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# Generalized $\mathcal{T}L$-fuzzy rough rings via $\mathcal{T}L$-fuzzy relational morphisms

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**Abstract**
In this paper, we introduce the notion of $\mathcal{T}L$-fuzzy (full) relational morphisms of rings and investigate some properties of $\mathcal{I}$-lower and $\mathcal{T}$-upper fuzzy rough approximations on rings with respect to them.

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**Keywords:** rough sets; $\mathcal{I}$-lower $L$-fuzzy rough approximations; $\mathcal{T}$-upper $L$-fuzzy rough approximations; $\mathcal{T}L$-fuzzy (full) relational morphisms; rings; $\mathcal{T}L$-fuzzy rough rings

## 1 Introduction

The theory of rough sets proposed by Pawlak [1] is a new mathematical approach for the study of incomplete or imprecise information. The usefulness of rough sets has been demonstrated by some successful applications in many areas such as knowledge discovery, machine learning, data analysis, approximate classification, conflict analysis, and so on. Equivalence relation is a key notion in Pawlak's rough set model. The equivalence classes are employed to construct lower and upper approximations. Some constructive and algebraic properties of rough sets are investigated by using equivalence relations. However, the requirement of an equivalence relation in Pawlak's rough set models seems to be a very restrictive condition that may limit the applications. Thus one of the aims of many researchers working on rough set theory has been to generalize this theory. Pawlak's model has been generalized by use of arbitrary relations in place of equivalence relations, or by use of covering or neighborhood systems or set-valued mappings in place of equivalence classes (see [2–8]).

Fuzzy set theory which was introduced by Zadeh [9] in 1965 and was generalized by Goguen [10] in 1967 is another mathematical tool to cope with the vague concepts via grading the turbidity. Dubois and Prade [11] introduced the problem communicating with the fuzzy sets and the rough sets. Many researchers have dealt with the integration of rough sets and fuzzy sets which are two distinct and complementary theories. For a deep study of fuzzy rough sets, the reader is referred to [11–16].

Biswas and Nanda [17] applied the notion of rough sets to algebra and introduced the notion of rough subgroups. Kuroki [18] introduced the notion of a rough ideal in a semigroup. In [19, 20], Davvaz analyzed a relationship between rough sets and ring theory considering a ring as a universal set and introduced the notion of rough ideals and rough subrings with respect to an ideal of a ring. By considering a ring as a universal set, Li *et al.*

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[21] studied $(\nu, \mathcal{T})$-fuzzy rough approximation operators with respect to a $\mathcal{T}L$-fuzzy ideal of the ring. Li and Yin [22] generalized lower and upper approximations to $\nu$-lower and $\mathcal{T}$-upper fuzzy rough approximations with respect to $\mathcal{T}$-congruence $L$-fuzzy relation on a semigroup. In order to have a more flexible tool for analysis of an information system, recently, Davvaz has studied the concept of generalized rough sets called by him $\mathcal{T}$-rough sets [23]. This is another generalization of rough sets. In this type of generalized rough sets, instead of equivalence relations, we require set-valued maps. This technique is useful, where it is difficult to find an equivalence relation among the elements of the universe set. In this generalized rough sets, a set-valued map gives rise to lower and upper generalized approximation operators. Ali *et al.* [24] studied some topological properties of the sets which are fixed by these operators. They also studied the degree of accuracy (DAG) for generalized rough sets and some properties of fuzzy sets which are induced by DAG. Davvaz, in [23], also introduced the concept of set-valued homomorphisms for groups, which is a generalization of an ordinary homomorphism. Yamak *et al.* [25, 26] investigated some properties of rough approximations with respect to set-valued homomorphisms of rings and modules in the perspective of set-valued homomorphisms. In [27], Ali *et al.* initiated the study of roughness in hemirings with respect to the Pawlak approximation space and also with respect to the generalized approximation space.

Recently, Ignjatović *et al.* [28] have introduced the notion of a (relational morphism) fuzzy relational morphism which is more general than a (congruence relation) fuzzy congruence relation. Set-valued homomorphisms and relational morphisms are related closely because a set-valued homomorphism defines a relational morphism and *vice versa*. This paper, in one respect, is an attendance in the sense of fuzzy generalization of the ideas presented in references [26]. This study offers a wider perspective than [21] in terms of using $\mathcal{T}L$-fuzzy relational morphisms which are $L$-fuzzy relations from any ring to any other ring instead of $\mathcal{T}$-congruence $L$-fuzzy relation on a semigroup. In this paper, we investigate some properties of the $\mathcal{T}L$-fuzzy relational morphisms looking from the side of homomorphisms of rings and designate them to construct an $L$-fuzzy approximation space and investigate some properties of this $L$-fuzzy approximation space with respect to rings.

## 2 Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence they are presented in brief.

Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice with the least element 0 and the greatest element 1. A triangular norm [29], or $t$-norm in short, is an increasing, associative and commutative mapping $\mathcal{T} : L \times L \to L$ that satisfies the boundary condition: for all $\alpha \in L$, $\mathcal{T}(\alpha, 1) = \alpha$. A $t$-norm $\mathcal{T}$ on $L$ is called $\vee$-distributive if $\mathcal{T}(\alpha, \beta_1 \vee \beta_2) = \mathcal{T}(\alpha, \beta_1) \vee \mathcal{T}(\alpha, \beta_2)$ for all $\alpha, \beta_1, \beta_2 \in L$. $\mathcal{T}$ is also called infinitely $\vee$-distributive if $\mathcal{T}(\alpha, \bigvee_{i\in\Lambda} \beta_i) = \bigvee_{i\in\Lambda} \mathcal{T}(\alpha, \beta_i)$ for all $\alpha, \beta_i \in L$, where $\Lambda$ is an index set. An implicator is a function $\mathcal{I} : L \times L \to L$ satisfying the conditions $\mathcal{I}(1, 0) = 0$ and $\mathcal{I}(1, 1) = \mathcal{I}(0, 1) = \mathcal{I}(0, 0) = 1$ (see [14]). The minimum $t$-norm $\mathcal{T}_M$ and the drastic product $t$-norm $\mathcal{T}_D$ on $L$ are defined as follows:

$$\mathcal{T}_M(\alpha, \beta) = \alpha \wedge \beta, \qquad \mathcal{T}_D(\alpha, \beta) = \begin{cases} \beta & \text{if } \alpha = 1; \\ \alpha & \text{if } \beta = 1; \\ 0 & \text{otherwise} \end{cases} \qquad \forall \alpha, \beta \in L.$$

An implicator $\mathcal{I}$ defined as

$$\mathcal{I}(x,y) = \bigvee_{\mathcal{T}(x,\alpha)\leq y} \alpha, \quad \alpha \in L$$

for all $x,y \in L$ is called an $R$-implicator (residual implicator) based on the $t$-norm $\mathcal{T}$. For an $R$-implicator $\mathcal{I}$ based on the $t$-norm $\mathcal{T}$, the following statements hold: $\forall a,b,c,d,a_i \in L$ $(i \in K)$ [21, 30],

(0) $a\mathcal{T}b \leq c \Leftrightarrow a \leq \mathcal{I}(b,c)$,
(1) $\mathcal{I}(a,1) = 1$ and $\mathcal{I}(1,a) = a$,
(2) $a \leq b \Rightarrow \mathcal{I}(a,b) = 1$,
(3) $a \leq b \Rightarrow \mathcal{I}(a,c) \geq \mathcal{I}(b,c)$ and $\mathcal{I}(c,a) \leq \mathcal{I}(c,b)$,
(4) $\mathcal{I}(a,b)\mathcal{T}\mathcal{I}(c,d) \leq \mathcal{T}(a,c)\mathcal{I}\mathcal{T}(b,d)$,
(5) $\mathcal{I}(\mathcal{T}(a,b),c) \leq \mathcal{I}(a,\mathcal{I}(b,c))$,
(6) $\mathcal{T}(\mathcal{I}(a,b),c) \leq \mathcal{I}(a,\mathcal{T}(b,c))$,
(7) $\mathcal{I}(\bigvee_{i\in K} a_i, b) = \bigwedge_{i\in K} \mathcal{I}(a_i,b)$,
(8) $\mathcal{I}(b, \bigwedge_{i\in K} a_i) = \bigwedge_{i\in K} \mathcal{I}(a_i,b)$.

Throughout this paper, unless otherwise indicated, $L$ is referred to as any lattice, and $\mathcal{T}$ and $\mathcal{I}$ are referred to as any $t$-norm and any implicator on $L$, respectively.

### 2.1 *L*-fuzzy subsets

In this subsection, we give some basic notions and results (see [9, 10, 14, 16, 30–33]).

Let $X$ be a non-empty set called the universe of discourse. An $L$-fuzzy subset of $X$ is any function from $X$ into $L$ (see [10]). The class of all subsets and $L$-fuzzy subsets of $X$ will be denoted by $\mathcal{P}(X)$ and $\mathcal{F}(X,L)$, respectively. In particular, if $L = [0,1]$ (where $[0,1]$ is the unit interval), then it is appropriate to replace a fuzzy subset with an $L$-fuzzy subset. In this case the set of all fuzzy subsets of $X$ is denoted by $\mathcal{F}(X)$. For any $\mu \in \mathcal{F}(X,L)$, the $\alpha$-cut (or level) set of $\mu$ will be denoted by $\mu_\alpha$, that is, $\mu_\alpha = \{x \in X \mid \mu(x) \geq a\}$, where $\alpha \in L$. In what follows, $\alpha_y$ will denote the fuzzy singleton with value $\alpha$ at $y$ and 0 elsewhere. For $A \subseteq X$ and $\alpha \in L$, $\alpha_A \in \mathcal{F}(X,L)$ is defined by $\alpha_A(x) = \alpha$ if $x \in A$ and $\alpha_A(x) = 0$ otherwise. $1_A$ is called a characteristic function of the set $A \subseteq X$. Let $\mu$ and $\nu$ be any two $L$-fuzzy subsets of $X$. The symbols $\mu \vee \nu$, $\mu \wedge \nu$ and $\mu\mathcal{T}\nu$ will mean the following $L$-fuzzy subsets of $X$, for all $x$ in $X$,

$$(\mu \vee \nu)(x) = \mu(x) \vee \nu(x),$$

$$(\mu \wedge \nu)(x) = \mu(x) \wedge \nu(x),$$

$$(\mu\mathcal{T}\nu)(x) = \mathcal{T}\big(\mu(x), \nu(x)\big).$$

### 2.2 *L*-fuzzy relations

Let $X$, $Y$ and $Z$ be non-empty sets.

**Definition 2.1** An $L$-fuzzy subset $\Theta \in \mathcal{F}(X \times Y, L)$ is referred to as an $L$-fuzzy relation from $X$ to $Y$, $\Theta(x,y)$ is the degree of relation between $x$ and $y$, where $(x,y) \in X \times Y$. If for each $x \in X$, there exists $y \in X$ such that $\Theta(x,y) = 1$, then $\Theta$ is referred to as a serial $L$-fuzzy relation from $X$ to $Y$. If $X = Y$, then $\Theta$ is referred to as an $L$-fuzzy relation on $X$; $\Theta$ is referred to as a reflexive $L$-fuzzy relation if $\Theta(x,x) = 1$ for all $x \in X$; $\Theta$ is referred to as a symmetric $L$-fuzzy relation if $\Theta(x,y) = \Theta(y,x)$ for all $x,y \in X$; $\Theta$ is referred to as a $\mathcal{T}$-transitive

$L$-fuzzy relation if $\Theta(x,z) \geq \bigvee_{y\in X}(\Theta(x,y)\mathcal{T}\Theta(y,z))$ for all $x,z \in X$. Let $\Theta \in \mathcal{F}(X \times Y, L)$ and $\Theta^{-1} \in \mathcal{F}(Y \times X, L)$ be $L$-fuzzy relations which satisfy the condition $\Theta^{-1}(y,x) = \Theta(x,y)$ for all $x \in X$ and $y \in Y$. Then $\Theta^{-1}$ is called the inverse $L$-fuzzy relation of $\Theta$.

The $\mathcal{T}$-compositions of the $L$-fuzzy relations $\Theta \in \mathcal{F}(X \times Y, L)$ and $\Phi \in \mathcal{F}(Y \times Z, L)$ is an $L$-fuzzy relation $\Phi \circ_{\mathcal{T}} \Theta : X \times Z \to L$ defined by $(\Phi \circ_{\mathcal{T}} \Theta)(x,z) = \bigvee_{y\in Y}(\Theta(x,y)\mathcal{T}\Phi(y,z))$ for all $(x,z) \in (X,Z)$ (see [32]).

## 2.3 Generalized *L*-fuzzy rough approximation operators

Let $X$ and $Y$ be two non-empty sets and $\Theta$ be an $L$-fuzzy relation from $X$ to $Y$. The triple $(X,Y,\Theta)$ is called a generalized $L$-fuzzy approximation space. If $\Theta$ is an $L$-fuzzy relation on $X$, then it is denoted by the pair $(X,\Theta)$, and especially if $L = [0,1]$, then the triple $(X,Y,\Theta)$ is called a generalized fuzzy approximation space. Let $\mathcal{T}$ be a $t$-norm and $\mathcal{I}$ be an implicator on $L$. For any $L$-fuzzy subset $\mu$ of $Y$, the $\mathcal{T}$-upper and $\mathcal{I}$-lower $L$-fuzzy rough approximations of $\mu$, denoted as $\overline{\Theta}^{\mathcal{T}}(\mu)$ and $\underline{\Theta}_{\mathcal{I}}(\mu)$ respectively, are two $L$-fuzzy sets of $X$ whose membership functions are defined respectively by

$$\overline{\Theta}^{\mathcal{T}}(\mu)(x) = \bigvee_{y\in Y}\mathcal{T}\big(\Theta(x,y),\mu(y)\big), \qquad \underline{\Theta}_{\mathcal{I}}(\mu)(x) = \bigwedge_{y\in Y}\mathcal{I}\big(\Theta(x,y),\mu(y)\big) \quad \forall x \in X.$$

The operators $\overline{\Theta}^{\mathcal{T}}$ and $\underline{\Theta}_{\mathcal{I}}$ from $\mathcal{F}(Y,L)$ to $\mathcal{F}(X,L)$ are referred to as $\mathcal{T}$-upper and $\mathcal{I}$-lower $L$-fuzzy rough approximation operators of $(X,Y,\Theta)$ respectively, and the pair $(\underline{\Theta}_{\mathcal{I}}(\mu), \overline{\Theta}^{\mathcal{T}}(\mu))$ is called the $(\mathcal{I},\mathcal{T})$-$L$-fuzzy rough set of $\mu$ with respect to $(X,Y,\Theta)$.

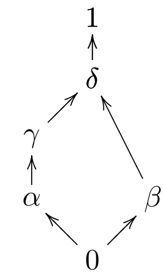
In [14], Wu *et al.* studied some properties of $\mathcal{T}$-upper and $\mathcal{I}$-lower fuzzy rough approximation operators of the generalized approximation space $(X,Y,\Theta)$, where $\mathcal{T}$ is a continuous $t$-norm and $\mathcal{I}$ is a hybrid monotonic implicator on $[0,1]$. The same properties for the $\mathcal{T}$-upper and the $\mathcal{I}$-lower $L$-fuzzy rough approximation operators can be obtained.

**Example 2.2** Let $L = \{0,\alpha,\beta,\gamma,1\}$ be a lattice whose Hasse diagram is depicted in Figure 1.

Let $\Theta : \mathbb{Z}_4 \times \mathbb{Z}_3 \to L$ be defined by

$$\Theta(x,y) = \begin{cases} 1 & \text{if } x = \overline{0}; \\ 0 & \text{if } x = \overline{1}; \\ \beta & \text{if } x = \overline{2}; \\ 0 & \text{if } x = \overline{3} \end{cases}$$

**Figure 1 Lattice *L*.**



for all $x \in \mathbb{Z}_4$ and $y \in \mathbb{Z}_3$, and let $\mu$, $\nu$, $\eta$ be $L$-fuzzy subsets of $\mathbb{Z}_3$ indicated in the following table:

| $x$ | $\overline{0}$ | $\overline{1}$ | $\overline{2}$ |
|---|---|---|---|
| $\mu(x)$ | 1 | 1 | 1 |
| $\nu(x)$ | 1 | $\alpha$ | $\alpha$ |
| $\eta(x)$ | $\gamma$ | $\beta$ | $\beta$ |

Then $\mathcal{T}_\mathcal{D}$-upper and $\mathcal{I}$-lower $L$-fuzzy rough approximations of them are obtained as follows, where $\mathcal{I}$ is an $R$-implicator based on the $t$-norm $\mathcal{T}_\mathcal{D}$:

| $x$ | $\overline{0}$ | $\overline{1}$ | $\overline{2}$ | $\overline{3}$ |
|---|---|---|---|---|
| $\overline{\Theta}^{\mathcal{T}}(\mu)(x)$ | 1 | 0 | $\beta$ | 0 |
| $\overline{\Theta}^{\mathcal{T}}(\nu)(x)$ | 1 | 0 | $\beta$ | 0 |
| $\overline{\Theta}^{\mathcal{T}}(\eta)(x)$ | $\delta$ | 0 | 0 | 0 |

| $x$ | $\overline{0}$ | $\overline{1}$ | $\overline{2}$ | $\overline{3}$ |
|---|---|---|---|---|
| $\underline{\Theta}_{\mathcal{I}}(\mu)(x)$ | 1 | 1 | 1 | 1 |
| $\underline{\Theta}_{\mathcal{I}}(\nu)(x)$ | $\alpha$ | 1 | $\delta$ | 1 |
| $\underline{\Theta}_{\mathcal{I}}(\eta)(x)$ | $\beta$ | 1 | $\delta$ | 1 |

### 2.4 $\mathcal{T}L$-fuzzy subrings

Let $(R,+,\cdot)$ be a ring and $I$ be a non-empty subset of $R$. Then we say that $I$ is a subring whenever $a,b \in I$ then $a-b, ab \in I$. A subring $I$ of $R$ which satisfies the condition $ra \in I$ and $ar \in I$ for all $r \in R$, $a \in I$ is said to be an ideal of $R$. Let $R$ and $S$ be rings. A function $f: R \to S$ is a homomorphism provided $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a,b \in R$.

**Definition 2.3** Let $R$ be a ring and $\mu \in \mathcal{F}(R,L)$. If, for all $x,y \in R$, $\mu$ satisfies the following conditions:

(i) $\mu(x) \leq \mu(-x)$,
(ii) $\mu(x)\mathcal{T}\mu(y) \leq \mu(x+y)$,
(iii) $\mu(x)\mathcal{T}\mu(y) \leq \mu(xy)$,

then $\mu$ is called a $\mathcal{T}L$-fuzzy subring of $R$. Moreover, a $\mathcal{T}L$-fuzzy subring $\mu$ is called a $\mathcal{T}L$-fuzzy ideal of $R$ if $\mu(x) \vee \mu(y) \leq \mu(xy)$.

**Definition 2.4** Let $R$ be a ring and $\mu, \nu \in \mathcal{F}(R,L)$. Define $\mu +_\mathcal{T} \nu$, $\mu \cdot_\mathcal{T} \nu$ and $\mu \odot_\mathcal{T} \nu$ as follows:

$$(\mu +_\mathcal{T} \nu)(x) = \bigvee_{x=a+b} \mu(a)\mathcal{T}\nu(b);$$

$$(\mu \cdot_\mathcal{T} \nu)(x) = \bigvee_{x=ab} \mu(a)\mathcal{T}\nu(b);$$

$$(\mu \odot_\mathcal{T} \nu)(x) = \bigvee_{x=\sum_{i=1}^n a_i b_i, n\in\mathbb{N}} \mathcal{T}_{i=1}^n \mu(a_i)\mathcal{T}\nu(b_i).$$

The $L$-fuzzy sets $\mu +_\mathcal{T} \nu$, $\mu \cdot_\mathcal{T} \nu$ and $\mu \odot_\mathcal{T} \nu$ are called the $\mathcal{T}$-sum, $\cdot_\mathcal{T}$-product and $\odot_\mathcal{T}$-product, respectively.

## 3 $\mathcal{T}L$-fuzzy relational morphism of rings

Throughout this paper, unless otherwise stated, $R$, $S$, and $K$ will be referred to as rings. Recall that a fuzzy relation $\Theta$ on a semigroup $A$ is called fuzzy compatible iff

$\min\{\Theta(a,b),\Theta(c,d)\} \leq \Theta(ac,bd)$ for all $a,b,c,d \in A$ (see [30, 34]). We will give a more general definition for rings in the light of the reference [28].

**Definition 3.1** Let $R$ and $S$ be two rings and $\Theta \in \mathcal{F}(R \times S, L)$. If, for all $(x,y),(a,b) \in R \times S$, $\Theta$ satisfies the following conditions:

(i) $\Theta(x,y)\mathcal{T}\Theta(a,b) \leq \Theta(x+a,y+b)$,

(ii) $\Theta(x,y) \leq \Theta(-x,-y)$,

(iii) $\Theta(x,y)\mathcal{T}\Theta(a,b) \leq \Theta(xa,yb)$,

then $\Theta$ is called a $\mathcal{T}L$-fuzzy relational morphism from $R$ to $S$. Moreover, a $\mathcal{T}L$-fuzzy relational morphism $\Theta$ is said to be full if $\Theta(x,y) \vee \Theta(a,b) \leq \Theta(xa,yb)$. The set of all the $\mathcal{T}L$-fuzzy (full) relational morphisms from $R$ to $S$ is denoted by $\mathrm{Hom}(R \times S,\mathcal{T},L)$ (respectively, $\mathrm{Hom}[R \times S,\mathcal{T},L]$). It is obvious that $\mathrm{Hom}[R \times S,\mathcal{T},L] \subseteq \mathrm{Hom}(R \times S,\mathcal{T},L)$ and if $\Theta \in \mathrm{Hom}(R \times S,\mathcal{T},L)$, then $\Theta^{-1} \in \mathrm{Hom}(S \times R,\mathcal{T},L)$.

**Example 3.2** Let $R$, $S$, $R_1$, $R_2$, $S_1$ and $S_2$ be rings.

(1) Let $\mu$ and $\nu$ be $\mathcal{T}L$-fuzzy subrings of $R$ and $S$, respectively. Let a $\mathcal{T}L$-fuzzy relation $\Theta$ be defined by $\Theta(a,b) = \mu(a)\mathcal{T}\nu(-b)$ for all $(a,b) \in R \times S$. Then $\Theta$ is a $\mathcal{T}L$-fuzzy relational morphism from $R$ to $S$. Moreover, if $\mathcal{T}$ is a $\vee$-distributive $t$-norm and $\mu$ and $\nu$ are $\mathcal{T}L$-fuzzy ideals of $R$ and $S$, respectively, then $\Theta$ is full.

(2) Let $\Phi \in \mathrm{Hom}(R_2 \times S_2,\mathcal{T},L)$ and $f : R_1 \to R_2$ and $g : S_1 \to S_2$ be homomorphisms of rings. Then $\Theta : R_1 \times S_1 \to L$ defined by $\Theta(a,b) = \Phi(f(a),g(b))$ for all $(a,b) \in R_1 \times S_1$ is a $\mathcal{T}L$-fuzzy relational morphism from $R_1$ to $S_1$. In particular, if $\Phi$ is full, then $\Theta$ is full.

(3) Let $f : R_1 \to R_2$ be a homomorphism of rings and $\mu$ be a $\mathcal{T}L$-fuzzy subring of $R_2$. Then $\Theta : R_1 \times R_2 \to L$ defined by $\Theta(a,b) = \mu(f(a))\mathcal{T}\mu(b)$ for all $(a,b) \in R_1 \times R_2$ is a $\mathcal{T}L$-fuzzy relational morphism from $R_1$ to $R_2$.

(4) Let $f : R_1 \to R_2$ be a homomorphism of rings and $\alpha,\beta \in L$ such that $\alpha \leq \beta$. Then $\Theta : R_1 \times R_2 \to L$ defined by

$$\Theta(a,b) = \begin{cases} \beta & \text{if } f(a) = b; \\ \alpha & \text{if } f(a) \neq b \end{cases}$$

for all $(a,b) \in R_1 \times R_2$ is a $\mathcal{T}L$-fuzzy relational morphism from $R_1$ to $R_2$.

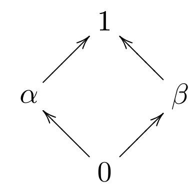
(5) $\mathbb{Z}$ is a ring under the usual operations of addition and multiplication. Let $L$ be any lattice with the least element 0 and the greatest element 1. $\Theta : \mathbb{Z} \times \mathbb{Z} \to L$ defined by

$$\Theta(a,b) = \begin{cases} 1 & \text{if } a = b; \\ 0 & \text{if } a \neq b \end{cases}$$

is a $\mathcal{T}L$-fuzzy relational morphism on $\mathbb{Z}$ but it is not full since $\Theta(3,3) \vee \Theta(1,2) = 1 \vee 0 \nleq 0 = \Theta(3,6)$.

(6) Let $L = \{0,\alpha,\beta,1\}$ be the lattice, see Figure 2.

**Figure 2 Lattice *L*.**




Let $\Theta : \mathbb{Z} \times \mathbb{Z}_2 \to L$ be defined by

$$\Theta(x,y) = \begin{cases} 0 & \text{if } x \text{ is odd;} \\ \alpha & \text{if } x \text{ is even and } x \neq 0; \\ 1 & \text{if } x = 0 \end{cases}$$

is a $\mathcal{T}L$-fuzzy full relational morphism.

**Proposition 3.3** *Let $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm.*

(i) *If $\Theta \in \mathrm{Hom}(R \times S, \mathcal{T}, L)$, then for all $x, a \in R$ and $y, b \in S$, $\Theta(x,y)\mathcal{T}\Theta(a,b) \leq \Theta(x-a, y-b)$.*

(ii) *Let $\Theta \in \mathrm{Hom}(R \times S, \mathcal{T}_1, L)$ and $\mathcal{T}_2 \leq \mathcal{T}_1$. Then $\Theta \in \mathrm{Hom}(R \times S, \mathcal{T}_2, L)$.*

(iii) *Let $\Theta, \Phi \in \mathrm{Hom}(R \times S, \mathcal{T}, L)$. Then $\Theta\mathcal{T}\Phi \in \mathrm{Hom}(R \times S, \mathcal{T}, L)$.*

*Proof* It is straightforward. □

By Proposition 3.3(iii), it is clear that $\mathrm{Hom}(R \times S, \mathcal{T}, L)$ is a monoid.

**Theorem 3.4** *Let $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm. If $\Theta \in \mathrm{Hom}(R \times S, \mathcal{T}, L)$ and $\Phi \in \mathrm{Hom}(S \times K, \mathcal{T}, L)$, then $\Phi \circ_{\mathcal{T}} \Theta \in \mathrm{Hom}(R \times K, \mathcal{T}, L)$.*

*Proof* Let $(r_1, k_1), (r_2, k_2) \in R \times K$. Thus

$$\begin{aligned} &(\Phi \circ_{\mathcal{T}} \Theta)(r_1, k_1)\mathcal{T}(\Phi \circ_{\mathcal{T}} \Theta)(r_2, k_2) \\ &\quad = \Bigl(\bigvee_{a \in S} \Theta(r_1, a)\mathcal{T}\Phi(a, k_1)\Bigr)\mathcal{T}\Bigl(\bigvee_{b \in S} \Theta(r_2, b)\mathcal{T}\Phi(b, k_2)\Bigr) \\ &\quad = \bigvee_{a \in S}\bigvee_{b \in S} \Theta(r_1, a)\mathcal{T}\Theta(r_2, b)\mathcal{T}\Phi(a, k_1)\mathcal{T}\Phi(b, k_2) \\ &\quad \leq \bigvee_{a \in S}\bigvee_{b \in S} \Theta(r_1 + r_2, a + b)\mathcal{T}\Phi(a + b, k_1 + k_2) \\ &\quad = \bigvee_{s \in S} \Theta(r_1 + r_2, s)\mathcal{T}\Phi(s, k_1 + k_2) \\ &\quad = (\Phi \circ_{\mathcal{T}} \Theta)(r_1 + r_2, k_1 + k_2), \\ &(\Phi \circ_{\mathcal{T}} \Theta)(r_1, k_1)\mathcal{T}(\Phi \circ_{\mathcal{T}} \Theta)(r_2, k_2) \\ &\quad = \Bigl(\bigvee_{a \in S} \Theta(r_1, a)\mathcal{T}\Phi(a, k_1)\Bigr)\mathcal{T}\Bigl(\bigvee_{b \in S} \Theta(r_2, b)\mathcal{T}\Phi(b, k_2)\Bigr) \\ &\quad = \bigvee_{a \in S}\bigvee_{b \in S} \Theta(r_1, a)\mathcal{T}\Theta(r_2, b)\mathcal{T}\Phi(a, k_1)\mathcal{T}\Phi(b, k_2) \end{aligned}$$

$$
\begin{aligned}
&\leq \bigvee_{a\in S}\bigvee_{b\in S}\Theta(r_1r_2,ab)\mathcal{T}\Phi(ab,k_1k_2)\\
&\leq \bigvee_{s\in S}\Theta(r_1r_2,s)\mathcal{T}\Phi(s,k_1k_2)\\
&= (\Phi\circ_{\mathcal{T}}\Theta)(r_1r_2,k_1k_2),
\end{aligned}
$$

$$
\begin{aligned}
(\Phi\circ_{\mathcal{T}}\Theta)(r_1,k_1) &= \bigvee_{s\in S}\Theta(r_1,s)\mathcal{T}\Phi(s,k_1)\\
&\leq \bigvee_{s\in S}\Theta(-r_1,-s)\mathcal{T}\Phi(-s,-k_1)\\
&= (\Phi\circ_{\mathcal{T}}\Theta)(-r_1,-k_1).
\end{aligned}
$$

So, $\Phi\circ_{\mathcal{T}}\Theta$ is a $\mathcal{T}L$-fuzzy relational morphism of rings. It is clear to see that if $\Theta\in \mathrm{Hom}[R\times S,\mathcal{T},L]$ and $\Phi\in\mathrm{Hom}[S\times K,\mathcal{T},L]$, then $\Phi\circ_{\mathcal{T}}\Theta\in\mathrm{Hom}[R\times K,\mathcal{T},L]$. □

**Definition 3.5** Let $\Theta\in\mathrm{Hom}(R\times S,\mathcal{T},L)$. Then the kernel and image of $\Theta$ (denoted by $\mathrm{Ker}\,\Theta$ and $\mathrm{Im}\,\Theta$, respectively) are defined to be the $L$-fuzzy subsets of $R$ and $S$, respectively that satisfy $\mathrm{Ker}\,\Theta(r)=\Theta(r,0)$ and $\mathrm{Im}\,\Theta(s)=\bigvee_{r\in R}\Theta(r,s)$ for all $r\in R$, $s\in S$.

**Remark** Let $f:R\to S$ be a homomorphism of rings and let $\Theta\in\mathcal{F}(R\times S,L)$ be defined by

$$
\Theta(a,b)=\begin{cases}1 & \text{if } f(a)=b;\\ 0 & \text{if } f(a)\neq b.\end{cases}
$$

Then $\Theta\in\mathrm{Hom}(R\times S,\mathcal{T},L)$ and $\mathrm{Ker}\,\Theta=1_{\mathrm{Ker}f}$ and $\mathrm{Im}\,R=1_{\mathrm{Im}f}$.

**Proposition 3.6** *Let* $\Theta\in\mathrm{Hom}(R\times S,\mathcal{T},L)$. *Then*:

(i) $\mathrm{Ker}\,\Theta$ *is a* $\mathcal{T}L$*-fuzzy subring of* $R$.
(ii) *If* $\Theta\in\mathrm{Hom}[R\times S,\mathcal{T},L]$, *then* $\mathrm{Ker}\,\Theta$ *is a* $\mathcal{T}L$*-fuzzy ideal of* $R$.
(iii) $\mathrm{Im}\,\Theta$ *is a* $\mathcal{T}L$*-fuzzy subring of* $S$ *if* $\mathcal{T}$ *is an infinitely* $\vee$*-distributive t-norm on* $L$.

*Proof*

(i) Let $x,y\in R$. Then

$$
\begin{aligned}
\mathrm{Ker}\,\Theta(x)\mathcal{T}\,\mathrm{Ker}\,\Theta(y) &= \Theta(x,0)\mathcal{T}\Theta(y,0)\\
&\leq \Theta(x+y,0)\\
&\leq \mathrm{Ker}\,\Theta(x+y),\\
\mathrm{Ker}\,\Theta(x)\mathcal{T}\,\mathrm{Ker}\,\Theta(y) &= \Theta(x,0)\mathcal{T}\Theta(y,0)\\
&= \Theta(xy,0)\\
&= \mathrm{Ker}\,\Theta(xy),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ker}\,\Theta(x) &= \Theta(x,0)\\
&\leq \Theta(-x,0)\\
&= \mathrm{Ker}\,\Theta(-x).
\end{aligned}
$$

(ii) If $\Theta \in \mathrm{Hom}[R \times S, \mathcal{T}, L]$, then it follows immediately from (i).

(iii) Let $x, y \in S$. Then

$$\begin{aligned}
\operatorname{Im}\Theta(x)\mathcal{T}\operatorname{Im}\Theta(y) &= \Big(\bigvee_{a\in R}\Theta(a,x)\Big)\mathcal{T}\Big(\bigvee_{b\in R}\Theta(b,y)\Big)\\
&= \bigvee_{a\in R}\bigvee_{b\in R}\Theta(a,x)\mathcal{T}\Theta(b,y)\\
&\leq \bigvee_{a\in R}\bigvee_{b\in R}\Theta(a+b,x+y)\\
&= \bigvee_{a,b\in R}\Theta(a+b,x+y)\\
&\leq \bigvee_{r\in R}\Theta(r,x+y)\\
&= \operatorname{Im}\Theta(x+y),\\
\operatorname{Im}\Theta(x)\mathcal{T}\operatorname{Im}\Theta(y) &= \Big(\bigvee_{a\in R}\Theta(a,x)\Big)\mathcal{T}\Big(\bigvee_{b\in R}\Theta(b,y)\Big)\\
&= \bigvee_{a\in R}\bigvee_{b\in R}\Theta(a,x)\mathcal{T}\Theta(b,y)\\
&\leq \bigvee_{a\in R}\bigvee_{b\in R}\Theta(ab,xy)\\
&= \bigvee_{a,b\in R}\Theta(ab,xy)\\
&\leq \bigvee_{r\in R}\Theta(r,xy)\\
&= \operatorname{Im}\Theta(xy),
\end{aligned}$$

$$\begin{aligned}
\operatorname{Im}\Theta(x) &= \bigvee_{a\in R}\Theta(a,x)\\
&\leq \bigvee_{a\in R}\Theta(-a,-x)\\
&= \bigvee_{-a\in R}\Theta(-a,-x)\\
&= \bigvee_{r\in R}\Theta(r,-x)\\
&= \operatorname{Im}\Theta(-x).
\end{aligned}$$

Thus $\operatorname{Im}\Theta$ is a $\mathcal{T}L$-fuzzy subring of $S$. □

Let $\Theta \in \mathcal{F}(R \times S, L)$ and $\mathcal{T}$ be any $t$-norm on $L$. For an $\alpha \in L$, let some $\alpha$-level sets be defined as follows:

$$(\Theta_S)_\alpha = \{s \in S \mid \exists r \in R : \Theta(r,s)\mathcal{T}\alpha \geq \alpha\},$$

$$(\Theta_R)_\alpha = \{r \in R \mid \exists s \in S : \Theta(r,s)\mathcal{T}\alpha \geq \alpha\},$$

$$\big(\Theta_S^*\big)_\alpha = \big\{s \in S \mid \Theta(0,s)\mathcal{T}\alpha \geq \alpha\big\},$$
$$\big(\Theta_R^*\big)_\alpha = \big\{r \in R \mid \Theta(r,0)\mathcal{T}\alpha \geq \alpha\big\}.$$

**Theorem 3.7** *Let* $\Theta \in \mathrm{Hom}(R \times S, \mathcal{T}, L)$. *Then*:
(i) *If* $(\Theta_S)_\alpha \neq \emptyset$, *then* $(\Theta_S)_\alpha$ *is a subring of* $S$.
(ii) *If* $(\Theta_R)_\alpha \neq \emptyset$, *then* $(\Theta_R)_\alpha$ *is a subring of* $R$.
(iii) *If* $(\Theta_S^*)_\alpha \neq \emptyset$, *then* $(\Theta_S^*)_\alpha$ *is a subring of* $(\Theta_S)_\alpha$.
(iv) *If* $(\Theta_R^*)_\alpha \neq \emptyset$, *then* $(\Theta_R^*)_\alpha$ *is a subring of* $(\Theta_R)_\alpha$.

*Proof* Let $a, b \in (\Theta_S)_\alpha$. Then there exist $x, y \in R$ such that $\Theta(x,a)\mathcal{T}\alpha \geq \alpha$ and $\Theta(y,b)\mathcal{T} \times \alpha \geq \alpha$. So, we have $\Theta(x,a)\mathcal{T}\Theta(y,b)\mathcal{T}\alpha \geq \alpha$. Since $\Theta \in \mathrm{Hom}(R \times S, \mathcal{T}, L)$, then $\Theta(x-y, a-b)\mathcal{T}\alpha \geq \alpha$ by Proposition 3.3(i) and $\Theta(xy, ab)\mathcal{T}\alpha \geq \alpha$. Therefore $a-b, ab \in (\Theta_S)_\alpha$. Hence $(\Theta_S)_\alpha$ is a subring of $S$. For (ii), the proof is similar to (i). For (iii), it is similar to (i) that $(\Theta_S^*)_\alpha$ is a subring of $S$. Since $(\Theta_S^*)_\alpha \subseteq (\Theta_S)_\alpha$, then $(\Theta_S^*)_\alpha$ is a subring of $(\Theta_S)_\alpha$. For (iv), the proof is similar to (iii). □

**Definition 3.8** Let $\mu$ be a $\mathcal{T}L$-fuzzy ideal of $R$. The $L$-fuzzy subset $y + \mu$ of $R$ defined by $(y+\mu)(x) = \mu(x-y)$ is called a coset of the $\mathcal{T}L$-fuzzy ideal $\mu$ (see [35]).

**Proposition 3.9** *Let* $\mu$ *be a* $\mathcal{T}L$*-fuzzy ideal of* $R$ *which satisfies* $\mu(0) = 1$ *and let* $x$, $y$, $u$, $v$ *be any elements in* $R$. *If* $x + \mu = u + \mu$ *and* $y + \mu = v + \mu$, *then*
(i) $(x+y) + \mu = (u+v) + \mu$ *and*
(ii) $(xy) + \mu = (uv) + \mu$.

*Proof* It is similar to the proof of Proposition 3.4 in [35]. □

Let $\mu$ be a $\mathcal{T}L$-fuzzy ideal of $R$ which satisfies $\mu(0) = 1$. Then Proposition 3.9 allows us to define two binary operations '+' and '$\cdot$' on the set $R/\mu$ of all cosets of $\mu$ as follows:

$$(x+\mu) + (y+\mu) = (x+y) + \mu,$$
$$(x+\mu) \cdot (y+\mu) = (xy) + \mu.$$

It is straightforward to see that $R/\mu$ is a ring under these binary operations with additive identity $\mu$, multiplicative identity $1 + \mu$ and $-(x+\mu) = (-x) + \mu$.

**Theorem 3.10** *Let* $\mu$ *be a* $\mathcal{T}L$*-fuzzy ideal of* $R$ *which satisfies* $\mu(0) = 1$ *and* $\Theta : R \times R/\mu \to L$ *be defined by* $\Theta(x, y+\mu) = \mu(x-y)$ *for all* $x, y \in R$. *Then* $\Theta \in \mathrm{Hom}(R \times R/\mu, \mathcal{T}, L)$.

*Proof* Let $x, a \in R$ and $y+\mu, b+\mu \in R/\mu$. Thus

$$\begin{aligned}
\Theta(x, y+\mu)\mathcal{T}\Theta(a, b+\mu) &= \mu(x-y)\mathcal{T}\mu(a-b) \\
&\leq \mu(x-y+a-b) \\
&= \mu\big(x+a-(y+b)\big) \\
&= \Theta(x+a, y+b+\mu) \\
&= \Theta\big(x+a, (y+\mu)+(b+\mu)\big),
\end{aligned}$$

$$\begin{aligned}
\Theta(x, y+\mu) &= \mu(x-y) \\
&\leq \mu\bigl(-(x-y)\bigr) \\
&= \mu(-x+y) \\
&= \mu\bigl(-x-(-y)\bigr) \\
&= \Theta(-x, -y+\mu), \\
\Theta(x, y+\mu)\mathcal{T}\Theta(a, b+\mu) &= \mu(x-y)\mathcal{T}\mu(a-b) \\
&\leq \mu\bigl((x-y)a\bigr)\mathcal{T}\mu\bigl(y(a-b)\bigr) \\
&= \mu(xa-ya)\mathcal{T}\mu(ya-yb) \\
&\leq \mu(xa-ya+ya-yb) \\
&= \mu(xa-yb) \\
&= \Theta(xa, yb+\mu) \\
&= \Theta\bigl(xa, (y+\mu)(b+\mu)\bigr).
\end{aligned}$$

So, $\Theta$ is a $\mathcal{T}L$-fuzzy relational morphism of rings. □

Let $\Theta \in \mathcal{F}(R\times S, L)$. For any $r \in R$ and $s \in S$, $\Theta_{(r,S)} \in \mathcal{F}(S, L)$ and $\Theta_{(R,s)} \in \mathcal{F}(R, L)$ is defined as follows:

$$\Theta_{(r,S)}(s_1) = \Theta(r, s_1) \quad \forall s_1 \in S,$$

$$\Theta_{(R,s)}(r_1) = \Theta(r_1, s) \quad \forall r_1 \in R.$$

$\Theta_{(r,S)}$ and $\Theta_{(R,s)}$ are called the $(r, S)$- and $(R, s)$-restriction of $\Theta$, respectively (see [32]).

**Theorem 3.11** *Let $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm. If $\Theta$ is a $\mathcal{T}L$-fuzzy* (*full*) *relational morphism from $R$ to $S$, then $\Theta_{(R,0)}$ and $\Theta_{(0,S)}$ are $\mathcal{T}L$-fuzzy subrings* (*ideals*) *of $S$ and $R$, respectively.*

*Proof* Since $\Theta_{(R,0)} = \operatorname{Ker}\Theta$, then it is obvious that $\Theta_{(R,0)}$ is a $\mathcal{T}L$-fuzzy subgroup of $R$ by Proposition 3.6. Similarly, $\Theta_{(0,S)}$ is $\mathcal{T}L$-fuzzy subgroup of $S$ since $\Theta^{-1} \in \operatorname{Hom}(S\times R, \mathcal{T}, L)$ and $\Theta_{(0,S)} = \operatorname{Ker}\Theta^{-1}$. Thus if $\Theta \in \operatorname{Hom}[S\times R, \mathcal{T}, L]$, then it is easy to see $\Theta_{(R,0)}$ and $\Theta_{(0,S)}$ are $\mathcal{T}L$-fuzzy ideals of $S$ and $R$, respectively. □

## 4 $\mathcal{I}$-lower and $\mathcal{T}$-upper fuzzy rough approximations on a ring

In this section, we study the properties of $\mathcal{I}$-lower and $\mathcal{T}$-upper fuzzy rough approximation operators with respect to a $\mathcal{T}L$-fuzzy (full) relational morphism of rings.

**Theorem 4.1** *Let $\mu, \nu \in \mathcal{F}(S, L)$, $\Theta \in \operatorname{Hom}(R\times S, \mathcal{T}, L)$ and $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm. Then*:

(i) $\overline{\Theta}^{\mathcal{T}}(\mu) +_{\mathcal{T}} \overline{\Theta}^{\mathcal{T}}(\nu) \leq \overline{\Theta}^{\mathcal{T}}(\mu +_{\mathcal{T}} \nu)$.
(ii) $\overline{\Theta}^{\mathcal{T}}(\mu) \cdot_{\mathcal{T}} \overline{\Theta}^{\mathcal{T}}(\nu) \leq \overline{\Theta}^{\mathcal{T}}(\mu \cdot_{\mathcal{T}} \nu)$.
(iii) $\overline{\Theta}^{\mathcal{T}}(\mu) \odot_{\mathcal{T}} \overline{\Theta}^{\mathcal{T}}(\nu) \leq \overline{\Theta}^{\mathcal{T}}(\mu \odot_{\mathcal{T}} \nu)$.

*Proof*

(i) Let $x \in R$. Then

$$\begin{aligned}
\big(\overline{\Theta}^{\mathcal{T}}(\mu) +_{\mathcal{T}} \overline{\Theta}^{\mathcal{T}}(\nu)\big)(x) &= \bigvee_{x=a+b} \big(\overline{\Theta}^{\mathcal{T}}(\mu)(a)\mathcal{T}\overline{\Theta}^{\mathcal{T}}(\nu)(b)\big) \\
&= \bigvee_{x=a+b} \Big(\bigvee_{k\in S} \Theta(a,k)\mathcal{T}\mu(k)\Big)\mathcal{T}\Big(\bigvee_{t\in S}\Theta(b,t)\mathcal{T}\nu(t)\Big) \\
&= \bigvee_{x=a+b}\bigvee_{k,t\in S}\big(\Theta(a,k)\mathcal{T}\Theta(b,t)\mathcal{T}\mu(k)\mathcal{T}\nu(t)\big) \\
&\leq \bigvee_{x=a+b}\bigvee_{s=k+t}\big(\Theta(a+b,k+t)\mathcal{T}\mu(k)\mathcal{T}\nu(t)\big) \\
&= \bigvee_{x=a+b}\Big(\Big(\bigvee_{s=k+t}\Theta(a+b,k+t)\Big)\mathcal{T}\Big(\bigvee_{s=k+t}\mu(k)\mathcal{T}\nu(t)\Big)\Big) \\
&= \bigvee_{x=a+b}\bigvee_{s\in S}\big(\Theta(a+b,s)\mathcal{T}(\mu +_{\mathcal{T}} \nu)(s)\big) \\
&= \bigvee_{s\in S}\big(\Theta(x,s)\mathcal{T}(\mu +_{\mathcal{T}} \nu)(s)\big) \\
&= \overline{\Theta}^{\mathcal{T}}(\mu +_{\mathcal{T}} \nu)(x).
\end{aligned}$$

So, we have $\overline{\Theta}^{\mathcal{T}}(\mu) +_{\mathcal{T}} \overline{\Theta}^{\mathcal{T}}(\nu) \leq \overline{\Theta}^{\mathcal{T}}(\mu \circ_{\mathcal{T}} \nu)$.

(ii) It is similar to the proof of (i).

(iii) Let $x \in R$. Then

$$\begin{aligned}
&\big(\overline{\Theta}^{\mathcal{T}}(\mu) \odot_{\mathcal{T}} \overline{\Theta}^{\mathcal{T}}(\nu)\big)(x) \\
&\quad= \bigvee_{x=\sum_{i=1}^n a_ib_i, n\in\mathbb{N}} \mathcal{T}_{i=1}^n\big(\overline{\Theta}^{\mathcal{T}}(\mu)(a_i)\mathcal{T}\overline{\Theta}^{\mathcal{T}}(\nu)(b_i)\big) \\
&\quad= \bigvee_{x=\sum_{i=1}^n a_ib_i, n\in\mathbb{N}} \mathcal{T}_{i=1}^n\Big(\Big(\bigvee_{k_i\in S}\Theta(a_i,k_i)\mathcal{T}\mu(k_i)\Big)\mathcal{T}\Big(\bigvee_{t_i\in S}\Theta(b_i,t_i)\mathcal{T}\nu(t_i)\Big)\Big) \\
&\quad= \bigvee_{x=\sum_{i=1}^n a_ib_i, n\in\mathbb{N}} \mathcal{T}_{i=1}^n\Big(\bigvee_{k_i,t_i\in S}\Theta(a_i,k_i)\mathcal{T}\mu(k_i)\mathcal{T}\Theta(b_i,t_i)\mathcal{T}\nu(t_i)\Big) \\
&\quad\leq \bigvee_{x=\sum_{i=1}^n a_ib_i, n\in\mathbb{N}} \mathcal{T}_{i=1}^n\Big(\bigvee_{k_i,t_i\in S}\Theta(a_ib_i,k_it_i)\mathcal{T}\mu(k_i)\mathcal{T}\nu(t_i)\Big) \\
&\quad= \bigvee_{x=\sum_{i=1}^n a_ib_i, n\in\mathbb{N}} \Big(\mathcal{T}_{i=1}^n\bigvee_{k_i,t_i\in S}\Theta(a_ib_i,k_it_i)\Big)\mathcal{T}\Big(\mathcal{T}_{i=1}^n\bigvee_{k_i,t_i\in S}\mu(k_i)\mathcal{T}\nu(t_i)\Big) \\
&\quad\leq \bigvee_{x=\sum_{i=1}^n a_ib_i, n\in\mathbb{N}} \Big(\bigvee_{k_i,t_i\in S}\Theta\Big(\sum_{i=1}^n a_ib_i, \sum_{i=1}^n k_it_i\Big)\Big)\mathcal{T}\Big(\mathcal{T}_{i=1}^n\bigvee_{k_i,t_i\in S}\mu(k_i)\mathcal{T}\nu(t_i)\Big) \\
&\quad\leq \bigvee_{p_i=k_it_i\in S}\Theta\Big(x,\sum_{i=1}^n p_i\Big)\mathcal{T}\Big(\mathcal{T}_{i=1}^n\bigvee_{p_i=k_it_i\in S}\mu(k_i)\mathcal{T}\nu(t_i)\Big) \\
&\quad\leq \bigvee_{p_i\in S}\Theta\Big(x,\sum_{i=1}^n p_i\Big)\mathcal{T}(\mu\odot_{\mathcal{T}}\nu)\Big(\sum_{i=1}^n p_i\Big)
\end{aligned}$$

$$\leq \bigvee_{s\in S} \Theta(x,s)\mathcal{T}(\mu \odot_{\mathcal{T}} \nu)(s)$$
$$= \overline{\Theta}^{\mathcal{T}}(\mu \odot_{\mathcal{T}} \nu)(x). \qquad \square$$

The following example shows that replacing '=' by '$\leq$' is not true in general in Theorem 4.1.

**Example 4.2** Let $L$ be the lattice which is given in Example 3.2(6) and $\mathcal{T} = \mathcal{T}_M$. Let $\Theta : \mathbb{Z}_4 \times \mathbb{Z}_3 \to L$ and $\Phi : \mathbb{Z} \times \mathbb{Z} \to L$ be defined by

$$\Theta(x,y) = \begin{cases} 1 & \text{if } x = \overline{0}; \\ 0 & \text{if } x = \overline{1}; \\ \beta & \text{if } x = \overline{2}; \\ 0 & \text{if } x = \overline{3}; \end{cases} \qquad \Phi(x,y) = \begin{cases} 1 & \text{if } x = y = 0; \\ \alpha & \text{if } x = 0, y \neq 0; \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then $\Theta \in \mathrm{Hom}(\mathbb{Z}_4 \times \mathbb{Z}_3, \mathcal{T}, L)$ and $\Phi \in \mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, \mathcal{T}, L)$. Let $\omega$ be an $L$-fuzzy subset of $\mathbb{Z}$ defined by

$$\omega(x) = \begin{cases} \alpha & \text{if } x = 0; \\ 1 & \text{if } x \neq 0. \end{cases}$$

Then $(\overline{\Phi}^{\mathcal{T}}(\omega) +_{\mathcal{T}} \overline{\Phi}^{\mathcal{T}}(\omega))(0) \leq \alpha$ and $\overline{\Phi}^{\mathcal{T}}(\omega +_{\mathcal{T}} \omega)(0) = 1$. Thus, $\overline{\Phi}^{\mathcal{T}}(\omega) +_{\mathcal{T}} \overline{\Phi}^{\mathcal{T}}(\omega) \neq \overline{\Phi}^{\mathcal{T}}(\mu +_{\mathcal{T}} \mu)$. Let $\mu$, $\nu$ be $L$-fuzzy subsets of $\mathbb{Z}_3$ defined by

| $x$ | $\overline{0}$ | $\overline{1}$ | $\overline{2}$ |
|---|---|---|---|
| $\mu(x)$ | 1 | $\alpha$ | $\alpha$ |
| $\nu(x)$ | $\beta$ | 0 | 0 |

Then $(\overline{\Theta}^{\mathcal{T}}(\mu) \cdot_{\mathcal{T}} \overline{\Theta}^{\mathcal{T}}(\nu))(\overline{2}) = 0$ and $\overline{\Theta}^{\mathcal{T}}(\mu \cdot_{\mathcal{T}} \nu)(\overline{2}) = \beta$. Thus, $\overline{\Theta}^{\mathcal{T}}(\mu) \cdot_{\mathcal{T}} \overline{\Theta}^{\mathcal{T}}(\nu) \neq \overline{\Theta}^{\mathcal{T}}(\mu \cdot_{\mathcal{T}} \nu)$.

**Definition 4.3** Let $(R,S,\Theta)$ be a $\mathcal{T}L$-fuzzy approximation space and $\mu$ be an $L$-fuzzy subset of $S$. $\mu$ is called a $\mathcal{I}$-lower $\mathcal{T}L$-fuzzy rough ring of $S$ if $\underline{\Theta}_{\mathcal{I}}(\mu)$ is a $\mathcal{T}L$-fuzzy subring of $R$ and $\mu$ is called a $\mathcal{T}$-upper $\mathcal{T}L$-fuzzy rough ring of $S$ if $\overline{\Theta}_{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy subring of $R$. The $(\mathcal{I},\mathcal{T})$-$L$-fuzzy rough set $(\underline{\Theta}_{\mathcal{I}}(\mu), \overline{\Theta}_{\mathcal{T}}(\mu))$ is called a $\mathcal{T}L$-fuzzy rough ring of $\mu$ if both $\underline{\Theta}_{\mathcal{I}}(\mu)$ and $\overline{\Theta}_{\mathcal{T}}(\mu)$ are $\mathcal{T}L$-fuzzy subrings of $R$.

**Theorem 4.4** *Let $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm, $\Theta \in \mathrm{Hom}(R \times S, \mathcal{T}, L)$ and $\mu$ be a $\mathcal{T}L$-fuzzy subring of $S$. Then $\mu$ is a $\mathcal{T}$-upper $\mathcal{T}L$-fuzzy rough ring of $S$.*

*Proof* Let $x, y \in R$. Then

$$\overline{\Theta}^{\mathcal{T}}(\mu)(x)\mathcal{T}\overline{\Theta}^{\mathcal{T}}(\mu)(y) = \left(\bigvee_{k\in S} \Theta(x,k)\mathcal{T}\mu(k)\right)\mathcal{T}\left(\bigvee_{t\in S} \Theta(y,t)\mathcal{T}\mu(t)\right)$$
$$= \bigvee_{k\in S}\bigvee_{t\in S} \Theta(x,k)\mathcal{T}\mu(k)\mathcal{T}\Theta(y,t)\mathcal{T}\mu(t)$$

$$
\begin{aligned}
&\leq \bigvee_{k\in S}\bigvee_{t\in S}\Theta(x+y,k+t)\mathcal{T}\mu(k+t)\\
&\leq \bigvee_{p\in S}\Theta(x+y,p)\mathcal{T}\mu(p)\\
&= \overline{\Theta}^{\mathcal{T}}(\mu)(x+y),
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Theta}^{\mathcal{T}}(\mu)(x)\mathcal{T}\overline{\Theta}^{\mathcal{T}}(\mu)(y) &= \Big(\bigvee_{k\in S}\Theta(x,k)\mathcal{T}\mu(k)\Big)\mathcal{T}\Big(\bigvee_{t\in S}\Theta(y,t)\mathcal{T}\mu(t)\Big)\\
&= \bigvee_{k\in S}\bigvee_{t\in S}\Theta(x,k)\mathcal{T}\mu(k)\mathcal{T}\Theta(y,t)\mathcal{T}\mu(t)\\
&\leq \bigvee_{k\in S}\bigvee_{t\in S}\Theta(xy,kt)\mathcal{T}\mu(kt)\\
&\leq \bigvee_{p\in S}\Theta(xy,p)\mathcal{T}\mu(p)\\
&= \overline{\Theta}^{\mathcal{T}}(\mu)(xy),
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Theta}^{\mathcal{T}}(\mu)(x) &= \bigvee_{k\in S}\Theta(x,k)\mathcal{T}\mu(k)\\
&\leq \bigvee_{k\in S}\Theta(-x,-k)\mathcal{T}\mu(-k)\\
&= \overline{\Theta}^{\mathcal{T}}(\mu)(-x).
\end{aligned}
$$

Hence we obtain $\overline{\Theta}^{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy subring of $R$. Thus $\mu$ is a $\mathcal{T}$-upper $\mathcal{T}L$-fuzzy rough ring of $S$. □

**Theorem 4.5** *Let $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm, $\Theta \in \mathrm{Hom}[R\times S,\mathcal{T},L]$ and $\mu$ be a $\mathcal{T}L$-fuzzy ideal of $S$. Then $\overline{\Theta}^{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy ideal of $R$.*

*Proof* Let $x, y \in R$. Then

$$
\begin{aligned}
\overline{\Theta}^{\mathcal{T}}(\mu)(x)\vee\overline{\Theta}^{\mathcal{T}}(\mu)(y) &= \Big(\bigvee_{k\in S}\Theta(x,k)\mathcal{T}\mu(k)\Big)\vee\Big(\bigvee_{t\in S}\Theta(y,t)\mathcal{T}\mu(t)\Big)\\
&= \bigvee_{k,t\in S}\big(\Theta(x,k)\mathcal{T}\mu(k)\big)\vee\big(\Theta(y,t)\mathcal{T}\mu(t)\big)\\
&\leq \bigvee_{k,t\in S}\big(\Theta(x,k)\vee\Theta(y,t)\big)\mathcal{T}\big(\mu(k)\vee\mu(t)\big)\\
&\leq \bigvee_{k,t\in S}\Theta(xy,kt)\mathcal{T}\mu(kt)\\
&\leq \bigvee_{p\in S}\Theta(xy,p)\mathcal{T}\mu(p)\\
&= \overline{\Theta}^{\mathcal{T}}(\mu)(xy).
\end{aligned}
$$

So, $\overline{\Theta}^{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy ideal of $R$ in conjunction with Theorem 4.4. □

The following example shows that $\mu$ is not a $\mathcal{I}$-lower $\mathcal{T}L$-fuzzy rough ring of $S$ in general under the condition of Theorem 4.4, and Theorem 4.5 may not be true for the $\mathcal{I}$-lower $L$-fuzzy rough approximation of $\mu$ even if $\Theta$ is full.

**Example 4.6** Consider $L$ and $\Theta$ as in Example 3.2(6). Let $\mathcal{T} = \mathcal{T}_M$. Then $\Theta$ is a $\mathcal{T}_M L$-fuzzy full relational morphism. Let a $L$-fuzzy subset $\mu$ of $\mathbb{Z}_2$ be defined by

| $x$ | $\overline{0}$ | $\overline{1}$ |
|---|---|---|
| $\mu(x)$ | 1 | $\alpha$ |

Then $\mu$ is a $\mathcal{T}_M L$-fuzzy ideal of $\mathbb{Z}_2$ and we obtain the $\mathcal{T}_M L$-fuzzy lower approximation operator of $\mu$ as follows:

$$\underline{\Theta}_{\mathcal{I}}(\mu)(x) = \begin{cases} \alpha & \text{if } x = 0; \\ 1 & \text{if } x \neq 0. \end{cases}$$

Since $\underline{\Theta}_{\mathcal{I}}(\mu)(2)\mathcal{T}_M\underline{\Theta}_{\mathcal{I}}(\mu)(-2) = 1\mathcal{T}_M 1 = 1 \nleq \alpha = \underline{\Theta}_{\mathcal{I}}(\mu)(0) = \underline{\Theta}_{\mathcal{I}}(\mu)(2 + (-2))$, then $\underline{\Theta}_{\mathcal{I}}(\mu)$ is not a $\mathcal{T}_M L$-fuzzy subring of $\mathbb{Z}$.

## 5 Conclusions

The generalized rough sets on algebraic sets such as group, ring and module were mainly studied by a set-valued homomorphism [23, 25, 26] which is a generalization of a congruence relation. In this paper, a definition of $\mathcal{T}L$-fuzzy (full) relational morphism is considered as a generalization of fuzzy congruence relations (or $\mathcal{T}$-congruence $L$-fuzzy relations) for rings. Then we obtained some new properties of a $\mathcal{T}L$-fuzzy (full) relational morphism to provide opportunity of putting reasonable interpretations and explored the features of generalized $\mathcal{I}$-lower and $\mathcal{T}$-upper fuzzy rough approximations of an $L$-fuzzy subset on rings. From these points of view, taking a fresh look at the generalized $(\mathcal{I},\mathcal{T})$-$L$-fuzzy rough sets on rings is an interesting research topic. Our further work on this topic will focus on the properties of generalized $(\mathcal{I},\mathcal{T})$-$L$-fuzzy rough sets on modules with respect to the $\mathcal{T}L$-fuzzy (full) relational morphism.

#### Competing interests
The authors declare that they have no competing interests.

#### Authors' contributions
The present study was proposed by CE, YC and SY. All authors read and approved the final manuscript.

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