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The uniqueness of a distance-regular graph with intersection array {32, 27, 8, 1; 1, 4, 27, 32} and related results

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Abstract It is known that, up to isomorphism, there is a unique distance-regular graph Δ with intersection array {32, 27; 1, 12} [equivalently, Δ is the unique strongly regular graph with parameters (105, 32, 4, 12)]. Here we investigate the distance-regular antipodal covers of Δ. We show that, up to isomorphism, there is just one distance-regular antipodal triple cover of Δ (a graph $\hat{\Delta}$ discovered by the author over 20 years ago), proving that there is a unique distance-regular graph with intersection array {32, 27, 8, 1; 1, 4, 27, 32}. In the process, we confirm an unpublished result of Steve Linton that there is no distance-regular antipodal double cover of Δ, and so no distance-regular graph with intersection array {32, 27, 6, 1; 1, 6, 27, 32}. We also show there is no distance-regular antipodal 4-cover of Δ, and so no distance-regular graph with intersection array {32, 27, 9, 1; 1, 3, 27, 32}, and that there is no distance-regular antipodal 6-cover of Δ that is a double cover of $\hat{\Delta}$.

Keywords Distance-regular graph · Strongly regular graph · Antipodal cover · Fundamental group

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1 Introduction

A graph Γ is *distance-regular* with *intersection array*

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$$

if Γ is connected with diameter d, and for $i = 0, 1, \ldots, d$, whenever vertices v, w are at distance i in Γ, there are exactly b_i vertices adjacent to w at distance $i + 1$ from v and exactly c_i vertices adjacent to w at distance $i - 1$ from v (with the convention that $b_d = c_0 = 0$). Distance-regular graphs occur in many areas of discrete mathematics, including coding theory, design theory and finite geometry. See, for example, the encyclopaedic reference [2] by Brouwer et al. The present paper is a contribution to the problem of determining the distance-regular graphs with a given intersection array.

We shall investigate the antipodal distance-regular covers of the Goethals-Seidel graph [9], the unique (up to isomorphism) distance-regular graph Δ with intersection array $\{32, 27; 1, 12\}$ [4,5]. The graph Δ can be constructed as the second subconstituent of the second subconstituent of the famous McLaughlin graph, the unique distance-regular graph with intersection array $\{112, 81; 1, 56\}$ (see [3]).

A distance-regular antipodal triple cover $\hat{\Delta}$ of Δ was constructed by the author in [14], but its uniqueness was not determined. The main purpose of this paper is to prove that, up to isomorphism, $\hat{\Delta}$ is the unique distance-regular graph with intersection array

$$\{32, 27, 8, 1; 1, 4, 27, 32\}.$$

In the process of doing this, we confirm an unpublished result of Steve Linton which shows that there is no distance-regular graph with intersection array $\{32, 27, 6, 1; 1, 6, 27, 32\}$. We also prove there is no distance-regular graph with intersection array $\{32, 27, 9, 1; 1, 3, 27, 32\}$, and that there is no distance-regular antipodal 6-cover of Δ that is a double cover of the distance-regular triple cover $\hat{\Delta}$.

We classify distance-regular antipodal r-covers of Δ by studying the r-fold topological covers of the 2-dimensional simplicial complex whose 0-, 1-, and 2-simplices are respectively the vertices, edges, and triangles of Δ. Our main tool is version 2.0 of the author's GAP program described in [12] for the computation of fundamental groups, certain quotients of fundamental groups, and covers of finite abstract 2-dimensional simplicial complexes. This program is freely available from [16], where we also provide a GAP/GRAPE [15,17] logfile of all the computations related to this paper. It is hoped that [16] and the methods of this paper will be useful in other classifications of covers of graphs. In particular, Theorem 3.1 should be of independent interest.

All graphs in this paper are finite and undirected, with no loops and no multiple edges, and have at least one vertex. Throughout, we follow [2] for graph-theoretical concepts and notation. An important new reference for distance-regular graphs, covering developments since [2] was published, is Van Dam et al. [18]. See also Brouwer et al. [1]. A good reference for the group-theoretical concepts used in this paper is Robinson [13]. We denote the commutator subgroup of a group G by $[G, G]$, the cyclic group of order n is denoted by C_n, and where p is a prime, the elementary abelian group of order p^k is denoted by C_p^k.

2 The fundamental group and covers

Let Γ be a connected graph. We also consider Γ to be an abstract 2-dimensional simplicial complex (or 2-complex) whose 0-, 1-, and 2-simplices are respectively the vertices, edges, and triangles of Γ. With this in mind, throughout this paper, by a *cover* of Γ, we mean a pair $(\tilde{\Gamma}, \theta)$, where $\tilde{\Gamma}$ is a graph and $\theta : V(\tilde{\Gamma}) \to V(\Gamma)$ is a surjection, called a *covering map*, such that the following hold:

- For every $v \in V(\Gamma)$, the *fibre* $v\theta^{-1}$ of v is a coclique (independent set) of $\tilde{\Gamma}$,
- The union of any two distinct fibres mapping under θ to a non-edge of Γ is a coclique of $\tilde{\Gamma}$,
- The induced subgraph on any two fibres mapping under θ to an edge of Γ is a perfect matching in $\tilde{\Gamma}$,
- The induced subgraph on any three fibres mapping under θ to a triangle of Γ consists of pairwise disjoint triangles in $\tilde{\Gamma}$.

Note that, as defined here, $\tilde{\Gamma}$ is a cover of Γ precisely when $\tilde{\Gamma}$ is a topological cover of Γ when they are both viewed as 2-complexes as described above.

When we do not need to specify the covering map explicitly, we may denote a cover $(\tilde{\Gamma}, \theta)$ simply by $\tilde{\Gamma}$. If each fibre of a cover $\tilde{\Gamma}$ of Γ has the same positive integer cardinality r, then we call $\tilde{\Gamma}$ an *r-cover* of Γ. A 2-cover is also called a *double cover*, and a 3-cover is also called a *triple cover*. If Γ is a non-complete graph, then an *antipodal r-cover* $\tilde{\Gamma}$ of Γ is a connected r-cover of Γ with the property that being equal or at maximum distance in $\tilde{\Gamma}$ is an equivalence relation on $V(\tilde{\Gamma})$, whose equivalence classes are the fibres.

The connected r-covers of Γ correspond to the transitive permutation representations of degree r of the fundamental group of Γ (regarded as a 2-complex), defined with respect to a fixed, but arbitrary, spanning tree of Γ. We shall explain this further in this section. For a more general, fuller and detailed exposition, see [12], on which our explanation is based.

Now, fix a spanning tree T of Γ. The choice of T does not matter, and in particular, does not affect the isomorphism class of the fundamental group of Γ. Then, for each arc (ordered edge) (v, w) of Γ, define an abstract group generator $g_{v,w}$. For our purposes, the fundamental group G of Γ is the finitely presented group whose generators are these $g_{v,w}$ and whose defining relations are:

- $g_{s,t} = 1$ for each arc (s, t) of T;
- $g_{v,w}g_{w,v} = 1$ for each edge $\{v, w\}$ of Γ;
- $g_{x,y}g_{y,z}g_{z,x} = 1$ for each triangle $\{x, y, z\}$ of Γ.

Suppose $(\tilde{\Gamma}, \theta)$ is any connected r-cover of Γ. Then $T\theta^{-1}$ consists of r disjoint copies $T_1, \ldots, T_r$ of T, with $T_i\theta = T$. We may thus label the vertices of $\tilde{\Gamma}$ by ordered pairs (v, i), with $v \in V(\Gamma)$ and $i \in I = \{1, \ldots, r\}$, so that the fibre of a vertex v of Γ is $\{(v, 1), \ldots, (v, r)\}$ and $(v, i) \in V(T_i)$ for each i. In particular, note that a different choice for the fixed spanning tree T would only result in a relabelling of vertices within the fibres of $\tilde{\Gamma}$.

Now for each arc (v, w) of Γ, let $\rho_{v,w}$ be the permutation of I defined by $i\rho_{v,w} = j$ if and only if $\{(v, i), (w, j)\}$ is an edge of $\tilde{\Gamma}$. If $\{x, y, z\}$ is a triangle of Γ, then $\{x, y, z\}\theta^{-1}$ is a set of disjoint triangles in $\tilde{\Gamma}$, and so $\rho_{x,y}\rho_{y,z}\rho_{z,x}$ is the identity permutation. It now follows that the map ρ defined on the generators of G by $(g_{v,w})\rho = \rho_{v,w}$ extends to a transitive permutation representation from G to the symmetric group S_r. The r-cover $\tilde{\Gamma}$ is completely defined by this representation.

Conversely, every transitive permutation representation $\rho : G \to S_r$ defines a connected r-cover, denoted Γ_ρ, of Γ. For each $v \in V(\Gamma)$, the fibre of v in Γ_ρ consists of the ordered

pairs (v, i), for $i \in I$, and (v, i) and (w, j) are joined by an edge in $\tilde{\Gamma}$ if and only if $\{v, w\}$ is an edge of Γ and $j = i((g_{v,w})\rho)$. The preimage in G of the stabilizer in $G\rho$ of a point $i \in I$ is isomorphic to the fundamental group of Γ_ρ (see [12]).

We consider two covers (Γ_1, θ_1) and (Γ_2, θ_2) of Γ to be *isomorphic* if there is a graph isomorphism from Γ_1 to Γ_2 which maps the fibres of Γ_1 to those of Γ_2. Thus, isomorphic covers differ only by a relabelling of the fibres and of the vertices within each fibre. In particular, every connected r-cover of Γ is isomorphic to a cover of the form Γ_ρ, for some transitive permutation representation $\rho : G \to S_r$, corresponding to the preimage in G of the stabilizer in $G\rho$ of the point 1.

Isomorphism of covers can be checked as follows using nauty [11], which can be called from within GRAPE [15]. Given the cover Γ_1 of Γ, we make a {red, blue}-vertex-coloured graph Γ_1^+. The red-coloured vertices of Γ_1^+ are the vertices of Γ_1, with two red vertices joined in Γ_1^+ precisely when those vertices are joined in Γ_1. The blue-coloured vertices of Γ_1^+ are in one-to-one correspondence with the fibres of Γ_1, and a blue vertex is joined in Γ_1^+ only to the red vertices in the corresponding fibre. Similarly, we make Γ_2^+ from the cover Γ_2 of Γ. Then Γ_1 and Γ_2 are isomorphic as covers of Γ if and only if Γ_1^+ is isomorphic to Γ_2^+ by a colour-preserving graph isomorphism.

We are usually interested in classifying covers up to isomorphism. Note that if (Γ_1, θ_1) and (Γ_2, θ_2) are isomorphic covers of Γ, and $(\tilde{\Gamma}_1, \tilde{\theta}_1)$ is any cover of Γ_1, then $(\tilde{\Gamma}_1, \tilde{\theta}_1\theta_1)$ is a cover of Γ isomorphic to $(\tilde{\Gamma}_2, \tilde{\theta}_2\theta_2)$ for some cover $(\tilde{\Gamma}_2, \tilde{\theta}_2)$ of Γ_2.

3 Imprimitivity and covers

We may sometimes be interested in a cover of a cover.

Let Γ be a connected graph, let (Γ_1, θ_1) be a connected m-cover of Γ, and let (Γ_2, θ_2) be a connected n-cover of Γ_1. Then clearly, $(\Gamma_2, \theta_2\theta_1)$ is a connected mn-cover of Γ. Given a spanning tree T of Γ, we take a spanning tree T_1 of Γ_1 containing the forest $T\theta_1^{-1}$, and define the fundamental group G of Γ with respect to T and the fundamental group G_1 of Γ_1 with respect to T_1. Then (Γ_1, θ_1) corresponds to a transitive representation $\rho_1 : G \to S_m$, with the fibre of v in Γ_1 being labelled $\{(v, i) : 1 \le i \le m\}$, as described previously, and (Γ_2, θ_2) corresponds to a transitive representation $\rho_2 : G_1 \to S_n$, with the fibre of (v, i) being labelled $\{(v, i, j) : 1 \le j \le n\}$. Now $(\Gamma_2, \theta_2\theta_1)$ corresponds to a transitive representation $\rho : G \to S_{mn}$, and if $m, n > 1$, then $G\rho$ acts imprimitively on the indices (i, j) of the fibre of a vertex of Γ, the blocks of imprimitivity being $\{(i, j) : 1 \le j \le n\}$, for $i = 1, \ldots, m$.

Conversely, suppose that $m, n > 1$ and $\rho : G \to S_{mn}$ is a transitive permutation representation of the fundamental group G of Γ such that $G\rho$ is an imprimitive group having m blocks of size n. Then, if $\sigma : G \to S_m$ is the transitive permutation action of $G\rho$ on the blocks of imprimitivity, we see that Γ_ρ is an n-cover of the m-cover Γ_σ of Γ.

We have the following useful result.

Theorem 3.1 *Let Γ be a non-complete distance-regular graph and suppose Γ_ρ is a distance-regular antipodal mn-cover of Γ corresponding to a transitive permutation representation ρ of the fundamental group G of Γ, such that $G\rho$ has $m > 1$ blocks of imprimitivity of size $n > 1$. Then Γ_ρ must be an n-cover of an antipodal distance-regular m-cover of Γ.*

Proof As above, we may suppose that $G\rho$ is a group of permutations of $\Omega := \{(i, j) : 1 \le i \le m, 1 \le j \le n\}$, with blocks of imprimitivity $B_i := \{(i, j) : 1 \le j \le n\}$, for $i = 1, \ldots, m$, and that the fibre of Γ_ρ mapping to the vertex v of Γ is labelled as $\{(v, i, j) : 1 \le i \le m, 1 \le j \le n\}$.

Now consider the partition

$$\pi := \{\{(v, i, j) : 1 \leq j \leq n\} : v \in V(\Gamma), \ 1 \leq i \leq m\}$$

of $V(\Gamma_\rho)$. Then each part in π is contained in a fibre of Γ_ρ, and π is an equitable (also called regular) partition of $V(\Gamma_\rho)$ (since each vertex in the part $\{(v, i, j) : 1 \leq j \leq n\}$ is joined to exactly one or no vertex in the part $\{(w, k, \ell) : 1 \leq \ell \leq n\}$, with a join to one vertex precisely when $\{v, w\} \in E(\Gamma)$ and $\rho_{v,w}$ maps the block B_i to B_k). We may thus apply Godsil and Hensel [8, Theorem 6.2] to deduce that Γ_ρ/π is an antipodal distance-regular m-cover of Γ (see also Theorem 7.3 in [7, Sect. 11], attributed to Brouwer, Cohen and Neumaier). □

4 On the distance-regular antipodal r-covers of Δ

Throughout this section, Δ denotes the unique distance-regular graph with intersection array $\{32, 27; 1, 12\}$ [equivalently, the unique strongly regular graph with parameters $(105, 32, 4, 12)$]. Suppose Γ is a distance-regular antipodal r-cover of Δ. Parameter feasibility conditions (see [2]) imply Γ has diameter 4, $r \in \{2, 3, 4, 6\}$, and Γ has intersection array

$$\{32, 27, 12(r-1)/r, 1; 1, 12/r, 27, 32\}. \tag{1}$$

Conversely, if Γ is a distance-regular graph with intersection array (1), then Γ is an antipodal r-cover of a distance-regular graph with intersection array $\{32, 27; 1, 12\}$ (see [6]), so Γ is an antipodal distance-regular r-cover of (a graph isomorphic to) Δ.

To study covers of Δ, we explore quotients of the fundamental group of Δ, viewed as a 2-complex, with respect to a fixed spanning tree T of Δ. This fundamental group is denoted throughout this section by D. It was shown in [14] that D is infinite.

Theorem 4.1 *The abelianised fundamental group $D/[D, D]$ of Δ is isomorphic to $C_2^{16} \times C_3^2$.*

Proof We compute a presentation for $D/[D, D]$ using the program [16], and determine the abelian invariants of this group using GAP [17]. □

We are now in a position to classify the connected double covers of Δ, which was done independently by Steve Linton over 20 years ago, using his vector enumeration algorithm [10].

Theorem 4.2 *Up to isomorphism of covers, Δ has just 13 connected double covers, with each having its abelianised fundamental group isomorphic to $C_2^{15} \times C_3^2$, and none being distance-regular.*

Proof Each subgroup of index 2 in D contains the commutator subgroup $[D, D]$. We compute the $2^{16} - 1$ covers of Δ corresponding to the subgroups of index 2 in $D/[D, D]$, and using GRAPE calling nauty, we determine that, up to isomorphism of covers, there are just 13 such covers. We test each of these covers for distance-regularity using GRAPE, and find that none of them are distance-regular. We use the program [16] to compute each of their abelianised fundamental groups. □

Corollary 4.3 *There is no distance-regular graph with intersection array $\{32, 27, 6, 1; 1, 6, 27, 32\}$.*

We now consider the triple covers of Δ, and start by showing that D has no quotient isomorphic to the symmetric group S_3. This is a corollary of the following:

Proposition 4.4 *Let N be a subgroup of D of index 2, and suppose N has a subgroup M, such that M is normal in D and N/M is abelian. Then D/M is abelian.*

Proof The subgroup N of D is isomorphic to the fundamental group of some connected 2-cover of Δ, so by Theorem 4.2, $[N, N]$ has index $2^{15}3^2$ in N, and so $[N, N]$ has index $2^{16}3^2$ in D, as does $[D, D]$, by Theorem 4.1. Thus $[N, N] = [D, D]$, and since N/M is abelian, M contains $[N, N]$, and so D/M is abelian. $\qquad\square$

Corollary 4.5 *The group D has no quotient isomorphic to the symmetric group S_3.*

We are now in a position to classify the connected triple covers of Δ.

Theorem 4.6 *Up to isomorphism of covers, Δ has just two connected triple covers, Δ^* and $\hat{\Delta}$. The cover Δ^* has abelianised fundamental group isomorphic to $C_2^{16} \times C_3^2$ and is not distance-regular. The cover $\hat{\Delta}$ has abelianised fundamental group isomorphic to $C_2^{18} \times C_3^2$ and is distance-regular.*

Proof Each connected triple cover of Δ is isomorphic to a cover of the form Δ_ρ, for some transitive permutation representation $\rho : D \to S_3$. Since D has no quotient isomorphic to S_3, we must have $D\rho \cong C_3$, the cyclic group of order 3, and so each subgroup of index 3 in D is normal and contains $[D, D]$.

We compute the four covers of Δ corresponding to the subgroups of index 3 in $D/[D, D]$, and determine that, up to isomorphism of covers, there are just two such covers, Δ^* and $\hat{\Delta}$, and we calculate that these have the properties as stated in the theorem. $\qquad\square$

Corollary 4.7 *Up to isomorphism, there is a unique distance-regular graph with intersection array $\{32, 27, 8, 1; 1, 4, 27, 32\}$.*

Further properties of this distance-regular graph $\hat{\Delta}$ are given in [14] (where the graph is called Λ).

We now show there is no distance-regular antipodal 4-cover of Δ.

Theorem 4.8 *There is no distance-regular graph with intersection array $\{32, 27, 9, 1; 1, 3, 27, 32\}$.*

Proof A distance-regular graph with the intersection array of the theorem would be an antipodal 4-cover of Δ. Such a 4-cover would correspond to some transitive permutation representation ρ of degree 4 of D. The image $D\rho$ of D cannot be imprimitive, for otherwise, by Theorem 3.1, Δ would have a distance-regular antipodal 2-cover. We cannot have $D\rho \cong S_4$, since D has no quotient isomorphic to S_3, which is a quotient of S_4. This leaves $D\rho \cong A_4$ as the only possibility.

Suppose now D has a quotient isomorphic to A_4. Then there is a transitive permutation representation $\alpha : D \to S_6$, having three blocks of imprimitivity of size 2, with $D\alpha \cong A_4$. The corresponding cover Δ_α is a 2-cover of a 3-cover $\tilde{\Delta}$ of Δ. The 3-cover $\tilde{\Delta}$ corresponds to a normal subgroup N of D of index 3, and N has a subgroup M of index 4 that is normal in D, such $D/M \cong A_4$.

Suppose now $\tilde{\Delta}$ is isomorphic (as a cover of Δ) to Δ^*. By Theorem 4.6, $[N, N]$ has index $2^{16}3^2$ in N, and so $[N, N]$ has index $2^{16}3^3$ in D. By Theorem 4.1, $[D, D]$ has index $2^{16}3^2$ in D, so $[N, N]$ is a normal subgroup of index 3 in $[D, D]$. Since $N/M \cong C_2^2$, M contains $[N, N]$, and so either D/M is abelian or D/M has a normal subgroup of order 3, neither of which holds.

Thus, we must have $\tilde{\Delta}$ isomorphic to $\hat{\Delta}$. We computed all $2^{18} - 1$ connected 2-covers of $\hat{\Delta}$ as covers of the form Δ_ρ of Δ, for ρ a transitive permutation representation from D to S_6. Just three of these covers Δ_ρ have $D\rho$ isomorphic to A_4, corresponding to the three permutation isomorphic transitive representations of degree 6 of one quotient of D isomorphic to A_4. Now given a connected 6-cover Δ_ρ with $D\rho \cong A_4$, we construct the cover Δ_σ of Δ defined by the representation of $D\rho$ acting by right multiplication on the four (right) cosets of a Sylow 3-subgroup. Up to isomorphism of covers of Δ, there is only one such Δ_σ, which we find is not distance-regular.

□

Finally, we can prove the following:

Theorem 4.9 *There is no distance-regular antipodal* 6*-cover of* Δ *that is a double cover of* $\hat{\Delta}$.

Proof When we determined all $2^{18} - 1$ connected 2-covers of $\hat{\Delta}$, we found that none is distance-regular. □

Combining this result with Theorems 3.1, 4.2 and 4.6, we see that there is no distance-regular antipodal 6-cover of Δ corresponding to an imprimitive degree 6 permutation representation of D, but this still leaves open the possibility of a distance-regular antipodal 6-cover of Δ corresponding to a primitive such representation.

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