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Comments on large-N volume independence

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Abstract: We study aspects of the large-N volume independence on $\mathbb{R}^3 \times L^\Gamma$, where L^Γ is a Γ-site lattice for Yang-Mills theory with adjoint Wilson-fermions. We find the critical number of lattice sites above which the center-symmetry analysis on L^Γ agrees with the one on the continuum S^1. For Wilson parameter set to one and $\Gamma \geq 2$, the two analyses agree. One-loop radiative corrections to Wilson-line masses are finite, reminiscent of the UV-insensitivity of the Higgs mass in deconstruction/Little-Higgs theories. Even for theories with $\Gamma = 1$, volume independence in QCD(adj) may be guaranteed to work by tuning one low-energy effective field theory parameter. Within the parameter space of the theory, at most three operators of the 3d effective field theory exhibit one-loop UV-sensitivity. This opens the analytical prospect to study 4d non-perturbative physics by using lower dimensional field theories (d=3, in our example).

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JHEP01(2010)098

Contents

JHEP01(2010)098

1 Volume independence, regularization dependence, and adjoint fermions

Consider a large-N non-abelian asymptotically free gauge theory compactified toroidally on a four-manifold $M_4{=}\mathbb{R}^d{\times}(S^1)^{4-d}$. It is well-known since the early 1980s [1–3] that if center symmetry, or approximate center symmetry if center is absent, is preserved and if translational invariance is unbroken, to the extent possible in a compactified theory, these theories obey volume independence.[1] In a lattice regularized theory, the reduction to a one-site model is known as Eguchi-Kawai (EK) reduction. Despite the familiarity with the necessary and sufficient conditions which validate such an equivalence, it took a long time to find a working example of a four-dimensional gauge theory obeying volume independence in arbitrarily small volumes [5]. Retrospectively, the physical reason is clear. Until recently, for gauge theories formulated on $M_4{=}\mathbb{R}^3{\times}S^1$, only thermal compactifications were considered (with the notable exception of supersymmetric theories). Then, of course, for the class of theories that we have in mind, small-S^1 corresponds to a deconfined, center-symmetry broken phase, for which volume independence is invalid.

In order to find field theories for which volume independence may work all the way down to arbitrarily small radius, one must therefore abandon thermal compactification and the use of the partition function $Z(\beta){=}\mathrm{tr}(e^{-\beta H})$, where H is the Hamiltonian and β — the inverse temperature. Instead, one needs to consider non-thermal circle compactifications, for which fermions are endowed with translationally invariant periodic boundary conditions and the twisted partition function $Z(L){=}\mathrm{tr}[e^{-LH}(-1)^F]$, where F denotes fermion number and L is the circle size. Since thermal and quantum fluctuations are different in a gauge theory, different choices of boundary conditions for fermions may lead to a different realization of center symmetry at small radius. With this intuition, Kovtun, Yaffe and one of us (M.Ü) showed [5] that a subclass of large-N QCD-like gauge theories satisfies volume independence upon compactification on M_4. Examples of these theories are SU(N) QCD

[1] In the early literature, the necessity of unbroken translation symmetry is realized only in [2]. The possible breaking of translation symmetry is not of only formal interest, see [4]. If one views volume independence in the class of large-N orbifold equivalences [5], the spontaneous breaking of discrete translation symmetries in orbifold theories is ubiquitous.

with $N_f < 5.5$ massless adjoint Weyl fermions, for which a fermion-induced stabilization is at work; in what follows we call this class of theories QCD(adj).[2]

The analysis of stability of center symmetry in ref. [5] is done in a 4d continuum. Let the Wilson line winding around the S^1 be $\Omega(x){=}Pe^{i\int A_4 dx_4}$. The one-loop potential for its expectation value is:

$$V[\Omega] = (N_f - 1)\frac{2}{\pi^2 L^3}\sum_{n=1}^{\infty}\frac{1}{n^4}|\mathrm{tr}\Omega^n|^2\,, \tag{1.1}$$

and leads to a stable center symmetry for $N_f{>}1$. A similar continuum analysis on an $S^3{\times}S^1$ manifold also leads to an unbroken center symmetry in any domain where the one-loop analysis is reliable [6, 7]. In fact, since the first homotopy groups $\pi_1(S^3{\times}S^1){=}\pi_1(\mathbb{R}^3{\times}S^1){=}\mathbb{Z}$ coincide, in both cases the effective action or effective one-loop potential is expressed in terms of the Wilson line around the S^1 circle, and the result on $S^3{\times}S^1$ reproduces the $\mathbb{R}^3{\times}S^1$ result by taking an infinite S^3 radius.

However, the continuum analysis on $\mathbb{R}^3{\times}S^1$ or M_4, although suggestive, does not necessarily imply that volume independence will also hold in a lattice regularized [8] or partially lattice regularized theory [9]. If we consider replacing S^1 with a discrete lattice with Γ-sites:

$$M_{3,\Gamma} = \mathbb{R}^3 \times L^\Gamma \equiv \mathbb{R}^3 \times \{\Gamma - \text{sites lattice}\}, \tag{1.2}$$

then, in the continuum limit

$$\Gamma \to \infty, \qquad a \to 0, \qquad L = \Gamma a = \text{fixed}\,, \tag{1.3}$$

with $L\Lambda_{\rm QCD} \ll 1$ ($\Lambda_{\rm QCD}$ is the strong-coupling scale of the theory), one must reproduce the continuum one-loop result from lattice perturbation theory. Thus, at large Γ, the two analysis *must* agree. Indeed, as we will show explicitly, they do so. Ref. [9] studied the $\Gamma = 1$ theory with the intention of mapping the dynamics of a four-dimensional gauge theory to a three-dimensional effective field theory. This study has potential importance, since certain tools available for lower dimensional gauge theories can be put at work for the 4d gauge theory by using large-N volume independence. However, ref. [9] reached somehow puzzling results, arguing that adjoint fermions do not seem to be sufficient to stabilize the center symmetry of the theory on $M_{3,\Gamma=1}$. Ref. [8] studied the Γ-site model both analytically and numerically by using a lattice-regularized theory and its continuum limit in the non-compact dimensions. It clarified that the $\Gamma{=}1$ model studied in ref. [9] was incomplete in the sense of effective field theory, and for $\Gamma{=}1$ it concluded that the center symmetry was not broken by the order parameter with single-winding Wilson operators (for which ref. [9] reported otherwise). This implies that, at least a $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_N$ must be unbroken. The analysis of ref. [8] includes the chiral point, i.e., massless fermions, but does not address the possibility of breaking of the center to a subgroup larger than $\mathbb{Z}_2$. In this work, one of our goals is to fill this gap. A related numerical simulation for massive SU(3) QCD(adj), using staggered fermions, showed that a lattice system which

[2]Other examples are SO and Sp theories with adjoint representation fermions in the same range and SO (Sp) theories with $2 \leq N_f < 5.5$ symmetric (anti-symmetric) representation fermions.

essentially mimics the $\mathbb{R}^3\times\{\Gamma=2\}$-site model remains center symmetric for sufficiently light fermions [10].

In this work, prompted by refs. [8, 9], we address the following questions:

- How many lattice sites ($\Gamma_{\rm cr}$) along the compact direction are necessary in order for the *center-stability analysis* on $\mathbb{R}^3\times L^\Gamma$ to agree with the one on the continuum $\mathbb{R}^3\times S^1$?
- If $\Gamma_{\rm cr}$ is larger than one, is this detrimental for large-N volume independence for volume reduction down to $\Gamma\leq\Gamma_{\rm cr}$?

We hope that understanding the answers will eventually lead to developing analytical tools of practical utility to using volume independence of QCD(adj). On the numerical side, there is already important progress: Recent work by Bringoltz and Sharpe [11] studied QCD(adj) on a single-site lattice, and showed that center symmetry remains intact in all channels, including the cross-correlation of Wilson loops in different directions.[3] This provides numerical evidence for the equivalence of the single-site matrix model and infinite-volume field theory in the $N{=}\infty$ limit.[4]

We also would like to *a priori* state that the results of our analysis for $\Gamma_{\rm cr}$ are not new and are recently obtained in ref. [8] by using different methods. However, our one-loop potentials (2.13), (3.5), (3.11) on $M_{3,\Gamma}$ are new and are particularly useful to analyze center stability and to check limits, such as the continuum limit, which must agree with [5], and the supersymmetric limit for the $N_f{=}1$ theory, for which it must vanish. Restricted to $\Gamma{=}1$, these potentials coincide with the chiral limit of the expression given in ref. [26].

2 Pure Yang-Mills theory on $M_{3,\Gamma} = \mathbb{R}^3 \times L^\Gamma$

Consider a gauge theory with $[\mathrm{U}(N)]^\Gamma$ product gauge-group structure[5] on $M_{3,\Gamma}{=}\mathbb{R}^3\times L^\Gamma$, with action:

$$S = \sum_{I=1}^{\Gamma} \frac{L}{g^2} \int d^3x \left(\frac{1}{4}\mathrm{tr}|F_{\mu\nu,I}|^2 + \frac{1}{2a^2}\mathrm{tr}|D_\mu U_I|^2 + \ldots \right) \tag{2.1}$$

where $F_{\mu\nu}\equiv||(F_{\mu\nu})^i{}_j||$ is the three-dimensional field strength (one for each of the Γ $\mathrm{U}(N)$ factors). $U_I(x)$ denote the group-valued link fields between sites $(I, I+1)$. The unitary link field $U_I(x)$ is local in $\mathbb{R}^3$, transforms in the $(\Box_I, \overline{\Box}_{I+1})$ representation of $[\mathrm{U}(N)]^\Gamma$, and can be represented as $U^i_{I\,j} \equiv (e^{i\Phi_I})^i_j$, with Φ_I — an arbitrary real $N{\times}N$ matrix. The trace

[3] It is indeed necessary to check such cross-correlations. The validity of volume independence requires that the entire center-symmetry group be unbroken. For example, the Quenched-EK model [3] suffers from an exotic breaking of center symmetry, $(\mathbb{Z}_N)^4{\to}(\mathbb{Z}_N)_{\rm Diag.}$ [12]. Thus, even though expectation values of Wilson loops $\langle \mathrm{tr}U_\mu^{n_\mu}\rangle{=}0$, loops in different directions are locked so that the center breaks down to diagonal, for example, $\langle \mathrm{tr}U_1U_2^\dagger\rangle{\neq}0$, invalidating the equivalence between the infinite volume theory and the matrix model. Such exotic breaking is absent in QCD(adj).

[4] For proposals which implement EK-reduction by using non-compact matrix models, see for example [13–16]. For a study which takes advantage of the confined phases of YM theory, see ref. [17].

[5] The reason we prefer to use $\mathrm{U}(N)$ notation is that it simplifies obtaining the spectrum. Note that the mass spectrum has a natural geometric interpretation in terms lengths of strings stretched between appropriately positioned D-branes. At large N, the $\mathrm{U}(1)$ factors are inessential.

JHEP01(2010)098

in (2.1) is over the indices $i, j = 1 \dots N$. The overall normalization of the coupling in (2.1) will not be essential in our one-loop analysis.

The product-group theory (2.1) may be viewed in several different ways, for example, as a non-linear sigma model in 3d with a product gauge-group structure (a.k.a. "theory space," "moose,"" or "quiver"). It may also be viewed as a latticization of the compact circle in $M_4 = \mathbb{R}^3 \times S^1$. As a three dimensional field theory, (2.1) is non-renormalizable. Thus, the ellipsis are also meaningful in the above formula, and stand for counter-terms which make the Lagrangian complete as an effective field theory (EFT). Indeed, we will see some counter-terms (and hence EFT parameters) that need to be added to the action.

Under a gauge rotation, the link fields transform as bifundamentals, $U_I(x) \rightarrow g_I(x) U_I(x) g^{\dagger}_{I+1}(x)$, $I = 1 \dots \Gamma$, and $\Gamma + 1 \equiv 1$. The covariant derivative in (2.1) is $D_\mu U_I = \partial_\mu U_I + i A_{\mu,I} U_I - i U_I A_{\mu,I+1}$. One can build a non-local lattice Wilson line:

$$\Omega(x) = U_1 U_2 \dots U_\Gamma \, , \tag{2.2}$$

transforming as an adjoint scalar from 3d-point of view: $\Omega(x) \rightarrow g_1(x) \Omega(x) g_1^{\dagger}(x)$. Thus $\text{tr}\Omega^n$ is gauge invariant for any integer n. In what follows, we will calculate the one-loop potential for the gauge invariant Wilson line. Our calculation follows closely the calculation on $M_{4,\Gamma} = \mathbb{R}^4 \times L^\Gamma$ by Georgi and Pais [18] and on $M_{d,\Gamma} = \mathbb{R}^d \times L^\Gamma$ for arbitrary-d by Neuberger [19].

Consider a gauge covariant background holonomy:

$$\Omega = \text{Diag}\left(e^{iv_1}, \dots, e^{iv_N}\right) \, . \tag{2.3}$$

The spectrum of gauge bosons in the background (2.3) can be calculated by studying the "hopping" terms in the Lagrangian (2.1). Although $\text{tr}\Omega^n$ are gauge independent, and we will derive a one-loop potential in terms of these operators, the hopping terms in the compact direction are written in terms of U_I, which are gauge dependent. We choose a gauge in which $\langle U_I \rangle = U = \Omega^{1/\Gamma}$, a democratic distribution of holonomy to each link field:

$$\langle U_I \rangle \equiv U = \text{Diag}\left(e^{iv_1/\Gamma}, \dots, e^{iv_N/\Gamma}\right) \, . \tag{2.4}$$

The evaluation of the spectrum of the gauge bosons in this background follows from expanding the hopping terms in the Lagrangian:

$$L \supset \sum_{I=1}^{\Gamma} \text{tr}|A_{\mu,I} U_I - U_I A_{\mu,I+1}|^2 = \sum_{I=1}^{\Gamma} \text{tr}|A_{\mu,I} - U A_{\mu,I+1} U^\dagger|^2 \, , \tag{2.5}$$

where in the second formula, we used our $\langle U_I \rangle = U$ gauge choice. Fourier transforming the real-space lattice to the momentum Brillouin-zone variable, gives the spectrum of gauge bosons:

$$M^2_{ij}(k) = \left[\frac{2}{a} \sin\left(\frac{2\pi k + v_{ij}}{2\Gamma}\right)\right]^2, \; v_{ij} \equiv v_i - v_j, \; k = 1, \dots \Gamma, \;\; i,j = 1, \dots N \, . \tag{2.6}$$

This result is rather intuitive: Setting $v_{ij} = 0$ gives the spectrum of bosons in lattice gauge theory; $M(k) = |\frac{2}{a} \sin \frac{\pi k}{\Gamma}|$, also familiar from deconstruction [20, 21]. Setting $\Gamma = 1$ gives the distance between the eigenvalues of the Wilson line; $M_{ij} = \frac{1}{a}|e^{iv_i} - e^{iv_j}| = |\frac{2}{a}\sin\frac{v_{ij}}{2}|$, the "$W$-boson" spectrum, familiar from geometric D-brane pictures; see, e.g., [22].

We evaluate the one loop potential by a hard cutoff scheme in $d = 3$; recall that the fourth dimension is already latticized. The result is:

$$\begin{aligned} V[\Omega] &= \sum_{k=1}^{\Gamma}\sum_{i,j=1}^{N} I(v_{ij}) \,, \\ I(v_{ij}) &= \int_{\Lambda} \frac{d^3p}{(2\pi)^3} \ln\left[p^2 + M_{ij}^2(k)\right] \\ &= \frac{\Lambda |M_{ij}(k)|^2}{2\pi^2} - \frac{|M_{ij}(k)|^3}{6\pi} + \Lambda^3 \times \{M\text{−independent}\} \,. \end{aligned} \tag{2.7}$$

Zeta-function or dimensional regularization would set the divergent terms ($\sim\Lambda, \Lambda^3$) to zero, but produce identical results for the cubic term (the linearly divergent term in (2.7) can be recovered as a d=2 pole in the dimensionally regulated expression). Setting the cubic divergence to zero is not a problem as we want to know the holonomy dependence of the one-loop potential. However, setting the linear term to zero is *dangerous*, in the sense we will describe below.

There are some important aspects of the potential (2.7). Although there is a linear divergence for Γ=1, and although each term in the k-sum for $\Gamma\geq2$ has divergent holonomy-dependent contributions, it is straightforward to show that, for $\Gamma \geq 2$, $\sum_{k=1}^{\Gamma} M_{ij}^2(k)$ is a linearly-divergent constant, independent of v_{ij} [21]. This is, of course, the mechanism that the deconstruction and Little-Higgs theories use to solve the (little) hierarchy problem. Unlike supersymmetric theories, where loop corrections to the potential from particles of different spins cancel, here, the UV-sensitivity from particles of the same spin cancels, due to non-locality of the "Higgs-scalar" in theory space L^Γ. As in ref. [21], the one-loop divergent renormalization is to the cosmological constant, and the Higgs mass is UV-insensitive.

Since the potential (2.7) is finite for $\Gamma\geq2$, we rewrite it in a form more suitable to studying center-symmetry realizations:

$$V[\Omega,\Gamma] = \sum_{n=1}^{\infty} V_n(\Gamma,d)\ \ |\mathrm{tr}\Omega^n|^2, \tag{2.8}$$

where $V_n(\Gamma,d)$ is the mass for the Wilson line $\mathrm{tr}\Omega^n$. For convenience and to make the discussion more general, we give the general formula for $V_n(\Gamma,d)$ on $M_{d,\Gamma} = \mathbb{R}^d \times L^\Gamma$:

$$V_n(\Gamma,d) = (-1)\frac{c(d)}{a^d}\begin{cases} \frac{1}{n(n^2\Gamma^2-4)(n^2\Gamma^2-1)} & \mathbb{R}^4\times L^\Gamma \\ \frac{\Gamma}{(n^2\Gamma^2-\frac{9}{4})(n^2\Gamma^2-\frac{1}{4})} & \mathbb{R}^3\times L^\Gamma \\ \frac{1}{n(n^2\Gamma^2-1)} & \mathbb{R}^2\times L^\Gamma \\ \frac{\Gamma}{(n^2\Gamma^2-\frac{1}{4})} & \mathbb{R}^1\times L^\Gamma \end{cases}, \tag{2.9}$$

where $c(d)$ are (unimportant) positive prefactors depending on dimensionality. Eq. (2.9) generalizes the d=4 results of [18] (given on the top line) and can be obtained by Fourier transforming the potential (2.7) with respect to v_{ij} (see, e.g., the Γ=1 calculation after eq. (2.15)). It is clear that the Fourier coefficients $V_n(\Gamma, d)$ have poles at:

$$n\Gamma = \frac{d}{2}, \qquad n\Gamma = \frac{d-2}{2}, \tag{2.10}$$

These poles are an indication that center-symmetry realizations may have UV-sensitivity for low Γ; we note that the same result was also obtained in [8] by a different method. We now consider several d=3 cases, starting from the continuum Γ=∞, a multi-site lattice with Γ>3/2, and the one-site Γ=1 case.

Continuum on $\mathbb{R}^3\times S^1$: the continuum one-loop potential for pure YM theory on $\mathbb{R}^3 \times S^1$ is:

$$V[\Omega] = -\frac{2}{\pi^2 L^3} \sum_{n=1}^{\infty} \frac{1}{n^4} |\mathrm{tr}\Omega^n|^2 \ . \tag{2.11}$$

Note that the "masses" for all Wilson lines are negative,

$$V_n < 0, \qquad 1 \leq n < \infty, \qquad \Longrightarrow \qquad \mathbb{Z}_N \to \mathbb{Z}_1 \,, \tag{2.12}$$

which implies that the $\mathbb{Z}_N$ center symmetry with action $\mathbb{Z}_N$: $\mathrm{tr}\Omega \to e^{\frac{2\pi i}{N}} \mathrm{tr}\Omega$ is spontaneously broken to the trivial group $\mathbb{Z}_1$.

Γ>$\frac{3}{2}$-site models: The one-loop potential for the Γ-site model is given by:

$$\begin{aligned} V[\Omega, \Gamma] &= \sum_{k=1}^{\Gamma} \sum_{i,j=1}^{N} -\frac{1}{6\pi} \left| \frac{2}{a} \sin\left(\frac{2\pi k + v_{ij}}{2\Gamma} \right) \right|^3 \\ &= -\frac{2}{\pi^2 a^3} \sum_{n=1}^{\infty} \frac{\Gamma}{(n^2\Gamma^2 - \frac{9}{4})(n^2\Gamma^2 - \frac{1}{4})} |\mathrm{tr}\Omega^n|^2 \ . \end{aligned} \tag{2.13}$$

That it has the correct continuum limit follows from comparing its infinite-Γ, L=$a\Gamma$-fixed, limit to (2.11). For Γ>$\frac{3}{2}$, the masses of all Wilson lines are negative and this implies the same breaking pattern as in the continuum:

$$V_n\left(\Gamma > \frac{3}{2}\right) < 0, \qquad 1 \leq n < \infty, \qquad \Longrightarrow \qquad \mathbb{Z}_N \to \mathbb{Z}_1 \ . \tag{2.14}$$

This shows that for the pure-YM theory (2.1), the center-symmetry realization has no one-loop UV-sensitivity for Γ>$\frac{3}{2}$. Thus, the non-renormalizable theory defined by (2.1) is one-loop complete, in the sense of EFT.

As already noted, the continuum limit is continuously connected to this multiple-site domain, and the result (2.13) reduces to (2.11) in $\Gamma \to \infty$ limit. Moreover, even for fixed Γ, if one takes a large winding number, $n \gg 1$, in the lattice dispersion relations, only low-lying modes which possess a quasi-continuum dispersion relation, will dominate in their effect. Thus, also in this regime, we expect the continuum behavior to be produced and indeed, it is. The same effect is discussed for massive QCD(adj) in ref. [26].

JHEP01(2010)098

Γ=1-site model: the analytic continuation of eq. (2.13) to the $\Gamma<\frac{3}{2}$ domain is equivalent to the result of the zeta-function and dimensional regularization schemes. This can be also seen more directly by starting with the potential (2.7) in these regularization schemes, given by:

$$V[\Omega,\Gamma=1] \;=-\frac{4}{3\pi a^3}\sum_{i,j=1}^{N}\left|\sin\frac{v_{ij}}{2}\right|^3 \;. \tag{2.15}$$

Now, we rewrite (2.15), using the Fourier expansion:

$$\left|\sin\frac{x}{2}\right|^3=\frac{4}{3\pi}+\frac{3}{2\pi}\sum_{n=1}^{\infty}\frac{1}{(n^2-\frac{9}{4})(n^2-\frac{1}{4})}\cos nx\;, \tag{2.16}$$

and dropping holonomy-independent terms, in a way more useful for analyzing the center symmetry:

$$\begin{aligned} V[\Omega,\Gamma=1] &= -\frac{2}{\pi^2a^3}\sum_{n=1}^{\infty}\frac{1}{(n^2-\frac{9}{4})(n^2-\frac{1}{4})}\sum_{i,j=1}^{N}\cos nv_{ij} \\ &= \sum_{n=1}^{\infty}V_n(\Gamma=1,3)\;\;|\mathrm{tr}\Omega^n|^2. \end{aligned} \tag{2.17}$$

As already mentioned, both these regularization schemes are dangerous as they set terms dependent on the Wilson line to zero. This is an indication that for $\Gamma=1$, the non-renormalizable theory defined by (2.1) is not a complete theory in the sense of EFT, even at the one-loop order of the calculation.

With this warning, let us continue our examination of the symmetry realization of the Γ=1 theory, by using (2.17). Then, we would deduce:

$$V_1(\Gamma=1)>0,\;\;V_{n\geq 2}(\Gamma=1)<0 \qquad\Longrightarrow\qquad \mathbb{Z}_N\to\mathbb{Z}_2\,, \tag{2.18}$$

which implies:

$$\left\langle\frac{1}{N}\mathrm{tr}\Omega^{2n+1}\right\rangle=0,\qquad\left\langle\frac{1}{N}\mathrm{tr}\Omega^{2n}\right\rangle\neq 0. \tag{2.19}$$

This conclusion is, of course, strange. We expect center symmetry to be broken completely, not only partially, in the deconfined phase of pure-YM theory. Clearly, this must be an artifact of Γ=1. But why?

This question is already answered by Bringoltz [8]. Here, we follow his arguments. For the Γ=1 site theory, the operator with n=1 winding number, $|\mathrm{tr}\Omega|^2$ has a one-loop UV-sensitivity — its mass is linearly UV-divergent. On the other hand, for the rest of the operators $|\mathrm{tr}\Omega^n|^2$ which satisfy $\Gamma n>\frac{3}{2}$, one does not expect any one-loop divergences. This means that the Γ=1 theory viewed as an effective field theory (EFT) is one-loop incomplete. One needs to add a counter-term:

$$\delta L_{c.t.}=(c_1\Lambda+b_1)|\mathrm{tr}\Omega|^2\;, \tag{2.20}$$

with coefficient c_1 chosen to absorb the linear divergence and b_1 — a low-energy parameter. By a choice of b_1, we can make the sign of the pre-factor of the $|\mathrm{tr}\Omega|^2$ operator the same as in the continuum or $\Gamma>\frac{3}{2}$ theories, ensuring that:

$$[b_1 + V_1(\Gamma = 1)] < 0, \quad V_n(\Gamma = 1) < 0, \qquad 2 \leq n < \infty, \qquad \Longrightarrow \qquad \mathbb{Z}_N \to \mathbb{Z}_1 \tag{2.21}$$

so that the unbroken $\mathbb{Z}_2$ symmetry reduces to $\mathbb{Z}_1$.

To summarize, the realization of the center symmetry for the $\Gamma=1$ theory is UV sensitive, i.e. depends on the regularization used. At one loop, this is reflected in the fact that whether the center is broken down to $\mathbb{Z}_2$ or $\mathbb{Z}_1$ depends on the sign of $b_1 + V_1$, where b_1 is a free parameter of the one-site theory. The continuum result is only reproduced for some values of b_1, reflecting this UV sensitivity.

3 QCD(adj) with Wilson fermions on $M_{3,\Gamma} = \mathbb{R}^3 \times L^\Gamma$

Continuum $\mathbb{R}^3 \times S^1$: in the continuum, the adjoint fermion endowed with periodic boundary conditions stabilizes the center symmetry. As (1.1) shows, for $N_f>1$ the masses of all Wilson lines are positive and this is the reason that the $\mathbb{Z}_N$ center symmetry is preserved at arbitrarily small S^1:

$$V_n > 0, \qquad 1 \leq n < \infty, \qquad \Longrightarrow \qquad \mathbb{Z}_N \text{ intact} . \tag{3.1}$$

In this section, we would like to study the effect of fermions for the theory on $M_{3,\Gamma}$. Following [8, 9], we use Wilson fermions. The spectrum of the adjoint fermions in the background holonomy (2.3), (2.4), using the same notation as for the bosons, is given by:

$$M_{ij}^2(k) = \left[m + \frac{r}{a} 2 \sin^2\left(\frac{2\pi k + v_{ij}}{2\Gamma}\right)\right]^2 + \frac{1}{a^2}\sin^2\left(\frac{2\pi k + v_{ij}}{\Gamma}\right),$$
$$k = 1, \ldots \Gamma, \quad i, j = 1, \ldots N \tag{3.2}$$

Here, m is the bare fermion mass, r is the Wilson parameter to lift the doublers present when $r=0$ (set, for example, $m=0$, $v_{ij}=0$ for convenience; then, apart from $k=0$, modes near $k=\frac{\Gamma}{2}$ also yield a gapless fermionic excitation). In general, for each discretized dimension, each corner of the Brillouin zone contributes an extra species. This, of course, can be circumvented by the (chiral symmetry violating) Wilson term, which guarantees that these states become infinitely heavy and do not contribute in the continuum limit.[6] The holonomy-dependent part of the one-loop potential is:

$$V[\Omega] = -N_f \sum_{k=1}^{\Gamma} \sum_{i,j=1}^{N} \left(\frac{M_{ij}^2(k)\Lambda}{2\pi^2} - \frac{|M_{ij}(k)|^3}{6\pi}\right). \tag{3.3}$$

There are two limits of this expression which are particularly easy to study, and since we want to demonstrate a matter of principle regarding the UV-sensitivity of the realization of center symmetry, we will restrict our attention to these simple cases (however, these cases capture general aspects of the answer).

[6] The price one pays is having to tune the bare mass to cancel linearly divergent contributions to the fermion mass. However, this issue will not be important for our one-loop consideration.

JHEP01(2010)098

Case I: first, take $m{=}0$ and $r{=}1$. The spectrum reduces to:

$$M_{ij}^2(k) = \left[\frac{2}{a}\sin\left(\frac{2\pi k + v_{ij}}{2\Gamma}\right)\right]^2 . \tag{3.4}$$

This spectrum coincides with the one for bosons, similar to the continuum analysis, and, because of the choice of $r{=}1$, it is doubler-free. This property of one-dimensional Wilson fermions was also observed in the context of deconstruction of supersymmetric theories [22] and is particularly useful in checking the cancellation of loop contributions of gauge boson and one-fermion species.

With this choice of parameters, the one-loop potential for the Γ-site QCD(adj) is:

$$V[\Omega,\Gamma] \;= (N_f-1)\frac{2}{\pi^2 a^3}\sum_{n=1}^{\infty}\frac{\Gamma}{(n^2\Gamma^2-\frac{9}{4})(n^2\Gamma^2-\frac{1}{4})}|\mathrm{tr}\Omega^n|^2 . \tag{3.5}$$

For $\Gamma{>}\frac{3}{2}$, the masses for all Wilson lines are positive and this implies an unbroken center symmetry at $N_f{>}1$, exactly as in the continuum:

$$V_{n\geq 1}\left(\Gamma > \frac{3}{2}\right) > 0 \qquad \Longrightarrow \qquad \mathbb{Z}_N \text{ intact} . \tag{3.6}$$

Thus, with this choice of parameters, there is no UV-sensitivity of the center symmetry realization, and the one-loop potential on $\mathbb{R}^3 \times L^\Gamma$ coincides with the one in continuum $\mathbb{R}^3 \times S^1$.

However, for $\Gamma < \frac{3}{2}$, as it was the case with the pure Yang-Mills theory (flipped in overall sign with respect to (2.18)), we have:

$$V_1(\Gamma = 1) < 0,\;\; V_{n\geq 2}(\Gamma = 1) > 0 \qquad \Longrightarrow \qquad \mathbb{Z}_N \to \mathbb{Z}_1 . \tag{3.7}$$

The reason for the complete breaking $\mathbb{Z}_N \to \mathbb{Z}_1$ is that the V_1 term dominates the sum. A sufficiently negative V_1 is capable of breaking the center completely even if $V_{n\geq 2} > 0$.[7] However, this can be cured by one counter-term and one judicious choice of the EFT parameter, of the form:

$$\delta L_{c.t.} = (c_1\Lambda + b_1)|\mathrm{tr}\Omega|^2 . \tag{3.8}$$

As in the pure gauge theory, see discussion in the end of section 2, the choice of the EFT parameter, which makes the result on $\Gamma = 1$-site model agree with the continuum, is the one for which $b_1{+}V_1(\Gamma = 1){>}0$ is positive. With this choice, we have:

$$[b_1 + V_1(\Gamma = 1)] > 0,\;\; V_{n\geq 2}(\Gamma = 1) > 0 \qquad \Longrightarrow \qquad \mathbb{Z}_N \text{ intact} . \tag{3.9}$$

[7] The converse statement is not true. If $V_{n\geq 2}$ are negative, regardless of the value of $V_1 > 0$, only a $\mathbb{Z}_2$ subgroup of the center can be restored, not $\mathbb{Z}_N$. In such a case, the full center symmetry can only be restored with the tower of the double-trace deformation of ref. [24].

Case II: take m=0 and r=0. Since r=0 and only one dimension is latticized, there is only one doubler and it is not lifted. The spectrum is:

$$M_{ij}^2(k) = \frac{1}{a^2}\sin^2\left(\frac{2\pi k + v_{ij}}{\Gamma}\right), \qquad k = 1, \dots \Gamma, \;\; i,j = 1, \dots N \,. \tag{3.10}$$

The gauge-boson- and fermion- induced one-loop potential is:

$$V[\Omega,\Gamma] = \frac{2}{\pi^2 a^3}\sum_{n=1}^{\infty}\left(\frac{\Gamma N_d N_f}{(n^2\Gamma^2-9)(n^2\Gamma^2-1)} - \frac{\Gamma}{(n^2\Gamma^2-\frac{9}{4})(n^2\Gamma^2-\frac{1}{4})}\right)|\mathrm{tr}\Omega^n|^2, \tag{3.11}$$

where N_d=2, and N_f is the number of the adjoint Majorana fermions. Generalizing this analysis to $\mathbb{R}^d \times L^\Gamma$, we observe that the fermionic contribution contains terms such as:

$$V_n(\Gamma) \supset \frac{1}{(n^2\Gamma^2-d^2)(n^2\Gamma^2-(d-2)^2)}\,, \tag{3.12}$$

for $d > 2$, and has poles at:

$$n\Gamma = d, \qquad n\Gamma = d-2. \tag{3.13}$$

The result for d=4 is also obtained long ago in ref. [18]. For d=3 and Γ=1, the operators with $n \leq 3$ will acquire divergences. These are $|\mathrm{tr}\Omega^n|^2$, $n \leq 3$. Examining the parameter space of the one-loop potential (3.3) in detail, this is the maximal amount of fine-tuning needed. For $\Gamma > 3$, the one-loop potential is UV-insensitive.

Remark: We expect that on an asymmetric lattice, which mimics the gauge theory on $\mathbb{R}^3 \times \{\Gamma = 1\}$-site and the Wilson parameter set to r=1, the center symmetry will not be broken. A way to argue this is the following: The lattice regularization is more similar to hard cut-off than dimensional or zeta-function regularizations. In the continuum, let us fix the UV cut-off and determine the center symmetry realization with it. Then, we obtain, for QCD(adj) with N_f flavors:

$$V[\Omega,\Gamma,\Lambda] = (N_f-1)\left\{\left(\frac{\Lambda}{\pi^2 a^2} - \frac{32}{15\pi^2 a^3}\right)|\mathrm{tr}\Omega|^2 + \frac{2}{\pi^2 a^3}\sum_{n=2}^{\infty}\frac{1}{(n^2-\frac{9}{4})(n^2-\frac{1}{4})}|\mathrm{tr}\Omega^n|^2\right\}. \tag{3.14}$$

For $\Lambda > \frac{32}{15a}$ and N_f>1, the $\mathbb{Z}_N$ center symmetry is stable, and for N_f=0, the center symmetry breaks down to $\mathbb{Z}_1$. Clearly, the realization of center-symmetry depends on the regulator; we expect that the lattice regulator is the one ensuring stability of the center in the 1^4-lattice theory for QCD(adj) found in [11].

4 Prospects and open problems

We have shown how center-stabilization is achieved on $\mathbb{R}^3 \times L^\Gamma$ in QCD(adj). For the particular choice of Wilson parameter r=1, the center symmetry is preserved for any $\Gamma \geq 2$ and N_f>1. For Γ=1, center symmetry is preserved by a choice of a single low energy parameter, reflecting the already mentioned regulator dependence of the center-symmetry realization on the lattice.

JHEP01(2010)098

Thus, for Γ=1 UV sensitivity of the potential for the holonomy sets in already at one loop, while the Γ>1 theory has no one-loop UV sensitivity. This result is in accord with past studies of deconstrucion of five-dimensional field theories [23], which noted that UV-insensitivity of the mass term in a theory with Γ sites holds up to Γ-loop level and that strict all-orders UV-insensitivity is only recovered in the $\Gamma \to \infty$ limit.

Center-symmetry stabilization, although established by a perturbative calculation, has important implications for the full, perturbative and non-perturbative, dynamics of the theory. Consider for example, the Γ=1 model. Inspecting the mass spectrum (2.6) by using a center symmetric holonomy with $v_i = \frac{2\pi}{N} i$, we find $M_{ij} = |\frac{2}{a} \sin \frac{\pi(i-j)}{N}|$, which is equivalent to the spectrum in a lattice gauge theory with an effective box size $L_{\text{eff}} = Na$. This observation is at the heart of volume independence. It demonstrates how the classical zero mode of the KK-tower of states along the compact dimension is capable, *at large-N*, of generating a Brillouin zone of its own. The "zero-mode" matrix mode, quantum-mechanically and to all orders in perturbation theory, generates states with energies $0, \frac{2\pi}{Na}, \frac{2(2\pi)}{Na}, \ldots \frac{\pi}{a}$. Although the high energy modes with energies of order $\frac{\pi}{a} \gg \Lambda_{\text{QCD}}$ are weakly coupled, the low energy modes with $\frac{2\pi}{aN} \ll \Lambda_{\text{QCD}}$ will necessarily be strongly coupled. This is how a lower dimensional theory on $\mathbb{R}^3 \times \{\Gamma=1\}$-site model can reproduce the non-perturbative dynamics of one-higher dimensional theory.

This discussion easily generalizes to a Γ-site model, and on a center symmetric background, the effective size of the space reads

$$L_{\text{eff}} = \Gamma a N = LN. \tag{4.1}$$

This combination makes it manifest that spatial size $L = \Gamma a$ and number of colors N are on the same footing, and taking either to infinity is equivalent to *decompactification*. So long as $\Gamma a N \Lambda \gg 1$, the center symmetric theory will produce the four-dimensional gauge dynamics. The importance of the $L_{\text{eff}} = LN$ scale was realized in [24] in determining the semi-classical domain and soon after in refs. [7, 25] in one-loop potentials of massive QCD(adj).

Unlike deconstruction, where 4d dynamics can be produced only in the large Γ limit, in our formalism, even with $\Gamma = 1$, the 4d dynamics can be produced in the large-N limit. However, in our case, the fields of our lower dimensional theory are independent of the coordinate of the compact space, and it is a genuinely lower dimensional (3d, in our example) theory, which is equivalent to the 4d theory. This possibility strictly relies on a quantum mechanical condition, i.e, unbroken center-symmetry, as opposed to being a classical effect, as in deconstruction. The fact that a 4d-theory is non-perturbatively equivalent to a 3d (or even lower dimensional) theory makes us optimistic for the analytical utility of our formalism.

Apart from aiming to show that center-stabilization and large-N volume independence work both in the continuum [5] and even for the 1^4-lattice theory [11] (which is the original Eguchi-Kawai dream), attempts to make analytical use of volume independence may further our understanding of gauge theories. Some topics of interest are:

- *Conformal field theories:* Consider a compactified CFT. How does volume independence avoid the only length scale in the problem? What are the implications of volume independence in the AdS/CFT context?

- *Perturbation theory:* How can results independent of compactification radius arise within perturbation theory?

- *Large-N matrix models and quantum mechanics:* The combination of refs. [5, 11] tells us that volume independence works both in continuum and lattice, and the reduced theories are equivalent to the corresponding gauge theory on $\mathbb{R}^4$. Can one build new analytical methods to study large-N matrix quantum mechanics, and perhaps, even solve it?

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Note added. A recent preprint by Bringoltz studies massive and massless QCD(adj) on $\mathbb{R}^3 \times \{\Gamma{=}1\}$ by using lattice regularization, and by taking the continuum limit in the non-compact directions [26]. The results of our $\Gamma{=}1$ limit are in agreement with his.

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JHEP01(2010)098