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# Log Fano varieties over function fields of curves

**Brendan Hassett**[1], **Yuri Tschinkel**[2,3]

[1] Department of Mathematics, Rice University, MS 136, Houston, TX 77251-1892, USA (e-mail: hassett@rice.edu)
[2] Mathematisches Institut, Bunsenstr. 3–5, 37073 Göttingen, Germany
[3] Courant Institute, New York University, 251 Mercer Str., New York, NY 10012, USA (e-mail: tschinkel@cims.nyu.edu)

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**Abstract.** Consider a smooth log Fano variety over the function field of a curve. Suppose that the boundary has positive normal bundle. Choose an integral model over the curve. Then integral points are Zariski dense, after removing an explicit finite set of points on the base curve.

## 1. Introduction

Let $k$ be an algebraically closed field of characteristic zero and $B$ a smooth projective curve over $k$ with function field $F = k(B)$.

Our point of departure is the following theorem, combining work of Graber–Harris–Starr and Kollár–Miyaoka–Mori [6, 14]: Let $X$ be a smooth projective rationally connected variety over $F$. Then $X(F)$ is Zariski dense in $X$. One central example is Fano varieties, i.e., varieties with ample anticanonical class, which are known to be rationally connected (see [13, V.2.13]). In this context, it is not necessary to pass to a field extension to get rational points.

When $F$ is a number field, it may be necessary to pass to a finite extension to get rational points; there exist Fano varieties over $\mathbb{Q}$ without rational points. Moreover, even for Fano threefolds potential Zariski density, i.e., density after a finite extension of $F$, is unknown in general. For some positive results in this direction see [8, 4, 9].

In this paper we study Zariski density of *integral* points. Consider pairs $(X, D)$ consisting of a variety $X$ and a divisor $D \subset X$, and fix integral models $\pi : (\mathcal{X}, \mathcal{D}) \to B$ (see Sect. 2 for the definition). An $F$-rational point $s \in X \setminus D$ gives rise to a section $s : B \to \mathcal{X}$ of $\pi$, meeting $\mathcal{D}$ in finitely many points. As we vary $s$,

$$s^{-1}(\mathcal{D}) = \pi(s(B) \cap \mathcal{D}) \subset B$$

may vary as well. Fixing a finite set $S \subset B$, an $S$-integral point of $(\mathcal{X}, \mathcal{D})$ is an $F$-rational point of $X$ such that $s^{-1}(\mathcal{D}) \subset S$ (as sets).

**Theorem 1.** *Let $F$ be the function field of a smooth projective curve $B/k$. Let $(X, D)$ be a pair over $F$ consisting of a smooth projective variety $X$ and a smooth divisor $D \subset X$. Assume that*

- *$D$ is rationally connected;*
- *the divisor class of the normal bundle $c_1(\mathcal{N}_{D/X})$ is numerically equivalent to a nontrivial effective divisor.*

*Given a model $\pi : (\mathcal{X}, \mathcal{D}) \to B$, let $S$ be a nonempty finite set of points in $B$ containing the images of the singularities of $\mathcal{X}$ and $\mathcal{D}$. Then $S$-integral points of $(\mathcal{X}, \mathcal{D})$ are Zariski dense.*

Note however that we allow points of bad reduction outside $S$. For example, let $\mathcal{X} = \mathbb{P}^2_{[x,y,z]} \times \mathbb{P}^1_{[s,t]}$ and

$$\mathcal{D} = \{s(x^2 + yz) + t(y^2 + xz) = 0\}.$$

While $\mathcal{X}$ and $\mathcal{D}$ are nonsingular the fiber $\mathcal{D}_{[s,t]}$ is singular when $s^3 + t^3 = 0$.

Let $K_X$ denote the canonical class of $X$ and $K_X + D$ the log canonical class of $(X, D)$. The pair $(X, D)$ is *log Fano* if $-(K_X + D)$ is ample. By adjunction

$$(K_X + D)|D = K_D$$

so $-K_D$ is ample. Thus $D$ is Fano hence rationally connected [14], [13, V.2.13].

**Corollary 2.** *Let $(X, D)$ be a log Fano variety over $F$ with $X$ and $D$ smooth. Assume that the divisor class $c_1(\mathcal{N}_{D/X})$ is numerically equivalent to a nontrivial effective divisor. For each integral model and collection of places as specified in Theorem 1, integral points are Zariski dense.*

We expect that the hypothesis on $\mathcal{N}_{D/X}$ is not needed. For example, our argument does not apply to the case $X = \mathbb{P}^1$ and $D = \infty$, where density of integral points is immediate. And there are log Fano varieties $(X, D)$ with $\mathcal{N}_{D/X}$ negative, e.g., $(X, D) = (\mathbb{F}_n, \Sigma_0)$ where $\mathbb{F}_n$ is the Hirzebruch surface admitting a section $\Sigma_0$ with self-intersection $-n < 0$.

Furthermore, the condition that $(X, D)$ be log Fano can be weakened. There are numerous examples of varieties $X$ with trivial canonical class and Zariski-dense set of rational points over number and function fields. However, the deformation-theoretic approach here is not directly applicable. It is an open problem to characterize pairs with potentially dense integral points; Campana [5] has a conjectural description of these, at least when $D = \emptyset$.

Our result is a partial converse to the function-field version of Vojta's conjectures: If $(X, D)$ is defined over a number field $F$ and $K_X + D$ is ample then integral points are not Zariski dense (see [16] for the number field case, with connections to value-distribution theory). Very few density results for integral points over number fields are available, most of them in dimension two (see [15, 3, 10]).

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## 2. Integral models and reduction to the smooth case

**Definition 3.** A *pair* $(X, D)$ consists of a smooth projective variety and a reduced effective divisor with normal crossings.

Let $B$ be a smooth projective curve defined over an algebraically closed field $k$ of characteristic zero and $F = k(B)$ its function field.

**Definition 4.** Let $(X, D)$ be a pair defined over $F$. An *integral model*

$$\pi : (\mathcal{X}, \mathcal{D}) \to B$$

consists of a flat proper morphism from a normal variety $\pi_{\mathcal{X}} : \mathcal{X} \to B$ with generic fiber $X$, and a closed subscheme $\mathcal{D} \subset \mathcal{X}$ such that $\pi_{\mathcal{D}} := \pi_{\mathcal{X}}|_{\mathcal{D}} : \mathcal{D} \to B$ is flat and has generic fiber $D$. A point $b \in B$ is of *good reduction* if the fibers $\mathcal{X}_b = \pi_{\mathcal{X}}^{-1}(b)$ and $\mathcal{D}_b = \pi_{\mathcal{D}}^{-1}(b)$ are smooth.

We emphasize that $\mathcal{D}$ has no irreducible components contained in fibers of $\pi_{\mathcal{X}}$.

While in applications the model is often specified *a priori*, for each projective embedding of $(X, D)$ we can construct a natural model: The properness of the Hilbert scheme yields extensions of $X$ and $D$ to schemes flat and projective over $B$. Locally on $B$, these are obtained by 'clearing denominators' in the defining equations of $X$ and $D$. Normalizing if necessary, we obtain a model of $(X, D)$.

**Definition 5.** Let $S$ be a finite subset of $B$. An *$S$-integral point of* $(\mathcal{X}, \mathcal{D})$ is a section $s : B \to \mathcal{X}$ such that $s^{-1}(\mathcal{D}) \subset S$ as sets.

Thus if $D = \emptyset$ then integral points are just sections of $\mathcal{X} \to B$, which are $F$-rational points of $X$.

The following proposition is straightforward:

**Proposition 6.** *Let* $(\mathcal{X}_1, \mathcal{D}_1)$ *and* $(\mathcal{X}_2, \mathcal{D}_2)$ *be integral models of* $(X, D)$. *Let* $T \subset B$ *denote the set over which the birational map*

$$(\mathcal{X}_1, \mathcal{D}_1) \dashrightarrow (\mathcal{X}_2, \mathcal{D}_2)$$

*fails to be an isomorphism. Then S-integral points of* $(\mathcal{X}_1, \mathcal{D}_1)$ *are mapped to* $(S \cup T)$*-integral points of* $(\mathcal{X}_2, \mathcal{D}_2)$*. If S-integral points of* $(\mathcal{X}_1, \mathcal{D}_1)$ *are Zariski dense then* $(S \cup T)$*-integral points of* $(\mathcal{X}_2, \mathcal{D}_2)$ *are Zariski dense.*

In particular, if we allow ourselves to enlarge the set $S$ then Zariski-density of integral points is independent of the model.

We discuss how Theorem 1 can be reduced to the case of nonsingular integral models:

**Definition 7.** A *good resolution* of an integral model is a birational proper morphism from a pair

$$\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{D}}) \to (\mathcal{X}, \mathcal{D})$$

such that

- $\tilde{\mathcal{X}}$ is smooth and $\tilde{\mathcal{D}}$ is normal crossings;
- $\rho^{-1}(\mathcal{D}) = \tilde{\mathcal{D}}$;
- $\rho$ is an isomorphism over the open subset of $(\mathcal{X}, \mathcal{D})$ where $\mathcal{X}$ is smooth and $\mathcal{D}$ is normal crossings.

*Remark 8.* (1) $\tilde{\mathcal{D}}$ may very well have components contained in fibers over $B$, so $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}})$ is *not* necessarily an integral model.
(2) The normality condition in Definition 4 of an integral model guarantees that for each $b \in B$ and each irreducible component of $\mathcal{X}_b$, the total space $\mathcal{X}$ is smooth at the generic point of that component. In particular, $\rho$ is an isomorphism over a dense open subset of each fiber of $\mathcal{X}$.

Let $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}) \to (\mathcal{X}, \mathcal{D})$ be a good resolution, $S \subset B$ a finite set containing the images of the singularities of $\mathcal{X}$ and $\mathcal{D}$, and $\tilde{\mathcal{D}}^\circ$ the union of the components of $\tilde{\mathcal{D}}$ dominating $B$. We have:

- $\tilde{\mathcal{D}}^\circ$ is normal crossings;
- images of $S$-integral points of $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}^\circ)$ under $\rho$ are $S$-integral points of $(\mathcal{X}, \mathcal{D})$.

We have a bijection

$$\rho : \tilde{\mathcal{X}} \setminus \tilde{\mathcal{D}} \to \mathcal{X} \setminus \mathcal{D},$$

so $S$-integral points of $(\mathcal{X}, \mathcal{D})$ correspond to sections

$$\{\tilde{s} : B \to \tilde{\mathcal{X}} : \tilde{s}^{-1}(\tilde{\mathcal{D}}) \subset S\}.$$

Since the fibral components of $\tilde{\mathcal{D}}$ lie over $S$, $S$-integral points of $(\mathcal{X}, \mathcal{D})$ are equal to $S$-integral points of $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}^\circ)$.

This analysis and resolution of singularities in characteristic zero reduces Theorem 1 to the following special case:

**Theorem 9 (Smooth case).** *Retain the assumptions of Theorem 1 and assume in addition that* $\mathcal{X}$ *and* $\mathcal{D}$ *are nonsingular. Then for* any *nonempty* $S \subset B$ *the S-integral points in* $(\mathcal{X}, \mathcal{D})$ *are Zariski dense.*

Here is a roadmap for the rest of the paper. Our main strategy is to produce 'good' sections *in the boundary* $s : B \to \mathcal{D}$ that deform to a Zariski-dense collection of $S$-integral points. The 'deformation of combs' technique of [6, §2] yields a section $s_1 : B \to \mathcal{D}$ such that the normal bundle $\mathcal{N}_{s_1(B)/\mathcal{D}}$ is globally generated with no higher cohomology. Since $c_1(\mathcal{N}_{D/X})$ is effective and nonzero, we may assume that the teeth of our combs have positive intersection numbers with $\mathcal{D}$. Thus we can construct a section $s_2 : B \to \mathcal{D}$ such that $\mathcal{N}_{s_2(B)/\mathcal{X}}$ is globally generated with no higher cohomology. Finally, for any prescribed point $p \in B$, we produce a section $s_3 : B \to \mathcal{D}$ such that $s_3^*\mathcal{O}_{\mathcal{X}}(\mathcal{D}) \simeq \mathcal{O}_B(Np)$ for some $N \gg 0$. This entails controlling the divisor class of the points over which we attach our teeth.

To obtain density of $S$-integral points, we control how deformations of $s_3(B)$ meet $\mathcal{D}$. The first step is to restrict to deformations $s_t : \mathcal{D} \to \mathcal{X}$ such that $s_t^*\mathcal{O}_{\mathcal{X}}(\mathcal{D}) \simeq \mathcal{O}_B(Np)$. The key technical tool is the 'Atiyah extension' classifying first-order deformations of varieties with a line bundle. Taking this into account, we introduce new notions of free and very free curves in Sect. 3. Section 4 deduces smoothing results for morphisms of curves respecting line bundles; these generalize the results for free and very free curves in [13, II.7]. Section 5 is devoted to the proof of Theorem 9.

## 3. Atiyah classes and free curves

Assume that $k$ is algebraically closed. Let $B$ be a smooth projective variety, $\psi : \mathcal{Y} \to B$ a smooth surjective morphism from a quasi-projective variety, and $L$ a line bundle on $\mathcal{Y}$.

Consider the dual Atiyah extension [2, p. 196], [11, p. 243]

$$0 \to \Omega^1_{\mathcal{Y}} \to \mathcal{F}_{\mathcal{Y},L} \to \mathcal{O}_{\mathcal{Y}} \to 0.$$

Up to sign, it is classified by the first Chern class

$$\pm c_1(L) \in H^1\big(\mathcal{Y}, \Omega^1_{\mathcal{Y}}\big) = \mathrm{Ext}^1\big(\mathcal{O}_Y, \Omega^1_Y\big).$$

There is an induced relative Atiyah extension

$$0 \to \Omega^1_{\mathcal{Y}/B} \to \mathcal{F}_{\mathcal{Y},L/B} \to \mathcal{O}_{\mathcal{Y}} \to 0.$$

These are all locally free as $\psi$ is smooth. Writing $\mathcal{E}_{\mathcal{Y},L/B}$ for the dual to $\mathcal{F}_{\mathcal{Y},L/B}$, we obtain

$$0 \to \mathcal{O}_{\mathcal{Y}} \to \mathcal{E}_{\mathcal{Y},L/B} \to \mathcal{T}_{\mathcal{Y}/B} \to 0.$$

**Definition 10.** Let $C$ be a nodal projective curve. A nonconstant morphism $g : C \to \mathcal{Y}$ is *$L$-free over $B$* if for each $q \in C$

$$H^1\big(C, g^*\mathcal{E}_{\mathcal{Y},L/B} \otimes \mathcal{I}_q\big) = 0.$$

It is *$L$-very free over $B$* if for each subscheme $\Sigma \subset C$ of length two

$$H^1\big(C, g^*\mathcal{E}_{\mathcal{Y},L/B} \otimes \mathcal{I}_\Sigma\big) = 0.$$

**Proposition 11.** *Assume that $k$ is of characteristic zero. Let $b \in B$ and $y \in \mathcal{Y}_b = \psi^{-1}(b)$. Suppose $\mathcal{Y}_b$ is a proper rationally connected variety. Then there exists a morphism $g : \mathbb{P}^1 \to \mathcal{Y}_b$ that is $L$-free over $B$, with image containing $y$. If $L|\mathcal{Y}_b$ is numerically equivalent to a nontrivial effective divisor then we can choose $g$ to be $L$-very free over $B$.*

*Proof.* There exists a very free morphism $g : \mathbb{P}^1 \to \mathcal{Y}_b$ [13, IV.3.9.4], with $g^*\mathcal{T}_{\mathcal{Y}/B}$ ample. Moreover, given any finite collection of points $y_1, \dots, y_m \in \mathcal{Y}$, we may assume the image of $g$ contains these points.

We have the extension

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to g^*\mathcal{E}_{\mathcal{Y},L/B} \to g^*\mathcal{T}_{\mathcal{Y}/B} \to 0.$$

It follows that each summand of $g^*\mathcal{E}_{\mathcal{Y},L/B}$ is nonnegative, which implies $L$-freeness.

We prove $L$-very freeness. Assume $H$ is an effective nonzero divisor corresponding to $L$. Choose $g$ such that its image contains $y$, some point $y'$ in the support of $H$, and some point $y''$ not in the support of $H$. In particular, the image is not contained in any component of $H$. It follows that $g^*L$ has positive degree.

If $\mathcal{O}_{\mathbb{P}^1}$ were a summand of $g^*\mathcal{E}_{\mathcal{Y},L/B}$ then we would have

$$g^*\mathcal{E}_{\mathcal{Y},L/B} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus g^*\mathcal{T}_{\mathcal{Y}/B},$$

i.e., the Atiyah extension would split after pull-back. The inclusion $\mathcal{T}_{\mathbb{P}^1} \hookrightarrow g^*\mathcal{T}_{\mathcal{Y}/B}$ induces the Atiyah extension on $\mathbb{P}^1$ associated with $g^*L$:

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{E}_{\mathbb{P}^1, g^*L} \to \mathcal{T}_{\mathbb{P}^1} \to 0$$

which splits as well. However, this is classified by

$$\pm c_1(g^*L) \in H^1\big(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}\big) = \mathrm{Ext}^1(\mathcal{T}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}),$$

which is nontrivial because $L$ is effective and nonzero. (Here we are using the assumption that the base field is of characteristic zero.) □

## 4. Spaces of morphisms and comb constructions

We retain the notation introduced in Sect. 3.

Let $C_0$ be a nodal projective connected curve. Fix a morphism $f_0 : C_0 \to \mathcal{Y}$ and write

$$g_0 = \psi \circ f_0 : C_0 \to B.$$

Let $(S, 0)$ be a smooth scheme with distinguished closed point. Consider a flat projective morphism $\varpi : \mathcal{C} \to S$ such that $\varpi^{-1}(0) \simeq C_0$ and an

$S$-morphism

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "g"] \arrow[dr, "\varpi"'] & & B \times S \arrow[dl, "\pi'_S"] \\ & S & \end{tikzcd}$$

such that $g|C_0 = g_0$. (Here $\pi'_S$ is the projection to $S$.) Fix a line bundle $\mathcal{M}$ on $\mathcal{C}$ such that $\mathcal{M}|C_0 \simeq f_0^* L$.

Let $\mathrm{Mor}_{B\times S}(\mathcal{C}, \mathcal{Y} \times S)$ denote the morphisms

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "h"] \arrow[dr, "g"'] & & \mathcal{Y} \times S \arrow[dl, "(\psi, \mathrm{Id}_S)"] \\ & B \times S & \end{tikzcd}$$

over $B \times S$; each connected component is a quasi-projective scheme over $S$. Let $\mathrm{Pic}_{\mathcal{C}/S}$ denote the relative Picard scheme [1, Theorem 7.3], which is locally of finite type over $S$. We have a morphism

$$\begin{aligned} \mathrm{Mor}_{B\times S}(\mathcal{C}, \mathcal{Y} \times S) &\to \mathrm{Pic}_{\mathcal{C}/S} \\ h &\mapsto (\pi_{\mathcal{Y}} \circ h)^* L \otimes \mathcal{M}^{-1}, \end{aligned}$$

where $\pi_{\mathcal{Y}} : \mathcal{Y} \times S \to \mathcal{Y}$ is the projection. The fiber over the zero-section is a closed subscheme

$$\mathrm{Mor}^{\bullet}_{B\times S}(\mathcal{C}, \mathcal{Y} \times S) \subset \mathrm{Mor}_{B\times S}(\mathcal{C}, \mathcal{Y} \times S).$$

Finally, we have

$$\begin{aligned} \mathrm{Mor}_{B\times S}\big((\mathcal{C}, \mathcal{M}), \big(\mathcal{Y} \times S, \pi_{\mathcal{Y}}^* L\big)\big) = \big\{ &(h, \iota) : h \in \mathrm{Mor}^{\bullet}_{B\times S}(\mathcal{C}, \mathcal{Y} \times S), \\ &\iota : (\pi_{\mathcal{Y}} \circ h)^* L \xrightarrow{\sim} \mathcal{M} \big\} \end{aligned}$$

which is a $\mathbb{G}_m$-torsor over $\mathrm{Mor}^{\bullet}_{B\times S}(\mathcal{C}, \mathcal{Y} \times S)$. This keeps track of a choice of identification between $\mathcal{M}$ and the pull-back of $L$.

**Proposition 12.** *The relative tangent space of*

$$\mathrm{Mor}_{B\times S}\big((\mathcal{C}, \mathcal{M}), \big(\mathcal{Y} \times S, \pi_{\mathcal{Y}}^* L\big)\big) \quad (\textit{resp.}\ \mathrm{Mor}_{B\times S}(\mathcal{C}, \mathcal{Y} \times S))$$

*over $S$ at $[f_0]$ is isomorphic to*

$$\Gamma\big(C_0, f_0^* \mathcal{E}_{\mathcal{Y},L/B}\big) \quad \big(\textit{resp.}\ \Gamma\big(C_0, f_0^* \mathcal{T}_{\mathcal{Y}/B}\big)\big).$$

*The obstruction space is contained in*

$$H^1\big(C_0, f_0^* \mathcal{E}_{\mathcal{Y},L/B}\big) \quad \big(\textit{resp.}\ H^1\big(C_0, f_0^* \mathcal{T}_{\mathcal{Y}/B}\big)\big).$$

*In particular, the morphism space is smooth over $S$ at $f_0$ provided the corresponding first cohomology group vanishes.*

*Proof.* The description of the tangent and obstruction spaces is an application of the technique of [11, §2.3]. Using the smoothness criterion of [7, 17.5.1–2], it suffices to show that the schemes are flat over $S$ with regular geometric fibers. The morphism spaces are flat over $S$ at $f_0$ provided their fibers over 0 have the expected dimension there [13, I.2.17]. When the first cohomology group vanishes, the dimension of the fiber equals the dimension of its tangent space. □

*Remark 13.* Instead of working with pairs consisting of varieties and line bundles relative to $B$, we could work with varieties relative to $B \times B\mathbb{G}_m$. Here $B\mathbb{G}_m$ is the classifying stack of the multiplicative group; morphisms $\mathcal{Y} \to B\mathbb{G}_m$ correspond to line bundles on $\mathcal{Y}$. The Atiyah extension of $\mathcal{Y}/B$ determined by $L$ coincides with the relative tangent bundle of the morphism $\mathcal{Y} \to B \times B\mathbb{G}_m$. (We are grateful to one of the referees for suggesting this point of view.)

**Definition 14.** A *comb* is a nodal projective curve

$$C_0 = B' \cup T_1 \cup \ldots \cup T_r$$

where $B'$ is smooth and connected and each $T_i$ is a smooth rational curve meeting $B'$ in one point $b'_i$. The $T_i$ are pairwise disjoint.

**Proposition 15.** *Let $B' \subset \mathcal{Y}$ be a smooth projective curve satisfying $H^1(B', \mathcal{E}_{\mathcal{Y},L/B}|B') = 0$; assume that $B'$ meets a proper separably rationally connected fiber of $\psi$. Suppose we are given*

- *a comb $C_0$ with handle $B'$ and teeth $T_i$;*
- *a morphism $f_0 : C_0 \to \mathcal{Y}$, such that*
  (1) *$f_0|B'$ extends the inclusion $B' \subset \mathcal{Y}$;*
  (2) *each restriction $F_i := f_0|T_i$ is $L$-free and contained in a proper fiber of $\psi$.*

*Then for any*

- *smooth scheme with distinguished closed point $(\mathbb{A}, 0)$;*
- *flat proper morphism $\varpi : \mathcal{C} \to \mathbb{A}$ with distinguished fiber $C_0$;*
- *morphism $g : \mathcal{C} \to B \times \mathbb{A}$ over $\mathbb{A}$ such that $g|C_0 = \psi \circ f_0$;*
- *line bundle $\mathcal{M}$ on $\mathcal{C}$ such that $\mathcal{M}|C_0 \simeq f_0^* L$;*

*there exists an étale neighborhood $(S, 0)$ of $(\mathbb{A}, 0)$ and a morphism $f : \mathcal{C} \to \mathcal{Y} \times S$ over $B \times S$ such that $f^*(\pi_{\mathcal{Y}}^* L) \simeq \mathcal{M}$ and $f|C_0 = f_0$.*

*Proof.* Since the teeth are $L$-free, $\mathcal{E}_{\mathcal{Y},L/B}$ pulls back to a semi-positive vector bundle on each $T_i$. An inductive argument [13, II.7.5] reduces the cohomology over the comb to the cohomology of the restriction to the handle

$$H^1(C_0, \mathcal{E}_{\mathcal{Y},L/B}|C_0) \simeq H^1\big(B', \mathcal{E}_{\mathcal{Y},L/B}|B'\big) = 0.$$

We then apply Proposition 12 to construct $f$. By construction,

$$f_0 \in \mathrm{Mor}_{B\times\mathbb{A}}\big((\mathcal{C},\mathcal{M}),\big(\mathcal{Y}\times\mathbb{A},\pi_{\mathcal{Y}}^*L\big)\big)$$

which is smooth over $\mathbb{A}$ because of the vanishing of the higher cohomology of $\mathcal{E}_{\mathcal{Y},L/B}|C_0$. Thus for a suitable étale neighborhood $(S,0)$ of the origin in $\mathbb{A}$ we have a section

$$[f]: S \to \mathrm{Mor}_{B\times S}\big((\mathcal{C},\mathcal{M}),\big(\mathcal{Y}\times S,\pi_{\mathcal{Y}}^*L\big)\big)$$

with $[f]_0 = f_0$. (We use the same notation for $\mathcal{C}$ and $\mathcal{M}$ and their restrictions over $S$.) The universal property of our morphism space gives $f:\mathcal{C}\to\mathcal{Y}\times S$ over $B\times S$ and an identification $\iota:(\pi_{\mathcal{Y}}\circ f)^*L \xrightarrow{\sim} \mathcal{M}|S$. □

*Remark 16.* Alternatively, we could have analyzed the deformations of $f_0: C_0 \to \mathcal{Y}$ using the *normal sheaf* $\mathcal{N}_{f_0}$. This works best when $f_0$ is an embedding, which can be achieved when the fibers of $\psi$ have dimension $\geq 3$. Then we can regard $f_0(C_0)$ as a point in the Hilbert scheme $\mathcal{H}ilb$ of nodal curves in $\mathcal{Y}$. Suppose in addition that $B' \subset \mathcal{Y}$ is a section of $\psi:\mathcal{Y}\to B$. Then there exists a morphism

$$\begin{aligned}\alpha : \mathcal{H}ilb &\to \mathrm{Pic}(B)\\ C_s &\mapsto \det\big(\mathbb{R}^\bullet\psi_*(L\otimes\mathcal{O}_{C_s})\big),\end{aligned}$$

where det is the determinant of cohomology, which is defined for perfect complexes (see [12, Theorem 2]). A deformation argument similar to ours can be used to produce smooth curves in suitable fibers of $\alpha$ corresponding to sections $\sigma: B\to\mathcal{Y}$ with $\sigma^*L = L|B'$. (We are grateful to one of the referees for pointing out this approach.)

**Proposition 17.** *Let $B' \subset \mathcal{Y}$ be a smooth projective curve. There exists an integer $n>0$ with the following property*:

*Suppose we are given*

- *a comb $C_0$ with handle $B'$ and teeth $T_1,\ldots,T_q$ attached at $b'_1,\ldots,b'_q$;*
- *a morphism $f_0: C_0\to\mathcal{Y}$, such that*
  (1) *$f_0$ is an embedding along $B'$ and extends the inclusion $B'\subset\mathcal{Y}$;*
  (2) *each $F_i := f_0|T_i$ is $L$-free and contained in a proper fiber of $\psi$;*

*Then there exists a subset $\{i_1,\ldots,i_r\}\subset\{1,\ldots,q\}$ with $r\geq q-n$ such that*

$$f'_0 := f_0|C'_0 : C'_0 \to \mathcal{Y}, \quad C'_0 := B'\cup T_{i_1}\cup\ldots\cup T_{i_r}\subset C_0$$

*deforms to a morphism $f'_t: B'\to\mathcal{Y}$ with*

$$\big(f'_t\big)^*L \simeq L|B'\otimes\mathcal{O}_{B'}\big(e_{i_1}b'_{i_1}+\ldots+e_{i_r}b'_{i_r}\big),\quad e_i = \deg F_i^*L$$

*and $\psi\circ f'_t = \psi|B'$. Furthermore, if the $F_i$ are assumed to be $L$-very free then for suitable $n>0$ we can take $f'_t$ to be $L$-free over $B$.*

*Proof.* We first construct a family $\varpi : \mathcal{C} \to \mathbb{A}^r$ deforming $C_0$ to $B'$ (see [13, p. 156]). Consider the smooth codimension-two subvariety

$$Z = \bigcup_{i=1}^{r} \big(\{b'_i\} \times \{t_i = 0\}\big) \subset B' \times \mathbb{A}^r$$

and the blow-up

$$\mathcal{C} := \mathrm{Bl}_Z(B' \times \mathbb{A}^r) \xrightarrow{\sigma} B' \times \mathbb{A}^r$$

with exceptional divisors $E_1, \ldots, E_r$. The composed morphism

$$\varpi : \mathcal{C} \to B' \times \mathbb{A}^r \to \mathbb{A}^r$$

is still flat with $\varpi^{-1}(0) \simeq C_0$; *every* fiber of $\varpi$ is a comb with handle $B'$. We introduce a line bundle on this family: Consider

$$L' = L|B' \otimes \mathcal{O}_{B'}\big(e_1 b'_1 + \ldots + e_r b'_r\big)$$

and write

$$\mathcal{M} = (\pi_{B'} \circ \sigma)^* L' \otimes \mathcal{O}_{\mathcal{C}}(-e_1 E_1 - \ldots - e_r E_r).$$

Note that $\mathcal{C}_s \simeq B'$ and $\mathcal{M}|\mathcal{C}_s \simeq L|B'$, i.e., our modification leaves the fibers away from the coordinate axes of $\mathbb{A}^r$ unchanged.

We apply the smoothing technique of [13, II.7.9] to the morphism space

$$\mathrm{Mor}_{B \times \mathbb{A}^r}\big((\mathcal{C}, \mathcal{M}), \big(\mathcal{Y} \times \mathbb{A}^r, \pi_{\mathcal{Y}}^* L\big)\big).$$

This gives a curve containing the origin $0 \in T \subset \mathbb{A}^r$, whose generic point is contained in at most $c \leq n$ of the coordinate hyperplanes, such that the restricted family $\mathcal{C}_T := \mathcal{C} \times_{\mathbb{A}^r} T \to T$ admits a morphism

$$\begin{tikzcd}
\mathcal{C}_T \arrow[rr, "f"] \arrow[dr] & & \mathcal{Y} \times T \arrow[dl] \\
& B \times T &
\end{tikzcd}$$

restricting to $f_0$ at $0 \in T$. Let $\mathcal{C}'_T$ denote the irreducible component of $\mathcal{C}_T$ containing $B' \subset C_0$; its distinguished fiber is a subcomb

$$C'_0 := B' \cup T_{i_1} \cup \ldots \cup T_{i_q} \subset C_0, \quad q = r - c.$$

The restriction $f' = f|\mathcal{C}'$ is our desired morphism.

When the teeth are $L$-very free, the argument of [13, II.7.10] yields a smoothing that is $L$-free over $B$. □

**Corollary 18.** *Retain the assumptions of Proposition 17 and assume k is of characteristic zero. Suppose in addition that $B'$ meets a proper ratio-*

*nally connected fiber of $\psi$. Then there exists a free $f'_s : B' \to \mathcal{Y}$ with $\psi \circ f'_s = \psi|B'$.*

*Proof.* We just need to construct the comb $C_0$ and the morphism $f_0$. Using Proposition 11, we choose

- points $b_1, \ldots, b_r \in B$ with $\mathcal{Y}_{b_i}$ proper;
- distinct points $b'_1, \ldots, b'_r \in B'$ with $\psi(b'_i) = b_i$; and
- $L$-very free curves $F_i : T_i \to \mathcal{Y}_{b_i}$ containing $b'_i$ as a smooth point.

Let $C_0 = B' \cup T_1 \cup \ldots \cup T_r$ denote the corresponding comb and $f_0 : C_0 \hookrightarrow \mathcal{Y}$ the morphism obtained by gluing the inclusion $B' \subset \mathcal{Y}$ with the $F_i$. □

*Remark 19.* Again, we could also work over the Hilbert scheme and apply the smoothing techniques of [6, §2] to establish this result.

Our main application is to sections of rationally connected fibrations.

**Theorem 20.** *Let $B$ be a smooth projective curve, $\psi : \overline{\mathcal{Y}} \to B$ a proper morphism from a smooth variety with rationally connected generic fiber, $\mathcal{Y} \subset \overline{\mathcal{Y}}$ the open subset where $\psi$ is smooth, and $L$ an invertible sheaf on $\mathcal{Y}$ restricting to a nontrivial effective divisor on the generic fiber of $\psi$. Fix an integer $N \gg 0$.*

*For each invertible sheaf $M \in \mathrm{Pic}^N(B)$, there exists a section $s : B \to \mathcal{Y}$ such that $s^*L = M$ and $s$ is $L$-free over $B$. In particular, the sheaves*

$$s^*\mathcal{E}_{\mathcal{Y},L/B} \quad \text{and} \quad s^*\mathcal{T}_{\mathcal{Y}/B} = \mathcal{N}_{s(B)/\mathcal{Y}}$$

*are both globally generated with no higher cohomology.*

*Proof of 20.* The Graber–Harris–Starr theorem [6] gives a section $s_1 : B \to \mathcal{Y}$. The exact sequence

$$0 \to \mathcal{T}_{\mathcal{Y}/B} \to \mathcal{T}_{\mathcal{Y}} \to \psi^*\mathcal{T}_B \to 0$$

induces

$$0 \to s_1^*\mathcal{T}_{\mathcal{Y}/B} \to s_1^*\mathcal{T}_{\mathcal{Y}} \to \mathcal{T}_B \to 0$$

which is split by the differential $ds_1 : \mathcal{T}_B \to s_1^*\mathcal{T}_{\mathcal{Y}}$. Thus we have

$$s_1^*\mathcal{T}_{\mathcal{Y}} = s_1^*\mathcal{T}_{\mathcal{Y}/B} \oplus \mathcal{T}_B$$

and the first term coincides with the normal bundle $\mathcal{N}_{s_1(B)/\mathcal{Y}}$.

Proposition 17 gives a section $s_2 : B \to \mathcal{Y}$ that is $L$-free over $B$. We assume this in what follows, so in particular

$$H^1\big(B, s_2^*\mathcal{E}_{\mathcal{Y},L/B}\big) = 0.$$

To complete the proof, we produce smoothings $f_t : B \to \mathcal{Y}$ of suitable combs

$$f_0 : C_0 = B \cup_{b_1} T_1 \ldots \cup_{b_r} T_r \to \mathcal{Y}$$

where $f_0|B = s_2$ $F_i = f_0|T_i : T_i \to \mathcal{Y}_{b_i}$ is $L$-very free. These are given by Proposition 15, but it is necessary to specify the points of attachment carefully to achieve the desired value of $f_t^*L$.

Choose $e$ sufficiently large so that for each smooth fiber $\mathcal{Y}_b$ and every point $y \in \mathcal{Y}_b$, there exists an $L$-very free curve $T \to \mathcal{Y}_b$ passing through $y$ with $L \cdot T = e$. We therefore may assume

$$e = T_1 \cdot L = T_2 \cdot L = \ldots = T_r \cdot L > 0.$$

Let $\mathcal{C} \to \mathbb{A}^r$ and $\mathcal{M}$ denote the families constructed in the proof of Proposition 17. Thus for generic $s \in \mathbb{A}^r$ we have $C_s \simeq B$ and

$$\mathcal{M}|C_s \simeq s_2^*L(eb_1 + \ldots + eb_r).$$

Let $U \subset B$ denote the locus over which $\mathcal{Y}_b$ is smooth and contains a $L$-very free curve $T$ of degree $e$. To complete the proof of Theorem 20, we require the following moving lemma, which governs the precise value of $N$:

**Lemma 21.** *Let $B$ be a smooth projective curve, $U \subset B$ a dense open subset, and $e \in \mathbb{N}$. Fix a line bundle $\Lambda$ on $B$ of degree $\ell$, $r \geq 2\mathsf{g}(B)+1$, and $N = er + \ell$. For any $M \in \mathrm{Pic}^N(B)$ there exist distinct points $b_1, \ldots, b_r \in U$ so that*

$$M \simeq \Lambda(e(b_1 + \ldots + b_r)).$$

This is an elementary application of Riemann–Roch. Choose an $e$th root of $M \otimes \Lambda^{-1}$, i.e., a line bundle $A$ with $A^{\otimes e} \otimes \Lambda = M$. Any line bundle of degree $r$ on $B$ is very ample so consider the embedding

$$\phi_A : B \hookrightarrow \mathbb{P}^{r-\mathsf{g}(B)}.$$

The divisors with any support along $B \setminus U$ form a finite union of hyperplanes in the linear system $|A|$. The divisors admitting points of multiplicity $> 1$ form a proper subvariety of $\Delta \subset |A|$ by the Bertini theorem. Any divisor in the complement of the hyperplanes and $\Delta$ can be expressed in the form $b_1 + \ldots + b_r$ with the $b_i$ distinct in $U$. □

## 5. Proof of the smooth case

In this section, we prove Theorem 9; take $S = \{p\}$ for some $p \in B$.

Apply Theorem 20 to $\overline{\mathcal{Y}} = \mathcal{D}$, $L = \mathcal{O}_{\mathcal{D}}(\mathcal{D}) = \mathcal{N}_{\mathcal{D}/\mathcal{X}}$, and $M = \mathcal{O}_B(Np)$ for some suitable $N \gg 0$. We obtain a section $s : B \to \mathcal{D}$ with the following properties:

- $s^*\mathcal{D} = \mathcal{N}_{\mathcal{D}/\mathcal{X}}|s(B) \simeq \mathcal{O}_B(Np)$;
- $\mathcal{E}_{\mathcal{D},\mathcal{O}(\mathcal{D})/B}|s(B)$ is globally generated with no higher cohomology.

Consider the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_B \arrow[r] \arrow[d, equal] & \mathcal{E}_{\mathcal{D},\mathcal{O}(\mathcal{D})/B}|s(B) \arrow[r] \arrow[d] & \mathcal{N}_{s(B)/\mathcal{D}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_B \arrow[r] & \mathcal{E}_{\mathcal{X},\mathcal{O}(\mathcal{D})/B}|s(B) \arrow[r] \arrow[d] & \mathcal{N}_{s(B)/\mathcal{X}} \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{N}_{\mathcal{D}/\mathcal{X}}|s(B) \arrow[r, equal] \arrow[d] & \mathcal{N}_{\mathcal{D}/\mathcal{X}}|s(B) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} .$$

The top two horizontal exact sequences are the Atiyah extensions defining $\mathcal{E}_{\mathcal{D},\mathcal{O}(\mathcal{D})/B}$ and $\mathcal{E}_{\mathcal{X},\mathcal{O}(\mathcal{D})/B}$ restricted to $s(B)$. We identify the relative tangent bundles of $\mathcal{D}$ and $\mathcal{X}$ over $B$ restricted to $s(B)$ with the normal bundles $\mathcal{N}_{s(B)/\mathcal{D}}$ and $\mathcal{N}_{s(B)/\mathcal{X}}$. The right vertical exact sequence is the standard normal bundle sequence for $s(B) \subset \mathcal{D} \subset \mathcal{X}$.

The terms in the bottom row are isomorphic to $\mathcal{O}_B(Np)$, which has no higher cohomology. Since the middle term in the upper row has vanishing higher cohomology, we deduce that $H^1(\mathcal{E}_{\mathcal{X},\mathcal{O}(\mathcal{D})/B}|s(B)) = 0$. Furthermore, the middle vertical exact sequence yields an exact sequence on global sections.

The inclusion $\mathcal{D} \subset \mathcal{X}$ induces an inclusion of morphism spaces

$$\mathrm{Mor}_B((B, M), (\mathcal{D}, \mathcal{O}_D(\mathcal{D}))) \hookrightarrow \mathrm{Mor}_B((B, M), (\mathcal{X}, \mathcal{O}_X(\mathcal{D}))). \tag{5.1}$$

These are smooth by Proposition 12. The image is precisely the indeterminacy of the rational map

$$\begin{aligned} G : \mathrm{Mor}_B((B, M), (\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}))) &\dashrightarrow \mathbb{P}(\Gamma(B, \mathcal{O}_B(Np))) \\ s_t &\longmapsto s_t^{-1}\mathcal{D}. \end{aligned}$$

We may interpret $S$-integral points as sections $s_t$ mapping to elements of $\Gamma(B, \mathcal{O}_B(\mathcal{D}))$ vanishing at $p$ to maximal order $N$. Thus we are interested in elements of $G^{-1}[\Gamma(B, \mathcal{O}_B)]$, where $\Gamma(B, \mathcal{O}_B) \subset \Gamma(B, \mathcal{O}_B(Np))$ corresponds to the constant functions, i.e., the image of the map on global sections induced by the inclusion of sheaves $\mathcal{O}_B \hookrightarrow \mathcal{O}_B(Np)$.

The indeterminacy of $G$ is resolved by blowing up the subscheme (5.1). The stalk of its normal bundle at $s(B)$ is canonically isomorphic to $\Gamma(\mathcal{O}_{\mathcal{X}}(\mathcal{D})|s(B))$. In particular, the proper transform of $G^{-1}[\Gamma(B, \mathcal{O}_B)]$ meets the exceptional fiber over $s(B)$ at the point

$$[\Gamma(B, \mathcal{O}_B)] \in \mathbb{P}(\Gamma(\mathcal{O}_{\mathcal{X}}(\mathcal{D})|s(B))) \simeq \mathbb{P}(\Gamma(B, \mathcal{O}_B(Np))).$$

Thus $s(B)$ deforms to $s_t(B) \in G^{-1}[\Gamma(B, \mathcal{O}_B)]$, corresponding to an $S$-integral point.

The sections thus produced are Zariski dense in $\mathcal{X}$. Indeed, our construction produces sections passing through the generic point of $\mathcal{D}$ that deform out of $\mathcal{D}$ to the generic point of $\mathcal{X}$.

*Remark 22.* Here is an alternate approach suggested by one of the referees. Fix a section $r \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}))$ inducing the canonical inclusion $\mathcal{O}_{\mathcal{X}} \hookrightarrow \mathcal{O}_{\mathcal{X}}(\mathcal{D})$. Since $M = \mathcal{O}_B(Np)$, we have a morphism

$$\begin{gathered}\tilde{G} : \mathrm{Mor}_B((B, M), (\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}))) \to \Gamma(B, \mathcal{O}_B(Np)) \\ \left(s_t, \iota : s_t^*\mathcal{O}_{\mathcal{X}}(\mathcal{D}) \xrightarrow{\sim} M\right) \mapsto \iota\left(s_t^* r\right)\end{gathered}$$

which yields $G$ on composition by

$$\Gamma(B, \mathcal{O}_B(Np)) \dashrightarrow \mathbb{P}(\Gamma(B, \mathcal{O}_B(Np))).$$

One can show that $\tilde{G}$ is smooth via deformation theory. The preimage of the curve $\Gamma(B, \mathcal{O}_B) \subset \Gamma(B, \mathcal{O}_B(Np))$ gives the desired sections of $\mathcal{X}$.

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