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The binomial sequence spaces which include the spaces ℓ_p and ℓ_∞ and geometric properties

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Abstract

In this work, we introduce the binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ which include the spaces ℓ_p and ℓ_∞, in turn. Moreover, we show that the spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ are *BK*-spaces and prove that these spaces are linearly isomorphic to the spaces ℓ_p and ℓ_∞, respectively. Furthermore, we speak of some inclusion relations and give the Schauder basis of the space $b_p^{r,s}$. Lastly, we determine the α-, β-, and γ-duals of those spaces and give some geometric properties of the space $b_p^{r,s}$.

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1 The basic information and notations

The set of all real (or complex) valued sequences is symbolized by w which becomes a vector space under point-wise addition and scalar multiplication. Any vector subspace of w is called a sequence space. The spaces of all bounded, null, convergent, and absolutely p-summable sequences are denoted by ℓ_∞, c_0, c, and ℓ_p, respectively, where $1 \leq p < \infty$.

A Banach sequence space is called a *BK*-space provided each of the maps $p_n : X \longrightarrow \mathbb{C}$ defined by $p_n = x_n$ is continuous for all $n \in \mathbb{N}$ [1]. By considering the notion of *BK*-space, one can say that the sequence spaces ℓ_∞, c_0, and c are *BK*-spaces according to their usual *sup-norm* defined by $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$ and ℓ_p is a *BK*-space according to its ℓ_p-*norm* defined by

$$\|x\|_{\ell_p} = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{\frac{1}{p}},$$

where $1 \leq p < \infty$.

For an arbitrary infinite matrix $A = (a_{nk})$ of real (or complex) entries and $x = (x_k) \in w$, the A-transform of x is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k \tag{1.1}$$

and is supposed to be convergent for all $n \in \mathbb{N}$ [2]. In terms of the ease of use, we prefer that the summation without limits runs from 0 to ∞.

Given two sequence spaces X and Y, and an infinite matrix $A = (a_{nk})$, the sequence space X_A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.2}$$

which is called the domain of an infinite matrix A. Also, by $(X : Y)$, we denote the class of all matrices such that $X \subset Y_A$. If $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n, k \in \mathbb{N}$, an infinite matrix $A = (a_{nk})$ is called a triangle. Also, a triangle matrix A uniquely has an inverse A^{-1} which is a triangle matrix.

Let the summation matrix $S = (s_{nk})$ be defined as follows:

$$s_{nk} = \begin{cases} 1, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then the spaces of all bounded and convergent series are defined by means of the summation matrix such that $bs = (\ell_\infty)_S$ and $cs = c_S$, respectively.

The theory of matrix transformation was set in motion by the theory of summability which was developed by Cesàro, Norlund, Riesz, *etc.* By taking into account this theory, many authors have constructed new sequence spaces. For example, $(\ell_\infty)_{N_q}$ and c_{N_q} in [3], X_p and X_∞ in [4], a_p^r and a_∞^r in [5]. Furthermore, many authors have used especially the Euler matrix for defining new sequence spaces. These are e_0^r and e_c^r in [6], e_p^r and e_∞^r in [7] and [8], $e_0^r(\Delta)$, $e_c^r(\Delta)$ and $e_\infty^r(\Delta)$ in [9], $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ in [10], $e_0^r(B^{(m)})$, $e_c^r(B^{(m)})$, and $e_\infty^r(B^{(m)})$ in [11], $e_0^r(\Delta, p)$, $e_c^r(\Delta, p)$, and $e_\infty^r(\Delta, p)$ in [12], $e_0^r(u, p)$ and $e_c^r(u, p)$ in [13].

In this work, we introduce the binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ which include the spaces ℓ_p and ℓ_∞, in turn. Moreover, we show that the spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ are BK-spaces and prove that these spaces are linearly isomorphic to the spaces ℓ_p and ℓ_∞, respectively. Furthermore, we speak of some inclusion relations and give the Schauder basis of the space $b_p^{r,s}$. Lastly, we determine the α-, β-, and γ-duals of those spaces and give some geometric properties of the space $b_p^{r,s}$.

2 The binomial sequence spaces which include the spaces ℓ_p and ℓ_∞

In this part, we define the binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ which include the spaces ℓ_p and ℓ_∞, respectively. Furthermore, we show that those spaces are BK-spaces and are linearly isomorphic to the spaces ℓ_p and ℓ_∞. Also, we show that the binomial sequence space $b_p^{r,s}$ is not a Hilbert space except the case $p = 2$, where $1 \leq p < \infty$.

Let r, s be nonzero real numbers with $r + s \neq 0$. Then the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is defined as follows:

$$b_{nk}^{r,s} = \begin{cases} \frac{1}{(s+r)^n}\binom{n}{k} s^{n-k} r^k, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}_0$. For $sr > 0$, one can easily check that the following properties hold for the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$:

(i) $\|B^{r,s}\| < \infty$,

(ii) $\lim_{n\to\infty} b^{r,s}_{nk} = 0$ (each $k \in \mathbb{N}$),
(iii) $\lim_{n\to\infty} \sum_k b^{r,s}_{nk} = 1$.

Thus, the binomial matrix is regular whenever $sr > 0$. Here and in the following, unless stated otherwise, we suppose that $sr > 0$.

By taking into account the binomial matrix $B^{r,s} = (b^{r,s}_{nk})$, the binomial sequence spaces $b^{r,s}_p$ and $b^{r,s}_\infty$ are defined by

$$b^{r,s}_p = \left\{ x = (x_k) \in w : \sum_n \left| \frac{1}{(s+r)^n} \sum_{k=0}^n \binom{n}{k} s^{n-k} r^k x_k \right|^p < \infty \right\}, \quad 1 \le p < \infty,$$

and

$$b^{r,s}_\infty = \left\{ x = (x_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(s+r)^n} \sum_{k=0}^n \binom{n}{k} s^{n-k} r^k x_k \right| < \infty \right\}.$$

By considering the notation of (1.2), the binomial sequence spaces $b^{r,s}_p$ and $b^{r,s}_\infty$ can be redefined by the matrix domain of $B^{r,s} = (b^{r,s}_{nk})$ as follows:

$$b^{r,s}_p = (\ell_p)_{B^{r,s}} \quad \text{and} \quad b^{r,s}_\infty = (\ell_\infty)_{B^{r,s}}. \tag{2.1}$$

Let us define a sequence $y = (y_k)$ as follows:

$$\left(B^{r,s}x\right)_k = y_k = \frac{1}{(s+r)^k} \sum_{j=0}^k \binom{k}{j} s^{k-j} r^j x_j \tag{2.2}$$

for all $k \in \mathbb{N}$. This sequence will be frequently used as the $B^{r,s}$-transform of x.

We would like to touch on a point, if we take $s + r = 1$, we obtain the Euler matrix $E^r = (e^r_{nk})$. So, the binomial matrix $B^{r,s} = (b^{r,s}_{nk})$ generalizes the Euler matrix.

Now, we want to continue with the following theorem which is needed in the next.

Theorem 2.1 *The binomial sequence spaces $b^{r,s}_p$ and $b^{r,s}_\infty$ are BK-spaces according to their norms defined by*

$$\|x\|_{b^{r,s}_p} = \left\|B^{r,s}x\right\|_{\ell_p} = \left(\sum_{n=1}^\infty \left|\left(B^{r,s}x\right)_n\right|^p \right)^{\frac{1}{p}}$$

and

$$\|x\|_{b^{r,s}_\infty} = \left\|B^{r,s}x\right\|_\infty = \sup_{n\in\mathbb{N}} \left|\left(B^{r,s}x\right)_n\right|,$$

where $1 \le p < \infty$.

Proof We know that the sequence spaces ℓ_p and ℓ_∞ are *BK*-spaces with their ℓ_p-norm and *sup-norm*, respectively, where $1 \le p < \infty$. Furthermore, (2.1) holds and the binomial matrix $B^{r,s} = (b^{r,s}_{nk})$ is a triangle matrix. By taking into account these three facts and Theorem 4.3.12 of Wilansky [2], we conclude that the binomial sequence spaces $b^{r,s}_p$ and $b^{r,s}_\infty$ are *BK*-spaces, where $1 \le p < \infty$. This completes the proof of the theorem. □

Theorem 2.2 *The binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$ are linearly isomorphic to the sequence spaces ℓ_p and ℓ_∞, in turn, where $1 \leq p < \infty$.*

Proof To refrain from the usage of similar statements, we prove the theorem for only the sequence space $b_p^{r,s}$, where $1 \leq p < \infty$. For the proof of the theorem, we need to show the existence of a linear bijection between the spaces $b_p^{r,s}$ and ℓ_p. Let L be a transformation such that $L : b_p^{r,s} \longrightarrow \ell_p$, $L(x) = B^{r,s}x$. By the definition of the binomial sequence space $b_p^{r,s}$, we conclude that, for all $x \in b_p^{r,s}$, $L(x) = B^{r,s}x \in \ell_p$. Furthermore, it is obvious that L is a linear transformation and $x = 0$ whenever $L(x) = 0$. Therefore, L is injective.

For given $y = (y_k) \in \ell_p$, let us define a sequence $x = (x_k)$ such that

$$x_k = \frac{1}{r^k} \sum_{j=0}^{k} \binom{k}{j} (-s)^{k-j} (s+r)^j y_j$$

for all $k \in \mathbb{N}$. Then we get

$$\begin{aligned}
\|x\|_{b_p^{r,s}} &= \|B^{r,s}x\|_{\ell_p} \\
&= \left(\sum_{n=1}^{\infty} \left| \left(B^{r,s}x\right)_n \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{n=1}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} r^k x_k \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{n=1}^{\infty} \left| \frac{1}{(s+r)^n} \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \sum_{j=0}^{k} \binom{k}{j} (-s)^{k-j} (s+r)^j y_j \right|^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{n=1}^{\infty} |y_n|^p \right)^{\frac{1}{p}} \\
&= \|y\|_{\ell_p} \\
&= \left\|L(x)\right\|_{\ell_p} < \infty.
\end{aligned}$$

Hence, we conclude that L is norm preserving and $x \in b_p^{r,s}$, namely L is surjective. As a consequence, L is a linear bijection. This means that the spaces $b_p^{r,s}$ and ℓ_p are linearly isomorphic, that is, $b_p^{r,s} \cong \ell_p$, where $1 \leq p < \infty$. This completes the proof of the theorem. □

Theorem 2.3 *The binomial sequence space $b_p^{r,s}$ is not a Hilbert space except the case $p = 2$, where $1 \leq p < \infty$.*

Proof Let $p = 2$. Remembering Theorem 2.1, one can say that $b_2^{r,s}$ is a BK-space according to its ℓ_2-norm defined by

$$\|x\|_{b_2^{r,s}} = \|B^{r,s}x\|_{\ell_2} = \left(\sum_{n=1}^{\infty} \left| \left(B^{r,s}x\right)_n \right|^2 \right)^{\frac{1}{2}}.$$

Moreover, this norm can be generated by an inner product such that

$$\|x\|_{b_2^{r,s}} = \langle B^{r,s}x, B^{r,s}x\rangle^{\frac{1}{2}}.$$

Therefore, $b_2^{r,s}$ is a Hilbert space.

Now, we assume that $1 \leq p < \infty$ and $p \neq 2$. We define two sequences $y = (y_k)$ and $z = (z_k)$ as follows:

$$y_k = \frac{-s + k(r+s)}{r}\left(-\frac{s}{r}\right)^{k-1} \quad \text{and} \quad z_k = -\frac{s + k(r+s)}{r}\left(-\frac{s}{r}\right)^{k-1}$$

for all $k \in \mathbb{N}$. Then we obtain

$$\|y + z\|^2_{b_p^{r,s}} + \|y - z\|^2_{b_p^{r,s}} = 8 \neq 2^{\frac{2}{p}+2} = 2\left(\|y\|^2_{b_p^{r,s}} + \|z\|^2_{b_p^{r,s}}\right).$$

Thus, the norm of the binomial sequence space $b_p^{r,s}$ does not satisfy the parallelogram equality. As a consequence, the norm cannot be generated by an inner product, that is, the binomial sequence space $b_p^{r,s}$ is not a Hilbert space whenever $p \neq 2$. This completes the proof of the theorem. □

3 The inclusion relations and Schauder basis

In this part, we speak of some inclusion relations and give the Schauder basis for the binomial sequence space $b_p^{r,s}$, where $1 \leq p < \infty$.

Theorem 3.1 *The inclusions $e_p^r \subset b_p^{r,s}$ and $e_\infty^r \subset b_\infty^{r,s}$ strictly hold, where e_p^r and e_∞^r are the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞, respectively.*

Proof If $r + s = 1$, one can easily see that $E^r = B^{r,s}$. Therefore, the inclusion $e_\infty^r \subset b_\infty^{r,s}$ holds. Suppose that $0 < r < 1$ and $s = 5$. Let us now consider a sequence $x = (x_k)$ such that $x_k = (-\frac{4}{r})^k$ for all $k \in \mathbb{N}$. Then it is clear that $x = (x_k) = ((-\frac{4}{r})^k) \notin \ell_\infty$, $E^r x = ((-3 - r)^k) \notin \ell_\infty$ and $B^{r,s}x = ((\frac{1}{5+r})^k) \in \ell_\infty$. As a result of this, $x = (x_k) \in b_\infty^{r,s} \setminus e_\infty^r$. This shows that the inclusion $e_\infty^r \subset b_\infty^{r,s}$ is strictly. We can prove the other part of the theorem by using a similar technique. This completes the proof of the theorem. □

Theorem 3.2 *The inclusion $\ell_p \subset b_p^{r,s}$ is strict, where $1 \leq p < \infty$.*

Proof First we assume that $1 < p < \infty$. From the definition of the space ℓ_p, we write

$$\sum_k |x_k|^p < \infty$$

for all $x = (x_k) \in \ell_p$. For given an arbitrary sequence $x = (x_k) \in \ell_p$, by taking into account the equality (2.2) and the Hölder inequality, we obtain

$$\begin{aligned} \left|\left(B^{r,s}x\right)_k\right|^p &= \left|\frac{1}{(s+r)^k}\sum_{j=0}^{k}\binom{k}{j}s^{k-j}r^j x_j\right|^p \\ &\leq \left(\frac{1}{|s+r|^k}\right)^p \left[\left(\sum_{j=0}^{k}\binom{k}{j}|s|^{k-j}|r|^j\right)^{p-1} \times \left(\sum_{j=0}^{k}\binom{k}{j}|s|^{k-j}|r|^j|x_j|^p\right)\right] \end{aligned}$$

$$= \frac{1}{|s+r|^k} \sum_{j=0}^{k} \binom{k}{j} |s|^{k-j} |r|^j |x_j|^p$$
$$= \sum_{j=0}^{k} \binom{k}{j} \left| \frac{s}{s+r} \right|^k \left| \frac{r}{s} \right|^j |x_j|^p,$$

where $1 \leq p < \infty$. And

$$\sum_k \left| \left(B^{r,s} x \right)_k \right|^p \leq \sum_k \sum_{j=0}^{k} \binom{k}{j} \left| \frac{s}{s+r} \right|^k \left| \frac{r}{s} \right|^j |x_j|^p$$
$$= \sum_j |x_j|^p \sum_{k=j}^{\infty} \binom{k}{j} \left| \frac{s}{s+r} \right|^k \left| \frac{r}{s} \right|^j$$
$$= \left| \frac{s+r}{s} \right| \sum_j |x_j|^p.$$

If we consider the comparison test, we conclude that $B^{r,s}x \in \ell_p$, namely $x \in b_p^{r,s}$. As a consequence $\ell_p \subset b_p^{r,s}$, where $1 < p < \infty$.

Now, we keep in view the sequence $v = (v_k)$ defined by $v_k = (-1)^k$ for all $k \in \mathbb{N}$. Then it is clear that $v = (v_k) \notin \ell_p$ and $B^{r,s}v = ((\frac{s-r}{s+r})^k) \in \ell_p$, namely $v = (v_k) \in b_p^{r,s}$. Because of $v = (v_k) \in b_p^{r,s} \setminus \ell_p$, the inclusion $\ell_p \subset b_p^{r,s}$ is strict. In case of $p = 1$, the theorem can be proved by using a similar method. This completes the proof of the theorem. □

Theorem 3.3 *The spaces $b_p^{r,s}$ and ℓ_∞ overlap but these spaces do not include each other, where $1 \leq p < \infty$.*

Proof It is obvious that $v = ((-1)^k) \in \ell_\infty$ and $v = ((-1)^k) \in b_p^{r,s}$. So, the spaces $b_p^{r,s}$ and ℓ_∞ overlap, where $1 \leq p < \infty$. Here, we consider the sequences $e = (1, 1, 1, \dots)$ and $u = (u_k)$ defined by $u_k = (-\frac{s}{r})^k$ for all $k \in \mathbb{N}$, where $|\frac{s}{r}| > 1$. Then we conclude that $e \in \ell_\infty$ but $B^{r,s}e = e \notin \ell_p$, that is, $e \notin b_p^{r,s}$ and $u \notin \ell_\infty$ but $B^{r,s}u = (1, 0, 0, \dots) \in \ell_p$, namely $u \in b_p^{r,s}$. As a consequence, $e \in \ell_\infty \setminus b_p^{r,s}$ and $u \in b_p^{r,s} \setminus \ell_\infty$. On account of this, $b_p^{r,s}$ and ℓ_∞ do not include each other, where $1 \leq p < \infty$. This completes the proof of the theorem. □

Theorem 3.4 *The inclusions $\ell_\infty \subset b_\infty^{r,s}$ and $b_p^{r,s} \subset b_\infty^{r,s}$ are strict, where $1 \leq p < \infty$.*

Proof The inequality

$$\|x\|_{b_\infty^{r,s}} = \sup_{k \in \mathbb{N}} \left| \frac{1}{(s+r)^k} \sum_{j=0}^{k} \binom{k}{j} s^{k-j} r^j x_j \right| \leq \|x\|_\infty$$

holds for all $x \in \ell_\infty$. In this way, the inclusion $\ell_\infty \subset b_\infty^{r,s}$ holds. Now, we consider the sequence $v = (v_k)$ defined by $v_k = (-\frac{s+r}{r})^k$ for all $k \in \mathbb{N}$. Then we conclude that $v = (v_k) \notin \ell_\infty$ but $B^{r,s}v = ((-\frac{r}{r+s})^k) \in \ell_\infty$, namely $v = (v_k) \in b_\infty^{r,s}$. Therefore, the inclusion $\ell_\infty \subset b_\infty^{r,s}$ strictly holds.

For given $x = (x_k) \in b_p^{r,s}$, where $1 \leq p < \infty$, by taking into account Theorem 2.2 and the inclusion $\ell_p \subset \ell_\infty$, we conclude that $B^{r,s}x \in \ell_\infty$, namely $x \in b_\infty^{r,s}$. Thus, the inclusion $b_p^{r,s} \subset$

$b_\infty^{r,s}$ holds. Also, it is clear that $e \in b_\infty^{r,s} \setminus b_p^{r,s}$. Hence, the inclusion $b_p^{r,s} \subset b_\infty^{r,s}$ is strict. This completes the proof of the theorem. □

Now, let us continue with the definition of the Schauder basis of a normed space. Let $(X, \|\cdot\|_X)$ be a normed sequence space and $d = (d_k)$ be a sequence in X. If for every $x \in X$, there exists a unique sequence of scalars $\lambda = (\lambda_k)$ such that

$$\lim_{n\to\infty}\left\| x - \sum_{k=0}^{n} \lambda_k d_k \right\|_X = 0$$

then $d = (d_k)$ is called a Schauder basis for X [1].

Theorem 3.5 *Let $\mu_k = \{B^{r,s}x\}_k$ be given for all $k \in \mathbb{N}$. We define the sequence $g^{(k)}(r,s) = \{g_n^{(k)}(r,s)\}_{n\in\mathbb{N}}$ of the elements of the binomial sequence space $b_p^{r,s}$ as follows:*

$$g_n^{(k)}(r,s) = \begin{cases} 0, & 0 \le n < k, \\ \frac{1}{r^n}\binom{n}{k}(-s)^{n-k}(s+r)^k, & n \ge k \end{cases}$$

for all fixed $k \in \mathbb{N}$. Then the sequence $\{g^{(k)}(r,s)\}_{k\in\mathbb{N}}$ is a Schauder basis for the binomial sequence space $b_p^{r,s}$, and every $x \in b_p^{r,s}$ has a unique representation of the form

$$x = \sum_k \mu_k g^{(k)}(r,s),$$

where $1 \le p < \infty$.

Proof Let $x = (x_k) \in b_p^{r,s}$ be given, where $1 \le p < \infty$. For all non-negative integer m, we define

$$x^{[m]} = \sum_{k=0}^{m} \mu_k g^{(k)}(r,s).$$

Then, if we apply the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ to $x^{[m]}$, we write

$$B^{r,s}x^{[m]} = \sum_{k=0}^{m} \mu_k B^{r,s} g^{(k)}(r,s) = \sum_{k=0}^{m} \left(B^{r,s}x\right)_k e^{(k)}$$

and

$$\left\{B^{r,s}\left(x - x^{[m]}\right)\right\}_n = \begin{cases} 0, & 0 \le n \le m, \\ (B^{r,s}x)_n, & n > m \end{cases}$$

for all $m, n \in \mathbb{N}$.

For any given $\epsilon > 0$, there exists a non-negative integer m_0 such that

$$\sum_{n=m_0+1}^{\infty} \left|\left(B^{r,s}x\right)_n\right|^p \le \left(\frac{\epsilon}{2}\right)^p$$

for all $m \geq m_0$. Thus,

$$\begin{aligned}\|x - x^{[m]}\|_{b_p^{r,s}} &= \left(\sum_{n=m+1}^{\infty} \left|(B^{r,s}x)_n\right|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{n=m_0+1}^{\infty} \left|(B^{r,s}x)_n\right|^p\right)^{\frac{1}{p}} \\ &\leq \frac{\epsilon}{2} < \epsilon\end{aligned}$$

for all $m \geq m_0$. This shows us that

$$x = \sum_k \mu_k g^{(k)}(r,s).$$

Lastly, we should show the uniqueness of this representation. For this purpose, assume that

$$x = \sum_k \lambda_k g^{(k)}(r,s).$$

Since the linear transformation L defined from $b_p^{r,s}$ to ℓ_p in the proof of Theorem 2.2 is continuous, we have

$$(B^{r,s}x)_n = \sum_k \lambda_k \{B^{r,s}g^{(k)}(r,s)\}_n = \sum_k \lambda_k e_n^{(k)} = \lambda_n$$

for every $n \in \mathbb{N}$, which contradicts the fact that $(B^{r,s}x)_n = \mu_n$ for every $n \in \mathbb{N}$. Therefore, every $x \in b_p^{r,s}$ has a unique representation. This completes the proof of the theorem. □

From Theorem 2.1, we know that $b_p^{r,s}$ is a Banach space, where $1 \leq p < \infty$. If we consider this fact and Theorem 3.5, we can give the next corollary.

Corollary 3.6 *The binomial sequence space $b_p^{r,s}$ is separable, where $1 \leq p < \infty$.*

4 The α-, β-, and γ-duals

In this part, we determine the α-, β-, and γ-duals of the binomial sequence spaces $b_p^{r,s}$ and $b_\infty^{r,s}$, where $1 \leq p < \infty$.

Now, we start with a definition. The multiplier space of the sequence spaces X and Y is denoted by $M(X,Y)$ and defined by

$$M(X,Y) = \{y = (y_k) \in w : xy = (x_k y_k) \in Y \text{ for all } x = (x_k) \in X\}.$$

By taking into account the definition of a multiplier space, the α-, β -,and γ-duals of a sequence space X are defined by

$$X^\alpha = M(X,\ell_1), \qquad X^\beta = M(X,cs) \quad \text{and} \quad X^\gamma = M(X,bs),$$

respectively.

For use in the next lemma, we now give some properties:

$$\sup_{n\in\mathbb{N}}\sum_k |a_{nk}|^q < \infty, \tag{4.1}$$

$$\sup_{n,k\in\mathbb{N}} |a_{nk}| < \infty, \tag{4.2}$$

$$\lim_{n\to\infty} a_{nk} = a_k \quad \text{for each } k\in\mathbb{N}, \tag{4.3}$$

$$\sup_{K\in\mathcal{F}}\sum_k \left|\sum_{n\in K} a_{nk}\right|^q < \infty, \tag{4.4}$$

$$\lim_{n\to\infty}\sum_k |a_{nk}| = \sum_k \left|\lim_{n\to\infty} a_{nk}\right|, \tag{4.5}$$

$$\sup_{k\in\mathbb{N}}\sum_n |a_{nk}| < \infty, \tag{4.6}$$

where $\mathcal{F}$ is the collection of all finite subsets of $\mathbb{N}$, $\frac{1}{p}+\frac{1}{q}=1$ and $1<p\le\infty$.

Lemma 4.1 (see [14]) *Let $A=(a_{nk})$ be an infinite matrix, then the following hold*:

(i) $A=(a_{nk})\in(\ell_1:\ell_1)\Leftrightarrow$ (4.6) *holds*,
(ii) $A=(a_{nk})\in(\ell_1:c)\Leftrightarrow$ (4.2) *and* (4.3) *hold*,
(iii) $A=(a_{nk})\in(\ell_1:\ell_\infty)\Leftrightarrow$ (4.2) *holds*,
(iv) $A=(a_{nk})\in(\ell_p:\ell_1)\Leftrightarrow$ (4.4) *holds with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $1<p\le\infty$,
(v) $A=(a_{nk})\in(\ell_p:c)\Leftrightarrow$ (4.1) *and* (4.3) *hold with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $1<p<\infty$,
(vi) $A=(a_{nk})\in(\ell_p:\ell_\infty)\Leftrightarrow$ (4.1) *holds with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $1<p<\infty$,
(vii) $A=(a_{nk})\in(\ell_\infty:c)\Leftrightarrow$ (4.3) *and* (4.5) *hold*,
(viii) $A=(a_{nk})\in(\ell_\infty:\ell_\infty)\Leftrightarrow$ (4.1) *holds with* $q=1$.

Theorem 4.2 *Let $v_1^{r,s}$ and $v_2^{r,s}$ be defined as follows*:

$$v_1^{r,s}=\left\{a=(a_k)\in w:\sup_{K\in\mathcal{F}}\sum_k\left|\sum_{n\in K}\binom{n}{k}(-s)^{n-k}r^{-n}(r+s)^k a_n\right|^q<\infty\right\}$$

and

$$v_2^{r,s}=\left\{a=(a_k)\in w:\sup_{k\in\mathbb{N}}\sum_n\left|\binom{n}{k}(-s)^{n-k}r^{-n}(r+s)^k a_n\right|<\infty\right\}.$$

Then $\{b_1^{r,s}\}^\alpha=v_2^{r,s}$ and $\{b_p^{r,s}\}^\alpha=v_1^{r,s}$, where $1<p\le\infty$.

Proof Let $a=(a_n)\in w$ be given. Remembering the sequence $x=(x_n)$, which is defined in the proof of Theorem 2.2, we have

$$a_n x_n=\sum_{k=0}^{n}\binom{n}{k}(-s)^{n-k}r^{-n}(r+s)^k a_n y_k=\left(H^{r,s}y\right)_n$$

for all $n\in\mathbb{N}$. Then, by considering the equality above, we deduce that $ax=(a_n x_n)\in\ell_1$ whenever $x=(x_k)\in b_1^{r,s}$ or $x=(x_k)\in b_p^{r,s}$ if and only if $H^{r,s}y\in\ell_1$ whenever $y=$

$(y_k) \in \ell_1$ or $y = (y_k) \in \ell_p$, respectively, where $1 < p \leq \infty$. This shows us that $a = (a_n) \in \{b_1^{r,s}\}^\alpha$ or $a = (a_n) \in \{b_p^{r,s}\}^\alpha$ if and only if $H^{r,s} \in (\ell_1 : \ell_1)$ or $H^{r,s} \in (\ell_p : \ell_1)$, respectively, where $1 < p \leq \infty$. If we combine these two facts and Lemma 4.1(i) and (iv), we obtain

$$a = (a_n) \in \{b_1^{r,s}\}^\alpha \Leftrightarrow \sup_{k \in \mathbb{N}} \sum_n \left| \binom{n}{k} (-s)^{n-k} r^{-n} (r+s)^k a_n \right| < \infty$$

or

$$a = (a_n) \in \{b_p^{r,s}\}^\alpha \Leftrightarrow \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \binom{n}{k} (-s)^{n-k} r^{-n} (r+s)^k a_n \right|^q < \infty,$$

respectively, where $1 < p \leq \infty$. Therefore, $\{b_1^{r,s}\}^\alpha = v_2^{r,s}$ and $\{b_p^{r,s}\}^\alpha = v_1^{r,s}$, where $1 < p \leq \infty$. This completes the proof of the theorem. □

Theorem 4.3 *Let $v_3^{r,s}$, $v_4^{r,s}$, $v_5^{r,s}$, $v_6^{r,s}$, and $v_7^{r,s}$ be defined as follows*:

$$v_3^{r,s} = \left\{ a = (a_k) \in w : \sum_{j=k}^{\infty} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$v_4^{r,s} = \left\{ a = (a_k) \in w : \sup_{n,k \in \mathbb{N}} \left| \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right| < \infty \right\},$$

$$v_5^{r,s} = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_k \left| \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right| = \sum_k \left| \sum_{j=k}^{\infty} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right| \right\},$$

$$v_6^{r,s} = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right|^q < \infty \right\}, \quad 1 < q < \infty,$$

$$v_7^{r,s} = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \binom{j}{k} (-s)^{j-k} r^{-j} (r+s)^k a_j \right| < \infty \right\}.$$

Then the following equalities hold:

(I) $\{b_1^{r,s}\}^\beta = v_3^{r,s} \cap v_4^{r,s}$,

(II) $\{b_p^{r,s}\}^\beta = v_3^{r,s} \cap v_6^{r,s}$, *where* $1 < p < \infty$,

(III) $\{b_\infty^{r,s}\}^\beta = v_3^{r,s} \cap v_5^{r,s}$,

(IV) $\{b_1^{r,s}\}^\gamma = v_4^{r,s}$,

(V) $\{b_p^{r,s}\}^\gamma = v_6^{r,s}$, *where* $1 < p < \infty$,

(VI) $\{b_\infty^{r,s}\}^\gamma = v_7^{r,s}$.

Proof To avoid the repetition of similar statements, we give the proof of the theorem for only the sequence space $b_p^{r,s}$, where $1 < p < \infty$.

Let $a = (a_k) \in w$ be given. By considering the sequence $x = (x_k)$, which is used in the proof of Theorem 2.2, we obtain

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left[\frac{1}{r^k}\sum_{j=0}^{k}\binom{k}{j}(-s)^{k-j}(r+s)^j y_j\right]a_k \\ &= \sum_{k=0}^{n}\left[\sum_{j=k}^{n}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^k a_j\right]y_k \\ &= \left(G^{r,s}y\right)_n\end{aligned}$$

for all $n \in \mathbb{N}$, where the matrix $G^{r,s} = (g_{nk}^{r,s})$ is defined by

$$g_{nk}^{r,s} = \begin{cases} \sum_{j=k}^{n}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^k a_j, & 0 \le k \le n, \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then:

(II) $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in b_p^{r,s}$ if and only if $G^{r,s}y \in c$ whenever $y = (y_k) \in \ell_p$, where $1 < p < \infty$. This fact shows that $a = (a_k) \in \{b_p^{r,s}\}^\beta$ if and only if $G^{r,s} \in (\ell_p : c)$, where $1 < p < \infty$. By combining this result and Lemma 4.1(v), we deduce that

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{n}\left|\sum_{j=k}^{n}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^k a_j\right|^q < \infty \tag{4.7}$$

and

$$\sum_{j=k}^{\infty}\binom{j}{k}(-s)^{j-k}r^{-j}(r+s)^k a_j \quad \text{exists for each } k \in \mathbb{N},$$

where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. As a result of this, we obtain $\{b_p^{r,s}\}^\beta = v_3^{r,s} \cap v_6^{r,s}$, where $1 < p < \infty$.

(V) By following a similar way, $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in b_p^{r,s}$ if and only if $G^{r,s}y \in \ell_\infty$ whenever $y = (y_k) \in \ell_p$, where $1 < p < \infty$. This says us that $a = (a_k) \in \{b_p^{r,s}\}^\gamma$ if and only if $G^{r,s} \in (\ell_p : \ell_\infty)$, where $1 < p < \infty$. By using this result and Lemma 4.1(vi), we conclude that (4.7) holds, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. As a consequence of this, we obtain $\{b_p^{r,s}\}^\gamma = v_6^{r,s}$, where $1 < p < \infty$. This completes the proof of the theorem. □

5 Geometric properties of the binomial sequence space $b_p^{r,s}$

In this part, we give some geometric properties of the binomial sequence space $b_p^{r,s}$. Let us start with some notions.

Let $(X, \|\cdot\|_X)$ be a Banach space. Then X is said to have the Banach-Saks property, if every bounded sequence $u = (u_n)$ contains a subsequence $v = (v_n)$ such that the Cesàro means $\frac{1}{n+1}\sum_{k=0}^{n} v_k$ are norm convergent [15].

X is said to have the weak Banach-Saks property, if every weakly null sequence $u = (u_n)$ contains a subsequence $v = (v_n)$ such that the Cesàro means $\frac{1}{n+1}\sum_{k=0}^{n} v_k$ are norm convergent [15].

X is said to have Banach-Saks type p, if every weakly null sequence $u = (u_n)$ has a subsequence $v = (v_n)$ such that, for some $M > 0$,

$$\left\| \sum_{k=0}^{n} v_k \right\|_X \leq M(n+1)^{\frac{1}{p}}$$

for all $n \in \mathbb{N}$, where $1 < p < \infty$ [16].

Let C be a weakly compact convex subset of X. Then X is said to have the weak fixed point property, if every self mapping $T : C \longrightarrow C$ that provides $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ has a fixed point [17].

Let X be a normed linear space and $S(X)$ be a unit sphere of X. Then the Gurarii modulus of convexity is defined as follows:

$$\beta_X(\epsilon) = \inf\left\{1 - \inf_{0 \leq \lambda \leq 1} \left\| \lambda x + (1-\lambda)y \right\| : x, y \in S(X), \|x - y\| = \epsilon\right\},$$

where $0 \leq \epsilon \leq 2$ [18].

Theorem 5.1 (see [19]) *A Banach space X has the weak fixed point property, if X provides the condition*

$$R(X) = \sup\left\{\liminf_{n\to\infty} \|x_n + x\|\right\} < 2,$$

where the supremum is taken over all weakly null sequences (x_n) of the unit ball and all points x of the unit ball.

Theorem 5.2 *The binomial sequence space $b_p^{r,s}$ is of the Banach-Saks type p.*

Proof Let (u_n) be a weakly null sequence in the $B(b_p^{r,s})$ unit ball of $b_p^{r,s}$. We suppose that (ϵ_n) is a sequence of positive numbers provided $\sum \epsilon_n \leq \frac{1}{2}$. Construct $v_0 = u_0 = 0$ and $v_1 = u_{n_1} = u_1$. Then we can find an $m_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i=m_1+1}^{\infty} v_1(i)e^{(i)} \right\|_{b_p^{r,s}} < \epsilon_1.$$

By virtue of $u_n \overset{w}{\longrightarrow} 0$ implying $u_n \longrightarrow 0$ coordinatewise, we can find an $n_2 \in \mathbb{N}$ such that

$$\left\| \sum_{i=0}^{m_1} u_n(i)e^{(i)} \right\|_{b_p^{r,s}} < \epsilon_1,$$

as $n \geq n_2$. Construct $v_2 = u_{n_2}$. Then we can find an $m_2 > m_1$ such that

$$\left\| \sum_{i=m_2+1}^{\infty} v_2(i)e^{(i)} \right\|_{b_p^{r,s}} < \epsilon_2.$$

If we use $x_n \longrightarrow 0$ coordinatewise one more time, we can find an $n_3 > n_2$ such that

$$\left\| \sum_{i=0}^{m_2} u_n(i)e^{(i)} \right\|_{b_p^{r,s}} < \epsilon_2,$$

as $n \geq n_3$.

By continuing this method, we can constitute two increasing sequences (m_k) and (n_k) such that

$$\left\| \sum_{i=0}^{m_k} u_n(i) e^{(i)} \right\|_{b_p^{r,s}} < \epsilon_k$$

for all $n \geq n_{k+1}$ and

$$\left\| \sum_{i=m_k+1}^{\infty} v_2(i) e^{(i)} \right\|_{b_p^{r,s}} < \epsilon_k,$$

where $v_k = u_{n_k}$. Thus

$$\begin{aligned} \left\| \sum_{k=0}^{n} v_k \right\|_{b_p^{r,s}} &= \left\| \sum_{k=0}^{n} \left(\sum_{i=0}^{m_{k-1}} v_k(i) e^{(i)} + \sum_{i=m_{k-1}+1}^{m_k} v_k(i) e^{(i)} + \sum_{i=m_k+1}^{\infty} v_k(i) e^{(i)} \right) \right\|_{b_p^{r,s}} \\ &\leq \left\| \sum_{k=0}^{n} \left(\sum_{i=m_{k-1}+1}^{m_k} v_k(i) e^{(i)} \right) \right\|_{b_p^{r,s}} + 2 \sum_{k=0}^{n} \epsilon_k \end{aligned}$$

and

$$\begin{aligned} \left\| \sum_{k=0}^{n} \sum_{i=m_{k-1}+1}^{m_k} v_k(i) e^{(i)} \right\|_{b_p^{r,s}}^p &= \sum_{k=0}^{n} \sum_{i=m_{k-1}+1}^{m_k} \left| \frac{1}{(s+r)^i} \sum_{j=0}^{i} \binom{i}{j} s^{i-j} r^j v_k(j) \right|^p \\ &\leq \sum_{k=0}^{n} \sum_{i=0}^{\infty} \left| \frac{1}{(s+r)^i} \sum_{j=0}^{i} \binom{i}{j} s^{i-j} r^j v_k(j) \right|^p \leq n+1. \end{aligned}$$

Thus we obtain

$$\left\| \sum_{k=0}^{n} v_k \right\|_{b_p^{r,s}} \leq (n+1)^{\frac{1}{p}} + 1 \leq 2(n+1)^{\frac{1}{p}}.$$

As a consequence, the binomial sequence space $b_p^{r,s}$ is of the Banach-Saks type p. This completes the proof of the theorem. □

We know from Theorem 2.2 that $b_p^{r,s}$ is linearly isomorphic to ℓ_p. So, it is clear that $R(b_p^{r,s}) = R(\ell_p) = 2^{\frac{1}{p}}$.

By combining this fact and Theorem 5.1, we can give the next theorem.

Theorem 5.3 *The binomial sequence space $b_p^{r,s}$ has the weak fixed point property, where $1 < p < \infty$.*

Theorem 5.4 *The inequality $\beta_{b_p^{r,s}}(\epsilon) \leq 1 - [1 - (\frac{\epsilon}{2})^p]^{\frac{1}{p}}$ holds, where $0 \leq \epsilon \leq 2$.*

Proof Let $0 \leq \epsilon \leq 2$ be given. By assuming the inverse of the binomial matrix $B^{r,s}$ is D, we construct two sequences u and v as follows:

$$u = \left(\left(D\left(1 - \left(\frac{\epsilon}{2} \right)^p \right) \right)^{\frac{1}{p}}, D\left(\frac{\epsilon}{2} \right), 0, 0, \ldots \right),$$

$$v = \left(\left(D\left(1 - \left(\frac{\epsilon}{2}\right)^p\right)\right)^{\frac{1}{p}}, D\left(-\frac{\epsilon}{2}\right), 0, 0, \dots \right).$$

Then we obtain

$$\left\| B^{r,s}u \right\|_{\ell_p} = \|u\|_{b_p^{r,s}} = 1 \quad \text{and} \quad \left\| B^{r,s}v \right\|_{\ell_p} = \|v\|_{b_p^{r,s}} = 1.$$

This shows that $u, v \in S(b_p^{r,s})$ and $\|B^{r,s}u - B^{r,s}v\|_{\ell_p} = \|u - v\|_{b_p^{r,s}} = \epsilon$.

For $0 \leq \lambda \leq 1$, we have

$$\begin{aligned} \left\| \lambda u + (1-\lambda)v \right\|_{b_p^{r,s}}^p &= \left\| \lambda B^{r,s}u + (1-\lambda)B^{r,s}v \right\|_{\ell_p}^p \\ &= 1 - \left(\frac{\epsilon}{2}\right)^p + |2\lambda - 1|\left(\frac{\epsilon}{2}\right)^p \end{aligned}$$

and

$$\inf_{0\leq\lambda\leq1} \left\| \lambda u + (1-\lambda)v \right\|_{b_p^{r,s}}^p = 1 - \left(\frac{\epsilon}{2}\right)^p. \tag{5.1}$$

Thus, we obtain

$$\beta_{b_p^{r,s}}(\epsilon) \leq 1 - \left[1 - \left(\frac{\epsilon}{2}\right)^p\right]^{\frac{1}{p}}.$$

This completes the proof of the theorem. □

By using the equality (5.1), we find two more results.

Corollary 5.5 *Since $\beta_{b_p^{r,s}}(\epsilon) = 1$, the binomial sequence space $b_p^{r,s}$ is strictly convex.*

Corollary 5.6 *Since $0 < \beta_{b_p^{r,s}}(\epsilon) \leq 1$, for $0 < \epsilon \leq 2$, the binomial sequence space $b_p^{r,s}$ is uniformly convex.*

6 Conclusion

By taking into account the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$, we conclude that $B^{r,s} = (b_{nk}^{r,s})$ reduces in the case $r + s = 1$ to $E^r = (e_{nk}^r)$ which is called the Euler matrix of order r. Therefore, our results obtained from the matrix domain of the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ are more general and more extensive than the results on the matrix domain of the Euler matrix of order r. Furthermore, the binomial matrix $B^{r,s} = (b_{nk}^{r,s})$ is not a special case of the weighed mean matrices. Thus, this paper has filled up a gap in the existent literature.

Competing interests
The author declares that they have no competing interests.

Author's contributions
The author read and approved the final manuscript.

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