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Nonlinear discrete fractional mixed type sum-difference equation boundary value problems in Banach spaces

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Abstract

This paper is concerned with the existence of a unique solution to a nonlinear discrete fractional mixed type sum-difference equation boundary value problem in a Banach space. Under certain suitable nonlinear growth conditions imposed on the nonlinear term, the existence and uniqueness result is established by using the Banach contraction mapping principle. Additionally, two representative examples are presented to illustrate the effectiveness of the main result.

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Keywords: discrete fractional mixed type sum-difference equation; boundary value problem; Banach space; existence and uniqueness; Banach contraction mapping principle

1 Introduction

For $a, b \in \mathbb{R}$, such that $b - a$ is a nonnegative integer, we define $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b = \{a, a+1, \ldots, b\}$ throughout this paper. It is also worth noting that, in what follows, for any Banach-valued function u defined on $\mathbb{N}_a$, we appeal to the convention $\sum_{s=k_1}^{k_2} u(s) = \theta$, where $k_1, k_2 \in \mathbb{N}_a$ with $k_1 > k_2$ and θ is the zero element of a given Banach space.

In this paper, we will consider the existence of a unique solution to the following discrete fractional mixed type sum-difference equation boundary value problem in the Banach space E:

$$\begin{cases} \Delta^{\alpha} u(t) + f(t+\alpha-1, u(t+\alpha-1), (Tu)(t), (Su)(t)) = \theta, & t \in \mathbb{N}_0, \\ u(\alpha-n) = \Delta u(\alpha-n) = \Delta^2 u(\alpha-n) = \cdots = \Delta^{n-2} u(\alpha-n) = \theta, \\ \Delta^{\alpha-1} u(\infty) = u_{\infty}, \end{cases} \tag{1.1}$$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}_2$, Δ^{α} denotes the discrete Riemann-Liouville fractional difference of order α, $f : \mathbb{N}_{\alpha-1} \times E \times E \times E \to E$ is continuous, θ represents the zero element of E, $\Delta^{\alpha-1} u(\infty) = \lim_{t \to +\infty} \Delta^{\alpha-1} u(t) = u_{\infty} \in E$ and

$$(Tu)(t) = \sum_{s=0}^{t} k(t,s) u(s+\alpha-1), \qquad (Su)(t) = \sum_{s=0}^{\infty} h(t,s) u(s+\alpha-1),$$

where $k : D \to \mathbb{R}$, $D = \{(t,s) \in \mathbb{N}_0 \times \mathbb{N}_0 : s \leq t\}$, $h : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$.

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Discrete fractional calculus is a generalization of ordinary difference and summation on arbitrary order that can be non-integer, and it has gained considerable popularity due mainly to its demonstrated applications in describing some real-world phenomena [1, 2]. Among all the topics, the branch of discrete fractional boundary value problems is currently undergoing active investigation; see, for example, [3–17] and the references therein.

Boundary value problems for differential equations in Banach spaces have been studied by many authors [18–36]. Especially for the study of nonlinear mixed type integro-differential equations which arise from many nonlinear problems in science [36], a series of excellent results have been obtained in recent years [21, 23, 30–36].

On the other hand, it is well known that discrete analogues of differential equations can be very useful in applications [37, 38], in particular for using computer to simulate the behavior of solutions for certain dynamic equations. However, compared to continuous case, significantly less is known about discrete difference calculus in Banach spaces [39–45]. Furthermore, as far as we know, the theory of discrete fractional mixed type sum-difference equations boundary value problems in Banach spaces is still a new research area. So, in this paper, we focus on this gap and provide some sufficient conditions for the existence and uniqueness of solutions to problem (1.1).

The remainder of this paper is organized as follows. Section 2 preliminarily presents some necessary basic knowledge for the theory of discrete fractional calculus in Banach spaces. In Section 3, the existence and uniqueness result for the solution to problem (1.1) will be established with the help of the contraction mapping principle. Finally, in Section 4, two concrete examples are provided to illustrate the possible applications of the established analytical result.

2 Preliminaries

In this section, we firstly present the definitions for the discrete Riemann-Liouville fractional difference and the discrete fractional sum for Banach-valued functions similar to the corresponding definitions for real-valued functions [46–49].

Definition 2.1 ([46]) For any t and ν, the falling factorial function is defined as

$$t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$$

provided that the right-hand side is well defined. We appeal to the convention that if $t+1-\nu$ is a pole of the gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.2 The νth discrete fractional sum of a function $f : \mathbb{N}_a \to E$, for $\nu > 0$, is defined by

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s), \quad t \in \mathbb{N}_{a+\nu}.$$

Also, we define the trivial sum $\Delta_a^{-0} f(t) = f(t)$, $t \in \mathbb{N}_a$.

Definition 2.3 The νth discrete Riemann-Liouville fractional difference of a function $f : \mathbb{N}_a \to E$, for $\nu > 0$, is defined by

$$\Delta_a^{\nu} f(t) = \Delta^n \Delta_a^{-(n-\nu)} f(t), \quad t \in \mathbb{N}_{a+n-\nu},$$

where n is the smallest integer greater than or equal to ν and Δ^n is the nth order forward difference operator. If $\nu = n \in \mathbb{N}_1$, then $\Delta_a^n f(t) = \Delta^n f(t)$.

Remark 2.1 From Definitions 2.2 and 2.3, it is easy to see that $\Delta_a^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$ and Δ_a^{ν} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+n-\nu}$, where n is the smallest integer greater than or equal to ν. Also, it is worth reminding the reader that the t in $\Delta_a^{\nu} f(t)$ (or $\Delta_a^{-\nu} f(t)$) represents an input for the function $\Delta_a^{\nu} f$ (or $\Delta_a^{-\nu} f$) and not for the function f. For ease of notation, throughout this paper we omit the subscript a in $\Delta_a^{\nu} f(t)$ and $\Delta_a^{-\nu} f(t)$ when it does not lead to domain confusion and general ambiguity.

Now, we present the following two results, which are analogues to the ordinary case for the real-valued function.

Lemma 2.1 *Let $f : \mathbb{N}_a \to E$ and $\nu, \mu > 0$. Then*

$$\Delta_{a+\mu}^{-\nu} \Delta_a^{-\mu} f(t) = \Delta_a^{-\nu-\mu} f(t) = \Delta_{a+\nu}^{-\mu} \Delta_a^{-\nu} f(t), \quad t \in \mathbb{N}_{a+\mu+\nu}.$$

Lemma 2.2 *Let $f : \mathbb{N}_a \to E$, $\nu > 0$ and p be a positive integer. Then*

$$\Delta_a^{-\nu} \Delta^p f(t) = \Delta^p \Delta_a^{-\nu} f(t) - \sum_{i=1}^{p} \frac{(t-a)^{\underline{\nu-i}}}{\Gamma(\nu - i + 1)} \Delta^{p-i} f(a).$$

Remark 2.2 Lemma 2.1 and Lemma 2.2 are natural analogues of Theorem 2.2 in [46] and Theorem 2.2 in [47] for real-valued functions. Their proofs are similar to the ordinary case. So, here we omit them. Additionally, by using Lemma 2.1, we can easily obtain the equality $\Delta^{\nu} \Delta^{-\nu} f(t) = f(t)$, $\nu > 0$ holds, for any Banach-valued function f.

At last, we need to state the following lemmas, which will be important in the sequel.

Lemma 2.3 *Let $\nu > 0$ and $f : \mathbb{N}_a \to E$. Then*

$$\begin{aligned} \Delta_{a+n-\nu}^{-\nu} \Delta_a^{\nu} f(t) ={}& f(t) + c_1 (t - a - n + \nu)^{\underline{\nu-1}} \\ & + c_2 (t - a - n + \nu)^{\underline{\nu-2}} + \cdots + c_n (t - a - n + \nu)^{\underline{\nu-n}}, \end{aligned} \tag{2.1}$$

where $c_i \in E$, $i = 1, 2, \ldots, n$, and n is the smallest integer greater than or equal to ν.

Proof By Definition 2.3, Lemma 2.1, Lemma 2.2 and Remark 2.2, we have

$$\begin{aligned} & \Delta_{a+n-\nu}^{-\nu} \Delta_a^{\nu} f(t) \\ &\quad = \left(\Delta_{a+n-\nu}^{-\nu} \Delta^n \Delta_a^{-(n-\nu)} f \right)(t) \end{aligned}$$

$$= \bigl(\Delta^{n}\Delta_{a+n-\nu}^{-\nu}\Delta_{a}^{-(n-\nu)}f\bigr)(t) - \sum_{i=1}^{n}\frac{(t-a-n+\nu)^{\underline{\nu-i}}}{\Gamma(\nu-i+1)}\bigl[\bigl(\Delta^{n-i}\Delta_{a}^{-(n-\nu)}f\bigr)(a+n-\nu)\bigr]$$

$$= f(t) - \sum_{i=1}^{n}\frac{(t-a-n+\nu)^{\underline{\nu-i}}}{\Gamma(\nu-i+1)}\bigl[\bigl(\Delta^{n-i}\Delta_{a}^{-(n-\nu)}f\bigr)(a+n-\nu)\bigr].$$

Setting $c_i = -\frac{(\Delta^{n-i}\Delta_a^{-(n-\nu)}f)(a+n-\nu)}{\Gamma(\nu-i+1)}$, $i = 1, 2, \ldots, n$; then we get (2.1). So the proof is complete. □

Lemma 2.4 ([48]) *Let $a \in \mathbb{R}$ and $\mu > 0$ be given. Then*

$$\Delta(t-a)^{\underline{\mu}} = \mu(t-a)^{\underline{\mu-1}}$$

for any t for which both sides are well defined. Furthermore, for $n-1 < \nu \leq n$, $n \in \mathbb{N}_1$ and $\mu \in \mathbb{R} \setminus (-\mathbb{N}_1)$,

$$\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \mu^{\underline{-\nu}}(t-a)^{\underline{\mu+\nu}}, \quad t \in \mathbb{N}_{a+\mu+\nu},$$

and

$$\Delta_{a+\mu}^{\nu}(t-a)^{\underline{\mu}} = \mu^{\underline{\nu}}(t-a)^{\underline{\mu-\nu}}, \quad t \in \mathbb{N}_{a+\mu+n-\nu}.$$

3 Main results

In this section, we establish the existence of a unique solution to problem (1.1). To accomplish this, we firstly list here the following conditions.

(C$_1$) There exist constants k^* and h^* such that

$$k^* = \sup_{t\in\mathbb{N}_0}\sum_{s=0}^{t}\bigl|k(t,s)\bigr| < +\infty,$$

$$h^* = \sup_{t\in\mathbb{N}_0}\frac{1}{1+(t+\alpha-1)^{\underline{\alpha-1}}}\sum_{s=0}^{\infty}\bigl|h(t,s)\bigr|\bigl[1+(s+\alpha-1)^{\underline{\alpha-1}}\bigr] < +\infty.$$

(C$_2$) $f^* = \sum_{t=\alpha-1}^{\infty}\|f(t,\theta,\theta,\theta)\| < +\infty$, and there exist nonnegative numbers a, b, c and a function $p : \mathbb{N}_{\alpha-1} \to [0,\infty)$ with $p^* = \sum_{t=\alpha-1}^{\infty}p(t)(1+t^{\underline{\alpha-1}}) < +\infty$ such that

$$\bigl\|f(t,u,v,w) - f(t,\overline{u},\overline{v},\overline{w})\bigr\| \leq p(t)\bigl(a\|u-\overline{u}\| + b\|v-\overline{v}\| + c\|w-\overline{w}\|\bigr)$$

for $t \in \mathbb{N}_{\alpha-1}$, $u, v, w, \overline{u}, \overline{v}, \overline{w} \in E$.

Next, we define

$$X = \left\{u : \mathbb{N}_{\alpha-n} \to E \,\middle|\, \sup_{t\in\mathbb{N}_{\alpha-n}}\frac{\|u(t)\|}{1+t^{\underline{\alpha-1}}} < +\infty\right\}$$

equipped with the norm

$$\|u\|_X = \sup_{t\in\mathbb{N}_{\alpha-n}}\frac{\|u(t)\|}{1+t^{\underline{\alpha-1}}}.$$

Furthermore, by means of the linear functional analysis theory, we can easily prove that $(X, \|\cdot\|_X)$ is a Banach space.

Next, we state and prove the following lemmas, which will be used to establish the existence result of solutions to problem (1.1).

Lemma 3.1 *If* (C_1) *and* (C_2) *hold, then, for any* $u \in X$,

$$\sum_{t=0}^{\infty} \bigl\|f\bigl(t+\alpha-1, u(t+\alpha-1), (Tu)(t), (Su)(t)\bigr)\bigr\| \le p^*\bigl(a + bk^* + ch^*\bigr)\|u\|_X + f^*. \tag{3.1}$$

Proof Setting $\overline{u} = \overline{v} = \overline{w} = \theta$ in (C_2), we have

$$\begin{aligned}&\bigl\|f(t,u,v,w)\bigr\| \le p(t)\bigl(a\|u\| + b\|v\| + c\|w\|\bigr) + \bigl\|f(t,\theta,\theta,\theta)\bigr\|,\\ &\quad (t,u,v,w) \in \mathbb{N}_{\alpha-1} \times E \times E \times E.\end{aligned}$$

So, for any $u \in X$, $t \in \mathbb{N}_0$, using (C_2) again produces

$$\begin{aligned}&\bigl\|f\bigl(t+\alpha-1, u(t+\alpha-1), (Tu)(t), (Su)(t)\bigr)\bigr\|\\ &\quad \le p(t+\alpha-1)\bigl(a\bigl\|u(t+\alpha-1)\bigr\| + b\bigl\|(Tu)(t)\bigr\| + c\bigl\|(Su)(t)\bigr\|\bigr) + \bigl\|f(t+\alpha-1,\theta,\theta,\theta)\bigr\|\\ &\quad = p(t+\alpha-1)\bigl[1+(t+\alpha-1)^{\underline{\alpha-1}}\bigr]\biggl(a\frac{\|u(t+\alpha-1)\|}{1+(t+\alpha-1)^{\underline{\alpha-1}}} + b\frac{\|(Tu)(t)\|}{1+(t+\alpha-1)^{\underline{\alpha-1}}}\\ &\qquad + c\frac{\|(Su)(t)\|}{1+(t+\alpha-1)^{\underline{\alpha-1}}}\biggr) + \bigl\|f(t+\alpha-1,\theta,\theta,\theta)\bigr\|\\ &\quad \le p(t+\alpha-1)\bigl[1+(t+\alpha-1)^{\underline{\alpha-1}}\bigr]\bigl(a\|u\|_X + bk^*\|u\|_X + ch^*\|u\|_X\bigr)\\ &\qquad + \bigl\|f(t+\alpha-1,\theta,\theta,\theta)\bigr\|.\end{aligned} \tag{3.2}$$

Summating both sides of (3.2), we can get (3.1). The proof is completed. □

Lemma 3.2 *Let* $h : \mathbb{N}_0 \to E$ *be given and* $n-1 < \alpha \le n$, $n \in \mathbb{N}_2$. *The unique solution of*

$$\begin{cases}\Delta^\alpha u(t) + h(t) = \theta, \quad t \in \mathbb{N}_0,\\ u(\alpha-n) = \Delta u(\alpha-n) = \Delta^2 u(\alpha-n) = \cdots = \Delta^{n-2}u(\alpha-n) = \theta,\\ \Delta^{\alpha-1}u(\infty) = u_\infty,\end{cases} \tag{3.3}$$

is

$$u(t) = \sum_{s=0}^{\infty} G(t,s)h(s) + \frac{u_\infty}{\Gamma(\alpha)}t^{\underline{\alpha-1}}, \quad t \in \mathbb{N}_{\alpha-n},$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\underline{\alpha-1}} - (t-s-1)^{\underline{\alpha-1}}, & s \in \mathbb{N}_0^{t-\alpha},\\ t^{\underline{\alpha-1}}, & s \in \mathbb{N}_{t-\alpha+1}.\end{cases} \tag{3.4}$$

Proof Suppose that $u : \mathbb{N}_{\alpha-n} \to E$ satisfies the equation of problem (3.3), then Lemma 2.3 implies that

$$u(t) = -\frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} h(s) + c_1 t^{\underline{\alpha-1}} + c_2 t^{\underline{\alpha-2}} + \cdots + c_n t^{\underline{\alpha-n}}$$

for some $c_i \in E$, $i = 1, 2, \ldots, n$, $t \in \mathbb{N}_{\alpha-n}$. By $u(\alpha - n) = \theta$, we get $c_n = \theta$.

Furthermore, in view of Lemma 2.4, we have

$$\begin{aligned} \Delta u(t) = {} & -\frac{1}{\Gamma(\alpha-1)} \sum_{s=0}^{t-(\alpha-1)} (t-s-1)^{\underline{\alpha-2}} h(s) \\ & + c_1(\alpha-1) t^{\underline{\alpha-2}} + c_2(\alpha-2) t^{\underline{\alpha-3}} + \cdots + c_{n-1}(\alpha-n+1) t^{\underline{\alpha-n}}. \end{aligned} \tag{3.5}$$

Substituting $\Delta u(\alpha - n) = \theta$ in (3.5) gives $c_{n-1} = \theta$.

Repeating the above steps with $\Delta^2 u(\alpha - n) = \cdots = \Delta^{n-2} u(\alpha - n) = \theta$, we can get

$$c_{n-2} = c_{n-3} = \cdots = c_2 = \theta.$$

Therefore,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} h(s) + c_1 t^{\underline{\alpha-1}}, \quad t \in \mathbb{N}_{\alpha-n}. \tag{3.6}$$

By virtue of Lemma 2.4 again, we have

$$\Delta^{\alpha-1} u(t) = -\sum_{s=0}^{t-1} h(s) + c_1 \Gamma(\alpha), \quad t \in \mathbb{N}_0. \tag{3.7}$$

Using the condition $\Delta^{\alpha-1} u(\infty) = u_\infty$ in (3.7), we obtain

$$c_1 = \frac{1}{\Gamma(\alpha)} \left(\sum_{s=0}^{\infty} h(s) + u_\infty \right).$$

Now, substitution of c_1 into (3.6) gives

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} h(s) + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{\infty} t^{\underline{\alpha-1}} h(s) + \frac{u_\infty}{\Gamma(\alpha)} t^{\underline{\alpha-1}} \\ &= \sum_{s=0}^{\infty} G(t,s) h(s) + \frac{u_\infty}{\Gamma(\alpha)} t^{\underline{\alpha-1}}, \quad t \in \mathbb{N}_{\alpha-n}, \end{aligned}$$

where $G(t,s)$ is defined by (3.4). The proof is complete. □

Remark 3.1 From the expression of $G(t,s)$, we can easily find that $G(t,s) \geq 0$ and $\frac{G(t,s)}{1+t^{\underline{\alpha-1}}} < \frac{1}{\Gamma(\alpha)}$ for $(t,s) \in \mathbb{N}_{n-\alpha} \times \mathbb{N}_0$.

With the above auxiliary results in hand, we now establish the main result as follows.

Theorem 3.1 *If* (C_1), (C_2) *hold and*

$$\sigma = \frac{p^*(a + bk^* + ch^*)}{\Gamma(\alpha)} < 1, \tag{3.8}$$

then problem (1.1) *has a unique solution* u *in* X.

Proof Define an operator $\mathcal{F} : X \to X$ by

$$(\mathcal{F}u)(t) = \sum_{s=0}^{\infty} G(t,s) f\bigl(s+\alpha-1, u(s+\alpha-1), (Tu)(s), (Su)(s)\bigr) + \frac{u_\infty}{\Gamma(\alpha)} t^{\underline{\alpha-1}},$$

where $t \in \mathbb{N}_{\alpha-n}$, and due to Lemma 3.1, we have

$$\begin{aligned} \frac{\|(\mathcal{F}u)(t)\|}{1 + t^{\underline{\alpha-1}}} &\le \sum_{s=0}^{\infty} \frac{G(t,s)}{1 + t^{\underline{\alpha-1}}} \bigl\| f\bigl(s+\alpha-1, u(s+\alpha-1), (Tu)(s), (Su)(s)\bigr)\bigr\| + \frac{\|u_\infty\| t^{\underline{\alpha-1}}}{\Gamma(\alpha)(1 + t^{\underline{\alpha-1}})} \\ &\le \frac{1}{\Gamma(\alpha)} \bigl\{ p^*\bigl(a + bk^* + ch^*\bigr)\|u\|_X + f^* + \|u_\infty\| \bigr\} \\ &= \sigma \|u\|_X + \varrho, \quad t \in \mathbb{N}_{\alpha-n}. \end{aligned}$$

Therefore,

$$\|\mathcal{F}u\|_X \le \sigma \|u\|_X + \varrho, \quad u \in X,$$

here $\varrho = (f^* + \|u_\infty\|)/\Gamma(\alpha)$ and σ is defined by (3.8). So, the operator $\mathcal{F}$ is well defined. Furthermore, from Lemma 3.2, we can transform problem (1.1) as an operator equation $u = \mathcal{F}u$, and it is clear to see that u is a solution of problem (1.1) is equivalent to a fixed point of $\mathcal{F}$.

Next, for any $u, v \in X$, we denote

$$\begin{aligned} A(u,v)(t) = {} & f\bigl(t+\alpha-1, u(t+\alpha-1), (Tu)(t), (Su)(t)\bigr) \\ & - f\bigl(t+\alpha-1, v(t+\alpha-1), (Tv)(t), (Sv)(t)\bigr), \quad t \in \mathbb{N}_0. \end{aligned}$$

In view of (C_2), we have

$$\begin{aligned} & \frac{\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\|}{1 + t^{\underline{\alpha-1}}} \\ & \quad \le \sum_{s=0}^{\infty} \frac{G(t,s)}{1 + t^{\underline{\alpha-1}}} \bigl\| A(u,v)(s) \bigr\| \\ & \quad \le \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{\infty} \bigl[p(s+\alpha-1)\bigl(a\bigl\| u(s+\alpha-1) - v(s+\alpha-1) \bigr\| + b\bigl\| (Tu)(s) - (Tv)(s) \bigr\| \\ & \qquad + c\bigl\| (Su)(s) - (Sv)(s) \bigr\| \bigr) \bigr] \\ & \quad \le \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{\infty} \bigl[p(s+\alpha-1)\bigl[1 + (s+\alpha-1)^{\underline{\alpha-1}}\bigr] \end{aligned}$$

$$
\begin{aligned}
&\quad \times \bigl(a\|u-v\|_X + bk^*\|u-v\|_X + ch^*\|u-v\|_X\bigr)\bigr] \\
&\le \frac{1}{\Gamma(\alpha)} p^*\bigl(a + bk^* + ch^*\bigr)\|u-v\|_X \\
&= \sigma\|u-v\|_X.
\end{aligned}
$$

So we get

$$
\|\mathcal{F}u - \mathcal{F}v\|_X \le \sigma\|u-v\|_X,
$$

which, together with the assumption that $\sigma < 1$, implies that $\mathcal{F}$ is a contraction mapping. By means of the Banach contraction mapping principle, we get that $\mathcal{F}$ has a unique fixed point in E; that is problem (1.1) has a unique solution. This completes the proof. □

4 Examples

In this section, we illustrate the possible application of the above established analytical result with the following two concrete examples.

Example 4.1 Consider the following problem:

$$
\begin{cases}
\Delta^{7/2}u_n(t) + \frac{3^{-(t+1)}}{n[1+(t+5/2)^{\underline{5/2}}]^2}\sin[n^2(t+5/2) + u_n(t+5/2)] \\
\quad + \frac{2^{-(t+1)}}{(n+2)^2[1+(t+5/2)^{\underline{5/2}}]^3}\ln\{1 + [\sum_{s=0}^{t}\frac{1}{(t+s+2)^{\underline{2}}}u_{3n}(s+5/2)]^2\} \\
\quad + \frac{e^{-(t+1)}}{\sqrt{n}[3+\sin(t+5/2)+(t+5/2)^{\underline{5/2}}]} \\
\quad \times \{\sum_{s=0}^{\infty}\frac{\cos(t^2 s)}{(s+2)^{\underline{2}}[1+(s+5/2)^{\underline{5/2}}]}u_{n+1}(s+5/2)\} = 0, \quad t \in \mathbb{N}_0, \\
u_n(-1/2) = \Delta u_n(-1/2) = \Delta^2 u_n(-1/2) = 0, \\
\Delta^{5/2}u_n(\infty) = \frac{1}{n!}, \quad n = 1, 2, 3, \ldots.
\end{cases}
\tag{4.1}
$$

Conclusion *Problem* (4.1) *has a unique solution* $\{u_n(t)\}$ *such that* $u_n(t) \to 0$ *as* $n \to \infty$ *for* $t \in \mathbb{N}_{-1/2}$.

Proof Let $E = c_0 = \{u = (u_1, u_2, \ldots, u_n, \ldots) : u_n \to 0\}$. Evidently, $(E, \|\cdot\|)$ is a Banach space with the norm $\|u\| = \sup_n |u_n|$ for any $u \in E$. Then the infinite discrete fractional difference system (4.1) can be regarded as a boundary value problem of the form (1.1) in the Banach space E. In this situation, $\alpha = 7/2$, $\theta = (0, 0, \ldots, 0, \ldots) \in E$, $u_\infty = (1, 1/2!, \ldots, 1/n!, \ldots) \in E$,

$$
k(t,s) = \frac{1}{(t+s+2)^{\underline{2}}}, \qquad h(t,s) = \frac{\cos(t^2 s)}{(s+2)^{\underline{2}}[1+(s+5/2)^{\underline{5/2}}]},
$$

and $f = (f_1, f_2, \ldots, f_n, \ldots)$, in which

$$
\begin{aligned}
f_n(t,u,v,w) = {} & \frac{3^{-(t-3/2)}}{n[1+t^{\underline{5/2}}]^2}\sin\bigl(n^2 t + u_n\bigr) + \frac{2^{-(t-3/2)}}{(n+2)^2[1+t^{\underline{5/2}}]^3}\ln\bigl(1+v_{3n}^2\bigr) \\
& + \frac{e^{-(t-3/2)}}{\sqrt{n}[3+\sin t + t^{\underline{5/2}}]}w_{n+1},
\end{aligned}
$$

where $t \in \mathbb{N}_{5/2}$ and $u = (u_1, u_2, \ldots, u_n, \ldots)$, $v = (v_1, v_2, \ldots, v_n, \ldots)$, $w = (w_1, w_2, \ldots, w_n, \ldots) \in E$. From the expression of f_n, it is easy to see that $f : \mathbb{N}_{5/2} \times E \times E \times E \to E$ is continuous.

Furthermore, for any $t \in \mathbb{N}_{5/2}$, $u, v, w, \overline{u}, \overline{v}, \overline{w} \in E$, we have

$$\begin{aligned}
&\bigl|f_n(t, u, v, w) - f_n(t, \overline{u}, \overline{v}, \overline{w})\bigr| \\
&\quad\le \frac{3^{-(t-3/2)}}{n[1 + t^{\underline{5/2}}]^2}\bigl|\sin\bigl(n^2 t + u_n\bigr) - \sin\bigl(n^2 t + \overline{u}_n\bigr)\bigr| \\
&\qquad + \frac{2^{-(t-3/2)}}{(n+2)^2[1 + t^{\underline{5/2}}]^3}\bigl|\ln\bigl(1 + v_{3n}^2\bigr) - \ln\bigl(1 + \overline{v}_{3n}^2\bigr)\bigr| \\
&\qquad + \frac{e^{-(t-3/2)}}{\sqrt{n}[3 + \sin t + t^{\underline{5/2}}]}|w_{n+1} - \overline{w}_{n+1}| \\
&\quad\le \frac{3^{-(t-3/2)}}{n[1 + t^{\underline{5/2}}]^2}|u_n - \overline{u}_n| + \frac{2^{-(t-3/2)}}{(n+2)[1 + t^{\underline{5/2}}]^3}|v_{3n} - \overline{v}_{3n}| \\
&\qquad + \frac{e^{-(t-3/2)}}{\sqrt{n}[3 + \sin t + t^{\underline{5/2}}]}|w_{n+1} - \overline{w}_{n+1}| \\
&\quad\le \frac{2^{-(t-3/2)}}{1 + t^{\underline{5/2}}}\bigl[|u_n - \overline{u}_n| + 1/3|v_{3n} - \overline{v}_{3n}| + |w_{n+1} - \overline{w}_{n+1}|\bigr],
\end{aligned}$$

and therefore,

$$\bigl\|f(t, u, v, w) - f(t, \overline{u}, \overline{v}, \overline{w})\bigr\| \le \frac{2^{-(t-3/2)}}{1 + t^{\underline{5/2}}}\bigl[\|u - \overline{u}\| + 1/3\|v - \overline{v}\| + \|w - \overline{w}\|\bigr],$$

where $a = c = 1$, $b = \frac{1}{3}$, $p(t) = \frac{2^{-(t-3/2)}}{1+t^{\underline{5/2}}}$, which imply that (C_2) holds together with the following facts:

$$p^* = \sum_{t=5/2}^{\infty} p(t)\bigl(1 + t^{\underline{5/2}}\bigr) = \sum_{t=5/2}^{\infty} 2^{-(t-3/2)} = 1 < \infty$$

and

$$f^* = \sum_{s=5/2}^{\infty} \bigl\|f(t, \theta, \theta, \theta)\bigr\| \le \sum_{s=5/2}^{\infty} 3^{-(t-3/2)} = 1/2 < \infty.$$

On the other hand, we can verify that

$$\begin{aligned}
k^* &= \sup_{t \in \mathbb{N}_0} \sum_{s=0}^{t} \frac{1}{(t+s+2)^{\underline{2}}} = \sup_{t \in \mathbb{N}_0} \frac{1}{2(t+1)} = \frac{1}{2} < \infty, \\
h^* &= \sup_{t \in \mathbb{N}_0} \frac{1}{1 + (t+5/2)^{\underline{5/2}}} \sum_{s=0}^{\infty} \frac{|\cos(t^2 s)|[1 + (s+5/2)^{\underline{5/2}}]}{[1 + (s+5/2)^{\underline{5/2}}](s+2)^{\underline{2}}} \\
&\le \sup_{t \in \mathbb{N}_0} \frac{1}{1 + (t+5/2)^{\underline{5/2}}} \sum_{s=0}^{\infty} \frac{1}{(s+2)^{\underline{2}}} \\
&\le \frac{1}{1 + \Gamma(7/2)} < \frac{1}{4} < \infty.
\end{aligned}$$

So (C_1) is also satisfied. Finally, by a simple calculation, we can obtain

$$\sigma = \frac{p^*(a + bk^* + ch^*)}{\Gamma(\alpha)} \le \frac{(1 + 1/6 + 1/4)}{\Gamma(7/2)} < 0.43 < 1.$$

Thus, all the conditions of Theorem 3.1 are satisfied and our conclusion follows from Theorem 3.1. ☐

Example 4.2 Consider the following problem:

$$\begin{cases} \Delta^{7/3}\omega(t,x) + \frac{2^{-(t+1)}}{4[1+(t+4/3)^{\underline{4/3}}]}\cos[\omega(t+4/3,x)] \\ \quad + \frac{3^{-(t+1)}}{e^2[1+(t+4/3)^{\underline{4/3}}]}[\sum_{s=0}^{t}\frac{1}{(t+s+2)^{\underline{2}}}\omega(s+4/3,x)] \\ \quad + \frac{e^{-(t+1)}}{e^3[2+\cos(t+4/3)+(t+4/3)^{\underline{4/3}}]}\{\sum_{s=0}^{\infty}\frac{\sin(t+e^s)}{(s+2)^{\underline{2}}[1+(s+4/3)^{\underline{4/3}}]}\omega(s+4/3,x)\} = 0, \\ \quad t \in \mathbb{N}_0, x \in [0,1], \\ \omega(-2/3,x) = \Delta_t\omega(-2/3,x) = 0, \qquad \Delta^{4/3}\omega(\infty,x) = x^2. \end{cases} \tag{4.2}$$

Here, $\Delta^{7/3}\omega(t,x)$ represents the discrete Riemann-Liouville fractional difference of order 7/3 for the function $\omega(t,x)$ with respect to its first variable t.

Conclusion *Problem* (4.2) *has a unique solution* $\omega : \mathbb{N}_{-2/3} \times [0,1] \to \mathbb{R}$ *such that for each given* $t \in \mathbb{N}_{-2/3}$, $\omega(t,x)$ *is continuous for* $x \in [0,1]$.

Proof Let $E = C[0,1] = \{g : [0,1] \to \mathbb{R} \text{ is continuous}\}$; then $(E, \|\cdot\|)$ is a Banach space equipped with the norm $\|g\| = \sup_{x\in[0,1]}|g(x)|, g \in E$. Define $u : \mathbb{N}_{-2/3} \to E$ by $u(t) = \omega(t,\cdot) \in E$; then the discrete fractional partial difference system (4.2) can be transformed into the form of problem (1.1), where $\theta = 0$, $u_\infty = x^2$,

$$k(t,s) = \frac{1}{(t+s+2)^{\underline{2}}}, \qquad h(t,s) = \frac{\sin(t+e^s)}{(s+2)^{\underline{2}}[1+(s+4/3)^{\underline{4/3}}]},$$

and

$$f(t,u,v,w) = \frac{2^{-(t-1/3)}}{4[1+t^{\underline{4/3}}]}\cos u + \frac{3^{-(t-1/3)}}{e^2[1+t^{\underline{4/3}}]}v + \frac{e^{-(t-1/3)}}{e^3[2+\cos t + t^{\underline{4/3}}]}w$$

for $(t,u,v,w) \in \mathbb{N}_{4/3} \times E \times E \times E$. It is obvious that f is continuous.

Choosing $a = \frac{1}{4}$, $b = \frac{1}{e^2}$, $c = \frac{1}{e^3}$ and $p(t) = \frac{2^{-(t-1/3)}}{1+t^{\underline{4/3}}}$, $t \in \mathbb{N}_{4/3}$; then we can verify that $p^* = 1$, $f^* < 1/4$, $k^* = 1/2$, $h^* < 0.4566$, $\sigma < 0.3631$ and

$$\left\|f(t,u,v,w) - f(t,\overline{u},\overline{v},\overline{w})\right\| \le p(t)\bigl(a\|u-\overline{u}\| + b\|v-\overline{v}\| + c\|w-\overline{w}\|\bigr)$$

holds for any $t \in \mathbb{N}_{4/3}$, $u,v,w,\overline{u},\overline{v},\overline{w} \in E$.

Clearly, all the conditions of Theorem 3.1 are fulfilled. Therefore, we can conclude that problem (4.2) has a unique solution. ☐

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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