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AdS/CFT and Landau Fermi liquids

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ABSTRACT: We study the field theory dual to a charged gravitational background in which the low temperature entropy scales linearly with the temperature. We exhibit the existence of a sound mode which is described by hydrodynamics, even at energies much larger than the temperature, and explain how this, and other properties of the field theory, are consistent with those of a (3+1)-dimensional Landau Fermi liquid, finely tuned to the Pomeranchuk critical point. We also discuss how one could engineer a higher-derivative gravitational Lagrangian which reproduces the correct low temperature behavior of shear viscosity in a generic Landau Fermi liquid.

Contents

1 Introduction and summary

Fermi liquid theory is one of the very few low energy effective theories at finite density that we understand well. Since the AdS/CFT correspondence provides a simple description of strongly coupled field theories at finite density, it is natural to ask whether a Fermi liquid description can be recovered at low energies in a holographic model. The possibility of observing a Fermi surface at leading order in the "$1/N$ expansion" was first raised in [1]. More recently, the logarithmic violation of the entanglement entropy was suggested as signifying the appearance of a Fermi surface [2, 3]. In [4], a certain singular behavior of the current-current correlator was suggested as an indicator of a Fermi surface. These models imply that the effective theory at low energy differs from the conventional Landau Fermi liquid theory, which is characterized (among other things) by a linear heat capacity at small temperatures and by the validity of Luttinger's theorem.[1]

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[1]Constructing Fermi liquids by the explicit inclusion of charged fermions in the bulk is another interesting direction [5–8].

On the other hand, the discovery of a gapless mode in holographic models at finite density and zero temperature [9] raised the possibility that the Landau Fermi liquid is not too far off: it contains a similar excitation called zero sound (subsequent work on various aspects of holographic zero sound includes [10–26]). It was pointed out in [10] that the equality between the speed of holographic zero sound and the speed of hydrodynamic sound is consistent with the Landau parameters, which control the interaction strength in a Fermi liquid, being parametrically large. In this paper we consider a particular background (reviewed in section 2) which exhibits a linear heat capacity at small temperatures and features a holographic zero sound mode. In section 3 we show that linearized hydrodynamics is completely sufficient to describe the holographic zero sound excitation to second order in the derivative expansion: it should therefore be identified with the usual hydrodynamic sound mode. The situation here is similar to [27], where a different background was considered.

In section 4 we consider Landau Fermi liquid theory and show that the observations in the holographic model can be explained by considering a regime of parametrically small Fermi velocity and taking the second Landau parameter to the stability bound $F_2 \to -5$ corresponding to the Pomeranchuk critical point. In particular, the quasiparticle lifetime $\tau \sim (F_2 + 5)$ becomes parametrically small, and as a result, linearized hydrodynamics is valid for energies that are much larger than the temperature, as long as they are small compared to the chemical potential. This parametrically small lifetime means that we are pushing the Landau Fermi liquid into a regime where it behaves like a non-Fermi liquid. This choice of parameters also resolves an apparent paradox: the viscosity/entropy density ratio for any model that involves Einstein-Hilbert gravity is $\eta/s = 1/4\pi$ [28–31]. At first sight, the viscosity/entropy ratio of a generic Fermi liquid diverges like $\eta/s \sim \mu^3/T^3$ for large ratios of chemical potential μ and temperature T. However, as we explain in section 4, the coefficient in front of the leading term in η/s vanishes for $F_2 = -5$, and so a Fermi liquid description may be compatible once we tune the Landau parameters accordingly.

This raises the question of whether we can have a holographic dual of a generic Landau Fermi liquid, where hydrodynamics breaks down at high energies and collisionless behavior takes over? One clear signature of any such model is that $\eta/s \sim \mu^3/T^3$. In section 5, we describe a procedure by which one could construct a model with precisely this behavior, by adding higher derivative terms to the gravitational Lagrangian. We discuss our results in section 6. The appendices contain some technical details related to sections 3 and 5.

2   The two-charge black hole

Here we briefly review the dilatonic black hole in AdS_5 recently explored in [32–35]. This black hole is usually referred to as the two-charge black hole, because it arises as a solution to the truncation of IIB supergravity on $AdS_5 \times S^5$ where two of the three U(1) charges are equal and non-vanishing while the third one is zero [36].

After this truncation one obtains the action [32, 35]

$$I_0 = \frac{1}{16\pi G} \int d^5x \sqrt{g} \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{8}{L^2}e^{\phi/\sqrt{6}} - \frac{4}{L^2}e^{-2\phi/\sqrt{6}} + 2e^{2\phi/\sqrt{6}}F_{ab}F^{ab} \right), \quad (2.1)$$

with ϕ a scalar field and F_{ab} the field strength of the Maxwell field A_b. The 2-charge black hole solution is [32, 35]

$$ds^2 = e^{2a(r)} \left(h(r)dt^2 - dx^2 - dy^2 - dz^2 \right) - \frac{e^{2b(r)}}{h(r)}dr^2 \,, \tag{2.2}$$

where

$$a(r) = \log \left(\frac{r}{L} \left(1 + \frac{Q^2}{r^2} \right)^{\frac{1}{3}} \right), \quad b(r) = -\log \left(\frac{r}{L} \left(1 + \frac{Q^2}{r^2} \right)^{\frac{2}{3}} \right), \quad h(r) = 1 - \frac{(r_H^2 + Q^2)^2}{(r^2 + Q^2)^2} \,, \tag{2.3}$$

$$\phi(r) = \sqrt{\frac{2}{3}} \log \left(1 + \frac{Q^2}{r^2} \right), \quad A_t(r) = \frac{Q}{2L} \left(1 - \frac{r_H^2 + Q^2}{r^2 + Q^2} \right). \tag{2.4}$$

The temperature T, chemical potential μ, entropy density s, charge density σ, energy density ε and pressure P of the dual field theory are given in terms of the parameters of the 2-charge black hole (2.2) via

$$T = \frac{r_H}{\pi L^2}, \quad \mu = \frac{\sqrt{2}Q}{L^2}, \quad s = \frac{r_H}{4GL^3}(r_H^2 + Q^2), \quad \sigma = \frac{\sqrt{2}Qs}{2\pi r_H}, \quad \varepsilon = 3P = \frac{3 \left(r_H^2 + Q^2 \right)^2}{16\pi GL^5}. \tag{2.5}$$

At large densities, this black hole is dual to a semi-local quantum liquid [37] which violates hyperscaling [38]. See [39–45] for related work.

3 Hydrodynamics

In common with other field theories, for small amplitude excitations around this equilibrium state with sufficiently small frequencies and momenta, local thermal equilibrium is maintained and one can describe the system using hydrodynamics [46]. In this hydrodynamic limit, the excitations of the system include hydrodynamic sound modes with dispersion relations [47]

$$\omega = \pm \sqrt{\frac{dP}{d\varepsilon}} k - i\frac{2\eta}{3 \left(\varepsilon + P \right)} k^2 + \dots, \tag{3.1}$$

where the ellipsis denotes higher order terms in k. The attenuation of this hydrodynamic sound mode is controlled by the viscosity η of the field theory. For the two-charge black hole, $\eta = s/4\pi$ [31], and thus the hydrodynamic sound mode dispersion relation is

$$\omega = \pm \frac{1}{\sqrt{3}} k - i\frac{r_H L^2}{6 \left(r_H^2 + Q^2 \right)} k^2 + \dots, = \pm \frac{1}{\sqrt{3}} k - i\frac{\pi T}{3 \left(\mu^2 + 2\pi^2 T^2 \right)} k^2 + \dots. \tag{3.2}$$

Note that the leading k^2 contribution to the attenuation vanishes linearly with T in the limit $T \to 0$. This is because $s \sim T$ at low T.

We are interested in the two-charge black hole solution in the large density limit $\mu \gg T, \omega, k$. Furthermore, we will assume $|\omega| \sim k$ from now, since this is true for the sound mode in which we are interested. Within this large density limit, the ratio k/T is still

arbitrary. We will now clarify our previous assertion of how "sufficiently small" k must be such that hydrodynamics is applicable. In a quasiparticle description, there is a mean free path l_η between (thermal) collisions and the regime of applicability of hydrodynamics is then $kl_\eta \ll 1$. For the two-charge black hole, we can identify a natural expansion parameter from the dispersion relation (3.2)

$$l_\eta = \frac{T}{\mu^2 + 2\pi^2 T^2}. \tag{3.3}$$

This suggests that hydrodynamics is valid provided that $k/\mu \ll \left(\mu/T + 2\pi^2 T/\mu\right)$. Within the large density limit, this inequality is *always* satisfied and hence the hydrodynamic result (3.2) should be true for arbitrary k within the large density limit. This is intimately related to the fact that $\eta \sim s$ for holographic theories, since from (3.1) we can write the range of applicability of hydrodynamics more abstractly as $|\omega| \sim k \ll l_\eta^{-1} \sim \eta^{-1} \sim s^{-1}$ and $s \sim T$ is always small (in units of μ) in the large density limit. In a Landau Fermi liquid, $l_\eta \sim \mu/T^2$ is the length scale over which individual fermions interact such that a hydrodynamic state is formed. For a generic holographic theory with $\eta \sim s$, the equivalent length scale is always small, indicating that a single-particle description is not applicable.

This applicability of hydrodynamics at arbitrarily low temperatures was shown analytically for the collective excitations of the Reissner-Nordstrom-AdS$_4$ black brane in [27]. To verify this for the two-charge black hole, we have calculated numerically the dispersion relation of the sound mode of the field theory dual to the two-charge black hole over a large range of temperatures. The details of this calculation are given in appendix A and the results are shown in figure 1. The dispersion relation agrees well with the hydrodynamic prediction (3.2) down to very low T. In the limit $T \to 0$, where the leading order hydrodynamic result (3.2) for the sound attenuation vanishes, the numerical result begins to differ from the (vanishing) hydrodynamic prediction. This difference is of order k^3 and thus the numerical results are consistent with the hydrodynamic prediction (3.2) to $\mathcal{O}\left(k^2\right)$ for all T in the large density limit. At higher orders in k, we expect that they will differ due to logarithmic terms present in the correlators of the semi-local quantum liquid state [27].

We emphasise here that this behaviour is totally different to that in a generic Landau Fermi liquid, which has $\eta \sim T^{-2}$ (since $\eta \neq s/4\pi$). For Landau Fermi liquids, hydrodynamics breaks down at low temperatures $|\omega| \sim \tau_\eta^{-1} \sim \eta^{-1} \sim T^2/\mu$ where the hydrodynamic sound attenuation becomes very large [48–50]. It is replaced by a collisionless regime where quantum interactions sustain a 'zero sound' mode with a different dispersion relation than (3.1). This has been confirmed experimentally [51].

4 Can Landau Fermi liquid theory describe the two-charge black hole?

In this section we explore the possibility that Landau Fermi liquid theory might explain the low energy physics of the collective excitations in the theory holographically dual to (2.2). As explained in the previous section, this can only happen for a very specific limit of Fermi liquid theory, since the gravity dual necessarily implies that $\eta/s = 1/4\pi$.
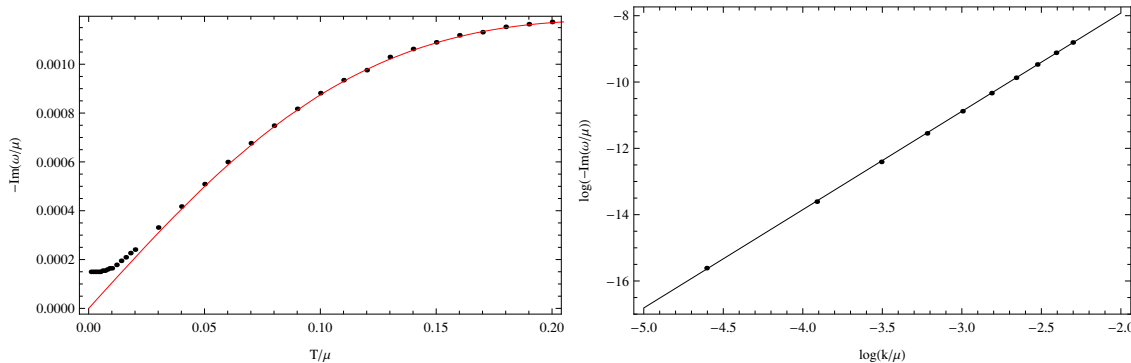
Figure 1. Numerical results for the imaginary part of the sound dispersion relation. The real part (not shown) is always $\approx k/\sqrt{3}$. *Left*: the sound attenuation at fixed $k/\mu = 0.1$ as a function of T/μ. The numerical results (shown as black dots) agree very well with the hydrodynamic dispersion relation (3.2) (shown as a red line) down to very small temperatures. *Right*: logarithmic plot of the sound attenuation at fixed, very small $T/\mu = 0.001$ as a function of k/μ. The best fit to the numerical results (shown as black dots) is a straight line of gradient ≈ 2.96 (shown as a black line), indicating a k^3 dependence.

4.1 Establishing the N-dependence of the Fermi liquid parameters

The low temperature heat capacity of the field theory holographically dual to (2.2) is given by

$$c_V = \frac{\pi L^3}{8G}\mu^2 T = \frac{N^2}{4}\mu^2 T,\tag{4.1}$$

where we have used $L^3/G = 2N^2/\pi$ (see e.g. [52]). This should be compared to the Landau Fermi liquid result

$$c_V = \frac{k_F m^*}{3}T.\tag{4.2}$$

Let us assume that Luttinger's theorem holds and therefore that the charge density is

$$\sigma = \alpha \int_0^{k_F} \frac{d\tau}{(2\pi)^3} = \alpha\frac{k_F^3}{6\pi^2}.\tag{4.3}$$

We assume that the numerical coefficient α is $\mathcal{O}(N^0)$ and counts the number of charged operators as well as their charge. This would be true, for example, if the charge in the state is carried by an $\mathcal{O}(1)$ number of fermionic operators. The validity of Luttinger's theorem has been widely debated in the literature (see e.g. [8, 53, 54]). In [54] it was shown that, by including magnetic monopole contributions, Friedel oscillations in the charge density are seen at a wavevector satisfying Luttinger's theorem for a (1+1)-dimensional theory. The validity of this result in higher dimensions (see [55] for progress in this direction) would be a useful check of our assumption.

With this assumption, we can deduce the N-dependence of the various Landau Fermi liquid parameters. We can rewrite one of the expressions in (2.5) as

$$\sigma = \frac{L^3}{16\pi G}\mu^3 = \frac{N^2}{8\pi^2}\mu^3,\tag{4.4}$$

at $T = 0$. Comparing this to (4.3), we deduce that

$$k_F = \left(\frac{3N^2}{4\alpha}\right)^{\frac{1}{3}} \mu.$$ (4.5)

Now comparing (4.1) and (4.2) and using (4.5) we obtain the following expressions for the effective quasiparticle mass and the Fermi velocity

$$m^* = \alpha^{\frac{1}{3}}\left(\frac{3N^2}{4}\right)^{\frac{2}{3}}\mu, \qquad \upsilon_F = \alpha^{-\frac{2}{3}}\left(\frac{3N^2}{4}\right)^{-\frac{1}{3}}.$$ (4.6)

We are therefore dealing with an extremely massive Fermi liquid whose Fermi velocity is parametrically small. We will now compare (4.6) with the formula for the effective quasiparticle mass in a relativistic Landau Fermi liquid

$$m^* = \mu\left(1 + \frac{F_1}{3}\right),$$ (4.7)

to deduce the Landau interaction parameter

$$\frac{F_1}{3} = \alpha^{\frac{1}{3}}\left(\frac{3N^2}{4}\right)^{\frac{2}{3}} = \alpha^{\frac{1}{3}}\epsilon^{-2},$$ (4.8)

where we have neglected terms that are $\mathcal{O}(N^{-2/3})$ and introduced an expansion parameter

$$\epsilon = \left(\frac{3N^2}{4}\right)^{-\frac{1}{3}} \ll 1.$$ (4.9)

We next consider the Landau Fermi liquid expression for the speed of hydrodynamic sound

$$c_1 = \frac{\upsilon_F}{\sqrt{3}}\left[(1 + F_0)\left(1 + \frac{F_1}{3}\right)\right]^{\frac{1}{2}} = \frac{1}{\sqrt{3}},$$ (4.10)

where the last equality is true for the theory holographically dual to (2.2). From (4.10) we deduce that

$$F_0 = \alpha - 1.$$ (4.11)

The main lesson of this equation is that $F_0 = \mathcal{O}(1)$, contrary to $F_1 = \mathcal{O}(N^{4/3})$ as given in equation (4.8).

   With these N-dependences, we can explain the results of various holographic computations, as we will show shortly. To explain the hydrodynamic results of the previous section, one more assumption is needed. We must tune $F_2 \to -5$ so that the quasiparticles are short-lived. This means that a hydrodynamic, rather than a single-particle, description is applicable at low T, as we found for the holographic theory in the previous section.

   In this limit, the speed of the zero sound mode c_0 is equal to the hydrodynamic value $c_1 = 1/\sqrt{3}$. To compute the speed of zero sound, one needs to solve the integral equation

$$\left(\frac{c_0}{\upsilon_F} - \cos\theta\right)\nu(\theta, \varphi) = \cos\theta \int F(\theta, \theta')\nu(\theta', \varphi')\frac{d\Omega'}{2\pi},$$ (4.12)

where $\nu(\theta, \varphi)$ parametrizes the deviation of the Fermi surface from the spherical form. For zero sound, this deviation is φ-independent and we can expand in Legendre polynomials,

$$\nu(\theta, \varphi) = \sum_l P_l(\cos\theta)\nu_l, \qquad \widetilde{\nu}_l = \frac{\nu_l}{2l+1}. \tag{4.13}$$

Equation (4.12) then becomes

$$\widetilde{\nu}_l + \sum_l \Omega_{ll'}\left(\frac{c_0}{v_F}\right)F_{l'}\widetilde{\nu}_{l'} = 0, \tag{4.14}$$

where

$$\Omega_{ll'}\left(\frac{c_0}{v_F}\right) = \frac{1}{2}\int_{-1}^{1} dy P_l(y)\frac{y}{y - \frac{c_0}{v_F}}P_{l'}(y). \tag{4.15}$$

We now substitute

$$\widetilde{s} \equiv \frac{v_F}{c_0} = \mathcal{O}(\epsilon), \tag{4.16}$$

into (4.14) and (4.15) and expand in powers of $\widetilde{s}$. This is an expansion in inverse powers of N. We start by computing the determinant of

$$A_{ll'} = \delta_{ll'} + \Omega_{ll'}\left(\frac{c_0}{v_F}\right)F_{l'}. \tag{4.17}$$

For (4.14) to have a non-trivial solution, it must be that $\det A = 0$.

At this point we will make an important assumption that

$$F_n = \mathcal{O}(1), \qquad n > 1, \tag{4.18}$$

which will simplify things considerably. We will only keep terms up to $\mathcal{O}(\widetilde{s}^2)$ in the matrix $A_{ll'}$ (4.17). After some algebra, we arrive at

$$\det A = 1 - \frac{F_0\widetilde{s}^2}{3} - \frac{F_1\widetilde{s}^2}{5} - \frac{F_0 F_1 \widetilde{s}^2}{9} - \frac{4}{225}F_1 F_2 \widetilde{s}^2 + \mathcal{O}(\widetilde{s}^2). \tag{4.19}$$

In (4.19), we have used (4.18) and $F_1, F_0 F_1 = \mathcal{O}(\widetilde{s}^{-2})$ to write only terms that can possibly be $\mathcal{O}(1)$ (or smaller). It is now clear that keeping terms subleading to $\mathcal{O}(\widetilde{s}^2)$ in (4.15) and (4.17) will only modify subleading terms in (4.19).

We now substitute

$$\widetilde{s}^2 = \frac{3}{(1+F_0)(1+F_1/3)}, \tag{4.20}$$

into (4.19) (i.e. impose that $c_0 = c_1$) and demand $\det A = 0$, which gives rise to

$$\frac{4}{25}\frac{F_1(5+F_2)}{(1+F_0)(3+F_1)} = \mathcal{O}(\widetilde{s}^2), \tag{4.21}$$

which implies

$$F_2 = -5 + \mathcal{O}(\widetilde{s}^2). \tag{4.22}$$

We also need to determine the values of $\widetilde{\nu}_l$. This computation is technically more involved than calculating the determinant: we now need to compute the eigenvector of A associated with the vanishing eigenvalue. We have done this calculation with $F_n \neq 0, n \leq 3$. The result is

$$\widetilde{\nu}_0 = 1, \quad \widetilde{\nu}_1 = \frac{\alpha^{\frac{1}{3}}\epsilon}{\sqrt{3}}, \quad \widetilde{\nu}_2 = \frac{2}{5}, \quad \widetilde{\nu}_3 = \mathcal{O}(\epsilon^2). \tag{4.23}$$

4.2 Observables

We start by considering observables that present some obvious contradictions with a Landau Fermi liquid description and we will see how these are resolved by counting powers of N. We will then consider the fate of collective excitations at small temperatures, where the Fermi liquid description predicts a particular low temperature behavior.

Friedel oscillations. It has been argued that the two-point functions of the currents should develop a non-analytic behavior at $k = 2k_F$ and $\omega \to 0$. As is clear from (4.5), such non-analytic structure cannot be observed from the usual linear fluctuations in supergravity, since the equations of motion for these fluctuations have $k \sim \mathcal{O}(N^0)$. To detect non-analytic behavior at $k_F \sim \mathcal{O}(N^{2/3})$, one should include the effects of magnetic monopoles along the lines of [54, 55].

Particle-hole continuum; Landau damping. A related problem is a non-trivial structure of the spectral function associated with the particle-hole continuum. On the ω axis, the upper edge of the continuum is defined, for small ω and k, by $\omega = v_F k$. The physics of this has been recently reviewed in [20], where it was proposed that the continuum is not seen because of a parametrically small v_F. Precisely such a situation is implied by eq. (4.6). See [4] for related work.

Entanglement entropy. It has been argued that the presence of a Fermi surface leads to a logarithmic violation of the area law

$$S \simeq L^2 k_F^2 \log L + \dots . \tag{4.24}$$

Such a violation has not been observed (see e.g. [56]). The resolution of this puzzle is simple: according to (4.5), the $L^2 \log L$ term in the entanglement entropy must multiply a factor of $N^{4/3}$, while the tree level result [57, 58] only gives the leading term which is $\mathcal{O}(N^2)$.

Collective modes with azimuthal dependence. As well as zero sound, a Landau Fermi liquid can support other collective modes due to more complicated fluctuations of the Fermi surface. These have not yet been seen in any holographic theory. Including azimuthal dependence in the deformation of the distribution function (4.13) leads to simple modifications of eqs. (4.14) and (4.15). The analog of the matrix (4.17), whose determinant must vanish, is now

$$A_{ll'}^m = \delta_{ll'} + \Omega_{ll'}^m(s) F_{l'}, \tag{4.25}$$

where

$$\Omega_{ll'}^m(s) = \frac{1}{2} \frac{(l - |m|)!}{(l + |m|)!} \int_{-1}^{1} dy P_l^m(y) \frac{y}{y - s} P_{l'}^m(y), \tag{4.26}$$

and $m \leq l, l'$. Repeating the steps that led to (4.21), we obtain

$$1 - \frac{3F_1(5 + F_2)S_1^2}{25(1 + F_0)(3 + F_1)} = \mathcal{O}(\tilde{s}^2), \tag{4.27}$$

where S_1 is the speed of the $m = 1$ collective mode $c^{(m=1)}$ measured in units of c_0, $S_1 = c^{(m=1)}/c_0$. When $F_0 \sim \alpha \sim \mathcal{O}(1)$, eq. (4.27) implies that for generic values of F_2, in addition to the $m = 0$ zero sound mode, there is also another collective mode. However because of the special value of F_2 given in (4.22), there is actually no solution of (4.27) with $c^{(m=1)}$ of order one. We have also checked that for higher modes ($m > 1$) the analog of eq. (4.27) has no solutions with $c^{(m>1)} = \mathcal{O}(1)$.

Shear viscosity. In the holographic model that we consider, the shear viscosity satisfies the universal law [31]

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{4.28}$$

This, at first sight, presents a puzzle, since the Fermi liquid result is

$$\eta = \frac{1}{5}\rho k_F \upsilon_F \tau_\eta \sim \rho k_F \upsilon_F \tau, \tag{4.29}$$

where

$$\frac{\tau_\eta}{\tau} = \frac{2}{\pi^2} \sum_{\nu=1,3,5,\ldots} \frac{2\nu+1}{\nu(\nu+1)[\nu(\nu+1)-2\lambda_\eta]}, \tag{4.30}$$

and $\lambda_\eta = \mathcal{O}(1)$. The quasiparticle lifetime, τ in (4.29) is related to the scattering probabilities of the quasiparticles $W(\theta, \varphi)$ via

$$\tau = \frac{8\pi}{m^{*3}\langle W \rangle T^2}, \tag{4.31}$$

where

$$\langle W \rangle = \int \frac{d\Omega}{4\pi} \frac{W(\theta, \varphi)}{\cos(\theta/2)}. \tag{4.32}$$

The probabilities are related to the scattering amplitudes which, in turn, can be related to the Landau parameters, if assumed to be φ-independent:

$$W(\theta, \varphi) = 2\pi |A(\theta, \varphi)|^2 = \frac{2\pi^5}{m^{*2}k_F^2} |\sum_l \mathcal{F}_l P_l(\cos\theta)|^2, \tag{4.33}$$

where

$$\mathcal{F}_l = \frac{F_l}{1 + F_l/(2l+1)}. \tag{4.34}$$

From (4.34) it is clear that generally $\mathcal{F}_l = \mathcal{O}(1)$ and therefore $\tau \sim k_F^2/(m^*T^2)$, which gives $\eta \sim N^2\mu^5/T^2$ and $\eta/s \simeq \mu^3/T^3$ in sharp contradiction with (4.28). However the value of F_2 in (4.22) is very non-generic! Precisely for this value, the denominator in (4.34) vanishes (as $\mathcal{O}(\epsilon^2)$) and τ receives an extra factor of ϵ^4. Hence, once (4.22) is taken into account, the leading order result (in powers of N^2) for η/s is no longer μ^3/T^3, and one needs to consider higher order terms in T/μ. Field theoretic calculations indicate that, in the $F_2 \to -5$ limit, marginal Fermi liquid-type excitations exist [59], and it would be interesting to determine the viscosity in models like this.

The applicability of hydrodynamics. In section 3 we observed that the characteristic length scale where a hydrodynamic description of the sound mode breaks down is given by eq. (3.3). As explained above, this is consistent with Landau Fermi liquid theory with $F_2 = -5$, in which the divergence of the viscosity at low T — which ensures the breakdown of hydrodynamics — is suppressed.

5 Higher derivative corrections

In the previous section we have shown that in the limit $F_2 \to -5$, many of the generic properties of a Landau Fermi liquid are no longer realised. It is important that we can recover these generic properties by altering the gravitational theory such that F_2 is no longer fine-tuned in this way. In this section we will discuss a procedure by which one may be able to recover the expected $\eta/s \sim \mu^3/T^3$ behavior of a Landau Fermi liquid, by including specific higher-derivative terms in the gravitational Lagrangian.

5.1 Entropy

To compute the entropy of the field theory dual to a gravitational solution, we use the Wald formula

$$S = -2\pi \int d\Omega \frac{\delta \mathcal{L}}{\delta R_{abcd}} E_{ab} E_{cd} \,, \tag{5.1}$$

where $\mathcal{L}$ is the Lagrangian, $d\Omega = d^3x \sqrt{g_3}$ is the horizon volume element, $E_{ab} = \sqrt{|g_2|}\epsilon_{ab}$ is the antisymmetric tensor with indices taking values t, r, and $g_2 = g_{tt}g_{rr}$. We will consider a set of three different four-derivative gravitational terms with coefficients c_i, $i = 1, 2, 3$. After setting the gravitational constant $G = 1$ and the AdS radius $L = 1$, the terms in the Lagrangian which contribute to the Wald formula are

$$\mathcal{L}_R = \frac{1}{16\pi} \left(R + (\beta_1 e^{\gamma_1 \phi} + \beta_2 e^{\gamma_2 \phi})(c_1 R^2 + c_2 R_{ab}R^{ab} + c_3 R_{abcd}R^{abcd}) \right), \tag{5.2}$$

where $\beta_1, \beta_2, \gamma_1$ and γ_2 are constants. We wish to fix the various constants to satisfy two requirements: that the viscosity is that of a Landau Fermi liquid in the low-temperature limit $r_H/Q \ll 1$, as discussed in the next subsection, and that the metric and dilaton have the same near-boundary asymptotics as in the two-derivative theory, as discussed in appendix B.

The Wald formula (5.1) gives the following expression for the entropy:

$$S = \frac{A}{4} \left(1 + (\beta_1 e^{\gamma_1 \phi} + \beta_2 e^{\gamma_2 \phi}) \left(2c_1 R + c_2(R_t^t + R_r^r) + 4c_3 g^{tt} g^{rr} R_{trtr} \right) |_{r=r_H} \right), \tag{5.3}$$

where A is the area of the horizon. For the two-charge black hole (2.2) we obtain

$$S = \frac{A}{4} \left(1 + \frac{8(\beta_1 e^{\gamma_1 \phi} + \beta_2 e^{\gamma_2 \phi})}{3r_H^2 \left(1 + \frac{Q^2}{r_H^2} \right)^{2/3}} \left[(11c_1 + 4c_2 + 5c_3)Q^2 + 24(5c_1 + c_2 - c_3)r_H^2 \right] \right). \tag{5.4}$$

To retain the Fermi liquid result $s \sim T$, we do not want the entropy to get corrections from these higher-derivative terms. For this purpose we choose

$$c_1 = 1 \,, \quad c_2 = -4 \,, \quad c_3 = 1 \,, \tag{5.5}$$

which corresponds to the Gauss-Bonnet term. The entropy density is therefore given by (2.5).

Note that in addition to the explicit corrections to the entropy density due to the higher-derivative contributions to the Wald formula, there is also a correction Δs coming from the $\mathcal{O}(\beta)$ correction to the classical background (2.2). The two-derivative term gives a contribution $s = (g_{xx}(r_H))^{3/2}/4$ to the entropy, and therefore $\Delta s = 3\sqrt{g_{xx}(r_H)}\Delta g_{xx}(r_H)/8$. However, we can define a new radial coordinate ρ (see appendix B) such that, as in [60], $g_{xx}(\rho) = -\rho^2$. Then we can choose the constants of integration of the higher-derivative equations of motion for the metric such that the horizon radius ρ_H is unchanged, and therefore $\Delta g_{xx}(\rho_H) = 0$ and $\Delta s = 0$.

5.2 Viscosity to entropy ratio

The total action we study $I = I_0 + I_{hd} + I_a + I_{pot}$ is a sum of the two-derivative terms I_0, given by (2.1), the higher-derivative Gauss-Bonnet term with dilaton coupling (see (5.2) and (5.5))

$$I_{hd} = \frac{1}{16\pi} \int d^5x \sqrt{g} (\beta_1 e^{\gamma_1 \phi} + \beta_2 e^{\gamma_2 \phi})(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}) \,, \tag{5.6}$$

the extra dilaton terms

$$I_a = \frac{1}{16\pi} \int d^5x \sqrt{g} \left(\beta_1 e^{\gamma_1 \phi}(a_1 + b_1 g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi) + \beta_2 e^{\gamma_2 \phi}(a_2 + b_2 g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi) \right) \,, \tag{5.7}$$

and the extra dilaton potential terms

$$I_{pot} = \frac{1}{16\pi} \int d^5x \sqrt{g} (g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi)^2 \beta_1 e^{\gamma_1 \phi} \left(c_1 + d_1 e^{w_1 \phi} + d_2 e^{w_2 \phi} + d_2 e^{w_2 \phi} \right) \,. \tag{5.8}$$

Note that the terms (5.7) and (5.8) do not contribute to the Wald formula (5.1). As we will explain, the terms (5.7) and (5.8) do not contribute to the viscosity either.

There are two reasons for adding the terms (5.7) and (5.8) to the action. Firstly, by choosing

$$a_1 = a_2 = -120 \,, \quad b_1 = b_2 = 4 \,, \tag{5.9}$$

the near-boundary asymptotics of the metric and dilaton are the same as in the two-derivative theory. The additional parameters c_1, $d_{1,2,3}$ and $w_{1,2,3}$ are required so that, by fine tuning them, we still have

$$T \simeq r_H \,, \quad \mu \simeq Q \,, \tag{5.10}$$

in the limit $r_H/Q \ll 1$, as in the two-derivative theory (2.5). We elaborate on both of these points in appendix B.

Consider the fluctuation $h_x^y(r, t, x)$ of the metric tensor in the momentum representation

$$h_x^y = \int \frac{d^2k}{(2\pi)^2} \phi_k(r) e^{-i\omega t + ikz} \,. \tag{5.11}$$

The action $I = I_0 + I_{hd}$, expanded up to second order in fluctuations, has the general form [60–62]

$$I = \frac{1}{16\pi} \int \frac{d^2k}{(2\pi)^2} \left[A(r)\phi_k''\phi_{-k} + B(r)\phi_k'\phi_{-k}' + C(r)\phi_k'\phi_{-k} \right.$$
$$\left. + D(r)\phi_k\phi_{-k} + E(r)\phi_k''\phi_{-k}'' + F(r)\phi_k''\phi_{-k}' \right], \quad (5.12)$$

plus an appropriate Gibbons-Hawking boundary term. The viscosity can be computed from the equation [60, 62]

$$\eta = \frac{1}{8\pi} \left(\kappa_2(r_H) + \kappa_4(r_H) \right), \tag{5.13}$$

where

$$\kappa_2(r) = \sqrt{-\frac{g_{rr}}{g_{tt}}} \left(A(r) - B(r) + \frac{F'(r)}{2} \right), \quad \kappa_4(r) = \left(E(r) \left(\sqrt{-\frac{g_{rr}}{g_{tt}}} \right)' \right)', \tag{5.14}$$

where a prime denotes a derivative with respect to r, and where the momentum k and frequency ω are set to zero. The terms I_a and I_{pot}, given in (5.7) and (5.8), do not involve derivatives of h_x^y and so do not affect the viscosity. We can separately compute the contribution to the viscosity η_0 from the two-derivative terms in the action I_0, and the contribution to the viscosity η_1 from the higher-derivative terms in the action I_{hd}. The total viscosity is given by $\eta = \eta_0 + \eta_1$.

Consider first the higher-derivative term I_{hd}. We can expand I_{hd} up to second order in fluctuations and extract the corresponding coefficients A_1, B_1, E_1, F_1, before using equations (5.13) and (5.14) to give

$$\eta_1 = \frac{1}{8\pi} \left(\kappa_2^{(1)}(r_H) + \kappa_4^{(1)}(r_H) \right)$$
$$= -\frac{\beta_1(Q^2(1 - 2\sqrt{6}\gamma_1) + 3r_H^2)}{6\pi} \frac{(r_H^2 + Q^2)^{\sqrt{2/3}\gamma_1 + 1/3}}{r_H^{2\sqrt{2/3}\gamma_1 - 1/3}}$$
$$- \frac{\beta_2(Q^2(1 - 2\sqrt{6}\gamma_2) + 3r_H^2)}{6\pi} \frac{(r_H^2 + Q^2)^{\sqrt{2/3}\gamma_2 + 1/3}}{r_H^{2\sqrt{2/3}\gamma_2 - 1/3}}. \tag{5.15}$$

Now we will determine the contribution to the viscosity from the two-derivative term I_0. The most general two-derivative term which contributes to the entropy density and to the shear viscosity is of the form

$$S = \int d^5x \sqrt{g}(R + L_m), \tag{5.16}$$

where L_m is the Lagrangian for matter. We denote the corresponding viscosity as η_0 and the corresponding entropy density as s_0. If the classical background is corrected by

higher-derivative terms, then the viscosity receives a correction $\Delta\eta$ and the entropy density receives a correction Δs. In addition to these, the higher-derivative terms, evaluated on the classical background, contribute η_1 and s_1 to the viscosity and the entropy density respectively, as previously explained.

The question is whether we need to compute $\Delta\eta$ and Δs to determine η/s. To leading order in β, the effect of these terms is

$$\frac{\eta_0 + \Delta\eta + \eta_1}{s_0 + \Delta s + s_1} - \frac{\eta_0 + \eta_1}{s_0 + s_1} \sim \frac{\eta_0 + \Delta\eta}{s_0 + \Delta s} - \frac{\eta_0}{s_0} \sim s_0 \Delta\eta - \eta_0 \Delta s. \tag{5.17}$$

But it is known [28, 31, 63] that η/s, when computed from the two-derivative action (5.16), does not depend on the background. In fact since $\eta = g_{xx}^{3/2}/(16\pi)$ (one can derive this in the formalism of [60], that is using equation (5.13) for the action (5.16) with arbitrary background metric), and $s = g_{xx}^{3/2}/4$, one always has $s_0 \Delta\eta - \eta_0 \Delta s = 0$ for any correction to the background. Therefore as long as we are interested in η/s, we do not have to compute $\Delta\eta$ and Δs [60, 62]. In fact, as we noted in the previous subsection, we have $\Delta s = 0$ and $\Delta\eta = 0$ satisfied separately, simply because of the $\Delta g_{xx}(r_H) = 0$ equation.

We choose

$$\gamma_1 = \frac{7}{2\sqrt{6}}, \quad \beta_1 = 3\beta, \quad \gamma_2 = -\frac{7}{\sqrt{6}}, \quad \beta_2 = \frac{3}{2}\beta. \tag{5.18}$$

The parameters $\beta_{1,2}$ and $\gamma_{1,2}$ then satisfy

$$\frac{\beta_1}{\beta_2}\frac{\gamma_1}{\gamma_2} = -1, \tag{5.19}$$

which, as shown in appendix B, is crucial to obtain the correct asymptotic behavior of g_{tt} and the dilaton. Therefore in the limit $r_H/Q \ll 1$, we obtain

$$\eta \simeq \frac{Q^5}{r_H^2}. \tag{5.20}$$

This behavior agrees with the low-temperature result of Landau Fermi liquid theory, provided that the black hole solution has $T \sim r_H$ and $\mu \sim Q$ when $r_H/Q \ll 1$, as in the two-derivative theory. This is a non-trivial requirement which needs fine tuning of the coefficients c_1, $d_{1,2,3}$ and $w_{1,2,3}$. In appendix B, we describe a numerical procedure by which one can tune these coefficients and change the leading functional dependence of T and μ on r_H and Q. It is clear from this analysis that it is not easy to produce a gravitational dual of these generic Landau Fermi liquid properties. This further underlines how different holographic metals are, compared to their conventional counterparts.

6 Discussion

In section 4 we described a scenario where a Landau Fermi liquid with specific parameters was responsible for the properties of the two-charge black hole. The two crucial assumptions were the existence of a single or, at most, $\mathcal{O}(1)$ Fermi surfaces and the applicability of Luttinger's theorem. There is some tension with the observation of singularities in the two-point function of the gauge-invariant fermionic operators [33], which would signify

that $k_F \sim \mu$ (as opposed to (4.5)). If one takes the singularities at $k_F \sim \mu$ seriously, two logical possibilities present themselves: i) the Fermi surface is formed by the N^2 species of gauginos, and Luttinger's theorem is almost satisfied [33], or ii) Luttinger's theorem is strongly violated in holographic models where charge is sourced by black hole horizons due to fractionalization in the dual field theory [3, 53].

In this paper we raise a third possibility, which explains all other observable data (such as the absence of Friedel oscillations, the particle-hole continuum and the logarithmic violation of entanglement entropy, as well as the behavior of collective excitations). To make this picture more precise, it would be interesting to identify possible effects responsible for (4.5), and also to check more thoroughly whether one can depart from the $F_2 = -5$ limit and recover a generic Landau Fermi liquid. In this case, there are precise predictions for the behavior of the sound mode outside of the hydrodynamic regime.[2] We took the first steps in this direction in section 5 by describing how one could, in principle, construct a higher derivative gravitational theory which reproduces the viscosity of a Landau Fermi liquid.

It would be interesting to check our assumption of Luttinger's theorem by a calculation along the lines of [54] (work in this direction was undertaken in [55]). We do not have anything to say at the moment regarding the microscopic origin of the parametrically large k_F. It is known that the naive large N counting breaks down in vector models at finite chemical potential [64, 65], and the recent works [66–70] have suggested other ways to construct controlled microscopic descriptions of related systems. It would be interesting to see whether the power law in (4.5) can be naturally reproduced. Finally, let us comment on the nature of the $F_2 = -5$ limit. It is known that for $F_l < -2l - 1$ the Landau Fermi liquid becomes unstable. If our scenario is correct, it would be interesting to identify this instability from a bulk perspective (see [71] for related work).

Acknowledgments

A Quasinormal modes of the two-charge black hole

The dispersion relations of the collective excitations in the field theory dual to a black hole solution are given by the dispersion relations of the quasinormal modes of that black hole [72]. These quasinormal modes are solutions of the linearised equations of motion for perturbations of the fields around the background solution (2.2) which are infalling at the

---

[2]We remind the reader that hydrodynamics fixes the speed of the sound mode c_1 to be equal to the thermodynamic value, $c_1^2 = \partial P / \partial \varepsilon$, which is different from the speed of zero sound in a generic Fermi liquid.

horizon $r = r_H$ and whose leading term vanishes at the boundary $r \to \infty$. It is convenient to go to Fourier space and perturb the background fields as follows

$$\phi\left(r\right) \to \phi\left(r\right) + \int \frac{d\omega dk}{\left(2\pi\right)^2} e^{-i\omega t + ikz} \varphi\left(r, \omega, k\right),$$

$$A_\mu\left(r\right) \to A_\mu\left(r\right) + \int \frac{d\omega dk}{\left(2\pi\right)^2} e^{-i\omega t + ikz} a_\mu\left(r, \omega, k\right),$$

$$g_{\mu\nu}\left(r\right) \to g_{\mu\nu}\left(r\right) + \int \frac{d\omega dk}{\left(2\pi\right)^2} e^{-i\omega t + ikz} h_{\mu\nu}\left(r, \omega, k\right). \tag{A.1}$$

The equations of motion for these perturbations can be obtained from the action (2.1) but they are very lengthy and we will not present them here.

The sound mode in which we are most interested is associated with longitudinal density oscillations in the field theory and thus is realised in the gravitational dual as a quasinormal mode of h_{tt} (which is dual to the component of the field theory energy-momentum tensor T^{tt}). At linear order, h_{tt} couples to $h_{zz}, h_{xx} + h_{yy}, h_{rr}, h_{tr}, h_{zr}, h_{zt}, a_t, a_z, a_r$, and φ. Within this set of fields there are, in fact, only three independent degrees freedom which are invariant under infinitesimal diffeomorphisms and U(1) gauge transformations. We choose the basis

$$Z_h = 2\omega k h_t^z + \omega^2 h_z^z - k^2 h h_t^t - \frac{1}{2}\left[\omega^2 - k^2\left(h + \frac{rgh'}{2g + rg'}\right)\right]\left(h_x^x + h_y^y\right),$$

$$Z_a = \omega a_z + k a_t - k\frac{rgA_t'}{2\left(2g + rg'\right)}\left(h_x^x + h_y^y\right),$$

$$Z_\varphi = \varphi - \frac{\sqrt{\frac{3}{2}}rg'}{2\left(2g + rg'\right)}\left(h_x^x + h_y^y\right), \tag{A.2}$$

and write the equations of motion in the form

$$Z_h'' + C_1 Z_h' + C_2 Z_a' + C_3 Z_\varphi' + C_4 Z_h + C_5 Z_a + C_6 Z_\varphi = 0,$$

$$Z_a'' + C_7 Z_h' + C_8 Z_a' + C_9 Z_\varphi' + C_{10} Z_h + C_{11} Z_a + C_{12} Z_\varphi = 0,$$

$$Z_\varphi'' + C_{13} Z_h' + C_{14} Z_a' + C_{15} Z_\varphi' + C_{16} Z_h + C_{17} Z_a + C_{18} Z_\varphi = 0, \tag{A.3}$$

where the coefficients C_i will be given shortly. Near the spacetime boundary $r \to 0$, the solutions of these equations are of the form

$$Z_h = Z_h^{(1)}\left[\frac{Z_h^{(0)}}{Z_h^{(1)}}\left(1 + \ldots\right) + r^{-4}\left(1 + \ldots\right)\right],$$

$$Z_a = Z_a^{(1)}\left[\frac{Z_a^{(0)}}{Z_a^{(1)}}\left(1 + \ldots\right) + r^{-2}\left(1 + \ldots\right)\right],$$

$$Z_\varphi = Z_\varphi^{(1)}\left[\frac{Z_\varphi^{(0)}}{Z_\varphi^{(1)}}r^{-2}\log\left(\frac{r}{r_0}\right)\left(1 + \ldots\right) + r^{-2}\left(1 + \ldots\right)\right], \tag{A.4}$$

where the ratios $Z^{(0)}/Z^{(1)}$ are fixed by imposing infalling boundary conditions at the horizon. Quasinormal modes are solutions to these equations of motion which are infalling at

the horizon and for which $Z^{(0)}/Z^{(1)}$ vanishes for all three fields. These modes exist only for a discrete set of quasinormal frequencies $\omega\,(k,T,\mu)$. To determine these quasinormal frequencies numerically we follow the 'determinant method' described in [73] with one difference. Since the source term in the near-boundary expansion of φ is the logarithmic term [74], we extract it by fitting the near-boundary expansion of the numerical solution to the form in (A.4) and construct the determinant with this. We note that this method does not give any information about the residue of the pole, only its location in the complex frequency plane. It would be interesting to determine the corresponding residue but we will not address this issue here.

Without further ado, we now list the coefficients C_i in the equations of motion (A.3)

$$
\begin{aligned}
C_1 = \Big[& k^4\left(r^2 - r_H^2\right)\left(8Q^{10}\left(6r^4 - r^2 r_H^2 + r_H^4\right) + 4Q^8\left(53r^6 - 19r^4 r_H^2 + 23r^2 r_H^4 + 3r_H^6\right)\right. \\
& + 2Q^6\left(262r^8 - 181r^6 r_H^2 + 83r^4 r_H^4 + 73r^2 r_H^6 + 3r_H^8\right) \\
& + Q^4\left(563r^{10} - 181r^8 r_H^2 - 324r^6 r_H^4 + 348r^4 r_H^6 + 73r^2 r_H^8 + r_H^{10}\right) \\
& + 6Q^2 r^2\left(44r^{10} + 13r^8 r_H^2 - 49r^6 r_H^4 + r^4 r_H^6 + 29r^2 r_H^8 + 2r_H^{10}\right) \\
& + 3r^4\left(15r^{10} + 15r^8 r_H^2 - 16r^6 r_H^4 - 16r^4 r_H^6 + 9r^2 r_H^8 + 9r_H^{10}\right)\Big) \\
& - 2k^2\omega^2\left(Q^2 + r^2\right)\left(2Q^{10}\left(5r^4 - 5r^2 r_H^2 + 2r_H^4\right) + Q^8\left(109r^6 - 120r^4 r_H^2 + 35r^2 r_H^4 + 4r_H^6\right)\right. \\
& + Q^6\left(303r^8 - 288r^6 r_H^2 + 16r^4 r_H^4 + 40r^2 r_H^6 + r_H^8\right) \\
& + 2Q^4 r^2\left(193r^8 - 156r^6 r_H^2 - 36r^4 r_H^4 + 38r^2 r_H^6 + 5r_H^8\right) \\
& + Q^2 r^4\left(219r^8 - 102r^6 r_H^2 - 156r^4 r_H^4 + 72r^2 r_H^6 + 19r_H^8\right) + 3r^6\left(15r^8 - 17r^4 r_H^4 + 6r_H^8\right)\Big) \\
& + \omega^4\left(Q^2 + r^2\right)^4\left(Q^2 + 3r^2\right)^2\left(Q^2\left(6r^2 - 2r_H^2\right) + 5r^4 - r_H^4\right)\Big] \Big/ \Big[r\left(Q^2 + r^2\right)\left(Q^2 + 3r^2\right) \\
& \left(r^2 - r_H^2\right)\left(2Q^2 + r^2 + r_H^2\right)\left(k^2\left(r^2 - r_H^2\right)\left(2Q^2 + r^2 + r_H^2\right) - \omega^2\left(Q^2 + r^2\right)^2\right) \\
& \left(k^2\left(Q^2\left(4r^2 - 2r_H^2\right) + 3r^4 - r_H^4\right) - \omega^2\left(Q^4 + 4Q^2 r^2 + 3r^4\right)\right)\Big],
\end{aligned}
$$

(A.5)

$$
C_2 = -\frac{16kLQ\left(Q^2 + r_H^2\right)\left(k^2\left(Q^2\left(2r_H^2 - 4r^2\right) - 3r^4 + r_H^4\right) + \omega^2\left(Q^4 + 4Q^2 r^2 + 3r^4\right)\right)}{r\left(Q^2 + r^2\right)\left(Q^2 + 3r^2\right)\left(k^2\left(r^2 - r_H^2\right)\left(2Q^2 + r^2 + r_H^2\right) - \omega^2\left(Q^2 + r^2\right)^2\right)},
$$

(A.6)

$$
C_3 = 0,
$$

(A.7)

$$
\begin{aligned}
C_4 = \Big[& k^4 L^4\left(Q^2 + 3r^2\right)\left(Q^2 + r^2\right)^2\left(r^2 - r_H^2\right)\left(Q^4\left(8r^2 - 4r_H^2\right) + 2Q^2\left(5r^4 + r^2 r_H^2 - 2r_H^4\right)\right. \\
& + 3r^6 + 3r^4 r_H^2 - r^2 r_H^4 - r_H^6\Big) - 2k^2\Big(Q^{12}\left(r^2\left(3L^4\omega^2 - 32r_H^2\right) - 2L^4 r_H^2\omega^2 + 32r^4\right) \\
& + Q^{10}\left(5r^4\left(5L^4\omega^2 + 16r_H^2\right) - 16r^2\left(L^4 r_H^2\omega^2 + 9r_H^4\right) - L^4 r_H^4\omega^2 + 64r^6\right) \\
& + Q^8 r^2\left(r^4\left(81L^4\omega^2 + 256r_H^2\right) - 24r^2\left(2L^4 r_H^2\omega^2 + r_H^4\right) - 8\left(L^4 r_H^4\omega^2 + 32r_H^6\right) + 24r^6\right) \\
& + 4Q^6\left(3r^8\left(11L^4\omega^2 + 8r_H^2\right) + r^6\left(96r_H^4 - 17L^4 r_H^2\omega^2\right) - 2r^4\left(3L^4 r_H^4\omega^2 + 32r_H^6\right) - 56r^2 r_H^8\right) \\
& + Q^4 r^2\left(115L^4 r^8\omega^2 + 2r^6\left(72r_H^4 - 23L^4 r_H^2\omega^2\right) + r^4\left(256r_H^6 - 34L^4 r_H^4\omega^2\right) - 304r^2 r_H^8 - 96r_H^{10}\right) \\
& + Q^2 r^2\left(51L^4 r^{10}\omega^2 - 12L^4 r^8 r_H^2\omega^2 + r^6\left(96r_H^6 - 23L^4 r_H^4\omega^2\right) + 64r^4 r_H^8 - 144r^2 r_H^{10} - 16r_H^{12}\right) \\
& + 3r^4\left(3L^4 r^{10}\omega^2 - 2L^4 r^6 r_H^4\omega^2 + 8r^4 r_H^8 - 8r_H^{12}\right)\Big) + L^4\omega^4\left(Q^2 + 3r^2\right)^2\left(Q^2 + r^2\right)^5\Big] \Big/ \Big[\left(Q^2 + r^2\right)^2 \\
& \left(Q^2 + 3r^2\right)\left(r^2 - r_H^2\right)^2\left(2Q^2 + r^2 + r_H^2\right)^2\left(k^2\left(Q^2\left(2r_H^2 - 4r^2\right) - 3r^4 + r_H^4\right)\right. \\
& + \omega^2\left(Q^4 + 4Q^2 r^2 + 3r^4\right)\Big)\Big],
\end{aligned}
$$

(A.8)

$$C_5 = \Big[64kLQ\left(Q^2+r_H^2\right)^3\Big(2k^4r^2\left(2Q^2+3r^2\right)\left(r^2-r_H^2\right)\left(2Q^2+r^2+r_H^2\right)$$
$$+ k^2\omega^2\Big(Q^6\left(2r_H^2-4r^2\right)+Q^4\left(-27r^4+16r^2r_H^2+r_H^4\right)+Q^2\left(-40r^6+18r^4r_H^2+8r^2r_H^4\right)$$
$$- 15r^8+9r^4r_H^4\Big)+\omega^4\left(Q^4+4Q^2r^2+3r^4\right)^2\Big)\Big/\Big[\left(Q^2+r^2\right)^2\left(Q^2+3r^2\right)\left(r^2-r_H^2\right)$$
$$\left(2Q^2+r^2+r_H^2\right)\left(\omega^2\left(Q^2+r^2\right)^2-k^2\left(r^2-r_H^2\right)\left(2Q^2+r^2+r_H^2\right)\right)\Big(k^2\Big(Q^2\left(2r_H^2-4r^2\right)$$
$$- 3r^4+r_H^4\Big)+\omega^2\left(Q^4+4Q^2r^2+3r^4\right)\Big)\Big], \tag{A.9}$$

$$C_6 = -\Big[8\sqrt{\tfrac{2}{3}}k^2Q^2\left(Q^2+r_H^2\right)^2\Big(k^4\left(-\left(r^2-r_H^2\right)^2\right)\left(2Q^2+r^2+r_H^2\right)^2\Big(2Q^2\left(6r^2-r_H^2\right)$$
$$+15r^4-r_H^4\Big)+2k^2\omega^2\Big(2Q^8\left(3r^4-7r^2r_H^2+2r_H^4\right)+Q^6\left(57r^6-106r^4r_H^2+29r^2r_H^4+4r_H^6\right)$$
$$+ Q^4\left(110r^8-158r^6r_H^2-13r^4r_H^4+36r^2r_H^6+r_H^8\right)$$
$$+ Q^2r^2\left(72r^8-58r^6r_H^2-79r^4r_H^4+40r^2r_H^6+9r_H^8\right)+r^4\left(15r^8-29r^4r_H^4+10r_H^8\right)\Big)$$
$$+ \omega^4\left(Q^2+r^2\right)^2\Big(2Q^6\left(r^2+r_H^2\right)+Q^4\left(-13r^4+32r^2r_H^2+r_H^4\right)$$
$$+ Q^2\left(-42r^6+54r^4r_H^2+16r^2r_H^4\right)-15r^8+27r^4r_H^4\Big)\Big)\Big]\Big/\Big[\left(Q^2+r^2\right)^3\left(Q^2+3r^2\right)$$
$$\left(r^2-r_H^2\right)\left(2Q^2+r^2+r_H^2\right)\left(\omega^2\left(Q^2+r^2\right)^2-k^2\left(r^2-r_H^2\right)\left(2Q^2+r^2+r_H^2\right)\right)$$
$$\left(k^2\left(Q^2\left(2r_H^2-4r^2\right)-3r^4+r_H^4\right)+\omega^2\left(Q^4+4Q^2r^2+3r^4\right)\right)\Big], \tag{A.10}$$

$$C_7 = -\Big[kQr\left(Q^2+r_H^2\right)\Big(k^2\left(2Q^4\left(2r^2+r_H^2\right)+Q^2\left(5r^4+6r^2r_H^2+r_H^4\right)+3r^2\left(r^4+r_H^4\right)\right)$$
$$+ \omega^2\left(Q^2-3r^2\right)\left(Q^2+r^2\right)^2\Big)\Big]\Big/\Big[L\left(Q^2+3r^2\right)\Big(\omega^2\left(Q^2+r^2\right)^2$$
$$- k^2\left(r^2-r_H^2\right)\left(2Q^2+r^2+r_H^2\right)\Big)\Big(k^2\left(Q^2\left(2r_H^2-4r^2\right)-3r^4+r_H^4\right)$$
$$+ \omega^2\left(Q^4+4Q^2r^2+3r^4\right)\Big)\Big], \tag{A.11}$$

$$C_8 = \Big[k^2\left(r^2-r_H^2\right)\Big(4Q^6\left(r_H^2-5r^2\right)+4Q^4\left(2r^4-11r^2r_H^2+r_H^4\right)$$
$$+ Q^2\left(27r^6-9r^4r_H^2-35r^2r_H^4+r_H^6\right)+9r^2\left(r^2-r_H^2\right)\left(r^2+r_H^2\right)^2\Big)$$
$$- \omega^2\left(Q^2+r^2\right)^2\left(Q^2+3r^2\right)\left(2Q^2\left(r^2+r_H^2\right)+3r^4+r_H^4\right)\Big]\Big/\Big[r\left(Q^2+3r^2\right)\left(r^2-r_H^2\right)$$
$$\left(2Q^2+r^2+r_H^2\right)\Big(k^2\left(r^2-r_H^2\right)\left(2Q^2+r^2+r_H^2\right)-\omega^2\left(Q^2+r^2\right)^2\Big)\Big], \tag{A.12}$$

$$C_9 = \frac{\sqrt{\tfrac{2}{3}}kQr\left(Q^2+r_H^2\right)}{L\left(Q^2+r^2\right)^2}, \tag{A.13}$$

$$C_{10} = \Big[4kr^2\left(2Q^3+3Qr^2\right)\left(Q^2+r_H^2\right)^3\Big]\Big/\Big[L\left(Q^2+r^2\right)\left(Q^2+3r^2\right)\left(r^2-r_H^2\right)$$
$$\left(2Q^2+r^2+r_H^2\right)\left(k^2\left(Q^2\left(2r_H^2-4r^2\right)-3r^4+r_H^4\right)+\omega^2\left(Q^4+4Q^2r^2+3r^4\right)\right)\Big], \tag{A.14}$$

$$
\begin{aligned}
C_{11} = \Big[&- L^4 \left(Q^4 + 4r^2 Q^2 + 3r^4 \right) \left(r^2 - r_H^2 \right)^2 \left(2Q^2 + r^2 + r_H^2 \right)^2 \left(3r^4 - r_H^4 + Q^2 \left(4r^2 - 2r_H^2 \right) \right) k^6 \\
&- \left(r^2 - r_H^2 \right) \left(2Q^2 + r^2 + r_H^2 \right) \Big(2 \left(3r_H^2 \omega^2 L^4 + 64r^4 - r^2 \left(5\omega^2 L^4 + 64r_H^2 \right) \right) Q^{10} \\
&+ \left(256r^6 + \left(64r_H^2 - 71L^4\omega^2 \right) r^4 - 40 \left(8r_H^4 - L^4 r_H^2 \omega^2 \right) r^2 + 3L^4 r_H^4 \omega^2 \right) Q^8 \\
&+ 4r^2 \left(24r^6 + 2 \left(64r_H^2 - 23L^4\omega^2 \right) r^4 + \left(23L^4 r_H^2 \omega^2 - 88r_H^4 \right) r^2 - 64r_H^6 + 5L^4 r_H^4 \omega^2 \right) Q^6 \\
&+ 2 \Big(3 \left(32r_H^2 - 37L^4\omega^2 \right) r^8 + 4 \left(11r_H^2 \omega^2 L^4 + 32r_H^4 \right) r^6 + \left(23L^4 r_H^4 \omega^2 - 192r_H^6 \right) r^4 \\
&- 32r_H^8 r^2 \Big) Q^4 + \left(-126L^4\omega^2 r^{10} + 6 \left(5r_H^2 \omega^2 L^4 + 16r_H^4 \right) r^8 + 44L^4 r_H^4 \omega^2 r^6 - 96r_H^8 r^4 \right) Q^2 \\
&+ 3L^4 r^8 \left(5r_H^4 - 9r^4 \right) \omega^2 \Big) k^4 + \omega^2 \Big(\left(6r_H^2 \omega^2 L^4 + 64r^4 + 32r_H^4 - 8r^2 \left(\omega^2 L^4 + 12r_H^2 \right) \right) Q^{14} \\
&+ \left(592r^6 - \left(77\omega^2 L^4 + 832r_H^2 \right) r^4 + 8 \left(7r_H^2 \omega^2 L^4 + 18r_H^4 \right) r^2 + 96r_H^6 + 3L^4 r_H^4 \omega^2 \right) Q^{12} \\
&+ 2 \Big(588r^8 - 6 \left(25\omega^2 L^4 + 32r_H^2 \right) r^6 + \left(103L^4 r_H^2 \omega^2 - 928r_H^4 \right) r^4 + 2 \left(240r_H^6 + 7L^4 \omega^2 r_H^4 \right) r^2 \\
&+ 52r_H^8 \Big) Q^{10} + \Big(784r^{10} + \left(1776r_H^2 - 617L^4\omega^2 \right) r^8 - 128 \left(26r_H^4 - 3L^4 r_H^2 \omega^2 \right) r^6 \\
&+ \left(103L^4 r_H^4 \omega^2 - 480r_H^6 \right) r^4 + 1200r_H^8 r^2 + 48r_H^{10} \Big) Q^8 + 2 \Big(84r^{12} + 28 \left(28r_H^2 - 13L^4\omega^2 \right) r^{10} \\
&+ \left(193L^4 r_H^2 \omega^2 - 132r_H^4 \right) r^8 - 32 \left(49r_H^6 - 3L^4 r_H^4 \omega^2 \right) r^6 + 540r_H^8 r^4 + 288r_H^{10} r^2 + 4r_H^{12} \Big) Q^6 \\
&+ r^2 \Big(96r_H^{12} + 720r^2 r_H^{10} - 784r^4 r_H^8 + r^{10} \left(336r_H^2 - 495L^4\omega^2 \right) + 8r^8 \left(25r_H^2 \omega^2 L^4 + 98r_H^4 \right) \\
&+ r^6 \left(193L^4 r_H^4 \omega^2 - 1152r_H^6 \right) \Big) Q^4 - 2r^4 \Big(-60r_H^{12} + 144r^4 r_H^8 - 50L^4 r^6 \omega^2 r_H^4 + 90L^4 r^{10} \omega^2 \\
&- 21r^8 \left(r_H^2 \omega^2 L^4 + 4r_H^4 \right) \Big) Q^2 + 3L^4 r^{12} \left(7r_H^4 - 9r^4 \right) \omega^2 \Big) k^2 \\
&- \left(Q^4 + 4r^2 Q^2 + 3r^4 \right)^2 \omega^4 \Big(\left(-\omega^2 L^4 + 16r^2 - 16r_H^2 \right) Q^8 + 4 \Big(2r^4 + \left(8r_H^2 - L^4\omega^2 \right) r^2 \\
&- 10r_H^4 \Big) Q^6 + 2 \left(-16r_H^6 + 8r^2 r_H^4 + r^4 \left(8r_H^2 - 3L^4\omega^2 \right) \right) Q^4 - 4 \left(2r_H^8 - 2r^4 r_H^4 + L^4 r^6 \omega^2 \right) Q^2 \\
&- L^4 r^8 \omega^2 \Big) \Big] \Big/ \Big[\left(Q^2 + r^2 \right) \left(Q^2 + 3r^2 \right) \left(r^2 - r_H^2 \right)^2 \left(2Q^2 + r^2 + r_H^2 \right)^2 \Big(\left(Q^2 + r^2 \right)^2 \omega^2 \\
&- k^2 \left(r^2 - r_H^2 \right) \left(2Q^2 + r^2 + r_H^2 \right) \Big) \Big(\left(-3r^4 + r_H^4 + Q^2 \left(2r_H^2 - 4r^2 \right) \right) k^2 \\
&+ \left(Q^4 + 4r^2 Q^2 + 3r^4 \right) \omega^2 \Big) \Big],
\end{aligned}
\tag{A.15}
$$

$$
\begin{aligned}
C_{12} = \Big[&4\sqrt{\tfrac{2}{3}} \, k Q r^2 \left(Q^2 + r_H^2 \right) \Big(k^4 \left(-Q^2 \right) \left(r^2 - r_H^2 \right)^2 \left(2Q^2 + r^2 + r_H^2 \right)^2 \Big(4Q^4 \\
&+ 2Q^2 \left(3r^2 + r_H^2 \right) + 3r^4 + r_H^4 \Big) - k^2 \omega^2 \left(Q^2 + r^2 \right)^2 \Big(2Q^8 \left(2r^2 + r_H^2 \right) \\
&+ Q^6 \left(-7r^4 + 50r^2 r_H^2 - 19r_H^4 \right) + Q^4 \left(-15r^6 + 58r^4 r_H^2 + 13r^2 r_H^4 - 20r_H^6 \right) \\
&+ Q^2 \left(-6r^8 + 18r^6 r_H^2 + 29r^4 r_H^4 - 12r^2 r_H^6 - 5r_H^8 \right) + 9r^6 r_H^4 - 3r^2 r_H^8 \Big) + 3\omega^4 \left(Q^2 + r^2 \right)^4 \\
&\left(Q^6 + Q^4 \left(r^2 + 4r_H^2 \right) + Q^2 \left(-r^4 + 6r^2 r_H^2 + 2r_H^4 \right) + 3r^2 r_H^4 \right) \Big) \Big] \Big/ \Big[L \left(Q^2 + r^2 \right)^3 \\
&\left(Q^2 + 3r^2 \right) \left(r^2 - r_H^2 \right) \left(2Q^2 + r^2 + r_H^2 \right) \left(\omega^2 \left(Q^2 + r^2 \right)^2 - k^2 \left(r^2 - r_H^2 \right) \left(2Q^2 + r^2 + r_H^2 \right) \right) \\
&\left(k^2 \left(Q^2 \left(2r_H^2 - 4r^2 \right) - 3r^4 + r_H^4 \right) + \omega^2 \left(Q^4 + 4Q^2 r^2 + 3r^4 \right) \right) \Big],
\end{aligned}
\tag{A.16}
$$

$$C_{13} = \left[4\sqrt{6}Q^2 r \left(k^2 - \omega^2\right) \left(Q^2 + r^2\right)^3 \right] / \left[\left(Q^2 + 3r^2\right) \left(\omega^2 \left(Q^2 + r^2\right)^2 \right. \right.$$
$$\left. - k^2 \left(r^2 - r_H^2\right) \left(2Q^2 + r^2 + r_H^2\right)\right) \left(k^2 \left(Q^2 \left(2r_H^2 - 4r^2\right) - 3r^4 + r_H^4\right) \right.$$
$$\left. \left. + \omega^2 \left(Q^4 + 4Q^2 r^2 + 3r^4\right)\right) \right], \tag{A.17}$$

$$C_{14} = \frac{8\sqrt{6}kLQ \left(Q^2 + r^2\right) \left(Q^2 + r_H^2\right)}{r \left(Q^2 + 3r^2\right) \left(k^2 \left(r^2 - r_H^2\right) \left(2Q^2 + r^2 + r_H^2\right) - \omega^2 \left(Q^2 + r^2\right)^2 \right)}, \tag{A.18}$$

$$C_{15} = \frac{Q^2 \left(6r^2 - 2r_H^2\right) + 5r^4 - r_H^4}{r \left(r^2 - r_H^2\right) \left(2Q^2 + r^2 + r_H^2\right)}, \tag{A.19}$$

$$C_{16} = - \left[4\sqrt{6}Q^2 \left(Q^2 + r^2\right) \left(Q^2 + r_H^2\right)^2 \right] / \left[\left(Q^2 + 3r^2\right) \left(r^2 - r_H^2\right) \left(2Q^2 + r^2 + r_H^2\right) \right.$$
$$\left. \left(k^2 \left(Q^2 \left(2r_H^2 - 4r^2\right) - 3r^4 + r_H^4\right) + \omega^2 \left(Q^4 + 4Q^2 r^2 + 3r^4\right)\right) \right], \tag{A.20}$$

$$C_{17} = \left[32\sqrt{6}kLQ^3 \left(k^2 - \omega^2\right) \left(Q^2 + r^2\right) \left(Q^2 + r_H^2\right) \right] / \left[\left(Q^2 + 3r^2\right) \left(\omega^2 \left(Q^2 + r^2\right)^2 \right. \right.$$
$$\left. - k^2 \left(r^2 - r_H^2\right) \left(2Q^2 + r^2 + r_H^2\right)\right) \left(k^2 \left(Q^2 \left(2r_H^2 - 4r^2\right) - 3r^4 + r_H^4\right) \right.$$
$$\left. \left. + \omega^2 \left(Q^4 + 4Q^2 r^2 + 3r^4\right)\right) \right], \tag{A.21}$$

$$C_{18} = \left[-L^4 \left(Q^4 + 4r^2 Q^2 + 3r^4\right) \left(r^2 - r_H^2\right)^2 \left(2Q^2 + r^2 + r_H^2\right)^2 \left(3r^4 - r_H^4 + Q^2 \left(4r^2 - 2r_H^2\right)\right) k^6 \right.$$
$$+ \left(r^2 - r_H^2\right) \left(2Q^2 + r^2 + r_H^2\right) \left(2 \left(-3r_H^2 \omega^2 L^4 + 48r^4 + 40r_H^4 + r^2 \left(5L^4 \omega^2 - 88r_H^2\right)\right) Q^{10} \right.$$
$$+ \left(200r^6 + \left(71L^4 \omega^2 - 144r_H^2\right) r^4 - 40 \left(r_H^2 \omega^2 L^4 + 5r_H^4\right) r^2 + 144r_H^6 - 3L^4 r_H^4 \omega^2\right) Q^8$$
$$+ 4\left(73r^8 + \left(46L^4 \omega^2 - 44r_H^2\right) r^6 - \left(23r_H^2 \omega^2 L^4 + 30r_H^4\right) r^4 - \left(28r_H^6 + 5L^4 \omega^2 r_H^4\right) r^2 \right.$$
$$\left. + 29r_H^8\right) Q^6 + \left(324r^{10} + \left(222L^4 \omega^2 - 256r_H^2\right) r^8 - 8 \left(11r_H^2 \omega^2 L^4 + 5r_H^4\right) r^6 \right.$$
$$- 2 \left(24r_H^6 + 23L^4 \omega^2 r_H^4\right) r^4 - 28r_H^8 r^2 + 48r_H^{10}\right) Q^4 + 2\left(90r^{12} + \left(63L^4 \omega^2 - 48r_H^2\right) r^{10} \right.$$
$$- \left(15r_H^2 \omega^2 L^4 + 64r_H^4\right) r^8 + \left(24r_H^6 - 22L^4 r_H^4 \omega^2\right) r^6 - 6r_H^8 r^4 + 4r_H^{12}\right) Q^2$$
$$\left. + 3r^6 \left(12r^8 + 9L^4 \omega^2 r^6 - 16r_H^4 r^4 - 5L^4 r_H^4 \omega^2 r^2 + 4r_H^8\right)\right) k^4 + \left(Q^2 + r^2\right) \omega^2 \left(2\left(3r_H^2 \omega^2 L^4 \right. \right.$$
$$+ 8r^4 - 16r_H^4 + r^2 \left(8r_H^2 - 4L^4 \omega^2\right)\right) Q^{12} + \left(-216r^6 + \left(720r_H^2 - 69L^4 \omega^2\right) r^4 \right.$$
$$\left. + 50 \left(L^4 r_H^2 \omega^2 - 12r_H^4\right) r^2 + 96r_H^6 + 3L^4 r_H^4 \omega^2\right) Q^{10} + \left(-672r^8 + \left(1216r_H^2 - 231L^4 \omega^2\right) r^6 \right.$$
$$+ 4 \left(39r_H^2 \omega^2 L^4 + 2r_H^4\right) r^4 + \left(25L^4 r_H^4 \omega^2 - 736r_H^6\right) r^2 + 184r_H^8\right) Q^8 + \left(-1096r^{10} \right.$$
$$+ 2 \left(720r_H^2 - 193L^4 \omega^2\right) r^8 + 4 \left(57r_H^2 \omega^2 L^4 + 64r_H^4\right) r^6 + \left(78L^4 r_H^4 \omega^2 - 352r_H^6\right) r^4$$
$$\left. - 344r_H^8 r^2 + 96r_H^{10}\right) Q^6 - 2\left(492r^{12} + \left(171L^4 \omega^2 - 488r_H^2\right) r^{10} - \left(79r_H^2 \omega^2 L^4 + 264r_H^4\right) r^8 \right.$$
$$\left. + \left(176r_H^6 - 57L^4 r_H^4 \omega^2\right) r^6 + 44r_H^8 r^4 + 48r_H^{10} r^2 - 8r_H^{12}\right) Q^4 + r^2 \left(-432r^{12} \right.$$
$$+ 3 \left(80r_H^2 - 51L^4 \omega^2\right) r^{10} + \left(42r_H^2 \omega^2 L^4 + 488r_H^4\right) r^8 + \left(79L^4 r_H^4 \omega^2 - 192r_H^6\right) r^6 - 88r_H^8 r^4$$
$$\left. \left. - 16r_H^{12}\right) Q^2 - 3r^8 \left(24r^8 + 9L^4 \omega^2 r^6 - 40r_H^4 r^4 - 7L^4 r_H^4 \omega^2 r^2 + 16r_H^8\right)\right) k^2$$
$$- \left(Q^2 + r^2\right)^4 \omega^4 \left(\left(-\omega^2 L^4 + 24r^2 - 24r_H^2\right) Q^8 - 4 \left(3r^4 + \left(2L^4 \omega^2 - 30r_H^2\right) r^2 + 27r_H^4\right) Q^6 \right.$$

$$
- 2 \left(42r^6 + \left(11L^4\omega^2 - 60r_H^2\right)r^4 - 30r_H^4 r^2 + 48r_H^6\right)Q^4 - 12\Big(9r^8 + \left(2L^4\omega^2 - 6r_H^2\right)r^6
$$
$$
- 5r_H^4 r^4 + 2r_H^8\Big)Q^2 - 9r^6\left(r^2\omega^2 L^4 + 4r^4 - 4r_H^4\right)\Big)\Big]\Big/\Big[\left(Q^2 + r^2\right)\left(Q^2 + 3r^2\right)\left(r^2 - r_H^2\right)^2
$$
$$
\left(2Q^2 + r^2 + r_H^2\right)^2\left(\left(Q^2 + r^2\right)^2\omega^2 - k^2\left(r^2 - r_H^2\right)\left(2Q^2 + r^2 + r_H^2\right)\right)\Big(\Big(-3r^4 + r_H^4
$$
$$
+ Q^2\left(2r_H^2 - 4r^2\right)\Big)k^2 + \left(Q^4 + 4r^2 Q^2 + 3r^4\right)\omega^2\Big)\Big]. \tag{A.22}
$$

B Black hole thermodynamics

In this appendix we discuss the thermodynamics of the charged black hole solution of the theory whose action is the sum of (2.1), (5.6), (5.7) and (5.8).

B.1 Adding Gibbons-Hawking boundary terms

The total action is

$$
I = I_0 + I_{hd} + I_a + I_{pot}, \tag{B.1}
$$

and must be supplemented by boundary terms which compensate for the variations of derivatives of the fields on the boundary of AdS, $r = \infty$. For the two-derivative action I_0, this is the Gibbons-Hawking term

$$
I_{GH} = -\frac{1}{8\pi}\int d^4 x\sqrt{|h|}\,K, \tag{B.2}
$$

where h_{ab} is the metric on the boundary, K is the trace of the extrinsic curvature tensor $K_a^b = \nabla_a n^b$, and the unit vector normal to the boundary is given by

$$
n^r = \sqrt{g^{rr}}, \quad n^a = 0, \ \ a \neq r. \tag{B.3}
$$

For the Gauss-Bonnet term I_{hd}, the necessary boundary term is (see e.g. [75, 76])

$$
I_{bd} = \frac{1}{4\pi}\int d^4 x\sqrt{-h}\,(\beta_1 e^{\gamma_1\phi} + \beta_2 e^{\gamma_2\phi})\left(KK_a^b K_b^a - \frac{1}{3}(K^3 + 2K_b^a K_c^b K_a^c)\right). \tag{B.4}
$$

Consider first the two-derivative action I_0. Evaluating it on the ansatz

$$
ds^2 = e^{2a(r)}(h(r)dt^2 - dx^2 - dy^2 - dz^2) - \frac{e^{2b(r)}}{h(r)}dr^2, \tag{B.5}
$$

with dilaton $\phi(r)$ and gauge potential $A_t(r)$, we obtain

$$
I_0 = \frac{1}{16\pi}\int d^5 x e^{4a+b}\left(-4e^{-\sqrt{\frac{2}{3}}\phi} - 8e^{\frac{\phi}{\sqrt{6}}} + \frac{1}{2}e^{-2b}h\left(8a'\left(5a' - 2b'\right) + \phi'^2 + 16a''\right) + \right.
$$
$$
\left. + e^{-2b}\left(-4e^{-2a+\sqrt{\frac{2}{3}}\phi}A_t'^2 + \left(9a' - b'\right)h' + h''\right)\right). \tag{B.6}
$$

Despite the presence of the second derivatives of the fields h'' and a'' in the action (B.6), we have a well-defined variational problem due to the boundary term (B.2). Now, evaluating the Gauss-Bonnet term I_{hd} on the ansatz (B.5), we obtain

$$
I_{hd} = \frac{1}{16\pi}\int d^5 x\, 12(\beta_1 e^{\gamma_1\phi} + \beta_2 e^{\gamma_2\phi})e^{4a-3b}a'\left(a'h'^2 + 2h^2 a'\left(5a'^2 - 4a'b' + 4a''\right) + \right.
$$
$$
\left. + h\left(h'\left(9a'^2 - 3a'b' + 2a''\right) + a'h''\right)\right). \tag{B.7}
$$

Again, the action depends on the second derivatives of the fields. Thanks to the boundary terms (B.4), we have a well-defined variational problem.

B.2 Scaling charge

Following [32], we note that the total action $I_{\rm tot}$ possesses a scaling symmetry

$$(x, y, z) \to c(x, y, z), \quad t \to t/c^3, \quad b \to b + 4 \log c, \quad a \to a - \log c, \quad h \to c^8 h, \quad A_t \to c^3 A_t. \tag{B.8}$$

The corresponding conserved charge is given by

$$\mathcal{Q} = -8C_0 A_t + e^{4a-b} h' + 12\beta e^{4a-3b} h a' h' \left(e^{\gamma_1 \phi} \left(a' + 2\gamma_1 \phi' \right) + \chi e^{\gamma_2 \phi} \left(a' + 2\gamma_2 \phi' \right) \right), \tag{B.9}$$

where we have used the equation of motion for the gauge field

$$A_t' = C_0 e^{-2a + b - \sqrt{2/3}\phi}, \tag{B.10}$$

and where C_0 is a constant. We have also denoted $\beta_1 = \beta$ and $\beta_2 = \chi\beta$. For our choice (5.18), $\chi = 1/2$. We choose the integration constant, which is related to the field theory charge density, to be

$$C_0 = Q(Q^2 + r_H^2). \tag{B.11}$$

which is consistent with the two-derivative solution.

To $\mathcal{O}(\beta)$, the charge conservation equation $\mathcal{Q}(r = r_H) = \mathcal{Q}(r = \infty)$ can be written

$$-8C_0 A_t(\infty) + (1 + 18\beta) r^5 h'|_{r=\infty} + 48(1 + \chi)\beta(r_H^2 + Q^2)^2 = 16\pi T s, \tag{B.12}$$

where we have used

$$s = \frac{1}{4} e^{3a(r_H)}, \quad T = \frac{1}{4\pi} e^{a-b} h'|_{r=r_H}, \quad A_t(r_H) = 0, \tag{B.13}$$

the asymptotic near-boundary behavior $e^{4a-b} = e^{\delta} r^5 = (1 + 18\beta) r^5$, and equation (5.18); the near-boundary behavior of δ is derived in the next subsection. At the two-derivative level, this is identical to the thermodynamic relation

$$-P = \varepsilon - Ts - \mu\sigma, \quad P = \varepsilon/3. \tag{B.14}$$

Our strategy will be to evaluate numerically this equation at $\mathcal{O}(\beta)$ and solve for $A_t(\infty)$ at this order. This allows us to determine how the chemical potential depends upon r_H and Q at $\mathcal{O}(\beta)$.

B.3 Temperature and chemical potential

In the remainder of this appendix we are going to describe how to numerically construct the black hole solution to $\mathcal{O}(\beta)$, and then explain how, by fine tuning the parameters c_1, $d_{1,2,3}$ and $w_{1,2,3}$ in the extra dilaton potential terms I_{pot}, one can change the functional dependence of T and μ on r_H and Q. It is convenient to choose the ansatz (as in [77])

$$ds^2 = B(\rho) e^{-2\delta(\rho)} dt^2 - \rho^2 (dx^2 + dy^2 + dz^2) - \frac{d\rho^2}{B(\rho)}, \tag{B.15}$$

where ρ is a new radial coordinate. In this ansatz the total Lagrangian, including the Gibbons-Hawking terms (B.2) and (B.4), is

$$L_{\text{tot}} = \frac{1}{2}e^{-\delta}\rho\left(2\rho\left(-4e^{-\sqrt{\frac{2}{3}}\phi}\rho - 8e^{\frac{\phi}{\sqrt{6}}}\rho - 4e^{2\delta+\sqrt{\frac{2}{3}}\phi}\rho A_t'^2 - 3B'\right) + B\left(-12 + 12\rho\delta' + \rho^2\phi'^2\right)\right)$$
$$+ 3\beta e^{-\delta+\gamma_1\phi}\left(a_1\rho^3 + B\left(-b_1\rho^3\phi'^2 - 4B'\left(1 + 3\gamma_1\rho\phi'\right)\right.\right.$$
$$+ B\left(-8\gamma_1\phi' + \left(c_1 + d_1e^{w_1\phi} + d_2e^{w_2\phi} + d_3e^{w_3\phi}\right)\rho^3\phi'^4 + 8\delta'\left(1 + 3\gamma_1\rho\phi'\right)\right)\right)\right)$$
$$+ 3\chi\beta e^{-\delta+\gamma_2\phi}\left(a_2\rho^3 + B\left(-b_2\rho^3\phi'^2 - 4B'\left(1 + 3\gamma_2\rho\phi'\right) + B\left(-8\gamma_2\phi' + 8\delta'\left(1 + 3\gamma_2\rho\phi'\right)\right)\right)\right).$$
(B.16)

It is convenient to work with the dimensionless parameter $\theta = r_H/Q$.

First of all, we solve for the gauge field

$$A_t' = C_0 e^{-\delta-\sqrt{\frac{2}{3}}\phi}\rho^{-3},$$
(B.17)

with C_0 given in (B.11). Then we solve the equations of motion for δ, B and ϕ perturbatively up to $\mathcal{O}(\beta)$. Denoting the background (two-derivative) solution as δ_0, B_0 and ϕ_0, the equations for the $\mathcal{O}(\beta)$ corrections to the background, which we denote as δ_1, B_1 and ϕ_1, are

$$3e^{\frac{7\phi_0}{\sqrt{6}}}\rho\,B_1\left(12 + \rho^2\phi_0'^2\right) - \sqrt{6}e^{\frac{7\phi_0}{\sqrt{6}}}\rho\phi_1\left(6\rho\left(4e^{\frac{\phi_0}{\sqrt{6}}}\rho - B_0'\right) - B_0\left(12 + \rho^2\phi_0'^2\right)\right)$$
$$+ 3\left(3\left(a_2 + 2a_1e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\right)\rho^3 + 6e^{\frac{7\phi_0}{\sqrt{6}}}\rho^2B_1' + B_0\left(\rho^3\phi_0'\left(2e^{\frac{7\phi_0}{\sqrt{6}}}\phi_1' - 3\left(b_2 + 2b_1e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\right)\phi_0'\right)\right.\right.$$
$$+ 18B_0'\left(2 - 7\sqrt{6}\rho\phi_0' + e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\left(4 + 7\sqrt{6}\rho\phi_0'\right)\right) + 6B_0\left(14\sqrt{6}\left(-1 + e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\right)\phi_0'\right.$$
$$+ 49\left(2 + e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\right)\rho\phi_0'^2 + e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\left(c_1 + d_1e^{w_1\phi_0} + d_2e^{w_2\phi_0} + d_3e^{w_3\phi_0}\right)\rho^3\phi_0'^4$$
$$+ 14\sqrt{6}\left(-1 + e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\right)\rho\phi_0''\right)\right)\right) = 0,$$
(B.18)

$$2e^{\frac{7\phi_0}{\sqrt{6}}}\rho^2\phi_1\left(72e^{\frac{\phi_0}{\sqrt{6}}}\rho + \sqrt{6}\left(\phi_0'\left(6\rho B_0' + B_0\left(18 + \rho^2\phi_0'^2\right)\right) + 6\rho B_0\phi_0''\right)\right)$$
$$+ 3\left(\rho B_0^2\phi_0'\left(18\left(32 + 3e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\left(5 + 8c_1 + 8d_1e^{w_1\phi_0} + 8d_2e^{w_2\phi_0} + 8d_3e^{w_3\phi_0}\right)\right)\rho\phi_0'^2\right.\right.$$
$$+ \left(56\sqrt{6} + e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\left(-56\sqrt{6} + 63\sqrt{6}c_1 + 9\left(d_1e^{w_1\phi_0}\left(7\sqrt{6} + 12w_2\right)\right.\right.\right.$$
$$+ d_2e^{w_2\phi_0}\left(7\sqrt{6} + 12w_2\right) + d_3e^{w_3\phi_0}\left(7\sqrt{6} + 12w_3\right)\right)\right)\right)\rho^2\phi_0'^3$$
$$+ 48e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\left(c_1 + d_1e^{w_1\phi_0} + d_2e^{w_2\phi_0} + d_3e^{w_3\phi_0}\right)\rho^3\phi_0'^4 - 84\sqrt{6}\left(-1 + e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\right)\rho\phi_0''$$
$$- 24\phi_0'\left(7\sqrt{6}\left(-1 + e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\right) - 18e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\left(c_1 + d_1e^{w_1\phi_0} + d_2e^{w_2\phi_0} + d_3e^{w_3\phi_0}\right)\rho^2\phi_0''\right)\right)$$
$$+ \rho\left(21\sqrt{6}a_2\rho^2 + 36B_0'\left(7\sqrt{6}B_0' - b_2\rho^2\phi_0'\right) - 3e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\left(7\sqrt{6}a_1\rho^2 + 84\sqrt{6}B_0'^2 + 24b_1\rho^2B_0'\phi_0'\right)\right.$$
$$+ 2e^{\frac{7\phi_0}{\sqrt{6}}}\rho\left(6\rho B_0'\phi_1' + \phi_0'\left(6\rho B_1' + B_1\left(18 + \rho^2\phi_0'^2\right)\right) + 6\rho B_1\phi_0''\right)$$

$$
\begin{aligned}
&- 3B_0\left(2B_0'\left(84\sqrt{6}\left(-1+e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\right)-\rho^2\phi_0'^2\left(-35\sqrt{6}\left(-1+e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\right)\right.\right.\right. \\
&+ \left.\left.48e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\left(c_1+d_1e^{w_1\phi_0}+d_2e^{w_2\phi_0}+d_3e^{w_3\phi_0}\right)\rho\phi_0'\right)\right)+\rho\left(-84\sqrt{6}B_0''\right. \\
&- 2e^{\frac{7\phi_0}{\sqrt{6}}}\rho\left(\phi_1'\left(6+\rho^2\phi_0'^2\right)+2\rho\phi_1''\right)+b_2\rho\left(36\phi_0'-7\sqrt{6}\rho\phi_0'^2+4\rho^2\phi_0'^3+12\rho\phi_0''\right) \\
&+ \left.\left.\left.\left.e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\left(84\sqrt{6}B_0''+b_1\rho\left(72\phi_0'+7\sqrt{6}\rho\phi_0'^2+8\rho^2\phi_0'^3+24\rho\phi_0''\right)\right)\right)\right)\right)\right)=0, \quad\text{(B.19)}
\end{aligned}
$$

$$
\begin{aligned}
-\delta_1' =\ &\frac{1}{3}\rho\phi_1'\phi_0'+\frac{1}{6\rho}e^{-\frac{7\phi_0}{\sqrt{6}}}\left(-3b_2\rho^2\phi_0'^2-6b_1e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\rho^2\phi_0'^2+288B_0\phi_0'^2+135e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}B_0\phi_0'^2\right. \\
&- 21\sqrt{6}\left(-1+e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\right)\rho B_0\phi_0'^3+12e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}\left(c_1+d_1e^{w_1\phi_0}+d_2e^{w_2\phi_0}+d_3e^{w_3\phi_0}\right)\rho^2B_0\phi_0'^4 \\
&- \left.42\sqrt{6}B_0\phi_0''+42\sqrt{6}e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0}B_0\phi_0''\right). \quad\text{(B.20)}
\end{aligned}
$$

The two-derivative solution can explicitly be written as

$$
B_0(\rho)=\frac{(3(r(\rho))^2+1)^2h(r(\rho))}{9(r(\rho))^2}\,,\ \phi_0(\rho)=\sqrt{\frac{2}{3}}\log\left(1+\frac{1}{(r(\rho))^2}\right)\,,\ \delta(\rho)=\log\frac{3(r(\rho))^2+1}{3\rho r(\rho)}\,, \quad\text{(B.21)}
$$

where

$$
r(\rho)=\frac{-2\cdot 3^{1/3}+2^{1/3}\left(9\rho^3+\sqrt{12+81\rho^6}\right)^{2/3}}{6^{2/3}\left(9\rho^3+\sqrt{12+81\rho^6}\right)^{1/3}}\,, \quad\text{(B.22)}
$$

is the solution to the equation

$$
e^{2a(r(\rho))}=\rho^2\,. \quad\text{(B.23)}
$$

Expanding near the boundary, $\rho\gg 1$, we find

$$
\phi_1=\frac{1}{\rho^2}(C_{f1}+C_{f2}\log\rho+C_{fp}\log^2\rho)+\dots, \quad\text{(B.24)}
$$

$$
B_1=36\rho^2+\frac{C_{b2}}{\rho^2}+\frac{C_{b3}\log\rho}{\rho^2}+\frac{C_{bp}\log^2\rho}{\rho^2}+\dots, \quad\text{(B.25)}
$$

where the expression (5.19) plays a key role. The constants C_{fp} and C_{bp} are independent of r_H and Q. By choosing

$$
a_1=a_2=-120\,,\quad b_1=b_2=4, \quad\text{(B.26)}
$$

the constants C_{fp} and C_{bp} are zero. We have verified this by fitting the numerical solution to the asymptotic formulae (B.24), (B.25) and making sure that the coefficients C_{f1}, C_{f2}, C_{b2}, C_{b3} are stable towards the change of the UV cut-off (the position of the boundary).

After we have solved for B_1 and ϕ_1, we can use this solution to integrate (B.20) to find δ_1. We fix the constant of integration in such a way that $g_{tt}/\rho^2=Be^{-2\delta}/\rho^2$ asymptotes to

one near the boundary, so that the speed of light is equal to one in the dual field theory. Using (B.24) and (B.25) we obtain for $\rho \gg 1$

$$\delta_1 = -\frac{-4 + 2\left(-2169 + 12b_1 + 6b_2 - 2\sqrt{6}C_{f1}\right)\beta}{36\rho^4} + 18\beta + \dots . \qquad (\text{B.27})$$

Therefore the near-boundary expansion of $g_{tt} = Be^{-2\delta}$ is given by

$$\frac{g_{tt}}{\rho^2} = \left(1 + \frac{-1 - 2\theta^2 - \theta^4}{\rho^4}\right) + \frac{\left(-1917 + 12b_1 + 6b_2 + 9C_{b2} - 2\sqrt{6}C_{f1} + 648\theta^2 + 324\theta^4\right)\beta}{9\rho^4}. \qquad (\text{B.28})$$

We want to extract $\delta g_{tt}^{(4)}$, the coefficient of the sub-leading $1/\rho^4$ term at $\mathcal{O}(\beta)$, as this will allow us to determine the chemical potential $A_t(\infty)$ from equation (B.12).[3] At $\mathcal{O}(\beta^0)$, this term, which controls the energy density of the dual field theory, is proportional to Q^4 at low T. It follows from (B.28) that, to find the $\mathcal{O}(\beta)$ correction to it, we need to find the coefficients C_{f1} and C_{b2} in the near-boundary expansions (B.24) and (B.25) of the dilaton and the metric.

Our numerical procedure is as follows. We solve equations (B.18) and (B.19) for ϕ_1 and B_1, with the horizon boundary conditions given by

$$B_1(\rho_H) = 0, \quad \phi_1(\rho_H) = c, \qquad (\text{B.30})$$

$$\begin{aligned}
\phi_1'(\rho_H) = \frac{1}{12\rho_H^2 B_0'(\rho_H)} e^{-\frac{7\phi_0(\rho_H)}{\sqrt{6}}} &\left(-21\sqrt{6}a_2\rho_H^2 + 21\sqrt{6}a_1 e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0(\rho_H)}\rho_H^2\right. \\
&- 48e^{4\sqrt{\frac{2}{3}}\phi_0(\rho_H)}\rho_H^2 c - 252\sqrt{6}B_0'(\rho_H)^2 + 252\sqrt{6}e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0(\rho_H)}B_0'(\rho_H)^2 \\
&+ 6a_2\rho_H^3\phi_0'(\rho_H) + 12a_1 e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0(\rho_H)}\rho_H^3\phi_0'(\rho_H) - 16\sqrt{6}e^{4\sqrt{\frac{2}{3}}\phi_0(\rho_H)}\rho_H^3 c\phi_0'(\rho_H) \\
&\left.+ 36b_2\rho_H^2 B_0'(\rho_H)\phi_0'(\rho_H) + 72b_1 e^{\frac{7}{2}\sqrt{\frac{3}{2}}\phi_0(\rho_H)}\rho_H^2 B_0'(\rho_H)\phi_0'(\rho_H)\right). \qquad (\text{B.31})
\end{aligned}$$

We then fix numerically the value of c such that $C_{f2} = C_{b3} = 0$ (we have verified numerically that both of these coefficients go to zero at the same time for the particular value of c). This ensures that ϕ_1 and B_1 behave, near the boundary, in the same way as ϕ_0 and B_0 respectively. For this value of c, we then extract numerically the coefficients C_{f1} and C_{b2} in the near-boundary expansions.

From (B.28) we find that the $\mathcal{O}(\beta)$ correction to the coefficient of ρ^{-4} in g_{tt} is

$$\frac{\delta g_{tt}^{(4)}}{Q^4} = -1917 + 12b_1 + 6b_2 + 9C_{b2} - 2\sqrt{6}C_{f1} + 648\theta^2 + 324\theta^4, \qquad (\text{B.32})$$

and the $\mathcal{O}(\beta)$ correction to the temperature is

$$\frac{\Delta T}{Q} = \widetilde{T}\left(\frac{B_1'(\rho_H)}{B_0'(\rho_H)} - \delta_1(\rho_H)\right), \qquad (\text{B.33})$$

where $\widetilde{T} = T/Q = \theta/\pi$ is the Q-normalized two-derivative result for the temperature.

---

[3]In equation (B.12) we have

$$r^5 h' = -4\left(-(1+k^2)^2 + \frac{\left(-1917 + 12b_1 + 6b_2 + 9C_{b2} - 2\sqrt{6}C_{f1} + 648\theta^2 + 324\theta^4\right)\beta}{9}\right). \qquad (\text{B.29})$$

After fixing w_1, w_2, w_3, we find numerically that, for generic c_1, $d_{1,2,3}$, one finds (for $\theta \ll 1$),

$$\frac{\delta g_{tt}^{(4)}}{Q^4} = \frac{C_1}{\theta} + \mathcal{O}(1), \tag{B.34}$$

$$\frac{\Delta T}{T} = \frac{C_2}{\theta^3} + \frac{C_3}{\theta^2} + \frac{C_4}{\theta} + \mathcal{O}(1). \tag{B.35}$$

By varying c_1, d_1, d_2, d_3, the four coefficients $C_{1,2,3,4}$ change. It is possible to make these coefficients numerically small, and we expect that by suitably fine tuning these coefficients, it is possible to produce a black hole solution with $C_{1,2,3,4} = 0$. However, this is a very expensive procedure to implement numerically and we have not been able to determine the required values of c_1 and $d_{1,2,3}$. Supposing that this solution exists, it would have $T \sim r_H$ and $\delta g_{tt}^{(4)} \sim Q^4$ to $\mathcal{O}(\beta)$ in the small T/μ limit. Equation (B.12) then implies that $\mu \sim Q$ in this limit, as required.

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