written as:

$$\Pi_X = \begin{array}{c} \\ I_1 \\ I_2 \\ \vdots \\ I_m \end{array} \begin{array}{c} \begin{array}{ccccc} p_0 & p_1 & p_2 & \cdots & p_n \end{array} \\ \begin{pmatrix} \pi_{1,0} & \pi_{1,1} & \pi_{1,2} & \cdots & \pi_{1,n} \\ \pi_{2,0} & \pi_{2,1} & \pi_{2,2} & \cdots & \pi_{2,n} \\ \vdots & \vdots & \vdots & & \vdots \\ \pi_{m,0} & \pi_{m,1} & \pi_{m,2} & \cdots & \pi_{m,n} \end{pmatrix} \end{array},$$

where $\pi_{j,k} = \pi_{I_j}(p_k)$. According to Eq. 20 we know that, if $|Q_j^-| = n_j$, then $\pi_{j,n_j} = 1$.

7 A Case to Verify the Reliability of Interior-Outer-Set Model

As an application, in this section we employ a distribution-cover of histograms to verify whether IOSM is reliable.

7.1 Samples Given by Using a Generator of Random Numbers

Running Program 2 in Huang and Shi (2002), a generator of random numbers, with MU=6.86, SIGMA=0.372, N=11, and SEED=82,495, we obtain 11 random numbers:

$$\begin{aligned} X &= \{x_1, x_2, \cdots, x_{11}\} \\ &= \{0.91, 6.59, 6.31, 6.50, 7.03, 6.49, \\ &\quad 7.27, 7.13, 6.72, 7.42, 6.34\}, \end{aligned} \qquad \text{Eq. 21}$$

whose population is a normal distribution $N(6.86, 0.372^2)$.

Again and again, running Program 2 with the same MU, SIGMA and N, but with another 90 different SEEDs, we obtain 90 samples, $X_1, X_2, \cdots, X_{90}$. For example, with SEED=876,905, we obtain:

$$X_3 = \{7.14, 6.98, 6.83, 7.00, 7.34, 6.47, 7.65, 6.99, 6.71, 7.47, 6.26\}.$$

7.2 Histograms

Considering the range and size of the samples, we employ intervals:

$$\begin{aligned} I_1 &= [5.65, 6.25), I_2 = [6.25, 6.85), \\ I_3 &= [6.85, 7.45), I_4 = [7.45, 8.05), \end{aligned} \qquad \text{Eq. 22}$$

to calculate RFHs. The midpoints of intervals lead to a discrete domain of definition:

$$U = \{u_1, u_2, u_3, u_4\} = \{5.95, 6.55, 7.15, 7.75\}. \qquad \text{Eq. 23}$$

Employing RFH formula in Eq. 11 with intervals in Eq. 22 and X_k, $k = 1, 2 \cdots, 90$, we obtain 90 RFHs $p_{X_1}^h(x), p_{X_2}^h(x), \cdots, p_{X_{90}}^h(x)$. For example, for X_3, we obtain:

$$\{p_{X_3}^h(u_1), p_{X_3}^h(u_2), p_{X_3}^h(u_3), p_{X_3}^h(u_4)\} = \{0.00, 0.36, 0.45, 0.18\}.$$

7.3 Distribution-Cover of Histograms

For an interval I_j, from 90 RFHs we obtain a probability sample W_{I_j}. For example, for $I_2 = [6.25, 6.85)$, we obtain:

$$
\begin{aligned}
W_{I_2} &= \{p^h_{X_1}(6.55), p^h_{X_2}(6.55), p^h_{X_3}(6.55), \cdots, p^h_{X_{90}}(6.55)\} \\
&= \{0.55, 0.55, 0.36, 0.45, 0.09, 0.45, 0.64, 0.45, 0.55, 0.27, 0.27, 0.55, 0.64, 0.36, 0.36, 0.73, 0.36, 0.45, \\
&\quad 0.36, 0.27, 0.64, 0.27, 0.45, 0.36, 0.36, 0.55, 0.64, 0.45, 0.45, 0.64, 0.55, 0.27, 0.45, 0.36, 0.45, 0.55, \\
&\quad 0.45, 0.36, 0.36, 0.45, 0.27, 0.45, 0.45, 0.45, 0.18, 0.36, 0.36, 0.27, 0.73, 0.55, 0.36, 0.45, 0.55, 0.64, \\
&\quad 0.64, 0.27, 0.55, 0.36, 0.64, 0.45, 0.36, 0.36, 0.36, 0.55, 0.36, 0.36, 0.18, 0.27, 0.27, 0.55, 0.55, 0.64, \\
&\quad 0.55, 0.45, 0.45, 0.45, 0.55, 0.09, 0.27, 0.36, 0.55, 0.45, 0.45, 0.73, 0.45, 0.36, 0.18, 0.36, 0.36, 0.36\}.
\end{aligned}
$$

Then, employing Eq. 15–Eq. 18, with samples W_{I_j}, $j = 1,2,3,4$ and controlling points $p_k = k/11, k = 0,1,\cdots,11$, we obtain a distribution-cover of histograms shown in Eq. 24.

$$
\Theta^D = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} \end{array} \\
\begin{pmatrix}
1.00 & 0.40 & 0.14 & 0.02 & 0.02 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.08 & 0.12 & 0.46 & 1.00 & 0.92 & 0.67 & 0.38 & 0.12 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.05 & 0.19 & 0.38 & 0.90 & 1.00 & 0.90 & 0.71 & 0.05 & 0.10 & 0.00 & 0.00 \\
1.00 & 0.45 & 0.17 & 0.06 & 0.02 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00
\end{pmatrix}
\end{array}. \qquad \text{Eq. 24}
$$

7.4 PPD

Employing IOSM in Eq. 20 with intervals in Eq. 22, the universe of discourse of probability in Eq. 14, and X in Eq. 21, we obtain a PPD:

$$
\Pi_X = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} \end{array} \\
\begin{pmatrix}
1.00 & 0.41 & 0.35 & 0.10 & 0.09 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.06 & 0.09 & 0.10 & 0.29 & 0.35 & 0.41 & 1.00 & 0.39 & 0.19 & 0.04 & 0.00 & 0.00 \\
0.04 & 0.19 & 0.19 & 0.39 & 0.45 & 1.00 & 0.29 & 0.06 & 0.00 & 0.00 & 0.00 & 0.00 \\
1.00 & 0.45 & 0.19 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00
\end{pmatrix}
\end{array}. \qquad \text{Eq. 25}
$$

7.5 Consistency Degree

In Eq. 24 and Eq. 25, the midpoint of I_j is u_j in Eq. 23, and the universes of discourse of probability are equal. Therefore, for an easier comparison, the cover in Eq. 24 and the PPD in Eq. 25 can be rewritten as:

$$
\Theta^D = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} \end{array} \\
\begin{pmatrix}
1.00 & 0.40 & 0.14 & 0.02 & 0.02 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.08 & 0.12 & 0.46 & 1.00 & 0.92 & 0.67 & 0.38 & 0.12 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.05 & 0.19 & 0.38 & 0.90 & 1.00 & 0.90 & 0.71 & 0.05 & 0.10 & 0.00 & 0.00 \\
1.00 & 0.45 & 0.17 & 0.06 & 0.02 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00
\end{pmatrix}
\end{array}, \qquad \text{Eq. 26}
$$

$$
\Pi = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} \end{array} \\
\begin{pmatrix}
1.00 & 0.41 & 0.35 & 0.10 & 0.09 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.06 & 0.09 & 0.10 & 0.29 & 0.35 & 0.41 & 1.00 & 0.39 & 0.19 & 0.04 & 0.00 & 0.00 \\
0.04 & 0.19 & 0.19 & 0.39 & 0.45 & 1.00 & 0.29 & 0.06 & 0.00 & 0.00 & 0.00 & 0.00 \\
1.00 & 0.45 & 0.19 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00
\end{pmatrix}
\end{array}. \qquad \text{Eq. 27}
$$

Both Θ^D and Π are defined on $\{5.95, 6.55, 7.15, 7.75\} \times \{0, 1/11, 2/11, \cdots, 1\}$. The Θ^D statistically provides confidence information about the probability that an event occurs in an interval. When the sample size $n \to \infty$, the confidence will be 1 and the matrix will become a continuous relation to represent the normal distribution in Eq. 3. The Π indicates that, with a small sample, it is impossible to accurately estimate a continuous probability distribution. And Π shows various possibilities that an event occurs with more than one probability value. When $n \to \infty$, the scattering of estimates will disappear and also the Π will be a relation to represent the normal distribution.

Using Eq. 8, we obtain the naive distance between Θ^D and Π:

$$d(\Theta^D,\Pi)=\frac{1}{\int_U\int_{U_p} du\,dp}\int_U\int_{U_p}|\Theta^D-\Pi|\,du\,dp$$
$$=\frac{1}{4\times 12}\sum_{1\le j\le 4,\,0\le k\le 11}|\theta^D_{I_j}(p_k)-\pi_{j,k}|$$
$$=\frac{1}{48}(0+0.01+0.21+0.08+0.07+0+0+0+0+0+0+0+0.06+0.01+0.02+0.13+0.65+0.51+0.33+0.01+0.07+0.04+0+0+0.04+0.14+0+0.01+0.45+0+0.61+0.65+0.05+0.10+0+0+0+0+0.02+0.06+0.02+0+0+0+0+0+0+0)$$
$$=\frac{4.37}{48}=0.091.$$

The kernels of Θ^D and Π are, respectively

$$A=\{(u_1,p_0),(u_2,p_4),(u_3,p_5),(u_4,p_0)\},$$
$$B=\{(u_1,p_0),(u_2,p_6),(u_3,p_5),(u_4,p_0)\}.$$

Then, the peak set of the cover and PPD is

$$Z=\{(u_1,p_0),(u_2,p_4),(u_2,p_6),(u_3,p_5),(u_4,p_0)\}.$$

The Z is a discrete set, with cardinal number 5.

Using Eq. 9, we obtain the extremal error between Θ^D and Π:

$$D(\Theta^D,\Pi)=\frac{1}{5}\sum_{(u_j,p_k)\in Z}|\theta^D_{I_j}(p_k)-\pi_{j,k}|$$
$$=\frac{1}{5}(|1-0.35|+|0.67-1|)=0.196.$$

Finally, using Eq. 10, we obtain the consistency degree of Π and Θ^D,

$$\alpha(\Theta^D,\Pi)=1-\frac{d(\Theta^D,\Pi)+D(\Theta^D,\Pi)}{2}$$
$$=1-\frac{0.091+0.196}{2}=0.857.$$

Notice that, for most $(u,p)\in\{5.95,6.55,7.15,7.75\}\times\{0,1/11,2/11,\cdots,1\}$, Θ^D is less than or equal to Π. According to Definition 4, we know that Π is consistent with Θ^D in degree 0.857.

Resimulating with other seed numbers, we have another 20 numerical experiments, and the consistency degrees are in [0.8332, 0.8789]. The degree about 0.86 is not so high and a development (Zong 2004) has been done for IOSM, but the applications of IOSM are successful (Huang 2002a; Zhang 2005). Therefore, we infer that IOSM is basically reliable.

The more sound conclusion at this stage would be based on the performance of the proposed method change in terms of the size of sample. It would be better to conduct a series of experiments based on different sample sizes to give a complete picture.

8 Conclusions

Except the pseudo risk that we are able to accurately predict by using system models and currently available data, what we can know about a risk is limited. In some cases, the risk would be represented with a fuzzy probablity. It is important to verify the reliability of a model for calculating fuzzy probabilities with a given sample.

The suggested approach consists of a hypothesis, a cover, and the interior-outer-set model. When a fuzzy probability depends on a given sample, all evidences lend support to the proposed hypothesis that a statistical result can approximately confirm a fuzzy probability. With a given sample we can have an estimation of the probability distribution of the population from which the sample is drawn. *N* samples lead to *N* estimations. The set of the estimations will cover the probability distribution. The set is a cover. We can employ the cover to check if a fuzzy probablity inferred by using a given sample from the same population approximately cover the probability distribution. Then we can employ the cover technique to verify the reliability of a model for calculating fuzzy probabilities.

In this article, we show a distribution-cover that, with consistent degree 0.857, confirms a possibility-probability distribution calculated by using the interior-outer-set model. The consistent degrees resulted from other numerical experiments are almost same. Although the degree, about 0.86, is not so high, we infer that the interior-outer-set model is basically reliable, because the applications of the model are successful (Huang 2002a; Zhang 2005).

Acknowledgment

This project was supported by National Basic Research Program of China (973 Program) (No.2012CB955402), and the State Key Laboratory of Earth Surface Processes and Resource Ecology.

References

de Cooman, G. 2005. A Behavioural Model for Vague Probability Assessments. *Fuzzy Sets and Systems* 154 (3): 305–58.

Dubois, D., and H. Prade. 1989. Fuzzy Sets, Probability, and Measurement. *European Journal of Operational Research* 40 (2): 135–54.

de Finetti, B. 1937. La Prévision: ses Lois Logiques, ses Sources Subjectives. *Annales de l'Institut Henri Poincaré* 7 (1): 1–68.

Freeling, A. N. S. 1980. Fuzzy Sets and Decision Analysis. *IEEE Transactions on Systems, Man and Cybernetics* SMC-10 (7): 341–54.

Huang, C. F. 1998. Concepts and Methods of Fuzzy Risk Analysis. In *Proceedings of the 1st China-Japan Conference on Risk Assessment and Management*, 12–23, Beijing, China, 1998.

——. 2000. Demonstration of Benefit of Information Distribution for Probability Estimation. *Signal Processing* 80 (6): 1037–48.

——. 2002a. An Application of Calculated Fuzzy Risk. *Information Sciences* 142 (1): 37–56.

——. 2002b. Information Diffusion Techniques and Small Sample Problem. *International Journal of Information Technology and Decision Making* 1 (2): 229–49.

Huang, C. F., and D. Y. Jia. 2008. Histogram-Covering Approach to Verify Model Reliability. *Journal of Multiple-Valued Logic and Soft Computing* 14 (3–5): 265–75.

Huang, C. F., and C. Moraga. 2002. A fuzzy Risk Model and its Matrix Algorithm. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 10 (4): 347–62.

Huang, C. F., and D. Ruan. 2008. Fuzzy Risks and an Updating Algorithm with New Observations. *Risk Analysis* 28 (3): 681–94.

Huang, C. F., and Y. Shi. 2002. *Towards Efficient Fuzzy Information Processing—Using the Principle of Information Diffusion*. Heidelberg: Physica-Verlag.

Karimi, I., and E. Hülermeier. 2007. Risk Assessment System of Natural Hazards: A New Approach Based on Fuzzy Probability. *Fuzzy Sets and Systems* 158 (9): 987–99.

Moeller, B., and M. Beer. 2003. *Fuzzy Randomness*. Berlin: Springer.

Moraga, C., and C. F Huang. 2003. Learning Subjective Probabilities from a Small Data Set. In *Proceedings of the 33rd International Symposium on Multiple-Valued Logic*, 355–60, Tokyo, Japan, 2003.

Otness, R. K., and L. Encysin. 1972. *Digital Time Series Analysis*. New York: John Wiley.

Ramsey, F. P. 1931. Truth and Probability. In *The Foundations of Mathematics*, edited by R. B. Braithwaite, 156–98. London: Routledge & Kegan Paul.

Tanaka, H., C. Fan, and K. Toguchi. 1983. Fault Tree Analysis by Fuzzy Probability. *IEEE Transactions on Reliability* R-32 (5): 453–57.

Walley, P. 1991. *Statistical Reasoning with Imprecise Probabilities*. London: Chapman & Hall.

Watson, S. R., J. J. Weiss, and M. L. Donnell. 1979. Fuzzy Decision Analysis. *IEEE Transactions on Systems, Man and Cybernetics* SMC-9 (1): 1–9.

Williams, P. M. 1975. *Notes on Conditional Previsions*. Technical Report. School of Mathematical and Physical Science, University of Sussex, UK.

Zadeh, L. A. 1968. Probability Measures of Fuzzy Events. *Journal of Mathematical Analysis and Applications* 23 (2): 421–27.

——. 1978. Fuzzy Sets as a Basis for a Theory of Possibility. *Fuzzy Sets and Systems* 1 (1): 3–28.

Zhang, J. X. 2005. *Study on Theory and Method of New Generation of Natural Disaster Risk Zoning—Taking Earthquake Disaster as an Example*. Ph.D. Dissertation. Beijing Normal University, Beijing (in Chinese).

Zong, T. 2004. Improvement on Interior-Outer-Set Model for Estimating Fuzzy Probability. In *Proceedings of 2004 Annual Meeting of the North American Fuzzy Information Processing Society*, 578–83. Banff, Canada, 2004.