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Research Article

Multiplicity of Nontrivial Solutions for Kirchhoff Type Problems

Bitao Cheng,[1] Xian Wu,[2] and Jun Liu[1]

[1] *College of Mathematics and Information Science, Qujing Normal University, Qujing, Yunnan 655011, China*
[2] *Department of Mathematics, Yunnan Normal University, Kunming, Yunnan 650092, China*

Correspondence should be addressed to Xian Wu, wuxian2001@yahoo.com.cn

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By using variational methods, we study the multiplicity of solutions for Kirchhoff type problems $-(a+b\int_\Omega |\nabla u|^2)\Delta u = f(x,u)$, in Ω; $u=0$, on $\partial\Omega$. Existence results of two nontrivial solutions and infinite many solutions are obtained.

1. Introduction

Consider the following Kirchhoff type problems

$$\begin{gathered} -\left(a+b\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u), \quad \text{in } \Omega, \\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in R^N $(N=1,2,\ \text{or}\ 3)$, $a,b>0$, and $f:\overline{\Omega}\times R^1 \mapsto R^1$ is a Carathéodory function that satisfies the subcritical growth condition

$$\left|f(x,t)\right| \le C\left(1+|t|^{p-1}\right) \quad \text{for some } 2<p<2^* = \begin{cases} \dfrac{2N}{N-2}, & N\ge 3, \\ +\infty, & N=1,2, \end{cases} \tag{1.2}$$

where C is a positive constant.

It is pointed out in [1] that the problem (1.1) model several physical and biological systems, where u describes a process which depends on the average of itself (e.g., population density). Moreover, this problem is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2\right)\Delta u = g(x,t), \tag{1.3}$$

proposed by Kirchhoff [2] as an extension of the classical D' Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Some early studies of Kirchhoff equations were Bernstein [3] and Poho$\hat{z}$aev [4]. However, (1.3) received much attention only after Lions [5] proposed an abstract framework to the problem. Some interesting results can be found, for example, in [6–13]. Specially, more recently, Alves et al. [14], Ma and Rivera [10], and He and Zou [9] studied the existence of positive solutions and infinitely many positive solutions of the problems by variational methods, respectively; Perera and Zhang [12] obtained one nontrivial solutions of (1.1) by Yang index theory; Zhang and Perera [13] and Mao and Zhang [11] got three nontrivial solutions (a positive solution, a negative solution, and a sign-changing solution) by invariant sets of descent flow.

In the present paper, we are interested in finding multiple nontrivial solutions of the problem (1.1). We will use a three-critical-point theorem due to Brezis and Nirenberg [15] and a Z_2 version of the Mountain Pass Theorem due to Rabinowitz [16] to study the existence of multiple nontrivial solutions of problem (1.1). Our results are different from the above theses.

2. Preliminaries

Let $X := H_0^1(\Omega)$ be the Sobolev space equipped with the inner product and the norm

$$(u,v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \qquad \|u\| = (u,u)^{1/2}. \tag{2.1}$$

Throughout the paper, we denote by $|\cdot|_r$ the usual L^r-norm. Since Ω is a bounded domain, it is well known that $X \hookrightarrow L^r(\Omega)$ continuously for $r \in [1, 2^*]$, compactly for $r \in [1, 2^*)$. Hence, for $r \in [1, 2^*]$, there exists γ_r such that

$$|u|_r \leq \gamma_r \|u\|, \quad \forall u \in X. \tag{2.2}$$

Recall that a function $u \in X$ is called a weak solution of (1.1) if

$$\left(a + b\|u\|^2\right)\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f(x,u) v \, dx, \quad \forall v \in X. \tag{2.3}$$

Seeking a weak solution of problem (1.1) is equivalent to finding a critical point of C^1 functional

$$\Phi(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \Psi(u), \tag{2.4}$$

where

$$\begin{aligned} \Psi(u) &:= \int_\Omega F(x,u)dx, \quad \forall u \in X, \\ F(x,t) &:= \int_0^t f(x,s)ds, \quad \forall (x,t) \in \overline{\Omega} \times R^1. \end{aligned} \tag{2.5}$$

Moreover,

$$\langle \Phi'(u), v \rangle = \left(a + b\|u\|^2\right) \int_\Omega \nabla u \nabla v - \int_\Omega f(x,u)v, \quad \forall u, v \in X. \tag{2.6}$$

Our assumptions lead us to consider the eigenvalue problems

$$\begin{aligned} -\Delta u &= \lambda u, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.7}$$

$$\begin{aligned} -\|u\|^2 \Delta u &= \mu u^3, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{2.8}$$

Denote by $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \cdots$ the distinct eigenvalues of the problem (2.7) and by $V_1, V_2, \ldots, V_k, \ldots$ the eigenspaces corresponding to these eigenvalues. It is well known that λ_1 can be characterized as

$$\lambda_1 = \inf\left\{ \|u\|^2 : u \in X,\ |u|_2 = 1 \right\}, \tag{2.9}$$

and λ_1 is achieved by $\varphi_1 > 0$.

μ is an eigenvalue of problem (2.8) means that there is a nonzero $u \in X$ such that

$$\|u\|^2 \int_\Omega \nabla u \nabla v \, dx = \mu \int_\Omega u^3 v \, dx, \quad \forall v \in X. \tag{2.10}$$

This u is called an eigenvector corresponding to eigenvalue μ. Set

$$I(u) = \|u\|^4, \quad u \in S := \left\{ u \in X : \int_\Omega u^4 = 1 \right\}. \tag{2.11}$$

Denote by $0 < \mu_1 < \mu_2 < \cdots$ all distinct eigenvalues of the problem (2.8). Then,

$$\mu_1 := \inf_{u \in S} I(u), \tag{2.12}$$

$\mu_1 > 0$ is simple and isolated, and μ_1 can be achieved at some $\varphi_1 \in S$ and $\varphi_1 > 0$ in Ω (see [12, 13]).

We need the following concept, which can be found in [17].

Definition 2.1. Let X be a Banach space and $\Phi \in C^1(X, R^1)$. We say that Φ satisfies the (PS) condition at the level $c \in R^1$ ($(PS)_c$ condition for short) if any sequence $\{u_n\} \subset X$ along with $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence. If Φ satisfies $(PS)_c$ condition for each $c \in R^1$, then we say that Φ satisfies the (PS) condition.

In this paper, the following theorems are our main tools, which are Theorem 4 in [15] and Theorem 9.12 in [16], respectively.

Theorem 2.2. *Let X be a real Banach space with a direct sum decomposition $X = X_1 \oplus X_2$, where $k = \dim X_2 < \infty$. Let $F \in C^1(X, R^1)$ and satisfy (PS) condition. Assume that there is $r > 0$ such that*

$$\begin{aligned} F(u) &\geq 0, \quad \text{for } u \in X_1, \ \|u\| \leq r, \\ F(u) &\leq 0, \quad \text{for } u \in X_2, \ \|u\| \leq r. \end{aligned} \tag{2.13}$$

Assume also that F is bounded below and

$$\inf_{u \in X} F(u) < 0. \tag{2.14}$$

Then F has at least two nonzero critical points.

Theorem 2.3. *Let X be an infinite dimensional real Banach space, and let $F \in C^1(X, R^1)$ be even and satisfy the PS condition and $F(0) = 0$. Let $X = X_1 \oplus X_2$, where X_2 is finite dimensional, and F satisfies that*

(i) *there exist constants $\rho, \alpha > 0$ such that $F|_{\partial B_\rho \cap X_1} \geq \alpha$, where*

$$\partial B_\rho = \{u \in X : \|u\| = \rho\}, \tag{2.15}$$

(ii) *for each finite dimensional subspace $E_1 \subset X$, the set $\{u \in E_1 : F(u) > 0\}$ is bounded.*

Then, F possesses an unbounded sequence of critical values.

3. Main Results

We need the following assumptions.

(f_1) $f(x, t)$ is odd in t for all $x \in \Omega$.

(f_2) There exist $\delta > 0$, $\epsilon > 0$ and $\lambda \in (\lambda_k, \lambda_{k+1})$, $k \in \mathbf{N}$, such that

$$a(\lambda_k + \epsilon)|t|^2 \le 2F(x,t) \le a\lambda |t|^2, \quad \forall x \in \Omega,\ |t| \le \delta, \tag{3.1}$$

where λ_k and λ_{k+1} are two consecutive eigenvalues of the problem (2.7).

(f_3) There exist $\delta > 0$ and $\lambda \in [\lambda_k, \lambda_{k+1})$, $k \in \mathbf{N}$ such that

$$2F(x,t) \le a\lambda |t|^2, \quad \forall x \in \Omega,\ |t| \le \delta, \tag{3.2}$$

where λ_k and λ_{k+1} are two consecutive eigenvalues of the problem (2.7).

(f_4)

$$\limsup_{|t|\to\infty} \frac{F(x,t) - (b/4)\mu_1 |t|^4}{|t|^\tau} < \alpha, \quad \text{uniformly in } x \in \Omega, \tag{3.3}$$

where $\tau \in [0,2]$ and $0 < 2\alpha < a\lambda_1$.

(f_5) $\exists \nu > 4$ such that $\nu F(x,t) \le t f(x,t)$, $|t|$ large.

Now, we are ready to state our main results.

Theorem 3.1. *If conditions (f_2) and (f_4) hold, then the problem* (1.1) *has at least two nontrivial solutions in X.*

Proof. Set

$$X_1 = \overline{\bigoplus_{i=k+1}^{\infty} V_i}, \qquad X_2 = \bigoplus_{i=1}^{k} V_i. \tag{3.4}$$

Then, X has a direct sum decomposition $X = X_1 \oplus X_2$ with $\dim X_2 < \infty$. Let M_r be such that

$$|u|_r \ge M_r \|u\|, \quad \forall u \in X_2. \tag{3.5}$$

Step 1. Φ is weakly lower semicontinuous.

Indeed, we only to show $\Psi : X \to R$ is weakly upper semicontinuous. Let $\{u_n\} \subset X$, $u \in X$, $u_n \rightharpoonup u$ in X. Then, we may assume that

$$u_n \longrightarrow u \quad \text{in } L^r(\Omega),\ r \in [1, 2^*). \tag{3.6}$$

We need to prove

$$\Psi(u) \ge \limsup_{n\to\infty} \Psi(u_n) = \inf_{k\in\mathbf{N}} \sup_{n\ge k} \Psi(u_n). \tag{3.7}$$

If this is false, then

$$\Psi(u) < \limsup_{n\to\infty} \Psi(u_n) = \inf_{k\in\mathbf{N}} \sup_{n\geq k} \Psi(u_n), \tag{3.8}$$

and hence there exist $\varepsilon_0 > 0$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\begin{aligned}
\varepsilon_0 &< \Psi(u_n) - \Psi(u) \\
&= \int_\Omega [F(x,u_n) - F(x,u)]\,dx \\
&= \int_\Omega \int_0^1 f(x, u + s(u_n - u))(u_n - u)\,ds\,dx \\
&\leq \int_\Omega \int_0^1 C\Big(|u + s(u_n - u)|^{p-1} + 1\Big)|u_n - u|\,ds\,dx \\
&\leq \int_\Omega C\Big[2^{p-1}\Big(|u|^{p-1} + |u_n - u|^{p-1}\Big) + 1\Big]|u_n - u|\,dx \\
&\leq \int_\Omega C2^{p-1}|u|^{p-1}|u_n - u|\,dx + \int_\Omega C2^{p-1}|u_n - u|^p\,dx + \int_\Omega C|u_n - u|\,dx \\
&\longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.9}$$

This is a contradiction. Hence, Ψ is weakly upper semicontinuous, and hence Φ is weakly lower semicontinuous.

Step 2. There exists $r > 0$, such that

$$\begin{aligned}
&\Phi(u) \geq 0, \quad \text{for } u \in X_1,\ \|u\| \leq r, \\
&\Phi(u) \leq 0, \quad \text{for } u \in X_2,\ \|u\| \leq r.
\end{aligned} \tag{3.10}$$

Particularly,

$$\Phi(u) < 0, \quad \text{for } u \in X_2,\ 0 < \|u\| \leq r. \tag{3.11}$$

Indeed, by (1.2) and (f_2), there exist two positive constants C_1, C_2 such that

$$F(x,t) \leq \frac{a}{2}\lambda|t|^2 + C_1|t|^p, \tag{3.12}$$

$$F(x,t) \geq \frac{a}{2}(\lambda_k + \epsilon)|t|^2 - C_2|t|^p. \tag{3.13}$$

Thus, for $u \in X_1$, the combination of (2.2) and (3.12) implies that

$$\begin{aligned}\Phi(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a}{2}\lambda\int_\Omega u^2 dx - C_1\int_\Omega |u|^p dx \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a}{2}\frac{\lambda}{\lambda_{k+1}}\|u\|^2 - C_1\gamma_p\|u\|^p \\ &= \frac{a}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_1\gamma_p\|u\|^p.\end{aligned} \tag{3.14}$$

Then, there exists $r_1 > 0$ such that

$$\Phi(u) \geq 0, \quad \text{for } u \in X_1,\ \|u\| \leq r_1, \tag{3.15}$$

due to $p > 2$ and $\lambda < \lambda_{k+1}$. Moreover, for $u \in X_2$, the combination of (2.2) and (3.13) implies that

$$\begin{aligned}\Phi(u) &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a}{2}(\lambda_k + \epsilon)\int_\Omega u^2 dx + C_2\int_\Omega |u|^p dx \\ &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a}{2}\left(\frac{\lambda_k + \epsilon}{\lambda_k}\right)\|u\|^2 + C_3\|u\|^p \\ &= -\frac{a}{2}\left(\frac{\lambda_k + \epsilon}{\lambda_k} - 1\right)\|u\|^2 + \frac{b}{4}\|u\|^4 + C_3\|u\|^p,\end{aligned} \tag{3.16}$$

where $C_3 = C_2\gamma_p$. Hence, there exists $r_2 > 0$ such that

$$\begin{gathered}\Phi(u) \leq 0, \quad \text{for } u \in X_2,\ \|u\| \leq r_2, \\ \Phi(u) < 0, \quad \text{for } u \in X_2,\ 0 < \|u\| \leq r_2.\end{gathered} \tag{3.17}$$

Lastly, the conclusion follows from choosing $r = \min\{r_1, r_2\}$.

Step 3. Φ is coercive on X, that is, $\Phi(u) \to +\infty$ as $n \to \infty$, and Φ is bounded from below. In fact, set

$$p(x,t) := F(x,t) - \frac{b}{4}\mu_1|t|^4. \tag{3.18}$$

Then,

$$\Phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{b}{4}\mu_1\int_\Omega u^4 dx - \int_\Omega p(x,u)dx, \quad \forall u \in X. \tag{3.19}$$

Condition (f_4) implies that

$$\limsup_{|t|\to\infty} \frac{p(x,t)}{|t|^\tau} < \alpha, \quad \text{uniformly in } x \in \Omega, \tag{3.20}$$

where $\tau \in [0,2]$ and $0 < 2\alpha < a\lambda_1$. By contradiction, if Φ is not coercive on X, then there exist a sequence $\{u_n\} \subset X$ and some constant $C_4 \in R^1$ such that

$$\|u_n\| \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \text{ but } \Phi(u_n) \le C_4. \tag{3.21}$$

By virtue of (3.20), there exist some constant $M > 1$ such that

$$-p(x,t) > -\alpha|t|^\tau, \quad \forall x \in \Omega, \ |t| > M. \tag{3.22}$$

Set $\Omega_n^1 = \{x \in \Omega : |u_n(x)| > M\}$ and $\Omega_n^2 = \{x \in \Omega : |u_n(x)| \le M\}$. Then, the combination of (3.19)–(3.22) and (1.2) implies that there exists $A = A(M) > 0$ such that

$$\begin{aligned}
C_4 \ge \Phi(u_n) &= \frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - \frac{b}{4}\mu_1 \int_\Omega u_n^4 dx - \int_\Omega p(x,u_n)dx \\
&= \frac{a}{2}\|u_n\|^2 + \frac{b}{4}\left(\|u_n\|^4 - \mu_1 \int_\Omega u_n^4 dx\right) + \int_{\Omega_n^1} -p(x,u_n)dx + \int_{\Omega_n^2} -p(x,u_n)dx \\
&\ge \frac{a}{2}\|u_n\|^2 - \int_{\Omega_n^1} \alpha|u_n(x)|^\tau dx - A \\
&\ge \frac{a}{2}\|u_n\|^2 - \int_{\Omega_n^1} \alpha|u_n(x)|^2 dx - A \\
&\ge \frac{a}{2}\|u_n\|^2 - \int_{\Omega} \alpha|u_n(x)|^2 dx - A \\
&\ge \left(\frac{a}{2} - \frac{\alpha}{\lambda_1}\right)\|u_n\|^2 - A \longrightarrow +\infty, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.23}$$

This is a contradiction. Therefore, Φ is coercive on X and so Φ is bounded from blew due to Φ is weakly lower semicontinuous.

Step 4. Φ satisfies (PS) condition; that is, any (PS) sequence has a convergent subsequence.

Indeed, let $\{u_n\} \subset X$ be a (PS) sequence of Φ. By the coerciveness of Φ we know that $\{u_n\}$ is bounded in X. By the reflexivity of X, we can assume that there exists $u \in X$ such that

$$u_n \rightharpoonup u \quad \text{in X}, \qquad u_n \longrightarrow u \quad \text{in } L^p(\Omega), \qquad u_n(x) \longrightarrow u(x) \quad \text{for a.e. } x \in \Omega. \tag{3.24}$$

Hence, by (1.2), we know that there is $C_5 > 0$ such that

$$\begin{aligned}\int_\Omega f(x,u_n)(u-u_n)dx &\le \left(\int_\Omega \left|f(x,u_n)\right|^{p/(p-1)}dx\right)^{(p-1)/p}\left(\int_\Omega |u-u_n|^p dx\right)^{1/p}\\ &\le 2C\left[\int_\Omega \left(|u_n|^p+1\right)dx\right]^{(p-1)/p}\cdot |u-u_n|_p \\ &\le C_5|u-u_n|_p \longrightarrow 0, \quad \text{as } n\longrightarrow\infty.\end{aligned} \tag{3.25}$$

Moreover, since

$$\begin{aligned}&\left(a+b\|u_n\|^2\right)\int_\Omega \nabla u_n \nabla(u-u_n) - \int_\Omega f(x,u_n)(u-u_n)dx\\ &\quad= \left\langle \Phi'(u_n),(u-u_n)\right\rangle \longrightarrow 0, \quad \text{as } n\longrightarrow\infty,\end{aligned} \tag{3.26}$$

then

$$\|u_n\| \longrightarrow \|u\|, \quad \text{as } n\longrightarrow\infty. \tag{3.27}$$

Hence, $u_n \to u$ in X due to the uniform convexity of X.

Now, the conclusion follows from Theorem 2.2. □

Corollary 3.2. *If conditions* (f_2) *and*

(f_4')

$$\lim_{|t|\to\infty}\left(F(x,t)-\frac{b}{4}\mu_1|t|^4\right) = -\infty, \quad \textit{uniformly in } x\in\Omega \tag{3.28}$$

hold, then the problem (1.1) *has at least two nontrivial solutions in* X.

Proof. Note that the condition (f_4') implies (f_4). Hence, the conclusion follows from Theorem 3.1. □

Remark 3.3. Perera and Zhang [12] only obtained one nontrivial solution of Kirchhoff type problem (1.1) by Yang index under the conditions

$$\lim_{t\to 0}\frac{f(x,t)}{at} = \lambda, \quad \lim_{|t|\to+\infty}\frac{f(x,t)}{bt^3} = \mu, \quad \text{uniformly in } x, \tag{3.29}$$

where $\lambda\in(\lambda_k,\lambda_{k+1})$ and $\mu\in(\mu_m,\mu_{m+1})$ is not an eigenvalue of (2.8), $k\neq m$. We point out the condition

$$\lim_{t\to 0}\frac{f(x,t)}{at} = \lambda, \quad \text{uniformly in } x \tag{3.30}$$

implies the condition (f_2), and as $m = 0$, that is, $\mu < \mu_1$, the condition

$$\lim_{|t| \to +\infty} \frac{f(x,t)}{bt^3} = \mu, \quad \text{uniformly in } x \tag{3.31}$$

implies the condition (f_4). Moreover, we allow $\mu \equiv \mu_1$ is an eigenvalue of (2.8). When $m \geq 1$, The following example shows that there are functions which satisfy (f_2) and (f_4) and do not satisfy the condition

(f_6) $\mu \in (\mu_m, \mu_{m+1})$ is not an eigenvalue of (2.8).

Example 3.4. Set

$$f(x,t) = \begin{cases} -s\tau|t|^{\tau-1} - br|t|^3 + s\tau + br - a\xi, & t < -1, \\ a\xi t, & |t| \leq 1, \\ s\tau|t|^{\tau-1} + br|t|^3 - s\tau - br + a\xi, & t > 1, \end{cases} \tag{3.32}$$

where $s < \alpha$, $\lambda_k < \xi < \lambda_{k+1}$, $\tau \in (1,2]$ and $r \leq \mu_1$. It is easy to verify $f(x,t)$ satisfies conditions (f_2) and (f_4), but

$$\lim_{|t| \to +\infty} \frac{f(x,t)}{bt^3} = r \leq \mu_1, \quad \text{uniformly in } x. \tag{3.33}$$

Certainly, our Theorem 3.1 cannot contain Theorem 1.1 in [12] completely.

Remark 3.5. Zhang and Perera [13] obtained a existence theorem (Theorem 1.1(ii)) of three solutions (a positive solution, a negative solution, and a sign-changing solution) for (1.1) under the conditions

$$\lim_{|t| \to +\infty} \frac{f(x,t)}{bt^3} = \mu < \mu_1, \quad \mu \neq 0, \tag{C_1}$$

$$\exists \lambda > \lambda_2 : F(x,t) \geq \frac{a\lambda}{2} t^2, \quad |t| \text{ small}. \tag{C_2}$$

But, our condition (f_4) is weaker than the condition (C_1) and the left hand of our condition (f_2) is weaker than the condition (C_2). Moreover, we allow $\mu \equiv \mu_1$ is an eigenvalue of (2.8). The above Example 3.4 with $k = 1$ (i.e, $\lambda_1 < \xi < \lambda_2$) shows that there are functions which satisfy all conditions of Theorem 3.1 and do not satisfy Theorem 1.1(ii) in [13]. Hence, Theorem 1.1(ii) in [13] cannot contain our Theorem 3.1.

Theorem 3.6. *Let conditions (f_1), (f_3), and (f_5) hold, then the problem (1.1) has infinite many solutions in X.*

Proof. Set

$$X_1 = \overline{\bigoplus_{i=k+1}^{\infty} V_i}, \qquad X_2 = \bigoplus_{i=1}^{k} V_i. \tag{3.34}$$

Then, X has a direct sum decomposition $X = X_1 \oplus X_2$ with $\dim X_2 < \infty$.

Step 1. There exist constants $\rho > 0$ and $\alpha > 0$ such that $\Phi|_{\partial B\rho \cap X_1} \geq \alpha$, where $B_\rho = \{u \in X : \|u\| = \rho\}$.

Indeed, for $u \in X_1$, by (1.2) and (f_3), we know (3.12) holds. Hence, by (2.2), we have

$$\begin{aligned}
\Phi(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a}{2}\lambda \int_\Omega u^2 dx - C_1 \int_\Omega |u|^p dx \\
&\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{a}{2}\frac{\lambda}{\lambda_{k+1}}\|u\|^2 - C_1\gamma_p\|u\|^p \\
&= \frac{a}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_1\gamma_p\|u\|^p.
\end{aligned} \tag{3.35}$$

Hence, we can choose small $\rho > 0$ such that

$$\Phi(u) \geq \frac{a}{4}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\rho^2 := \alpha > 0, \tag{3.36}$$

whenever $u \in X_1$ with $\|u\| = \rho$.

Step 2. For each finite dimensional subspace $E_1 \subset X$, the set $\{x \in E_1 : \Phi(x) \geq 0\}$ is bounded.

Indeed, by (1.2) and (f_5), we know that there exist constants $C_5, C_6 > 0$ such that

$$F(x,t) \geq C_5|t|^\nu - C_6. \tag{3.37}$$

Hence, for every $u \in E_1 \setminus \{0\}$, one has

$$\Phi(u) \leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - C_5 \int_\Omega |u|^\nu dx + C_6|\Omega|. \tag{3.38}$$

Since E_1 is finite dimensional, we can choosing $R = R(E_1) > 0$ such that

$$\Phi(u) < 0, \quad \forall u \in E_1 \setminus B_R. \tag{3.39}$$

Moreover, by Lemma 2.2(iii) in [13], we know that Φ satisfies PS condition, and Φ is even due to (f_1). Hence, the conclusion follows from Theorem 9.12 in [16]. □

Remark 3.7. Zhang and Perera [13] obtained an existence theorem of three solutions for (1.1) under the condition (f_5) and the condition

$$F(x,t) \leq \frac{a\lambda_1}{2}t^2, \quad |t| \text{ small}, \tag{3.40}$$

which implies our condition (f_3). Our Theorem 3.6 obtains the existence of infinite many solutions of (1.1) in the case adding the condition (f_1).

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References

[1] M. Chipot and B. Lovat, "Some remarks on nonlocal elliptic and parabolic problems," in *Proceedings of the 2nd World Congress of Nonlinear Analysts*, vol. 30, no. 7, pp. 4619–4627, 1997.

[2] G. Kirchhoff, *Mechanik*, Teubner, leipzig, Germany, 1883.

[3] S. Bernstein, "Sur une classe d'équations fonctionnelles aux dérivées partielles," *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, vol. 4, pp. 17–26, 1940.

[4] S. I. Pohožaev, "A certain class of quasilinear hyperbolic equations," *Matematicheskii Sbornik*, vol. 96(138), pp. 152–166, 1975.

[5] J.-L. Lions, "On some questions in boundary value problems of mathematical physics," in *Contemporary Developments in Continuum Mechanics and Partial Differential Equations*, vol. 30 of *North-Holland Mathematics Studies*, pp. 284–346, North-Holland, Amsterdam, The Netherlands, 1978.

[6] A. Arosio and S. Panizzi, "On the well-posedness of the Kirchhoff string," *Transactions of the American Mathematical Society*, vol. 348, no. 1, pp. 305–330, 1996.

[7] M. M. Cavalcanti, V. N. Domingos Cavalcanti, and J. A. Soriano, "Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation," *Advances in Differential Equations*, vol. 6, no. 6, pp. 701–730, 2001.

[8] P. D'Ancona and S. Spagnolo, "Global solvability for the degenerate Kirchhoff equation with real analytic data," *Inventiones Mathematicae*, vol. 108, no. 2, pp. 247–262, 1992.

[9] X. He and W. Zou, "Infinitely many positive solutions for Kirchhoff-type problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 3, pp. 1407–1414, 2009.

[10] T. F. Ma and J. E. Muñoz Rivera, "Positive solutions for a nonlinear nonlocal elliptic transmission problem," *Applied Mathematics Letters*, vol. 16, no. 2, pp. 243–248, 2003.

[11] A. Mao and Z. Zhang, "Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 3, pp. 1275–1287, 2009.

[12] K. Perera and Z. Zhang, "Nontrivial solutions of Kirchhoff-type problems via the Yang index," *Journal of Differential Equations*, vol. 221, no. 1, pp. 246–255, 2006.

[13] Z. Zhang and K. Perera, "Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow," *Journal of Mathematical Analysis and Applications*, vol. 317, no. 2, pp. 456–463, 2006.

[14] C. O. Alves, F. J. S. A. Corrêa, and T. F. Ma, "Positive solutions for a quasilinear elliptic equation of Kirchhoff type," *Computers & Mathematics with Applications*, vol. 49, no. 1, pp. 85–93, 2005.

[15] H. Brezis and L. Nirenberg, "Remarks on finding critical points," *Communications on Pure and Applied Mathematics*, vol. 64, no. 8-9, pp. 939–963, 1991.

[16] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *Cbms Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.

[17] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1989.