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Best proximity point results for modified Suzuki α-ψ-proximal contractions

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Abstract

In this paper, we introduce a modified Suzuki α-ψ-proximal contraction. Then we establish certain best proximity point theorems for such proximal contractions in metric spaces. As an application, we deduce best proximity and fixed point results in partially ordered metric spaces. The results presented generalize and improve various known results from best proximity and fixed point theory. Moreover, some examples are given to illustrate the usability of the obtained results.

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1 Introduction and Preliminaries

In the last decade, the answers of the following question has turned into one of the core subjects of applied mathematics and nonlinear functional analysis. Is there a point x_0 in a metric space (X,d) such that $d(x_0, Tx_0) = d(A,B)$ where A, B are non-empty subsets of a metric space X and $T : A \to B$ is a non-self-mapping where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$? Here, the point $x_0 \in X$ is called the best proximity point. The object of best proximity theory is to determine minimal conditions on the non-self-mapping T to guarantee the existence and uniqueness of a best proximal point. The setting of best proximity point theory is richer and more general than the metric fixed point theory in two senses. First, usually the mappings considered in fixed point theory are self-mappings, which is not necessary in the theory of best proximity. Secondly, if one takes $A = B$ in the above setting, the best proximity point becomes a fixed point. It is well known that fixed point theory combines various disciplines of mathematics, such as topology, operator theory, and geometry, to show the existence of solutions of the equation $Tx = x$ under proper conditions. On the other hand, if T is not a self-mapping, the equation $Tx = x$ could have no solutions and, in this case, it is of basic interest to determine an element x that is in some sense closest to Tx. One of the most interesting results in this direction is the following theorem due to Fan [1].

Theorem F *Let K be a non-empty compact convex subset of a normed space X and $T : K \to X$ be a continuous non-self-mapping. Then there exists an x such that $\|x - Tx\| = d(K, Tx) = \inf\{\|Tx - u\| : u \in K\}$.*

Many generalizations and extensions of this result have appeared in the literature (see [2–6] and references therein).

In fact best proximity point theory has been studied to find necessary conditions such that the minimization problem $\min_{x\in A} d(x, Tx)$ has at least one solution. For more details on this approach, we refer the reader to [7–13] and [5, 14–25].

One of the interesting generalizations of the Banach contraction principle which characterizes the metric completeness is due to Suzuki [26, 27] (see also [28, 29]). Recently, Abkar and Gabeleh [8] studied best proximity point results for Suzuki contractions. The aim of this paper is to introduce modified Suzuki α-ψ-proximal contractions and establish certain best proximity point theorems for such proximal contractions in metric spaces. As an application, we deduce best proximity and fixed point results in partially ordered metric spaces. The presented results generalize and improve various known results from best proximity and fixed point theory. Moreover, some examples are given to illustrate the usability of the obtained results.

We recollect some essential notations, required definitions and primary results to coherence with the literature. Suppose that A and B are two non-empty subsets of a metric space (X, d). We define

$$d(a,B) := \inf\{d(a,b) : b \in B\}, \quad a \in A,$$
$$A_0 := \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\},$$
$$B_0 := \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\}.$$

Under the assumption of $A_0 \neq \emptyset$, we say that the pair (A,B) has the P-property [20] if the following condition holds:

$$\begin{cases} d(x_1,y_1) = d(A,B), \\ d(x_2,y_2) = d(A,B), \end{cases} \quad \Longrightarrow \quad d(x_1,x_2) = d(y_1,y_2)$$

for all $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

In 2012, Samet *et al.* [24] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings in complete metric spaces.

Samet *et al.* [24] defined the notion of α-admissible mappings as follows.

Definition 1.1 Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Salimi *et al.* [22] modified and generalized the notion of α-admissible mappings in the following way.

Definition 1.2 [22] Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x,y) \geq \eta(x,y) \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Note that if we take $\eta(x,y) = 1$, then this definition reduces to Definition 1.1.

Definition 1.3 [14] A non-self-mapping T is called α-proximal admissible if

$$\begin{cases} \alpha(x_1,x_2) \geq 1, \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B), \end{cases} \implies \alpha(u_1,u_2) \geq 1$$

for all $x_1, x_2, u_1, u_2 \in A$, where $\alpha : A \times A \to [0,\infty)$.

Clearly, if $A = B$, T is α-proximal admissible implies that T is α-admissible.

Recently Hussain *et al.* [4] generalized the notion of α-proximal admissible as follows.

Definition 1.4 Let $T : A \to B$ and $\alpha, \eta : A \times A \to [0,\infty)$ be functions. Then T is called α-proximal admissible with respect to η if

$$\begin{cases} \alpha(x_1,x_2) \geq \eta(x_1,x_2), \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B), \end{cases} \implies \alpha(u_1,u_2) \geq \eta(u_1,u_2)$$

for all $x_1, x_2, u_1, u_2 \in A$. Note that if we take $\eta(x,y) = 1$ for all $x, y \in A$, then this definition reduces to Definition 1.3.

A function $\psi : [0,\infty) \to [0,\infty)$ is called Bianchini-Grandolfi gauge function [17, 18, 30] if the following conditions hold:

(i) ψ is non-decreasing;

(ii) there exist $k_0 \in \mathbb{N}$ and $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k,$$

for $k \geq k_0$ and any $t \in \mathbb{R}^+$.

In some sources, the Bianchini-Grandolfi gauge function is known as the (c)-comparison function (see *e.g.* [31]). We denote by Ψ the family of Bianchini-Grandolfi gauge functions. The following lemma illustrates the properties of these functions.

Lemma 1.1 (See [31]) *If* $\psi \in \Psi$, *then the following hold*:

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for all* $t \in \mathbb{R}^+$;

(ii) $\psi(t) < t$, *for any* $t \in (0,\infty)$;

(iii) ψ *is continuous at* 0;

(iv) *the series* $\sum_{k=1}^{\infty} \psi^k(t)$ *converges for any* $t \in \mathbb{R}^+$.

2 Best proximity point results in metric spaces

We start this section with the following definition.

Definition 2.1 Suppose that A and B are two non-empty subsets of a metric space (X,d). A non-self-mapping $T : A \to B$ is said to be modified Suzuki α-ψ-proximal contraction, if

$$d^*(x, Tx) \leq \alpha(x,y)d(x,y) \quad \Rightarrow \quad d(Tx, Ty) \leq \psi\big(d(x,y)\big) \tag{2.1}$$

for all $x, y \in A$ where $d^*(x,y) = d(x,y) - d(A,B)$, $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$.

The following is our first main result of this section.

Theorem 2.1 *Suppose that A and B are two non-empty closed subsets of a complete metric space (X,d) with $A_0 \neq \emptyset$. Let $T : A \to B$ be a modified Suzuki α-ψ-proximal contraction satisfying the following conditions*:

(i) $T(A_0) \subseteq B_0$ *and* (A,B) *satisfies the P-property*;

(ii) T *is* α*-proximal admissible with respect to* $\eta(x,y) = 2$;

(iii) *the elements* x_0 *and* x_1 *in* A_0 *with*

$$d(x_1, Tx_0) = d(A,B) \text{ satisfy } \alpha(x_0,x_1) \geq 2;$$

(iv) T *is continuous.*

Then T has a unique best proximity point.

Proof As A_0 is non-empty and $T(A_0) \subseteq B_0$, there exist elements x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A,B)$ and by (iii) $\alpha(x_0,x_1) \geq 2$. Owing to the fact that $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A,B).$$

Since T is α-proximal admissible, we have $\alpha(x_1,x_2) \geq 2$. Again, by using the fact that $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A,B).$$

So we conclude that

$$d(x_2, Tx_1) = d(A,B), \qquad d(x_3, Tx_2) = d(A,B), \qquad \alpha(x_1,x_2) \geq 2.$$

As T is α-proximal admissible, we derive that $\alpha(x_2,x_3) \geq 2$, that is,

$$d(x_3, Tx_2) = d(A,B), \qquad \alpha(x_2,x_3) \geq 2.$$

By repeating this process, we observe that

$$d(x_{n+1}, Tx_n) = d(A,B), \qquad \alpha(x_n,x_{n+1}) \geq 2 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

By the triangle inequality, we have

$$d(x_{n-1}, Tx_{n-1}) \leq d(x_n,x_{n-1}) + d(x_n, Tx_{n-1}) = d(x_n,x_{n-1}) + d(A,B),$$

which implies

$$d^*(x_{n-1}, Tx_{n-1}) \leq d(x_n,x_{n-1}) \leq 2d(x_n,x_{n-1}) \leq \alpha(x_n,x_{n-1})d(x_n,x_{n-1}).$$

From (2.1), we derive that

$$d(Tx_{n-1}, Tx_n) \leq \psi\bigl(d(x_{n-1},x_n)\bigr). \tag{2.3}$$

Due the fact that the pair (A, B) has the P-property together with (2.2), we conclude that

$$d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N}.$$

Consequently, from (2.3), we obtain

$$d(x_n, x_{n+1}) \le \psi\big(d(x_{n-1}, x_n)\big) \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then (2.2) implies that

$$d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B),$$

that is, x_{n_0} is a best proximity point of T. Hence, we assume that

$$d(x_{n+1}, x_n) > 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.5}$$

By using the fact that ψ is non-decreasing together with the assumption (2.1), inductively, we conclude that

$$d(x_n, x_{n+1}) \le \psi^n\big(d(x_1, x_0)\big) \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Fix $\epsilon > 0$; there exists $N \in \mathbb{N}$ such that

$$\sum_{n \ge N} \psi^n\big(d(x_0, x_1)\big) < \epsilon \quad \text{for all } n \in \mathbb{N}.$$

Let $m, n \in \mathbb{N}$ with $m > n \ge N$. By the triangle inequality, we have

$$d(x_n, x_m) \le \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \le \sum_{k=n}^{m-1} \psi^k\big(d(x_0, x_1)\big) < \sum_{n \ge N} \psi^n\big(d(x_0, x_1)\big) < \epsilon$$

which yields $\lim_{m,n,\to+\infty} d(x_n, x_m) = 0$. Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is $z \in X$ such that $x_n \to z$. By the continuity of T, we derive that $Tx_n \to Tz$ as $n \to \infty$. Hence, we get the desired result:

$$d(A, B) = \lim_{n \to \infty} d(x_{n+1}, Tx_n) = d(z, Tz).$$

We now show that T has a unique best proximity point. Suppose, on the contrary, that $y, z \in A_0$ are two best proximity points of T with $y \ne z$, that is,

$$d(y, Ty) = d(z, Tz) = d(A, B). \tag{2.6}$$

By applying the P-property and (2.6) we get

$$d(y, z) = d(Ty, Tz). \tag{2.7}$$

Also from (2.6) we get

$$d^*(y,Ty) = d(y,Ty) - d(A,B) = 0,$$

which implies that $d^*(y,Ty) = 0 \leq \alpha(y,z)d(y,z)$. Applying (2.1), we have

$$d(Ty,Tz) \leq \psi\bigl(d(y,z)\bigr).$$

From (2.7) we deduce

$$d(y,z) \leq \psi\bigl(d(y,z)\bigr) < d(y,z),$$

which is a contradiction. Hence, $y = z$. This completes the proof of the theorem. □

In the following theorem, we replace the continuity condition on Suzuki α-ψ-proximal contraction T by regularity of the space (X,d).

Theorem 2.2 *Suppose that A and B are two non-empty closed subsets of a complete metric space (X,d) with $A_0 \neq \emptyset$. Let $T: A \to B$ be a modified Suzuki α-ψ-proximal contraction satisfying the following conditions*:

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property*;

(ii) *T is α-proximal admissible with respect to $\eta(x,y) = 2$*;

(iii) *there exist elements x_0 and x_1 in A_0 with*

$$d(x_1,Tx_0) = d(A,B) \text{ satisfying } \alpha(x_0,x_1) \geq 2;$$

(iv) *if $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1}) \geq 2$ and $x_n \to x \in A$ as $n \to \infty$, then $\alpha(x_n,x) \geq 2$ for all $n \in \mathbb{N}$.*

Then T has a unique best proximity point.

Proof Following the lines of proof of Theorem 2.1, we obtain a Cauchy sequence $\{x_n\}$ which converges to $z \in X$. Suppose that the condition (iv) holds, that is, $\alpha(x_n,z) \geq 2$ for all $n \in \mathbb{N}$. From (2.4) and (2.5) we obtain

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n)$$

for all $n \in \mathbb{N}$. By using (2.2), we have

$$\begin{aligned} d^*(x_n,Tx_n) &= d(x_n,Tx_n) - d(A,B) \\ &\leq d(x_n,x_{n+1}) + d(x_{n+1},Tx_n) - d(A,B) = d(x_n,x_{n+1}) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} d^*(x_{n+1},Tx_{n+1}) &= d(x_{n+1},Tx_{n+1}) - d(A,B) \\ &\leq d(Tx_n,Tx_{n+1}) + d(x_{n+1},Tx_n) - d(A,B) \\ &= d(Tx_n,Tx_{n+1}) = d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1}). \end{aligned} \tag{2.9}$$

Hence, (2.8) and (2.9) imply that

$$d^*(x_n, Tx_n) + d^*(x_{n+1}, Tx_{n+1}) < 2d(x_n, x_{n+1}). \tag{2.10}$$

We suppose that the inequalities

$$d^*(x_n, Tx_n) > \alpha(x_n, z)d(x_n, z) \geq 2d(x_n, z)$$

and

$$d^*(x_{n+1}, Tx_{n+1}) > \alpha(x_{n+1}, z)d(x_{n+1}, z) \geq 2d(x_{n+1}, z)$$

hold for some $n \in \mathbb{N}$. Then, by using (2.10) we can write

$$\begin{aligned} 2d(x_n, x_{n+1}) &\leq 2d(x_n, z) + 2d(x_{n+1}, z) \\ &< d^*(x_n, Tx_n) + d^*(x_{n+1}, Tx_{n+1}) \leq 2d(x_n, x_{n+1}), \end{aligned}$$

a contradiction. Hence, for all $n \in \mathbb{N}$, we have either

$$d^*(x_n, Tx_n) \leq \alpha(x_n, z)d(x_n, z),$$

or

$$d^*(x_{n+1}, Tx_{n+1}) \leq \alpha(x_{n+1}, z)d(x_{n+1}, z).$$

Using (2.1), we obtain either

$$d(Tx_n, Tz) \leq \psi\bigl(d(x_n, z)\bigr),$$

or

$$d(Tx_{n+1}, Tz) \leq \psi\bigl(d(x_{n+1}, z)\bigr).$$

If we take the limit as $n \to +\infty$ in each of these inequalities, we have

$$Tx_n \to Tz \quad \text{or} \quad Tx_{n+1} \to Tz \quad \text{as } n \to \infty.$$

Consequently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \to Tz$ as $x_{n_k} \to z$. Therefore,

$$d(A, B) = \lim_{k\to\infty} d(x_{n_k+1}, Tx_{n_k}) = d(z, Tz).$$

The uniqueness of best proximity point follows as in the proof of Theorem 2.1. □

Example 2.1 Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$ be a usual metric on X. Suppose $A = (-\infty, -1]$ and $B = [5/4, +\infty)$. Define $T : A \to B$ by

$$Tx = \begin{cases} -x + |x + 3||x + 4|e^{-x}, & \text{if } x \in (-\infty, -2), \\ -\frac{1}{4}x + 1, & \text{if } x \in [-2, -1]. \end{cases}$$

Also, define $\alpha : X^2 \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 4, & \text{if } x,y \in [-2,-1], \\ 0, & \text{otherwise} \end{cases}$$

and $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{1}{2}t$. Clearly, $d(A,B) = 9/4$. Now we have:

$$A_0 = \{x \in A : d(x,y) = d(A,B) = 9/4 \text{ for some } y \in B\} = \{-1\},$$
$$B_0 = \{y \in B : d(x,y) = d(A,B) = 9/4 \text{ for some } x \in A\} = \{5/4\}.$$

Also, $T(A_0) \subseteq B_0$ and clearly, the pair (A,B) has the P-property. Suppose

$$\begin{cases} \alpha(x_1,x_2) \geq 1, \\ d(u_1, Tx_1) = d(A,B) = 9/4, \\ d(u_2, Tx_2) = d(A,B) = 9/4, \end{cases}$$

then

$$\begin{cases} x_1,x_2 \in [-2,-1], \\ d(u_1, Tx_1) = 9/4, \\ d(u_2, Tx_2) = 9/4. \end{cases}$$

Note that $Tw \in [5/4,3/2]$ for all $w \in [-2,-1]$. Hence, $u_1 = u_2 = -1$, *i.e.*, $\alpha(u_1,u_2) \geq 2$. That is, T is a α-proximal admissible mapping with respect to $\eta(x,y) = 2$. Also, assume that $\alpha(x_n,x_{n+1}) \geq 2$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Then $\{x_n\} \subseteq [-2,-1]$ and hence $x \in [-2,-1]$. That is, $\alpha(x_n,x) \geq 2$ for all $n \in \mathbb{N} \cup \{0\}$. If $x,y \in [-1,-2]$, then

$$d(Tx, Ty) = \frac{1}{4}|x-y| \leq \frac{1}{2}|x-y| = \psi\big(d(x,y)\big).$$

Otherwise, $\alpha(x,y) = 0$. That is, $\frac{1}{2}d^*(x, Tx) > \alpha(x,y)d(x,y) = 0$. Hence,

$$d^*(x, Tx) \leq \alpha(x,y)d(x,y) \quad \Rightarrow \quad d(Tx, Ty) \leq \psi\big(d(x,y)\big).$$

All conditions of Theorem 2.2 hold for this example and there is a unique best proximity point $z = -1$ such that $d(-1, T(-1)) = d(A,B)$. Note that in this example the contractive condition of Theorems 3.1 and 3.2 of Jleli and Samet [14] is not satisfied and so these are not applicable here. Indeed, if, $x = -2$ and $y = -1$, then we have

$$\alpha(-2,-1)d\big(T(-2), T(-1)\big) = 4 \times \frac{1}{4} = 1 > 1/2 = \psi\big(d(-2,-1)\big).$$

The following results are nice consequences of Theorem 2.2.

Theorem 2.3 *Let A and B be non-empty closed subsets of a complete metric space (X,d) such that A_0 is non-empty. Assume $T : A \to B$ is a non-self-mapping satisfying the following* *assertions*:

(i) $T(A_0) \subseteq B_0$ *and* (A, B) *satisfies the P-property*;

(ii) *for a function* $\delta : [0,1) \to (0, 1/2]$, *there exists* $r \in [0,1)$ *such that*

$$\delta(r)d^*(x, Tx) \leq d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \leq \psi\big(d(x, y)\big) \tag{2.11}$$

for $x, y \in A$ *where* $d^*(x, y) = d(x, y) - d(A, B)$ *and* $\psi \in \Psi$.

Then T *has a unique best proximity point*.

Proof First, we fix r and define $\alpha_r : A \times A \to [0, \infty)$ by $\alpha_r(x, y) = \frac{1}{\delta(r)}$ for all $x, y \in A$. Since $\frac{1}{\delta(r)} \geq 2$ for all $r \in [0, 1)$, $\alpha_r(w, v) \geq 2$ for all $w, v \in A$. Now, since $\alpha_r(w, v)$ is constant and $\alpha_r(w, v) \geq 2$ for all $w, v \in A$, T is an α_r-proximal admissible mapping with respect to $\eta(x, y) = 2$ and hence conditions (ii)-(iv) of Theorem 2.2 hold. Furthermore, if

$$d^*(x, Tx) \leq \alpha_r(x, y)d(x, y),$$

then

$$\delta(r)d^*(x, Tx) \leq d(x, y)$$

and so by (2.11) we deduce $d(Tx, Ty) \leq \psi(d(x, y))$. Hence all conditions of Theorem 2.2 hold and T has a unique best proximity point. □

If we take $\psi(t) = rt$ in Theorem 2.3, where $0 \leq r < 1$, then we obtain the following result.

Corollary 2.1 *Let* A *and* B *be non-empty closed subsets of a complete metric space* (X, d) *such that* A_0 *is non-empty. Assume* $T : A \to B$ *is a non-self-mapping satisfying the following assertions*:

(i) $T(A_0) \subseteq B_0$ *and* (A, B) *satisfies the P-property*;

(ii) *for a function* $\delta : [0,1) \to (0, 1/2]$, *there exists* $r \in [0,1)$ *such that*

$$\delta(r)d^*(x, Tx) \leq d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \leq rd(x, y) \tag{2.12}$$

for $x, y \in A$.

Then T *has a unique best proximity point*.

Corollary 2.2 *Let* A *and* B *be non-empty closed subsets of a complete metric space* (X, d) *such that* A_0 *is non-empty. Assume* $T : A \to B$ *is a non-self-mapping satisfying the following assertions*:

(i) $T(A_0) \subseteq B_0$ *and* (A, B) *satisfies the P-property*;

(ii) *define a non-increasing function* $\theta : [0,1) \to (1/2, 1]$ *by*

$$\theta(r) = \begin{cases} 1 & \textit{if } 0 \leq r \leq (\sqrt{5} - 1)/2, \\ (1 - r)r^{-2} & \textit{if } (\sqrt{5} - 1)/2 < r < 2^{-1/2}, \\ (1 + r)^{-1} & \textit{if } 2^{-1/2} \leq r < 1. \end{cases} \tag{2.13}$$

Assume that there exists $r \in [0,1)$ such that

$$\frac{1}{2}\theta(r)d^*(x,Tx) \leq d(x,y) \quad \textit{implies} \quad d(Tx,Ty) \leq rd(x,y)$$

for $x, y \in A$ where $d^*(x,y) = d(x,y) - d(A,B)$.
Then T has a unique best proximity point.

Proof If we take $\delta(r) = \frac{1}{2}\theta(r)$ in Corollary 2.1, we obtain the required result. □

If we take $\delta(r) = \frac{1}{2(1+r)}$ in Corollary 2.1, we obtain the main result of [8] in the following form.

Corollary 2.3 *Let A and B be non-empty closed subsets of a complete metric space (X,d) such that A_0 is non-empty. Assume $T : A \to B$ is a non-self-mapping satisfying the following assertions*:

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property*;
(ii) *define a non-increasing function $\beta : [0,1) \to (1/2,1]$ by*

$$\beta(r) = \frac{1}{2(1+r)}. \tag{2.14}$$

Assume that there exists $r \in [0,1)$ such that

$$\beta(r)d^*(x,Tx) \leq d(x,y) \quad \textit{implies} \quad d(Tx,Ty) \leq rd(x,y)$$

for $x, y \in A$.
Then T has a unique best proximity point.

If we take $\delta(r) = \frac{1}{2}$ in Corollary 2.1 we have following result.

Corollary 2.4 *Let A and B be non-empty closed subsets of a complete metric space (X,d) such that A_0 is non-empty. Assume $T : A \to B$ is a non-self mapping satisfying the following assertions*:

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property*;
(ii)

$$\frac{1}{2}d^*(x,Tx) \leq d(x,y) \quad \Rightarrow \quad d(Tx,Ty) \leq rd(x,y)$$

for all $x, y \in A$.
Then T has a unique best proximity point.

3 Best proximity point results in partially ordered metric spaces

Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [2, 9, 32] and references therein). The existence of best proximity and fixed point results in partially ordered metric spaces has been considered recently by many authors [4, 7, 21, 33, 34]. The aim of this section is to deduce some best proximity and fixed point results in the context of partially ordered metric spaces. Moreover, we obtain certain recent fixed point results as corollaries in partially ordered metric spaces.

Definition 3.1 [21] A mapping $T : A \to B$ is said to be proximally order-preserving if and only if it satisfies the condition

$$\begin{cases} x_1 \preceq x_2, \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B), \end{cases} \implies u_1 \preceq u_2$$

for all $x_1, x_2, u_1, u_2 \in A$.

Clearly, if $B = A$, then the proximally order-preserving map $T : A \to A$ reduces to a non-decreasing map.

Theorem 3.1 *Let A and B be two non-empty closed subsets of a partially ordered complete metric space $(X, d, \preceq)$ with $A_0 \neq \emptyset$. Suppose that $T : A \to B$ is a non-self-mapping satisfying the following conditions*:

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property*;

(ii) *T is proximally order-preserving*;

(iii) *there exist elements x_0 and x_1 in A_0 with*

$$d(x_1, Tx_0) = d(A,B) \text{ satisfying } x_0 \preceq x_1;$$

(iv) *T is continuous*;

(v)

$$\frac{1}{2} d^*(x, Tx) \le d(x,y) \quad \Rightarrow \quad d(Tx, Ty) \le \psi\bigl(d(x,y)\bigr)$$

for all $x, y \in A$ with $x \preceq y$ where $d^*(x,y) = d(x,y) - d(A,B)$ and $\psi \in \Psi$.

Then T has a unique best proximity point.

Proof Define $\alpha : A \times A \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 2, & \text{if } x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

Now we prove that T is a α-proximal admissible mapping with respect to $\eta(x,y) = 2$. For this, assume

$$\begin{cases} \alpha(x,y) \ge 2, \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B). \end{cases}$$

So

$$\begin{cases} x \preceq y, \\ d(u, Tx) = d(A,B), \\ d(v, Ty) = d(A,B). \end{cases}$$

Now, since T is proximally order-preserving, $u \preceq v$. Thus, $\alpha(u,v) \geq 2$. Furthermore, by (iii) the elements x_0 and x_1 in A_0 with

$$d(x_1, Tx_0) = d(A,B) \text{ satisfy } \alpha(x_0,x_1) \geq 2.$$

Let $d^*(x,Tx) \leq \alpha(x,y)d(x,y)$. Then for all $x,y \in A$ with $x \preceq y$, we have $\alpha(x,y) \geq 2$, and hence

$$\frac{1}{2}d^*(x,Tx) \leq d(x,y).$$

From (v) we get $d(Tx,Ty) \leq \psi(d(x,y))$. That is, T is a modified Suzuki α-ψ-proximal contraction. Thus all conditions of Theorem 2.1 hold and T has a unique best proximity point. □

Corollary 3.1 *Let A and B be two non-empty closed subsets of a partially ordered complete metric space $(X,d,\preceq)$ with $A_0 \neq \emptyset$. Suppose that $T : A \to B$ be a non-self-mapping satisfying the following conditions*:

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property*;

(ii) *T is proximally ordered-preserving*;

(iii) *there exist elements x_0 and x_1 in A_0 with*

$$d(x_1, Tx_0) = d(A,B) \text{ satisfying } x_0 \preceq x_1;$$

(iv) *T is continuous*;

(v)

$$\frac{1}{2}d^*(x,Tx) \leq d(x,y) \quad \Rightarrow \quad d(Tx,Ty) \leq r\bigl(d(x,y)\bigr)$$

for all $x,y \in A$ with $x \preceq y$ where $d^*(x,y) = d(x,y) - d(A,B)$ and $0 \leq r < 1$.

Then T has a unique best proximity point.

Theorem 3.2 *Suppose that A and B are two non-empty closed subsets of partially ordered complete metric space $(X,d,\preceq)$ with $A_0 \neq \emptyset$. Let $T : A \to B$ be a non-self mapping satisfying the following conditions*:

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property*;

(ii) *T is proximally order-preserving*;

(iii) *the elements x_0 and x_1 in A_0 with*

$$d(x_1, Tx_0) = d(A,B) \text{ satisfy } x_0 \preceq x_1;$$

(iv) *if $\{x_n\}$ is a non-decreasing sequence in A such that $x_n \to x \in A$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$*;

(v)

$$\frac{1}{2}d^*(x,Tx) \leq d(x,y) \quad \Rightarrow \quad d(Tx,Ty) \leq \psi\bigl(d(x,y)\bigr) \tag{3.1}$$

for all $x,y \in A$ with $x \preceq y$ where $d^*(x,y) = d(x,y) - d(A,B)$ and $\psi \in \Psi$.

Then T has a unique best proximity point.

Proof Defining $\alpha : X \times X \to [0, \infty)$ as in the proof of Theorem 3.1, we find that T is an α-proximal admissible mapping with respect to $\eta(x,y) = 2$ and is modified Suzuki α-ψ-proximal contraction. Assume $\alpha(x_n, x_{n+1}) \geq 2$ for all $n \in \mathbb{N}$ such that $x_n \to x$ as $n \to \infty$. Then $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Hence, by (iv) we get $x_n \preceq x$ for all $n \in \mathbb{N}$ and so $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. That is, all conditions of Theorem 2.2 hold and T has a unique best proximity point. □

Corollary 3.2 *Suppose that A and B are two non-empty closed subsets of partially ordered complete metric space $(X, d, \preceq)$ with $A_0 \neq \emptyset$. Let $T : A \to B$ be a non-self-mapping satisfying the following conditions*:

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property*;
(ii) *T is proximally ordered-preserving*;
(iii) *there exist elements x_0 and x_1 in A_0 with*

$$d(x_1, Tx_0) = d(A, B) \text{ satisfying } x_0 \preceq x_1;$$

(iv) *if $\{x_n\}$ is a non-decreasing sequence in A such that $x_n \to x \in A$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$*;
(v)

$$\frac{1}{2} d^*(x, Tx) \leq d(x, y) \quad \Rightarrow \quad d(Tx, Ty) \leq r\bigl(d(x, y)\bigr) \tag{3.2}$$

for all $x, y \in A$ with $x \preceq y$ where $d^*(x, y) = d(x, y) - d(A, B)$ and $0 \leq r < 1$.

Then T has a unique best proximity point.

4 Applications

As an application of our results, we deduce new fixed point results for Suzuki-type contractions in the set up of metric and partially ordered metric spaces.

If we take $A = B = X$ in Theorems 2.1 and 2.2, then we deduce the following result.

Theorem 4.1 *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping with respect to $\eta(x, y) = 2$ such that*

$$d(x, Tx) \leq \alpha(x, y) d(x, y) \quad \Rightarrow \quad d(Tx, Ty) \leq \psi\bigl(d(x, y)\bigr)$$

for all $x, y \in X$ where $\psi \in \Psi$. Also suppose that the following assertions holds:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 2$*;
(ii) *either T is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 2$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 2$ for all $n \in \mathbb{N}$.*

Then T has a unique fixed point.

If we take $\psi(t) = kt$ in Theorem 4.1, where $0 \leq k < 1$, then we conclude to the following theorem.

Theorem 4.2 *Let (X, d) ba a complete metric space and let $T : X \to X$ be an α-admissible mapping with respect to $\eta(x, y) = 2$ such that*

$$d(x, Tx) \leq \alpha(x, y) d(x, y) \quad \Rightarrow \quad d(Tx, Ty) \leq k d(x, y)$$

for all $x, y \in X$ *where* $k \in [0,1)$. *Also suppose that the following assertions hold*:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 2$;

(ii) *either* T *is continuous or if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 2$ *and* $x_n \to x \in X$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq 2$ *for all* $n \in \mathbb{N}$.

Then T *has a unique fixed point*.

As a consequence of Theorem 4.2, by taking $\alpha(x,y) = 2/\theta(r)$, we derive the following theorem.

Theorem 4.3 *Let* (X,d) *be a complete metric space and* T *be a self-mapping on* X. *Define a non-increasing function* $\theta : [0,1) \to (1/2, 1]$ *by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 < r < 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \leq r < 1. \end{cases} \tag{4.1}$$

Assume that there exists $r \in [0,1)$ *such that*

$$\frac{1}{2}\theta(r)d(x,Tx) \leq d(x,y) \quad \text{implies} \quad d(Tx,Ty) \leq rd(x,y) \tag{4.2}$$

for all $x, y \in X$. *Then* T *has a unique fixed point*.

Furthermore, if we take $A = B = X$ in Theorems 3.1 and 3.2, then we deduce the following results.

Theorem 4.4 *Suppose that* $(X, d, \preceq)$ *is a partially ordered complete metric space and* $T : X \to X$ *is a mapping satisfying the following conditions*:

(i) T *is non-decreasing*;

(ii) *there exists* x_0 *in* X *such that* $x_0 \preceq Tx_0$;

(iii) T *is continuous*;

(iv)

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \Rightarrow \quad d(Tx,Ty) \leq \psi\big(d(x,y)\big) \tag{4.3}$$

for all $x, y \in X$ *with* $x \preceq y$ *where* $\psi \in \Psi$.

Then T *has a unique fixed point*.

Theorem 4.5 *Suppose that* $(X, d, \preceq)$ *is a partially ordered complete metric space and let* $T : X \to X$ *be a mapping satisfying the following conditions*:

(i) T *is non-decreasing*;

(ii) *there exists* x_0 *in* X *such that* $x_0 \preceq Tx_0$;

(iii) *if* $\{x_n\}$ *is a non-increasing sequence in* X *such that* $x_n \to x \in X$ *as* $n \to \infty$, *then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$;

(iv)

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \Rightarrow \quad d(Tx,Ty) \leq \psi\big(d(x,y)\big) \tag{4.4}$$

for all $x, y \in X$ *with* $x \preceq y$ *where* $\psi \in \Psi$.

Then T *has a unique fixed point.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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