*Living Rev. Relativity*, **8**, (2005), 12
http://www.livingreviews.org/lrr-2005-12

# Analogue Gravity

Carlos Barceló
Instituto de Astrofísica de Andalucía (CSIC)
Camino Bajo de Huetor 50,
18008 Granada, Spain
email: carlos@iaa.es
http://www.iaa.csic.es/

Stefano Liberati
International School for Advanced Studies and INFN
Via Beirut 2-4,
34014 Trieste, Italy
email: liberati@sissa.it
http://www.sissa.it/~liberati

Matt Visser
School of Mathematics, Statistics, and Computer Science
Victoria University of Wellington; PO Box 600
Wellington, New Zealand
email: matt.visser@mcs.vuw.ac.nz
http://www.mcs.vuw.ac.nz/~visser

Accepted on 7 November 2005
Published on 16 December 2005

*Living Reviews in Relativity*

Published by the
Max Planck Institute for Gravitational Physics
(Albert Einstein Institute)
Am Mühlenberg 1, 14424 Golm, Germany
ISSN 1433-8351

**Abstract**

Analogue models of (and for) gravity have a long and distinguished history dating back to the earliest years of general relativity. In this review article we will discuss the history, aims, results, and future prospects for the various analogue models. We start the discussion by presenting a particularly simple example of an analogue model, before exploring the rich history and complex tapestry of models discussed in the literature. The last decade in particular has seen a remarkable and sustained development of analogue gravity ideas, leading to some hundreds of published articles, a workshop, two books, and this review article. Future prospects for the analogue gravity programme also look promising, both on the experimental front (where technology is rapidly advancing) and on the theoretical front (where variants of analogue models can be used as a springboard for radical attacks on the problem of quantum gravity).

## How to cite this article

Owing to the fact that a *Living Reviews* article can evolve over time, we recommend to cite the article as follows:

Carlos Barceló, Stefano Liberati and Matt Visser,
"Analogue Gravity",
*Living Rev. Relativity*, **8**, (2005), 12. [Online Article]: cited [<date>],
http://www.livingreviews.org/lrr-2005-12

The date given as <date> then uniquely identifies the version of the article you are referring to.

## Article Revisions

*Living Reviews* supports two different ways to keep its articles up-to-date:

**Fast-track revision** A fast-track revision provides the author with the opportunity to add short notices of current research results, trends and developments, or important publications to the article. A fast-track revision is refereed by the responsible subject editor. If an article has undergone a fast-track revision, a summary of changes will be listed here.

**Major update** A major update will include substantial changes and additions and is subject to full external refereeing. It is published with a new publication number.

For detailed documentation of an article's evolution, please refer always to the history document of the article's online version at http://www.livingreviews.org/lrr-2005-12.

# Contents

# 1 Introduction

> And I cherish more than anything else the Analogies, my most trustworthy masters. They know all the secrets of Nature, and they ought least to be neglected in Geometry.
> — Johannes Kepler

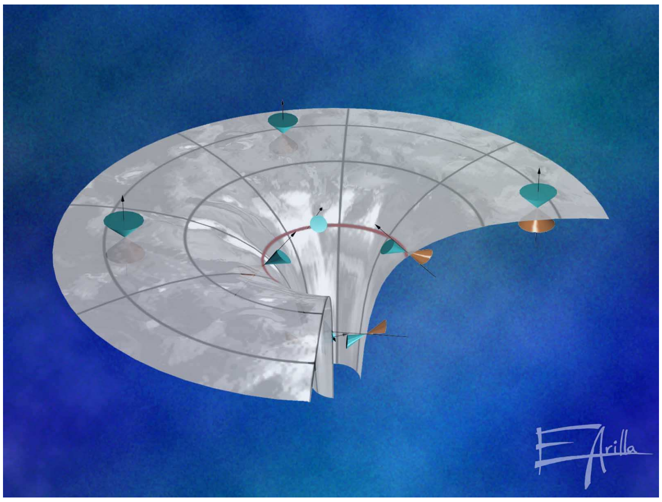

Figure 1: *Artistic impression of cascading sound cones (in the geometrical acoustics limit) forming an acoustic black hole when supersonic flow tips the sound cones past the vertical.*

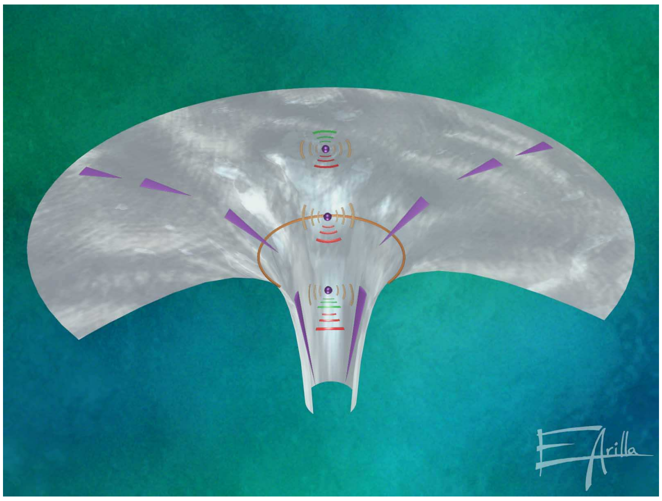

Figure 2: *Artistic impression of trapped waves (in the physical acoustics limit) forming an acoustic black hole when supersonic flow forces the waves to move downstream.*

Analogies have played a very important role in physics and mathematics – they provide new ways of looking at problems that permit cross-fertilization of ideas among different branches of science. A carefully chosen analogy can be extremely useful in focussing attention on a specific problem, and in suggesting unexpected routes to a possible solution. In this review article we will focus on "analogue gravity", the development of analogies (typically but not always based on condensed matter physics) to probe aspects of the physics of curved spacetime – and in particular to probe aspects of curved space quantum field theory.

The most well-known of these analogies is the use of sound waves in a moving fluid as an analogue for light waves in a curved spacetime. Supersonic fluid flow can then generate a "dumb hole", the acoustic analogue of a "black hole", and the analogy can be extended all the way to mathematically demonstrating the presence of phononic Hawking radiation from the acoustic horizon. This particular provides (at least in principle) a concrete laboratory model for curved-space quantum field theory in a realm that is technologically accessible to experiment.

There are many other "analogue models" that may be useful for this or other reasons – some of the analogue models are interesting for experimental reasons, others are useful for the way they provide new light on perplexing theoretical questions. The information flow is in principle bi-directional and sometimes insights developed within the context of general relativity can be used to understand aspects of the analogue model.

Of course analogy is not identity, and we are in no way claiming that the analogue models we consider are completely equivalent to general relativity – merely that the analogue model (in order to be interesting) should capture and accurately reflect a sufficient number of important features of general relativity (or sometimes special relativity). The list of analogue models is extensive, and in this review we will seek to do justice both to the key models, and to the key features of those models.

In the following chapters we shall:

- Discuss the flowing fluid analogy in some detail.
- Summarise the history and motivation for various analogue models.
- Discuss the many physics issues various researchers have addressed.
- Provide a (hopefully complete) catalogue of extant models.
- Discuss the main physics results obtained to date.
- Outline the many possible directions for future research.
- Summarise the current state of affairs.

By that stage the interested reader will have had a quite thorough introduction to the ideas, techniques, and hopes of the analogue gravity programme.

## 1.1 Going further

Apart from this present review article, and the references contained herein, there are several key items that stand out as starting points for any deeper investigation:

- The book "Artificial Black Holes", edited by Mario Novello, Matt Visser, and Grigori Volovik [284].
- The websites for the "Analogue models" workshop:
  - http://www.cbpf.br/~bscg/analog/
  - http://www.mcs.vuw.ac.nz/~visser/Analog/
  - http://www.physics.wustl.edu/~visser/Analog/
- The book "The Universe in a Helium droplet", by Grigori Volovik [418].
- The *Physics Reports* article, "Superfluid analogies of cosmological phenomena", by Grigori Volovik [413].

# 2 The Simplest Example of an Analogue Model

Acoustics in a moving fluid is the simplest and cleanest example of an analogue model [376, 387, 391, 389]. The basic physics is simple, the conceptual framework is simple, and specific computations are often simple (whenever, that is, they are not impossibly hard).[1]

## 2.1 Background

The basic physics is this: A moving fluid will drag sound waves along with it, and if the speed of the fluid ever becomes supersonic, then in the supersonic sound waves will never be able to fight their way back upstream [376, 387, 391, 389]. This implies the existence of a "dumb hole", a region from which sound can not escape.[2] Of course this sounds very similar, at the level of a non-mathematical *verbal* analogy, to the notion of a "black hole" in general relativity. The real question is whether this verbal analogy can be turned into a precise *mathematical* and *physical* statement – it is only after we have a precise mathematical and physical connection between (in this example) the physics of acoustics in a fluid flow and at least some significant features of general relativity that we can claim to have an "analogue model of (some aspects of) gravity". We (and the community at large) often abuse language by referring to such a model as "analogue gravity" for short.

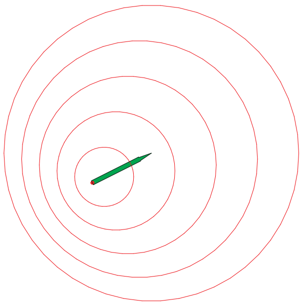

Figure 3: *A moving fluid will drag sound pulses along with it.*

Now the features of general relativity that one typically captures in an "analogue model" are the *kinematic* features that have to do with how fields (classical or quantum) are defined on curved spacetime, and the *sine qua non* of any analogue model is the existence of some "effective metric" that captures the notion of the curved spacetimes that arise in general relativity. (At the very least, one might wish to capture the notion of the Minkowski geometry of special relativity.) Indeed, the verbal description above (and its generalizations in other physical frameworks) *can* be converted into a precise mathematical and physical statement, which ultimately is the reason that analogue models are of physical interest. The analogy works at two levels;

- Geometrical acoustics.
- Physical acoustics.

[1] The need for a certain degree of caution regarding the allegedly straightforward physics of simple fluids might be inferred from the fact that the Clay Mathematics Institute is currently offering a US$1,000,000 Millennium Prize for significant progress on the question of existence and uniqueness of solutions to the Navier–Stokes equation. See http://www.claymath.org/millennium/ for details.

[2] In correct English, the word "dumb" means "mute", as in "unable to speak". The word "dumb" does *not* mean "stupid", though even many native English speakers get this wrong.

The advantage of geometrical acoustics is that the derivation of the precise mathematical form of the analogy is so simple as to be almost trivial, and that the derivation is extremely general. The disadvantage is that in the geometrical acoustics limit one can deduce only the causal structure of the spacetime, and does not obtain a unique effective metric. The advantage of physical acoustics is that while the derivation of the analogy holds in a more restricted regime, the analogy can do more for you in that it can now specify a specific effective metric and accommodate a wave equation for the sound waves.

## 2.2 Geometrical acoustics

At the level of geometrical acoustics we need only assume that:

- The speed of sound $c$, relative to the fluid, is well defined.
- The velocity of the fluid $\mathbf{v}$, relative to the laboratory, is well defined.

Then, relative to the laboratory, the velocity of a sound ray propagating, with respect to the fluid, along the direction defined by the unit vector $\mathbf{n}$ is

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = c\,\mathbf{n} + \mathbf{v}. \tag{1}$$

This defines a sound cone in spacetime given by the condition $\mathbf{n}^2 = 1$, i.e.,

$$-c^2\ \mathrm{d}t^2 + (\mathrm{d}\mathbf{x} - \mathbf{v}\,\mathrm{d}t)^2 = 0\ . \tag{2}$$

That is

$$-[c^2 - v^2]\ \mathrm{d}t^2 - 2\mathbf{v}\cdot\mathrm{d}\mathbf{x}\ \mathrm{d}t + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = 0\,. \tag{3}$$

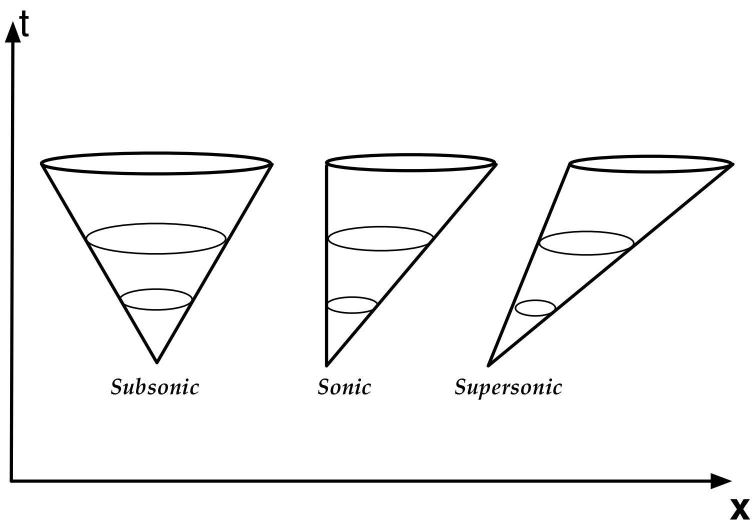


Figure 4: *A moving fluid will tip the "sound cones" as it moves. Supersonic flow will tip the sound cones past the vertical.*

Solving this quadratic equation for d$\mathbf{x}$ as a function of d$t$ provides a double cone associated with each point in space and time. This is associated with a conformal class of Lorentzian metrics [376, 387, 391, 389, 284]

$$g = \Omega^2 \left[ \begin{array}{c|c} -(c^2 - v^2) & -\mathbf{v}^T \\ \hline -\mathbf{v} & \mathbf{I} \end{array} \right] , \tag{4}$$

where $\Omega$ is an unspecified but non-vanishing function.

The virtues of the geometric approach are its extreme simplicity and the fact that the basic structure is dimension-independent. Moreover this logic rapidly (and relatively easily) generalises to more complicated physical situations.[3]

## 2.3 Physical acoustics

It is well known that for a static homogeneous inviscid fluid the propagation of sound waves is governed by the simple wave equation [219, 221, 264, 353]

$$\partial_t^2 \phi = c^2 \nabla^2 \phi. \tag{5}$$

Generalizing this result to a fluid that is non-homogeneous, or to a fluid that is in motion, possibly even in non-steady motion, is more subtle than it at first would appear. To derive a wave equation in this more general situation we shall start by adopting a few simplifying assumptions to allow us to derive the following theorem.

**Theorem.** *If a fluid is barotropic and inviscid, and the flow is irrotational (though possibly time dependent) then the equation of motion for the velocity potential describing an acoustic disturbance is identical to the d'Alembertian equation of motion for a minimally coupled massless scalar field propagating in a (3+1)-dimensional Lorentzian geometry*

$$\Delta\phi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \phi \right) = 0. \tag{6}$$

*Under these conditions, the propagation of sound is governed by an* acoustic metric *–* $g_{\mu\nu}(t, \mathbf{x})$*. This acoustic metric describes a (3+1)-dimensional Lorentzian (pseudo–Riemannian) geometry. The metric depends algebraically on the density, velocity of flow, and local speed of sound in the fluid. Specifically*

$$g_{\mu\nu}(t, \mathbf{x}) \equiv \frac{\rho}{c} \left[ \begin{array}{ccc} -(c^2 - v^2) & \vdots & -\mathbf{v}^T \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -\mathbf{v} & \vdots & \mathbf{I} \end{array} \right]. \tag{7}$$

*(Here* $\mathbf{I}$ *is the* $3 \times 3$ *identity matrix.) In general, when the fluid is non-homogeneous and flowing, the* acoustic Riemann tensor *associated with this Lorentzian metric will be nonzero.*

*Comment.* It is quite remarkable that even though the underlying fluid dynamics is Newtonian, nonrelativistic, and takes place in flat space plus time, the fluctuations (sound waves) are governed by a curved (3+1)-dimensional Lorentzian (pseudo-Riemannian) spacetime geometry. For practitioners of general relativity this observation describes a very simple and concrete physical model

[3]For instance, whenever one has a system of PDEs that can be written in first-order quasi-linear symmetric hyperbolic form, then it is an exact non-perturbative result that the matrix of coefficients for the first-derivative terms can be used to construct a conformal class of metrics that encodes the causal structure of the system of PDEs. For barotropic hydrodynamics this is briefly discussed in [80]. This analysis is related to the behaviour of characteristics of the PDEs, and ultimately can be linked back to the Fresnel equation that appears in the eikonal limit.

for certain classes of Lorentzian spacetimes, including (as we shall later see) black holes. On the other hand, this discussion is also potentially of interest to practitioners of continuum mechanics and fluid dynamics in that it provides a simple concrete introduction to Lorentzian differential geometric techniques.

*Proof.* The fundamental equations of fluid dynamics [219, 221, 264, 353] are the equation of continuity

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho\, \mathbf{v}) = 0, \tag{8}$$

and Euler's equation (equivalent to $\mathbf{F} = m\mathbf{a}$ applied to small lumps of fluid)

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \equiv \rho \left[\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}\right] = \mathbf{f}. \tag{9}$$

Start the analysis by assuming the fluid to be inviscid (zero viscosity), with the only forces present being those due to pressure.[4] Then for the force density we have

$$\mathbf{f} = -\boldsymbol{\nabla} p. \tag{10}$$

Via standard manipulations the Euler equation can be rewritten as

$$\partial_t \mathbf{v} = \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) - \frac{1}{\rho}\, \boldsymbol{\nabla} p - \boldsymbol{\nabla} \left( \frac{1}{2}\, v^2 \right). \tag{11}$$

Now take the flow to be *vorticity free*, that is, *locally irrotational*. Introduce the velocity potential $\phi$ such that $\mathbf{v} = -\boldsymbol{\nabla}\phi$, at least locally. If one further takes the fluid to be *barotropic* (this means that $\rho$ is a function of $p$ only), it becomes possible to define

$$h(p) = \int_0^p \frac{\mathrm{d}p'}{\rho(p')}; \qquad \text{so that} \qquad \boldsymbol{\nabla} h = \frac{1}{\rho}\, \boldsymbol{\nabla} p. \tag{12}$$

Thus the specific enthalpy, $h(p)$, is a function of $p$ only. Euler's equation now reduces to

$$-\partial_t \phi + h + \frac{1}{2} (\boldsymbol{\nabla}\phi)^2 = 0. \tag{13}$$

This is a version of Bernoulli's equation.

Now linearise these equations of motion around some assumed background $(\rho_0, p_0, \phi_0)$. Set

$$\rho = \rho_0 + \epsilon\rho_1 + O(\epsilon^2), \tag{14}$$
$$p = p_0 + \epsilon p_1 + O(\epsilon^2), \tag{15}$$
$$\phi = \phi_0 + \epsilon\phi_1 + O(\epsilon^2). \tag{16}$$

Sound is *defined* to be these linearised fluctuations in the dynamical quantities. Note that this is the *standard definition* of (linear) sound and more generally of acoustical disturbances. In principle, of course, a fluid mechanic might really be interested in solving the complete equations of motion for the fluid variables $(\rho, p, \phi)$. In practice, it is both traditional and extremely useful to separate the exact motion, described by the exact variables, $(\rho, p, \phi)$, into some average bulk motion, $(\rho_0, p_0, \phi_0)$, plus low amplitude acoustic disturbances, $(\epsilon\rho_1, \epsilon p_1, \epsilon\phi_1)$. See, for example [219, 221, 264, 353].

Since this is a subtle issue that we have seen cause considerable confusion in the past, let us be even more explicit by asking the rhetorical question: *"How can we tell the difference between a wind gust and a sound wave?"* The answer is that the difference is to some extent a matter of

[4] It is straightforward to add external forces, at least conservative body forces such as Newtonian gravity.

convention – sufficiently low-frequency long-wavelength disturbances (wind gusts) are conventionally lumped in with the average bulk motion. Higher-frequency, shorter-wavelength disturbances are conventionally described as acoustic disturbances. If you wish to be hyper-technical, we can introduce a high-pass filter function to define the bulk motion by suitably averaging the exact fluid motion. There are no deep physical principles at stake here – merely an issue of convention. The place where we are making a specific physical assumption that restricts the validity of our analysis is in the requirement that the amplitude of the high-frequency short-wavelength disturbances be small. This is the assumption underlying the linearization programme, and this is why sufficiently high-amplitude sound waves must be treated by direct solution of the full equations of fluid dynamics.

Linearizing the continuity equation results in the pair of equations

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \, \mathbf{v}_0) = 0, \tag{17}$$

$$\partial_t \rho_1 + \nabla \cdot (\rho_1 \, \mathbf{v}_0 + \rho_0 \, \mathbf{v}_1) = 0. \tag{18}$$

Now, the barotropic condition implies

$$h(p) = h(p_0 + \epsilon p_1 + O(\epsilon^2)) = h_0 + \epsilon \, \frac{p_1}{\rho_0} + O(\epsilon^2). \tag{19}$$

Use this result in linearizing the Euler equation. We obtain the pair

$$-\partial_t \phi_0 + h_0 + \frac{1}{2}(\nabla \phi_0)^2 = 0. \tag{20}$$

$$-\partial_t \phi_1 + \frac{p_1}{\rho_0} - \mathbf{v}_0 \cdot \nabla \phi_1 = 0. \tag{21}$$

This last equation may be rearranged to yield

$$p_1 = \rho_0 \, \left(\partial_t \phi_1 + \mathbf{v}_0 \cdot \nabla \phi_1\right). \tag{22}$$

Use the barotropic assumption to relate

$$\rho_1 = \frac{\partial \rho}{\partial p} \, p_1 = \frac{\partial \rho}{\partial p} \, \rho_0 \, (\partial_t \phi_1 + \mathbf{v}_0 \cdot \nabla \phi_1). \tag{23}$$

Now substitute this consequence of the linearised Euler equation into the linearised equation of continuity. We finally obtain, up to an overall sign, the wave equation:

$$-\, \partial_t \left( \frac{\partial \rho}{\partial p} \rho_0 \, (\partial_t \phi_1 + \mathbf{v}_0 \cdot \nabla \phi_1) \right) + \nabla \cdot \left( \rho_0 \nabla \phi_1 - \frac{\partial \rho}{\partial p} \rho_0 \mathbf{v}_0 \, (\partial_t \phi_1 + \mathbf{v}_0 \cdot \nabla \phi_1) \right) = 0. \tag{24}$$

This wave equation describes the propagation of the linearised scalar potential $\phi_1$. Once $\phi_1$ is determined, Equation (22) determines $p_1$, and Equation (23) then determines $\rho_1$. Thus this wave equation completely determines the propagation of acoustic disturbances. The background fields $p_0$, $\rho_0$ and $\mathbf{v}_0 = -\nabla \phi_0$, which appear as time-dependent and position-dependent coefficients in this wave equation, are constrained to solve the equations of fluid motion for a barotropic, inviscid, and irrotational flow. Apart from these constraints, they are otherwise permitted to have *arbitrary* temporal and spatial dependencies.

Now, written in this form, the physical import of this wave equation is somewhat less than pellucid. To simplify things algebraically, observe that the local speed of sound is defined by

$$c^{-2} \equiv \frac{\partial \rho}{\partial p}. \tag{25}$$

Now construct the symmetric $4 \times 4$ matrix

$$f^{\mu\nu}(t,\mathbf{x}) \equiv \frac{\rho_0}{c^2} \left[ \begin{array}{c|c} -1 & -v_0^j \\ \hline -v_0^i & (c^2\,\delta^{ij} - v_0^i v_0^j) \end{array} \right]. \tag{26}$$

(Greek indices run from 0–3, while Roman indices run from 1–3.) Then, introducing (3+1)-dimensional space-time coordinates, which we write as $x^\mu \equiv (t; x^i)$ , the above wave Equation (24) is easily rewritten as

$$\partial_\mu (f^{\mu\nu}\,\partial_\nu \phi_1) = 0. \tag{27}$$

This remarkably compact formulation is completely equivalent to Equation (24) and is a much more promising stepping-stone for further manipulations. The remaining steps are a straightforward application of the techniques of curved space (3+1)-dimensional Lorentzian geometry.

Now in any Lorentzian (i.e., pseudo–Riemannian) manifold the curved space scalar d'Alembertian is given in terms of the metric $g_{\mu\nu}(t,\mathbf{x})$ by (see, for example, [125, 266, 369, 265, 164, 422])

$$\Delta\phi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g}\; g^{\mu\nu}\,\partial_\nu \phi \right). \tag{28}$$

The inverse metric, $g^{\mu\nu}(t,\mathbf{x})$, is pointwise the matrix inverse of $g_{\mu\nu}(t,\mathbf{x})$, while $g \equiv \det(g_{\mu\nu})$. Thus one can rewrite the physically derived wave Equation (24) in terms of the d'Alembertian provided one identifies

$$\sqrt{-g}\; g^{\mu\nu} = f^{\mu\nu}. \tag{29}$$

This implies, on the one hand

$$\det(f^{\mu\nu}) = (\sqrt{-g})^4\; g^{-1} = g. \tag{30}$$

On the other hand, from the explicit expression (26), expanding the determinant in minors yields

$$\det(f^{\mu\nu}) = \left(\frac{\rho_0}{c^2}\right)^4 \cdot \left[(-1)\cdot(c^2 - v_0^2) - (-v_0)^2\right] \cdot \left[c^2\right] \cdot \left[c^2\right] = -\frac{\rho_0^4}{c^2}. \tag{31}$$

Thus

$$g = -\frac{\rho_0^4}{c^2}; \qquad \sqrt{-g} = \frac{\rho_0^2}{c}. \tag{32}$$

We can therefore pick off the coefficients of the *inverse* acoustic metric

$$g^{\mu\nu}(t,\mathbf{x}) \equiv \frac{1}{\rho_0 c} \left[ \begin{array}{c|c} -1 & -v_0^j \\ \hline -v_0^i & (c^2\,\delta^{ij} - v_0^i\; v_0^j) \end{array} \right]. \tag{33}$$

We could now determine the metric itself simply by inverting this $4 \times 4$ matrix (and if the reader is not a general relativist, proceeding in this direct manner is definitely the preferred option). On the other hand, for general relativists it is even easier to recognise that one has in front of one a specific example of the Arnowitt–Deser–Misner split of a (3+1)-dimensional Lorentzian spacetime metric into space + time, more commonly used in discussing initial value data in general relativity. (See, for example, [265, pages 505–508].) The acoustic metric is then read off by inspection

$$g_{\mu\nu} \equiv \frac{\rho_0}{c} \left[ \begin{array}{c|c} -(c^2 - v_0^2) & -v_0^j \\ \hline -v_0^i & \delta_{ij} \end{array} \right]. \tag{34}$$

Equivalently, the acoustic interval can be expressed as

$$\mathrm{d}s^2 \equiv g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \frac{\rho_0}{c}\left[-c^2\,\mathrm{d}t^2 + (\mathrm{d}x^i - v_0^i\,\mathrm{d}t)\,\delta_{ij}\,(\mathrm{d}x^j - v_0^j\,\mathrm{d}t)\right]. \tag{35}$$

This completes the proof of the theorem. □

We have presented the theorem and proof, which closely follows the discussion in [389], in considerable detail because it is a standard template that can be readily generalised in many ways. This discussion can then be used as a starting point to initiate the analysis of numerous and diverse physical models.

## 2.4 General features of the acoustic metric

A few brief comments should be made before proceeding further:

- Observe that the signature of this effective metric is indeed $(-,+,+,+)$, as it should be to be regarded as Lorentzian.
- Observe that in physical acoustics it is the inverse metric density,

$$f^{\mu\nu} = \sqrt{-g}\;g^{\mu\nu} \tag{36}$$

  that is of more fundamental significance for deriving the wave equation than is the metric $g_{\mu\nu}$ itself. (This observation continues to hold in more general situations where it is often significantly easier to calculate the tensor density $f^{\mu\nu}$ than it is to calculate the effective metric $g_{\mu\nu}$.)
- It should be emphasised that there are two distinct metrics relevant to the current discussion:
  - The *physical spacetime metric* is in this case just the usual flat metric of Minkowski space:

$$\eta_{\mu\nu} \equiv (\mathrm{diag}[-c_{\mathrm{light}}^2, 1, 1, 1])_{\mu\nu}. \tag{37}$$

    (Here $c_{\mathrm{light}}$ is the speed of light in vacuum.) The fluid particles couple only to the physical metric $\eta_{\mu\nu}$. In fact the fluid motion is completely non-relativistic, so that $||v_0|| \ll c_{\mathrm{light}}$, and it is quite sufficient to consider Galilean relativity for the underlying fluid mechanics.
  - Sound waves on the other hand, do not "see" the physical metric at all. Acoustic perturbations couple only to the effective *acoustic metric* $g_{\mu\nu}$.
- It is quite remarkable that (to the best of our knowledge) the acoustic metric was first derived and used in Moncrief's studies of the relativistic hydrodynamics of accretion flows surrounding black holes [268]. Indeed Moncrief was working in the more general case of a curved background "physical" metric, in addition to a curved "effective" metric. We shall come back to this work later on, in our historical section.
- The geometry determined by the acoustic metric does however inherit some key properties from the existence of the underlying flat physical metric. For instance, the *topology* of the manifold does not depend on the particular metric considered. The acoustic geometry inherits the underlying topology of the physical metric – ordinary $\Re^4$ – with possibly a few regions excised (due to whatever hard-wall boundary conditions one might wish to impose on the fluid). In systems constrained to have effectively less than 3 spacelike dimensions one can reproduce more complicated topologies (consider for example an effectively one-dimensional flow in a tubular ring).

- Furthermore, the acoustic geometry automatically inherits from the underlying Newtonian time parameter, the important property of "stable causality" [164, 422]. Note that

$$g^{\mu\nu}\,(\nabla_\mu t)\,(\nabla_\nu t) = -\frac{1}{\rho_0\;c} < 0. \tag{38}$$

  This precludes some of the more entertaining causality-related pathologies that sometimes arise in general relativity. (For a general discussion of causal pathologies in general relativity, see for example [164, 161, 162, 72, 163, 396]).

- Other concepts that translate immediately are those of "ergo-region", "trapped surface", "apparent horizon", and "event horizon". These notions will be developed more fully in the following subsection.

- The properly normalised four-velocity of the fluid is

$$V^\mu = \frac{(1;\; v_0^i)}{\sqrt{\rho_0\;c}}, \tag{39}$$

  so that

$$g_{\mu\nu}\,V^\mu\,V^\nu = g(V,V) = -1. \tag{40}$$

  This four-velocity is related to the gradient of the natural time parameter by

$$\nabla_\mu t = (1,0,0,0); \qquad\qquad \nabla^\mu t = -\frac{(1;\; v_0^i)}{\rho_0\;c} = -\frac{V^\mu}{\sqrt{\rho_0\;c}}. \tag{41}$$

  Thus the integral curves of the fluid velocity field are orthogonal (in the Lorentzian metric) to the constant time surfaces. The acoustic proper time along the fluid flow lines (streamlines) is

$$\tau = \int \sqrt{\rho_0\;c}\;\mathrm{d}t, \tag{42}$$

  and the integral curves are geodesics of the acoustic metric if and only if $\rho_0\,c$ is position independent.

- Observe that in a completely general (3+1)-dimensional Lorentzian geometry the metric has 6 degrees of freedom per point in spacetime. ($4 \times 4$ symmetric matrix $\Rightarrow$ 10 independent components; then subtract 4 coordinate conditions).

  In contrast, the acoustic metric is more constrained. Being specified completely by the three scalars $\phi_0(t,\mathbf{x})$, $\rho_0(t,\mathbf{x})$, and $c(t,\mathbf{x})$, the acoustic metric has at most 3 degrees of freedom per point in spacetime. The equation of continuity actually reduces this to 2 degrees of freedom, which can be taken to be $\phi_0(t,\mathbf{x})$ and $c(t,\mathbf{x})$.

  *Thus the simple acoustic metric of this section can at best reproduce some subset of the generic metrics of interest in general relativity.*

- A point of notation: Where the general relativist uses the word "stationary" the fluid dynamicist uses the phrase "steady flow". The general-relativistic word "static" translates to a rather messy constraint on the fluid flow (to be discussed more fully below).

- Finally, we should emphasise that in Einstein gravity the spacetime metric is related to the distribution of matter by the non-linear Einstein–Hilbert differential equations. In contrast, in the present context, the acoustic metric is related to the distribution of matter in a simple algebraic fashion.

## 2.5 Dumb holes – ergo-regions, horizons, and surface gravity

Let us start with the notion of an ergo-region: Consider integral curves of the vector

$$K^\mu \equiv (\partial/\partial t)^\mu = (1,0,0,0)^\mu. \tag{43}$$

If the flow is steady then this is the time translation Killing vector. Even if the flow is not steady the background Minkowski metric provides us with a natural definition of "at rest". Then[5]

$$g_{\mu\nu}\,(\partial/\partial t)^\mu\,(\partial/\partial t)^\nu = g_{tt} = -[c^2 - v^2]. \tag{44}$$

This quantity changes sign when $||\mathbf{v}|| > c$. Thus any region of supersonic flow is an ergo-region. (And the boundary of the ergo-region may be deemed to be the ergo-surface.) The analogue of this behaviour in general relativity is the ergosphere surrounding any spinning black hole – it is a region where space "moves" with superluminal velocity relative to the fixed stars [265, 164, 422].

A trapped surface in acoustics is defined as follows: Take any closed two-surface. If the fluid velocity is everywhere inward-pointing and the normal component of the fluid velocity is everywhere greater than the local speed of sound, then no matter what direction a sound wave propagates, it will be swept inward by the fluid flow and be trapped inside the surface. The surface is then said to be outer-trapped. (For comparison with the usual situation in general relativity see [164, pages 319–323] or [422, pages 310–311].) Inner-trapped surfaces (anti-trapped surfaces) can be defined by demanding that the fluid flow is everywhere outward-pointing with supersonic normal component. It is only because of the fact that the background Minkowski metric provides a natural definition of "at rest" that we can adopt such a simple and straightforward definition. In ordinary general relativity we need to develop considerable additional technical machinery, such as the notion of the "expansion" of bundles of ingoing and outgoing null geodesics, before defining trapped surfaces. That the above definition for acoustic geometries is a specialization of the usual one can be seen from the discussion on pages 262–263 of Hawking and Ellis [164]. The acoustic trapped region is now defined as the region containing outer trapped surfaces, and the acoustic (future) apparent horizon as the boundary of the trapped region. That is, the acoustic apparent horizon is the two-surface for which the normal component of the fluid velocity is everywhere equal to the local speed of sound. (We can also define anti-trapped regions and past apparent horizons but these notions are of limited utility in general relativity.)[6]

The event horizon (absolute horizon) is defined, as in general relativity, by demanding that it be the boundary of the region from which null geodesics (phonons) cannot escape. This is actually the future event horizon. A past event horizon can be defined in terms of the boundary of the region that cannot be reached by incoming phonons – strictly speaking this requires us to define notions of past and future null infinities, but we will simply take all relevant incantations as understood. In particular the event horizon is a null surface, the generators of which are null geodesics.

In all stationary geometries the apparent and event horizons coincide, and the distinction is immaterial. In time-dependent geometries the distinction is often important. When computing the surface gravity we shall restrict attention to stationary geometries (steady flow). In fluid flows of high symmetry (spherical symmetry, plane symmetry), the ergosphere may coincide with the acoustic apparent horizon, or even the acoustic event horizon. This is the analogue of the result in general relativity that for static (as opposed to stationary) black holes the ergosphere and event horizon coincide. For many more details, including appropriate null coordinates and Carter–Penrose diagrams, both in stationary and time-dependent situations, see [13].

[5] Henceforth, in the interests of notational simplicity, we shall drop the explicit subscript 0 on background field quantities unless there is specific risk of confusion.

[6] This discussion naturally leads us to what is perhaps the central question of analogue models – just how much of the standard "laws of black hole mechanics" [21, 423] carry over into these analogue models? Quite a lot but not everything – that's our main topic for the rest of the review.

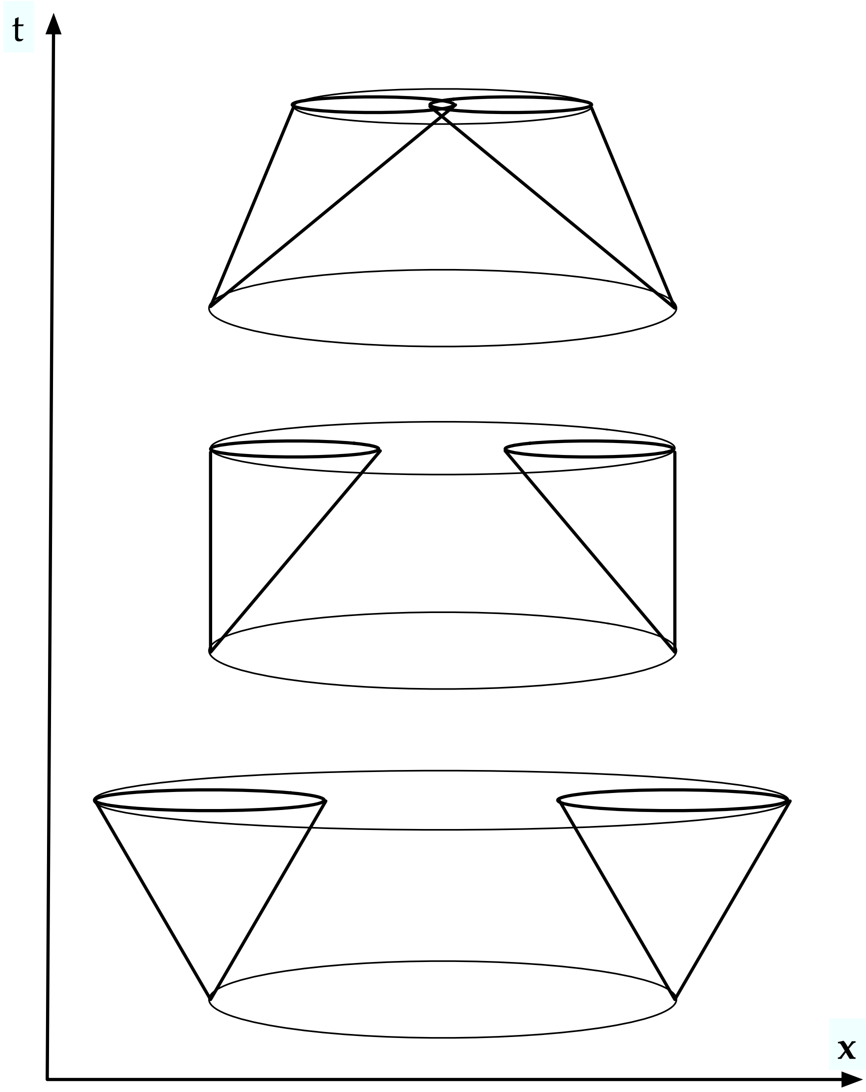


Figure 5: *A moving fluid can form "trapped surfaces" when supersonic flow tips the sound cones past the vertical.*

Because of the definition of event horizon in terms of phonons (null geodesics) that cannot escape the acoustic black hole, the event horizon is automatically a null surface, and the generators of the event horizon are automatically null geodesics. In the case of acoustics there is one particular parameterization of these null geodesics that is "most natural", which is the parameterization in terms of the Newtonian time coordinate of the underlying physical metric. This allows us to unambiguously define a "surface gravity" even for non-stationary (time-dependent) acoustic event horizons, by calculating the extent to which this natural time parameter fails to be an affine parameter for the null generators of the horizon. (This part of the construction fails in general relativity where there is no universal natural time-coordinate unless there is a timelike Killing vector – this is why extending the notion of surface gravity to non-stationary geometries in general relativity is so difficult.)

When it comes to explicitly calculating the surface gravity in terms of suitable gradients of the fluid flow, it is nevertheless very useful to limit attention to situations of steady flow (so that the acoustic metric is stationary). This has the added bonus that for stationary geometries the notion of "acoustic surface gravity" in acoustics is unambiguously equivalent to the general relativity definition. It is also useful to take cognizance of the fact that the situation simplifies considerably for static (as opposed to merely stationary) acoustic metrics.

To set up the appropriate framework, write the general stationary acoustic metric in the form

$$\mathrm{d}s^2 = \frac{\rho}{c}\left[-c^2\,\mathrm{d}t^2 + (\mathrm{d}\mathbf{x} - \mathbf{v}\,\mathrm{d}t)^2\right]. \tag{45}$$

The time translation Killing vector is simply $K^\mu = (1;\vec{0})$, with

$$K^2 \equiv g_{\mu\nu}K^\mu K^\nu \equiv -||\mathbf{K}||^2 = -\frac{\rho}{c}[c^2 - v^2]. \tag{46}$$

The metric can also be written as

$$\mathrm{d}s^2 = \frac{\rho}{c}\left[-(c^2 - v^2)\,\mathrm{d}t^2 - 2\mathbf{v}\cdot\mathrm{d}\mathbf{x}\,\mathrm{d}t + (\mathrm{d}\mathbf{x})^2\right]. \tag{47}$$

Now suppose that the vector $\mathbf{v}/(c^2 - v^2)$ is integrable, then we can define a new time coordinate by

$$\mathrm{d}\tau = \mathrm{d}t + \frac{\mathbf{v}\cdot\mathrm{d}\mathbf{x}}{c^2 - v^2}. \tag{48}$$

Substituting this back into the acoustic line element gives

$$\mathrm{d}s^2 = \frac{\rho}{c}\left[-(c^2 - v^2)\,\mathrm{d}\tau^2 + \left\{\delta_{ij} + \frac{v^i v^j}{c^2 - v^2}\right\}\mathrm{d}x^i\,\mathrm{d}x^j\right]. \tag{49}$$

In this coordinate system the absence of the time-space cross-terms makes manifest that the acoustic geometry is in fact static (there exists a family of spacelike hypersurfaces orthogonal to the timelike Killing vector). The condition that an acoustic geometry be static, rather than merely stationary, is thus seen to be

$$\boldsymbol{\nabla}\times\left\{\frac{\mathbf{v}}{(c^2 - v^2)}\right\} = 0, \tag{50}$$

that is, (since in deriving the existence of the effective metric we have already assumed the fluid to be irrotational),

$$\mathbf{v}\times\boldsymbol{\nabla}(c^2 - v^2) = 0. \tag{51}$$

This requires the fluid flow to be parallel to another vector that is not quite the acceleration but is closely related to it. (Note that, because of the vorticity free assumption, $\frac{1}{2}\boldsymbol{\nabla}v^2$ is just the three-acceleration of the fluid, it is the occurrence of a possibly position dependent speed of sound that complicates the above.)

Once we have a static geometry, we can of course directly apply all of the standard tricks [372] for calculating the surface gravity developed in general relativity. We set up a system of fiducial observers (FIDOS) by properly normalizing the time-translation Killing vector

$$\mathbf{V}_{\mathrm{FIDO}} \equiv \frac{\mathbf{K}}{||\mathbf{K}||} = \frac{\mathbf{K}}{\sqrt{(\rho/c)\;[c^2 - v^2]}}. \tag{52}$$

The four-acceleration of the FIDOS is defined as

$$\mathbf{A}_{\mathrm{FIDO}} \equiv (\mathbf{V}_{\mathrm{FIDO}} \cdot \boldsymbol{\nabla}) \mathbf{V}_{\mathrm{FIDO}}, \tag{53}$$

and using the fact that $\mathbf{K}$ is a Killing vector, it may be computed in the standard manner

$$\mathbf{A}_{\mathrm{FIDO}} = +\frac{1}{2} \frac{\boldsymbol{\nabla}||\mathbf{K}||^2}{||\mathbf{K}||^2}. \tag{54}$$

That is

$$\mathbf{A}_{\mathrm{FIDO}} = \frac{1}{2} \left[ \frac{\boldsymbol{\nabla}(c^2 - v^2)}{(c^2 - v^2)} + \frac{\boldsymbol{\nabla}(\rho/c)}{(\rho/c)} \right]. \tag{55}$$

The surface gravity is now defined by taking the norm $||\mathbf{A}_{\mathrm{FIDO}}||$, multiplying by the lapse function, $||\mathbf{K}|| = \sqrt{(\rho/c)\;[c^2 - v^2]}$, and taking the limit as one approaches the horizon: $|v| \to c$ (remember that we are currently dealing with the static case). The net result is

$$||\mathbf{A}_{\mathrm{FIDO}}|| \; ||\mathbf{K}|| = \frac{1}{2} \mathbf{v} \cdot \boldsymbol{\nabla}(c^2 - v^2) + O(c^2 - v^2), \tag{56}$$

so that the surface gravity is given in terms of a normal derivative by[7]

$$g_H = \frac{1}{2} \frac{\partial(c^2 - v^2)}{\partial n} = c \, \frac{\partial(c - v)}{\partial n}. \tag{57}$$

This is not quite Unruh's result [376, 377, 378] since he implicitly took the speed of sound to be a position-independent constant. The fact that prefactor $\rho/c$ drops out of the final result for the surface gravity can be justified by appeal to the known conformal invariance of the surface gravity [192]. Though derived in a totally different manner, this result is also compatible with the expression for "surface-gravity" obtained in the solid-state black holes of Reznik [319], wherein a position dependent (and singular) refractive index plays a role analogous to the acoustic metric. As a further consistency check, one can go to the spherically symmetric case and check that this reproduces the results for "dirty black holes" enunciated in [386].

Since this is a static geometry, the relationship between the Hawking temperature and surface gravity may be verified in the usual fast-track manner – using the Wick rotation trick to analytically continue to Euclidean space [147]. If you don't like Euclidean signature techniques (which are in any case only applicable to equilibrium situations) you should go back to the original Hawking derivations [159, 160].[8]

One final comment to wrap up this section: The coordinate transform we used to put the acoustic metric into the explicitly static form is perfectly good mathematics, and from the general relativity point of view is even a simplification. However, from the point of view of the underlying

[7] Because of the background Minkowski metric there can be no possible confusion as to the definition of this normal derivative.

[8] There are a few potential subtleties in the derivation of the existence Hawking radiation which we are for the time being glossing over, see Section 5.1 for details.

Newtonian physics of the fluid, this is a rather bizarre way of deliberately de-synchronizing your clocks to take a perfectly reasonable region – the boundary of the region of supersonic flow – and push it out to "time" plus infinity. From the fluid dynamics point of view this coordinate transformation is correct but perverse, and it is easier to keep a good grasp on the physics by staying with the original Newtonian time coordinate.

If the fluid flow does not satisfy the integrability condition which allows us to introduce an explicitly static coordinate system, then defining the surface gravity is a little trickier.

Recall that by construction the acoustic apparent horizon is in general defined to be a two-surface for which the normal component of the fluid velocity is everywhere equal to the local speed of sound, whereas the acoustic event horizon (absolute horizon) is characterised by the boundary of those null geodesics (phonons) that do not escape to infinity. In the stationary case these notions coincide, and it is still true that the horizon is a null surface, and that the horizon can be ruled by an appropriate set of null curves. Suppose we have somehow isolated the location of the acoustic horizon, then in the vicinity of the horizon we can split up the fluid flow into normal and tangential components

$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel; \qquad \text{where} \qquad \mathbf{v}_\perp = v_\perp\, \hat{\mathbf{n}}. \tag{58}$$

Here (and for the rest of this particular section) it is essential that we use the natural Newtonian time coordinate inherited from the background Newtonian physics of the fluid. In addition $\hat{\mathbf{n}}$ is a unit vector field that at the horizon is perpendicular to it, and away from the horizon is some suitable smooth extension. (For example, take the geodesic distance to the horizon and consider its gradient.) We only need this decomposition to hold in some open set encompassing the horizon and do not need to have a global decomposition of this type available. Furthermore, by definition we know that $v_\perp = c$ at the horizon. Now consider the vector field

$$L^\mu = (\,1;\; v_\parallel^i\,). \tag{59}$$

Since the spatial components of this vector field are by definition tangent to the horizon, the integral curves of this vector field will be generators for the horizon. Furthermore the norm of this vector (in the acoustic metric) is

$$||\mathbf{L}||^2 = -\frac{\rho}{c}\left[\,-(c^2-v^2) - 2\mathbf{v}_\parallel\cdot\mathbf{v} + \mathbf{v}_\parallel\cdot\mathbf{v}_\parallel\right] = \frac{\rho}{c}\,(c^2 - v_\perp^2). \tag{60}$$

In particular, on the acoustic horizon $L^\mu$ defines a null vector field, the integral curves of which are generators for the acoustic horizon. We shall now verify that these generators are geodesics, though the vector field $\mathbf{L}$ is not normalised with an affine parameter, and in this way shall calculate the surface gravity.

Consider the quantity $(\mathbf{L}\cdot\boldsymbol{\nabla})\mathbf{L}$ and calculate

$$L^\alpha\nabla_\alpha L^\mu = L^\alpha(\nabla_\alpha L_\beta - \nabla_\beta L_\alpha)g^{\beta\mu} + \frac{1}{2}\nabla_\beta(L^2)g^{\beta\mu}. \tag{61}$$

To calculate the first term note that

$$L_\mu = \frac{\rho}{c}\,(-[c^2 - v_\perp^2]; \mathbf{v}_\perp). \tag{62}$$

Thus

$$L_{[\alpha,\beta]} = -\left[\begin{array}{c|c} 0 & -\nabla_i\left[\frac{\rho}{c}(c^2-v_\perp^2)\right] \\ \hline +\nabla_j\left[\frac{\rho}{c}(c^2-v_\perp^2)\right] & \left(\frac{\rho}{c}\,v^\perp\right)_{[i,j]} \end{array}\right]. \tag{63}$$

And so:

$$L^\alpha L_{[\beta,\alpha]} = \left( \mathbf{v}_\parallel \cdot \boldsymbol{\nabla} \left[ \frac{\rho}{c}(c^2 - v_\perp^2) \right] ; \nabla_j \left[ \frac{\rho}{c}(c^2 - v_\perp^2) \right] + v_\parallel^i \left( \frac{\rho}{c}\, v^\perp \right)_{[j,i]} \right). \tag{64}$$

On the horizon, where $c = v_\perp$, this simplifies tremendously

$$(L^\alpha L_{[\beta,\alpha]})|_{\mathrm{horizon}} = -\frac{\rho}{c}\ \left(0; \nabla_j (c^2 - v_\perp^2)\right) = -\frac{\rho}{c}\ \frac{\partial(c^2 - v_\perp^2)}{\partial n}\ (0; \hat{n}_j)\,. \tag{65}$$

Similarly, for the second term we have

$$\nabla_\beta (L^2) = \left( 0; \nabla_j \left[ \frac{\rho}{c}(c^2 - v_\perp^2) \right] \right). \tag{66}$$

On the horizon this again simplifies

$$\nabla_\beta (L^2)|_{\mathrm{horizon}} = +\frac{\rho}{c}\ \left(0; \nabla_j (c^2 - v_\perp^2)\right) = +\frac{\rho}{c}\ \frac{\partial(c^2 - v_\perp^2)}{\partial n}\ (0; \hat{n}_j)\,. \tag{67}$$

There is partial cancellation between the two terms, and so

$$(L^\alpha \nabla_\alpha L_\mu)_{\mathrm{horizon}} = +\frac{1}{2}\ \frac{\rho}{c}\ \frac{\partial(c^2 - v_\perp^2)}{\partial n}\ (0; \hat{n}_j)\,, \tag{68}$$

while

$$(L_\mu)_{\mathrm{horizon}} = \frac{\rho}{c}\ (0; c\, \hat{n}_j)\,. \tag{69}$$

Comparing this with the standard definition of surface gravity [422] [9]

$$(L^\alpha \nabla_\alpha L_\mu)_{\mathrm{horizon}} = +\frac{g_H}{c}\ (L_\mu)_{\mathrm{horizon}}\,, \tag{70}$$

we finally have

$$g_H = \frac{1}{2} \frac{\partial(c^2 - v_\perp^2)}{\partial n} = c\ \frac{\partial(c - v_\perp)}{\partial n}\,. \tag{71}$$

This is in agreement with the previous calculation for static acoustic black holes, and insofar as there is overlap, is also consistent with results of Unruh [376, 377, 378], Reznik [319], and the results for "dirty black holes" [386]. From the construction it is clear that the surface gravity is a measure of the extent to which the Newtonian time parameter inherited from the underlying fluid dynamics fails to be an affine parameter for the null geodesics on the horizon.[10]

Again, the justification for going into so much detail on this specific model is that this style of argument can be viewed as a template – it will (with suitable modifications) easily generalise to more complicated analogue models.

### 2.5.1 Example: vortex geometry

As an example of a fluid flow where the distinction between ergosphere and acoustic event horizon is critical consider the "draining bathtub" fluid flow. We shall model a draining bathtub by a (3+1) dimensional flow with a linear sink along the z-axis. Let us start with the simplifying assumption that the background density $\rho$ is a position-independent constant throughout the flow

[9] There is an issue of normalization here. On the one hand we want to be as close as possible to general relativistic conventions. On the other hand, we would like the surface gravity to really have the dimensions of an acceleration. The convention adopted here, with one explicit factor of $c$, is the best compromise we have come up with. (Note that in an acoustic setting, where the speed of sound is not necessarily a constant, we cannot simply set $c \to 1$ by a choice of units.)

[10] There are situations in which this surface gravity is a lot larger than one might naively expect [239].

(which automatically implies that the background pressure $p$ and speed of sound $c$ are also constant throughout the fluid flow). The equation of continuity then implies that for the radial component of the fluid velocity we must have

$$v^{\hat{r}} \propto \frac{1}{r}\,. \tag{72}$$

In the tangential direction, the requirement that the flow be vorticity free (apart from a possible delta-function contribution at the vortex core) implies, via Stokes' theorem, that

$$v^{\hat{t}} \propto \frac{1}{r}\,. \tag{73}$$

(If these flow velocities are nonzero, then following the discussion of [401] there must be some external force present to set up and maintain the background flow. Fortunately it is easy to see that this external force affects only the background flow and does not influence the linearised fluctuations we are interested in.) For the background velocity potential we must then have

$$\phi(r,\theta) = -A\;\ln(r/a) - B\;\theta. \tag{74}$$

Note that, as we have previously hinted, the velocity potential is not a true function (because it has a discontinuity on going through $2\pi$ radians). The velocity potential must be interpreted as being defined patch-wise on overlapping regions surrounding the vortex core at $r = 0$. The velocity of the fluid flow is

$$\mathbf{v} = -\boldsymbol{\nabla}\phi = \frac{(A\;\hat{r} + B\;\hat{\theta})}{r}. \tag{75}$$

Dropping a position-independent prefactor, the acoustic metric for a draining bathtub is explicitly given by

$$\mathrm{d}s^2 = -c^2\;\mathrm{d}t^2 + \left(\mathrm{d}r - \frac{A}{r}\mathrm{d}t\right)^2 + \left(r\,\mathrm{d}\theta - \frac{B}{r}\mathrm{d}t\right)^2 + \mathrm{d}z^2. \tag{76}$$

Equivalently

$$\mathrm{d}s^2 = -\left(c^2 - \frac{A^2+B^2}{r^2}\right)\mathrm{d}t^2 - 2\frac{A}{r}\,\mathrm{d}r\,\mathrm{d}t - 2B\,\mathrm{d}\theta\,\mathrm{d}t + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \mathrm{d}z^2. \tag{77}$$

A similar metric, restricted to A=0 (no radial flow), and generalised to an anisotropic speed of sound, has been exhibited by Volovik [404], that metric being a model for the acoustic geometry surrounding physical vortices in superfluid $^3$He. (For a survey of the many analogies and similarities between the physics of superfluid $^3$He and the Standard Electroweak Model see [420], this reference is also useful as background to understanding the Lorentzian geometric aspects of $^3$He fluid flow.) Note that the metric given above is *not* identical to the metric of a spinning cosmic string, which would instead take the form [388]

$$\mathrm{d}s^2 = -c^2(\mathrm{d}t - \tilde{A}\,\mathrm{d}\theta)^2 + \mathrm{d}r^2 + (1-\tilde{B})r^2\mathrm{d}\theta^2 + \mathrm{d}z^2. \tag{78}$$

In conformity with previous comments, the vortex fluid flow is seen to possess an acoustic metric that is stably causal and which does not involve closed timelike curves. (At large distances it is possible to *approximate* the vortex geometry by a spinning cosmic string [404], but this approximation becomes progressively worse as the core is approached.)

The ergosphere forms at

$$r_{\text{ergosphere}} = \frac{\sqrt{A^2+B^2}}{c}. \tag{79}$$

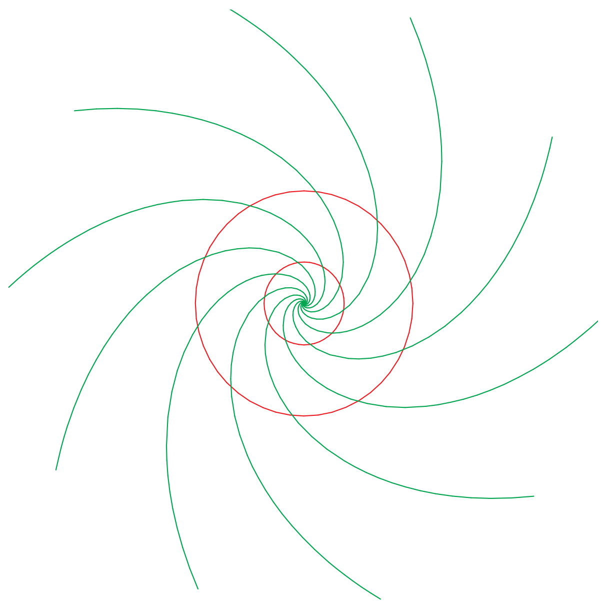

Figure 6: *A collapsing vortex geometry (draining bathtub): The green spirals denote streamlines of the fluid flow. The outer circle represents the ergo-surface while the inner circle represents the [outer] event horizon.*

Note that the sign of $A$ is irrelevant in defining the ergosphere and ergo-region: It does not matter if the vortex core is a source or a sink.

The acoustic event horizon forms once the radial component of the fluid velocity exceeds the speed of sound, that is at

$$r_{\text{horizon}} = \frac{|A|}{c}. \tag{80}$$

The sign of $A$ now makes a difference. For $A < 0$ we are dealing with a future acoustic horizon (acoustic black hole), while for $A > 0$ we are dealing with a past event horizon (acoustic white hole).

### 2.5.2 Example: slab geometry

A popular model for the investigation of event horizons in the acoustic analogy is the one-dimensional slab geometry where the velocity is always along the $z$ direction and the velocity profile depends only on $z$. The continuity equation then implies that $\rho(z)\, v(z)$ is a constant, and the acoustic metric becomes

$$\mathrm{d}s^2 \propto \frac{1}{v(z)\, c(z)} \left[ -c(z)^2\, \mathrm{d}t^2 + \{\mathrm{d}z - v(z)\, \mathrm{d}t\}^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \right]. \tag{81}$$

That is

$$\mathrm{d}s^2 \propto \frac{1}{v(z)\, c(z)} \left[ - \left\{ c(z)^2 - v(z)^2 \right\} \mathrm{d}t^2 - 2v(z)\, \mathrm{d}z\, \mathrm{d}t + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \right]. \tag{82}$$

If we set $c = 1$ and ignore the conformal factor we have the toy model acoustic geometry discussed by Unruh [378, page 2828, equation (8)], Jacobson [188, page 7085, equation (4)], Corley and Jacobson [88], and Corley [86]. (In this situation one must again invoke an external force to set up and maintain the fluid flow. Since the conformal factor is regular at the event horizon, we know that the surface gravity and Hawking temperature are independent of this conformal factor [192].) In the general case it is important to realise that the flow can go supersonic for either of two reasons: The fluid could speed up, or the speed of sound could decrease. When it comes to calculating the "surface gravity" both of these effects will have to be taken into account.

#### 2.5.3 Example: Painlevé–Gullstrand geometry

To see how close the acoustic metric can get to reproducing the Schwarzschild geometry it is first useful to introduce one of the more exotic representations of the Schwarzschild geometry: the Painlevé–Gullstrand line element, which is simply an unusual choice of coordinates on the Schwarzschild spacetime.[11] In modern notation the Schwarzschild geometry in ingoing (+) and outgoing (−) Painlevé–Gullstrand coordinates may be written as:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \left(\mathrm{d}r \pm \sqrt{\frac{2GM}{r}}\mathrm{d}t\right)^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{83}$$

Equivalently

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 \pm \sqrt{\frac{2GM}{r}}\,\mathrm{d}r\,\mathrm{d}t + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right). \tag{84}$$

This representation of the Schwarzschild geometry was not (until the advent of the analogue models) particularly well-known, and it has been independently rediscovered several times during the 20th century. See for instance Painlevé [293], Gullstrand [154], Lemaître [228], the related discussion by Israel [183], and more recently, the paper by Kraus and Wilczek [218]. The Painlevé–Gullstrand coordinates are related to the more usual Schwarzschild coordinates by

$$t_{\mathrm{PG}} = t_{\mathrm{S}} \pm \left[4M\ \operatorname{arctanh}\left(\sqrt{\frac{2GM}{r}}\right) - 2\sqrt{2GMr}\right]. \tag{85}$$

Or equivalently

$$\mathrm{d}t_{\mathrm{PG}} = \mathrm{d}t_{\mathrm{S}} \pm \frac{\sqrt{2GM/r}}{1-2GM/r}\,\mathrm{d}r. \tag{86}$$

With these explicit forms in hand, it becomes an easy exercise to check the equivalence between the Painlevé–Gullstrand line element and the more usual Schwarzschild form of the line element. It should be noted that the + sign corresponds to a coordinate patch that covers the usual asymptotic region plus the region containing the future singularity of the maximally extended Schwarzschild spacetime. It thus covers the future horizon and the black hole singularity. On the other hand the − sign corresponds to a coordinate patch that covers the usual asymptotic region plus the region containing the past singularity. It thus covers the past horizon and the white hole singularity.

As emphasised by Kraus and Wilczek, the Painlevé–Gullstrand line element exhibits a number of features of pedagogical interest. In particular the constant time spatial slices are completely flat – the curvature of space is zero, and all the spacetime curvature of the Schwarzschild geometry has been pushed into the time–time and time–space components of the metric.

Given the Painlevé–Gullstrand line element, it might seem trivial to force the acoustic metric into this form: Simply take $\rho$ and $c$ to be constants, and set $v = \sqrt{2GM/r}$? While this certainly forces the acoustic metric into the Painlevé–Gullstrand form the problem with this is that this

[11] The Painlevé–Gullstrand line element is sometimes called the Lemaître line element.

assignment is incompatible with the continuity equation $\nabla\cdot(\rho\mathbf{v})\neq 0$ that was used in deriving the acoustic equations.

The best we can actually do is this: Pick the speed of sound $c$ to be a position independent constant, which we normalise to unity ($c=1$). Now set $v=\sqrt{2GM/r}$, and use the continuity equation $\nabla\cdot(\rho\mathbf{v})=0$ to deduce $\rho|\mathbf{v}|\propto 1/r^2$ so that $\rho\propto r^{-3/2}$. Since the speed of sound is taken to be constant we can integrate the relation $c^2=\mathrm{d}p/\mathrm{d}\rho$ to deduce the equation of state must be $p=p_\infty+c^2\rho$ and that the background pressure satisfies $p-p_\infty\propto c^2 r^{-3/2}$. Overall the acoustic metric is now

$$\mathrm{d}s^2\propto r^{-3/2}\left[-\mathrm{d}t^2+\left(\mathrm{d}r\pm\sqrt{\frac{2GM}{r}}\mathrm{d}t\right)^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right)\right]. \tag{87}$$

So we see that the net result is conformal to the Painlevé–Gullstrand form of the Schwarzschild geometry but not identical to it. For many purposes this is quite good enough: We have an event horizon, we can define surface gravity, we can analyse Hawking radiation.[12] Since surface gravity and Hawking temperature are conformal invariants [192] this is sufficient for analysing basic features of the Hawking radiation process. The only way in which the conformal factor can influence the Hawking radiation is through backscattering off the acoustic metric. (The phonons are minimally coupled scalars, not conformally coupled scalars so there will in general be effects on the frequency-dependent greybody factors.)

If we focus attention on the region near the event horizon, the conformal factor can simply be taken to be a constant, and we can ignore all these complications.

## 2.6 Regaining geometric acoustics

Up to now, we have been developing general machinery to force acoustics into Lorentzian form. This can be justified either with a view to using fluid mechanics to teach us more about general relativity, or to using the techniques of Lorentzian geometry to teach us more about fluid mechanics.

For example, given the machinery developed so far, taking the short wavelength/high frequency limit to obtain geometrical acoustics is now easy. Sound rays (phonons) follow the *null geodesics* of the acoustic metric. Compare this to general relativity where in the geometrical optics approximation light rays (photons) follow *null geodesics* of the physical spacetime metric. Since null geodesics are insensitive to any overall conformal factor in the metric [265, 164, 422] one might as well simplify life by considering a modified conformally related metric

$$h_{\mu\nu}\equiv\left[\begin{array}{c|c}-(c^2-v_0^2) & -v_0^j\\ \hline -v_0^i & \delta^{ij}\end{array}\right]. \tag{88}$$

This immediately implies that, in the geometric acoustics limit, sound propagation is insensitive to the density of the fluid. In this limit, acoustic propagation depends only on the local speed of sound and the velocity of the fluid. It is only for specifically wave related properties that the density of the medium becomes important.

We can rephrase this in a language more familiar to the acoustics community by invoking the Eikonal approximation. Express the linearised velocity potential, $\phi_1$, in terms of an amplitude, $a$, and phase, $\varphi$, by $\phi_1\sim ae^{i\varphi}$. Then, neglecting variations in the amplitude $a$, the wave equation reduces to the *Eikonal equation*

$$h^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi=0. \tag{89}$$

[12] Similar constructions work for the Reissner–Nordstrom geometry [239], as long as one does not get too close to the singularity. Likewise certain aspects of the Kerr geometry can be emulated in this way [401].

This Eikonal equation is blatantly insensitive to any overall multiplicative prefactor (conformal factor).

As a sanity check on the formalism, it is useful to re-derive some standard results. For example, let the null geodesic be parameterised by $x^\mu(t) \equiv (t; \mathbf{x}(t))$. Then the null condition implies

$$\begin{aligned} & h_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t} = 0 \\ \Longleftrightarrow\ & -(c^2 - v_0^2) - 2v_0^i \frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t} = 0 \\ \Longleftrightarrow\ & \left\| \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} - \mathbf{v}_0 \right\| = c. \end{aligned} \tag{90}$$

Here the norm is taken in the flat physical metric. This has the obvious interpretation that the ray travels at the speed of sound, $c$, relative to the moving medium.

Furthermore, if the geometry is stationary one can do slightly better. Let $x^\mu(s) \equiv (t(s); \mathbf{x}(s))$ be some null path from $\mathbf{x}_1$ to $\mathbf{x}_2$, parameterised in terms of physical arc length (i.e., $||\mathrm{d}\mathbf{x}/\mathrm{d}s|| \equiv 1$). Then the tangent vector to the path is

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = \left( \frac{\mathrm{d}t}{\mathrm{d}s}; \frac{\mathrm{d}x^i}{\mathrm{d}s} \right). \tag{91}$$

The condition for the path to be null (though not yet necessarily a null geodesic) is

$$g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0. \tag{92}$$

Using the explicit algebraic form for the metric, this can be expanded to show

$$-(c^2 - v_0^2) \left( \frac{\mathrm{d}t}{\mathrm{d}s} \right)^2 - 2v_0^i \left( \frac{\mathrm{d}x^i}{\mathrm{d}s} \right) \left( \frac{\mathrm{d}t}{\mathrm{d}s} \right) + 1 = 0. \tag{93}$$

Solving this quadratic

$$\left( \frac{\mathrm{d}t}{\mathrm{d}s} \right) = \frac{-v_0^i \left( \frac{\mathrm{d}x^i}{\mathrm{d}s} \right) + \sqrt{c^2 - v_0^2 + \left( v_0^i \, \frac{\mathrm{d}x^i}{\mathrm{d}s} \right)^2}}{c^2 - v_0^2}. \tag{94}$$

Therefore, the total time taken to traverse the path is

$$T[\gamma] = \int_{\mathbf{x}_1}^{\mathbf{x}_2} (\mathrm{d}t/\mathrm{d}s)\, \mathrm{d}s = \int_\gamma \frac{1}{c^2 - v_0^2} \left\{ \sqrt{(c^2 - v_0^2)\mathrm{d}s^2 + (v_0^i \, \mathrm{d}x^i)^2} - v_0^i \, \mathrm{d}x^i \right\}. \tag{95}$$

If we now recall that extremising the total time taken is Fermat's principle for sound rays, we see that we have checked the formalism for stationary geometries (steady flow) by reproducing the discussion on page 262 of Landau and Lifshitz [221].

## 2.7 Generalizing the physical model

There are a large number of ways in which the present particularly simple analogue model can be generalised. Obvious issues within the current physical framework are:

- Adding external forces.
- Working in (1+1) or (2+1) dimensions.
- Adding vorticity, to go beyond the irrotational constraint.

Beyond these immediate questions, we could also seek similar effects in other physical or mathematical frameworks.

### 2.7.1 External forces

Adding external forces is easy, an early discussion can be found in [389] and more details are available in [401]. The key point is that with an external force one can to some extent shape the background flow (see for example the discussion on [149]). Upon linearization, the fluctuations are however insensitive to any external force.

### 2.7.2 The role of dimension

The role of spacetime dimension in these acoustic geometries is sometimes a bit surprising and potentially confusing. This is important because there is a real physical distinction, for instance, between truly (2+1)-dimensional systems and effectively (2+1)-dimensional systems in the form of (3+1)-dimensional systems with cylindrical symmetry. Similarly there is a real physical distinction between a truly (1+1)-dimensional system and a (3+1)-dimensional system with transverse symmetry. We emphasise that in cartesian coordinates the wave equation

$$\frac{\partial}{\partial x^\mu}\left(f^{\mu\nu}\frac{\partial}{\partial x^\nu}\,\phi\right)=0, \tag{96}$$

where

$$f^{\mu\nu}=\left[\begin{array}{c|c} -\rho/c^2 & -\rho\, v^j/c^2 \\ \hline -\rho\, v^i/c^2 & \rho\,\{\delta^{ij}-v^iv^j/c^2\} \end{array}\right], \tag{97}$$

holds *independent* of the dimensionality of spacetime. It depends only on the Euler equation, the continuity equation, a barotropic equation of state, and the assumption of irrotational flow [376, 387, 391, 389].

Introducing the inverse acoustic metric $g^{\mu\nu}$, defined by

$$f^{\mu\nu}=\sqrt{-g}\,g^{\mu\nu}; \qquad g=\frac{1}{\det(g^{\mu\nu})} \tag{98}$$

the wave Equation (96) corresponds to the d'Alembertian wave equation in a curved space-time with contravariant metric tensor:

$$g^{\mu\nu}=\left(\frac{\rho}{c}\right)^{-2/(d-1)}\left[\begin{array}{c|c} -1/c^2 & -\mathbf{v}^T/c^2 \\ \hline -\,\mathbf{v}/c^2 & \mathbf{I}_{d\times d}-\mathbf{v}\otimes\mathbf{v}^T/c^2 \end{array}\right], \tag{99}$$

where $d$ is the dimension of *space* (not spacetime). The covariant acoustic metric is then

$$g_{\mu\nu}=\left(\frac{\rho}{c}\right)^{2/(d-1)}\left[\begin{array}{c|c} -\left(c^2-v^2\right) & -\mathbf{v}^T \\ \hline -\mathbf{v} & \mathbf{I}_{d\times d} \end{array}\right]. \tag{100}$$

**d = 3:** The acoustic line element for three space and one time dimension reads

$$g_{\mu\nu}=\left(\frac{\rho}{c}\right)\left[\begin{array}{c|c} -\left(c^2-v^2\right) & -\mathbf{v}^T \\ \hline -\mathbf{v} & \mathbf{I}_{3\times 3} \end{array}\right]. \tag{101}$$

**d = 2:** The acoustic line element for two space and one time dimension reads

$$g_{\mu\nu}=\left(\frac{\rho}{c}\right)^{2}\left[\begin{array}{c|c} -\left(c^2-v^2\right) & -\mathbf{v}^T \\ \hline -\mathbf{v} & \mathbf{I}_{2\times 2} \end{array}\right]. \tag{102}$$

This situation would be appropriate, for instance, when dealing with surface waves or excitations confined to a particular substrate.

**d = 1:** The naive form of the acoustic metric in (1+1) dimensions is ill-defined, because the conformal factor is raised to a formally infinite power. This is a side effect of the well-known conformal invariance of the Laplacian in 2 dimensions. The wave equation in terms of the densitised inverse metric $f^{\mu\nu}$ continues to make good sense; it is only the step from $f^{\mu\nu}$ to the effective metric that breaks down.

Note that this issue only presents a difficulty for physical systems that are *intrinsically* one-dimensional. A three-dimensional system with plane symmetry, or a two-dimensional system with line symmetry, provides a perfectly well behaved model for (1+1) dimensions, as in the cases $d = 3$ and $d = 2$ above.

#### 2.7.3 Adding vorticity

For the preceding analysis to hold it is necessary and sufficient that the flow locally be vorticity free, $\boldsymbol{\nabla} \times \mathbf{v} = 0$, so that velocity potentials exist on an atlas of open patches. Note that the irrotational condition is automatically satisfied for the super-fluid component of physical superfluids. (This point has been emphasised by Comer [84], who has also pointed out that in superfluids there will be multiple acoustic metrics – and multiple acoustic horizons – corresponding to first and second sound.) Even for normal fluids, vorticity free flows are common, especially in situations of high symmetry. Furthermore, the previous condition enables us to handle vortex filaments, where the vorticity is concentrated into a thin vortex core, provided we do not attempt to probe the vortex core itself. It is not necessary for the velocity potential $\phi$ to be globally defined.

Though physically important, dealing with situations of distributed vorticity is much more difficult, and the relevant wave equation is more complicated in that the velocity scalar is now insufficient to completely characterise the fluid flow.[13] An approach similar to the spirit of the present discussion, but in terms of Clebsch potentials, can be found in [307]. The eikonal approximation (geometrical acoustics) leads to the same conformal class of metrics previously discussed, but in the realm of physical acoustics the wave equation is considerably more complicated than a simple d'Alembertian. (Roughly speaking, the vorticity becomes a source for the d'Alembertian, while the vorticity evolves in response to gradients in a generalised scalar potential. This seems to take us outside the realm of models of direct interest to the general relativity community.)[14]

### 2.8 Simple Lagrangian meta-model

As a first (and rather broad) example of the very abstract ways in which the notion of an acoustic metric can be generalised, we start from the simple observation that irrotational barotropic fluid mechanics can be described by a Lagrangian, and ask if we can extend the notion of an acoustic metric to all (or at least some wide class of) Lagrangian systems?

Indeed, suppose we have a single scalar field $\phi$ whose dynamics is governed by some generic Lagrangian $\mathcal{L}(\partial_\mu\phi, \phi)$, which is some arbitrary function of the field and its first derivatives (here we will follow the notation and ideas of [15]). In the general analysis that follows the previous irrotational and inviscid fluid system is included as a particular case; the dynamics of the scalar field $\phi$ is now much more general. We want to consider linearised fluctuations around some background

[13] Vorticity is automatically generated, for instance, whenever the background fluid is non-barotropic, and in particular when $\nabla\rho \times \nabla p \neq 0$. Furthermore, it has been argued in [343] that quantum back-reaction can also act as a source for vorticity.

[14] In references [141, 143, 139, 140, 145, 144, 138, 142] the author has attempted to argue that vorticity can be related to the concept of torsion in a general affine connexion. We disagree. Although deriving a wave equation in the presence of vorticity very definitely moves one beyond the realm of a simple Riemannian spacetime, adding torsion to the connexion is not sufficient to capture the relevant physics.

solution $\phi_0(t, \mathbf{x})$ of the equations of motion, and to this end we write

$$\phi(t, \mathbf{x}) = \phi_0(t, \mathbf{x}) + \epsilon\phi_1(t, \mathbf{x}) + \frac{\epsilon^2}{2}\phi_2(t, \mathbf{x}) + O(\epsilon^3). \tag{103}$$

Now use this to expand the Lagrangian around the classical solution $\phi_0(t, \mathbf{x})$:

$$\begin{aligned}
\mathcal{L}(\partial_\mu\phi, \phi) = \mathcal{L}(\partial_\mu\phi_0, \phi_0) &+ \epsilon\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu\phi_1 + \frac{\partial\mathcal{L}}{\partial\phi}\,\phi_1\right] \\
&+ \frac{\epsilon^2}{2}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu\phi_2 + \frac{\partial\mathcal{L}}{\partial\phi}\,\phi_2\right] \\
&+ \frac{\epsilon^2}{2}\left[\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\,\partial_\mu\phi_1\,\partial_\nu\phi_1 + 2\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial\phi}\,\partial_\mu\phi_1\,\phi_1\right. \\
&\qquad\left. + \frac{\partial^2\mathcal{L}}{\partial\phi\,\partial\phi}\,\phi_1\,\phi_1\right] + O(\epsilon^3).
\end{aligned} \tag{104}$$

It is particularly useful to consider the action

$$S[\phi] = \int \mathrm{d}^{d+1}x\ \mathcal{L}(\partial_\mu\phi, \phi), \tag{105}$$

since doing so allows us to integrate by parts. (Note that the Lagrangian $\mathcal{L}$ is taken to be a scalar density, not a true scalar.) We can now use the Euler–Lagrange equations for the background field

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0, \tag{106}$$

to discard the linear terms (remember we are linearizing around a solution of the equations of motion) and so we get

$$\begin{aligned}
S[\phi] = S[\phi_0] + \frac{\epsilon^2}{2}\int \mathrm{d}^{d+1}x &\left[\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}\,\partial_\mu\phi_1\,\partial_\nu\phi_1\right. \\
&\left. + \left(\frac{\partial^2\mathcal{L}}{\partial\phi\,\partial\phi} - \partial_\mu\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial\phi}\right\}\right)\,\phi_1\,\phi_1\right] + O(\epsilon^3).
\end{aligned} \tag{107}$$

Having set things up this way, the equation of motion for the linearised fluctuation is now easily read off as

$$\partial_\mu\left(\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}\partial_\nu\phi_1\right) - \left(\frac{\partial^2\mathcal{L}}{\partial\phi\,\partial\phi} - \partial_\mu\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial\phi}\right\}\right)\phi_1 = 0. \tag{108}$$

This is a second-order differential equation with position-dependent coefficients (these coefficients all being implicit functions of the background field $\phi_0$).

This can be given a nice clean geometrical interpretation in terms of a d'Alembertian wave equation – provided we *define* the effective spacetime metric by

$$\sqrt{-g}\; g^{\mu\nu} \equiv f^{\mu\nu} \equiv \left.\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}\right|_{\phi_0}. \tag{109}$$

Note that this is another example of a situation in which calculating the inverse metric density is easier than calculating the metric itself.

Suppressing the $\phi_0$ except when necessary for clarity, this implies [in (d+1) dimensions, d space dimensions plus 1 time dimension]

$$(-g)^{(d-1)/2} = -\det\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}. \tag{110}$$

Therefore

$$g^{\mu\nu}(\phi_0) = \left(-\det\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}\right)^{-1/(d-1)}\Bigg|_{\phi_0} \left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}\Bigg|_{\phi_0}. \tag{111}$$

And, taking the inverse

$$g_{\mu\nu}(\phi_0) = \left(-\det\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}\right)^{1/(d-1)}\Bigg|_{\phi_0} \left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}^{-1}\Bigg|_{\phi_0}. \tag{112}$$

We can now write the equation of motion for the linearised fluctuations in the geometrical form

$$[\Delta(g(\phi_0)) - V(\phi_0)]\,\phi_1 = 0, \tag{113}$$

where $\Delta$ is the d'Alembertian operator associated with the effective metric $g(\phi_0)$, and $V(\phi_0)$ is the background-field-dependent (and so in general position-dependent) "mass term":

$$V(\phi_0) = \frac{1}{\sqrt{-g}}\left(\frac{\partial^2\mathcal{L}}{\partial\phi\,\partial\phi} - \partial_\mu\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial\phi}\right\}\right) \tag{114}$$

$$= \left(-\det\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}\right)^{-1/(d-1)} \times \left(\frac{\partial^2\mathcal{L}}{\partial\phi\,\partial\phi} - \partial_\mu\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial\phi}\right\}\right). \tag{115}$$

Thus $V(\phi_0)$ is a true scalar (not a density). Note that the differential Equation (113) is automatically formally self-adjoint (with respect to the measure $\sqrt{-g}\ \mathrm{d}^{d+1}x$).

It is important to realise just how general the result is (and where the limitations are): It works for *any* Lagrangian depending only on a single scalar field and its first derivatives. The linearised PDE will be *hyperbolic* (and so the linearised equations will have wave-like solutions) if and only if the effective metric $g_{\mu\nu}$ has Lorentzian signature $\pm[-,+^d]$. Observe that if the Lagrangian contains nontrivial second derivatives you should not be too surprised to see terms beyond the d'Alembertian showing up in the linearised equations of motion.

As a specific example of the appearance of effective metrics due to Lagrangian dynamics we reiterate the fact that inviscid irrotational barotropic hydrodynamics naturally falls into this scheme (which is why, with hindsight, the derivation of the acoustic metric presented earlier in this review was so relatively straightforward). In inviscid irrotational barotropic hydrodynamics the lack of viscosity (dissipation) guarantees the existence of a Lagrangian; which a priori could depend on several fields. Since the flow is irrotational $\mathbf{v} = -\boldsymbol{\nabla}\phi$ is a function only of the velocity potential, and the Lagrangian is a function only of this potential and the density. Finally the equation of state can be used to eliminate the density leading to a Lagrangian that is a function only of the single field $\phi$ and its derivatives. [15]

## 2.9 Going further

The class of analogue models based on fluid mechanics is now quite large and the literature is extensive. Most of the relevant discussion will be deferred until subsequent sections, so for the time being we shall just mention reasonably immediate generalizations such as:

- Working with specific fluids.
  - Superfluids.
  - Bose–Einstein condensates.
- Abstract generalizations.
  - Normal modes in generic systems.
  - Multiple signal speeds.

We next turn to a brief historical discussion, seeking to place the work of the last decade into its proper historical perspective.

# 3 History and Motivation

From the point of view of the general relativity community the history of analogue models can reasonably neatly (but superficially) be divided into a "historical" period (essentially pre-1981) and a "modern" period (essentially post-1981).

## 3.1 Modern period

### 3.1.1 The years 1981–1999

The key event in the "modern" period (though largely unrecognised at the time) was the 1981 publication of Unruh's paper "Experimental black hole evaporation" [376], which implemented an analogue model based on fluid flow, and then used the power of that analogy to probe fundamental issues regarding Hawking radiation from "real" general relativity black holes.

We believe that Unruh's 1981 article represents the first observation of the now widely established fact that Hawking radiation has nothing to do with general relativity *per se*, but that Hawking radiation is instead a fundamental curved-space quantum field theory phenomenon that occurs whenever a horizon is present in an effective geometry.[15] Though Unruh's 1981 paper was seminal in this regard, it lay largely unnoticed for many years.

Some 10 years later Jacobson's article "Black-hole evaporation and ultrashort distances" [185] used Unruh's analogy to build a physical model for the "trans-Planckian modes" believed to be relevant to the Hawking radiation process. Progress then sped up with the relatively rapid appearance of [186] and [377, 378]. (This period also saw the independent rediscovery of the fluid analogue model by one of the present authors [387], and the first explicit consideration of superfluids in this regard [84].)

The later 1990's then saw continued work by Jacobson and his group [187, 188, 88, 90, 198], with new and rather different contributions coming in the form of the solid state models considered by Reznik [319, 318].[16] This period also saw the introduction of the more general class of superfluid models considered by Volovik and his collaborators [402, 403, 213, 110, 407, 405, 406, 199, 409, 410], more precise formulations of the notions of horizon, ergosphere, and surface gravity in analogue models [389, 391], and discussions of the implications of analogue models regarding Bekenstein–Hawking entropy [390, 391]. Finally, analogue spacetimes based on *special* relativistic acoustics were considered in [33].

By the year 2000, articles on one or another aspect of analogue gravity were appearing at the rate of over 20 per year, and it becomes impractical to summarise more than a selection of them.

### 3.1.2 The year 2000

Key developments in 2000 were the introduction, by Garay and collaborators, of the use of Bose–Einstein condensates as a working fluid [136, 137], and the extension of those ideas by the present authors [14]. Further afield, the trans-Planckian problem also reared its head in the context of cosmological inflation, and analogue model ideas previously applied to Hawking radiation were reused in that context [205, 273].

That year also marked the appearance of a review article on superfluid analogues [413], more work on "near-horizon" physics [123], and the transference of the idea of analogue-inspired "multiple metric" theories into cosmology where they can be used as the basis for a precise definition

[15] We emphasise: To get Hawking radiation you need an effective geometry, a horizon, and a suitable quantum field theory on that geometry.

[16] Reference [172] is an attempt at connecting Hawking evaporation with the physics of collapsing bubbles. This was part of a more general programme aimed at connecting black hole thermodynamics with perfect fluid thermodynamics [173].

of what is meant by a VSL ("variable speed of light") cosmology [28]. Models based on nonlinear electrodynamics were investigated in [11], $^3$He – A based models were reconsidered in [193, 411], and "slow light" models in quantum dielectrics were considered in [235, 236, 231].

The most radical proposal to appear in 2000 was that of Laughlin *et al.* [76]. Based on taking a superfluid analogy rather literally they mooted an actual physical breakdown of general relativity at the horizon of a black hole [76].

Additionally, the workshop on "Analogue models of general relativity", held at CBPF (Rio de Janeiro) gathered some 20 international participants and greatly stimulated the field, leading ultimately to the publication of the book [284] in 2002.

#### 3.1.3 The year 2001

This year saw more applications of analogue-inspired ideas to cosmological inflation [107, 263, 262, 207, 275], to neutron star cores [66], and to the cosmological constant [414, 416].

Closer to the heart of the analogue programme were the development of a "normal mode" analysis in [15, 16, 398], the development of dielectric analogues in [342], speculations regarding the possibly emergent nature of Einstein gravity [20, 398], and further developments regarding the use of $^3$He – A [106] as an analogue for electromagnetism. Experimental proposals were considered in [19, 398, 331].

Vorticity was discussed in [307], and the use of BECs as a model for the breakdown of Lorentz invariance in [397]. Analogue models based on nonlinear electrodynamics were discussed in [101]. Acoustics in an irrotational vortex were investigated in [120].

The excitation spectrum in superfluids, specifically the fermion zero modes, were investigated in [412, 182], while the relationship between rotational friction in superfluids and super-radiance in rotating spacetimes was discussed in [57]. More work on "slow light" appeared in [48]. The possible role of Lorentz violations at ultra-high energy was emphasised in [190].

#### 3.1.4 The year 2002

"What did we learn from studying acoustic black holes?" was the title and theme of Parentani's article in 2002 [300], while Schützhold and Unruh developed a rather different fluid-based analogy based on gravity waves in shallow water [344, 345]. Super-radiance was investigated in [27], while the propagation of phonons and quasiparticles was discussed in [122, 121]. More work on "slow light" appeared in [124, 311].

The stability of an acoustic white hole was investigated in [234], while further developments regarding analogue models based on nonlinear electrodynamics were presented by Novello and collaborators in [102, 103, 282, 278, 126]. Analogue spacetimes relevant to braneworld cosmologies were considered in [12].

Though analogue models lead naturally to the idea of high-energy violations of Lorentz invariance, it must be stressed that definite observational evidence for violations of Lorentz invariance is lacking – in fact there are rather strong constraints on how strong any possible Lorentz violating effect might be [195, 194].

#### 3.1.5 The year 2003

That year saw further discussion of analogue-inspired models for black hole entropy and the cosmological constant [419, 421], and the development of analogue models for FRW geometries [115, 114, 17, 105, 242]. There were several further developments regarding the foundations of BEC-based models in [18, 116], while analogue spacetimes in superfluid neutron stars were further investigated in [67].

Effective geometry was the theme in [280], while applications of nonlinear electrodynamics (and its effective metric) to cosmology were presented in [281]. Super-radiance was further investigated in [26, 24], while the limitations of the "slow light" analogue were explained in [379]. Vachaspati argued for an analogy between phase boundaries and acoustic horizons in [381]. Emergent relativity was again addressed in [227].

The review article by Burgess [53], emphasised the role of general relativity as an effective field theory – the *sine qua non* for any attempt at interpreting general relativity as an emergent theory. The lecture notes by Jacobson [191] give a nice introduction to Hawking radiation and its connection to analogue spacetimes.

#### 3.1.6 The year 2004

The year 2004 saw the appearance of some 30 articles on (or closely related to) analogue models. Effective geometries in astrophysics were discussed by Perez Bergliaffa [306], while the physical realizability of acoustic Hawking radiation was addressed in [95, 382]. More cosmological issues were raised in [382, 424], while a specifically astrophysical use of the acoustic analogy was invoked in [96, 97, 98].

BEC-based horizons were again considered in [149, 148], while backreaction effects were the focus of attention in [10, 9, 208]. More issues relating to the simulation of FRW cosmologies were raised in [118, 119].

Unruh and Schützhold discussed the universality of the Hawking effect [380], and a new proposal for possibly detecting Hawking radiation in a electromagnetic wave guide [347]. The causal structure of analogue spacetimes was considered in [13], while quasinormal modes attracted attention in [31, 237, 64, 269]. Two dimensional analogue models were considered in [55].

There were attempts at modelling the Kerr geometry [401], and generic "rotating" spacetimes [77], a proposal for using analogue models to generate massive phonon modes in BECs [400], and an extension of the usual formalism for representing weak-field gravitational lensing in terms of an analogue refractive index [38].

Finally we mention the development of yet more strong observational bounds on possible ultra high energy Lorentz violation [196, 197].

#### 3.1.7 The year 2005

The first few months of 2005 have seen continued and vigourous activity on the analogue model front.

More studies of the super-resonance phenomenon have appeared [25, 113, 209, 354], and a mini-survey was presented in [63]. Quasinormal modes have again received attention in [78], while the Magnus force is reanalysed in terms of the acoustic geometry in [432]. Singularities in the acoustic geometry are considered in [56], while back-reaction has received more attention in [343].

Interest in analogue models is intense and shows no signs of abating.

We shall in the next subsection focus more precisely on the early history of analogue models, and specifically those that seem to us to have had a direct historical connection with the sustained burst of work carried out in the last 15 years.

### 3.2 Historical Period

Of course the division into pre-1981 and post-1981 articles is at a deeper level somewhat deceptive. There have been several analogue models investigated over the years, with different aims, different levels of sophistication, and ultimately different levels of development. Armed with a good library and some hindsight it is possible to find interesting analogues in a number of places.[17]

### 3.2.1 Optics

Perhaps the first paper to seriously discuss analogue models and effective metric techniques was that of Gordon (yes, he of the Klein–Gordon equation) [151]. Note that Gordon seemed largely interested in trying to describe dielectric media by an "effective metric". That is: Gordon wanted to use a gravitational field to mimic a dielectric medium. What is now often referred to as the Gordon metric is the expression

$$[g_{\text{effective}}]_{\mu\nu} = \eta_{\mu\nu} + \left[1 - n^{-2}\right] V_\mu V_\nu, \tag{116}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric, $n$ is the refractive index, and $V_\mu$ is the 4-velocity of the medium.

After that, there was sporadic interest in effective metric techniques. One historically important contribution was one of the *problems* in the well-known book "The classical theory of fields" by Landau and Lifshitz [222]. See the end of chapter 10, paragraph 90, and the problem immediately thereafter: "Equations of electrodynamics in the presence of a gravitational field". Note that in contrast to Gordon, here the interest is in using dielectric media to mimic a gravitational field.

In France the idea was taken up by Pham Mau Quan [308], who showed that (under certain conditions) Maxwell's equations can be expressed directly in terms of the effective metric specified by the coefficients

$$[g_{\text{effective}}]_{\mu\nu} = g_{\mu\nu} + \left[1 - \frac{1}{\epsilon\mu}\right] V_\mu V_\nu, \tag{117}$$

where $g_{\mu\nu}$ is the ordinary spacetime metric, $\epsilon$ and $\mu$ are the permeability and permittivity, and $V_\mu$ is the 4-velocity of the medium. The trajectories of the electromagnetic rays are interpreted in this case as geodesics of null length of this new effective metric.

Three articles that directly used the dielectric analogy to analyse specific physics problems are those of Skrotskii [352], Balazs [8], and Winterberg [427]. The general formalism was more fully developed in articles such as those by Peblanski [304, 303], and good summary of this classical period can be found in the article by de Felice [100].

In summary and with the benefit of hindsight: An arbitrary gravitational field can always be represented as an equivalent optical medium, but subject to the somewhat unphysical restriction that

$$[\text{magnetic permitivity}] \propto [\text{electric permeability}]. \tag{118}$$

If an optical medium does not satisfy this constraint (with a position independent proportionality constant) then it is not completely equivalent to a gravitational field. For a position dependent proportionality constant complete equivalence can be established in the geometric optics limit, but for wave optics the equivalence is not complete.

### 3.2.2 Acoustics

There were several papers in the 1980's using an acoustic analogy to investigate the propagation of shockwaves in astrophysical situations, most notably those of Moncrief [268] and Matarrese [259,

[17] Indeed historically, though not of direct relevance to general relativity, analogue models played a key role in the development of electromagnetism – Maxwell's derivation of his equations for the electromagnetic field was guided by a rather complicated analogue model in terms of spinning vortices of aether. Of course, once you have the equations in hand you can treat them in their own right and forget the model that guided you – which is exactly what happened in this particular case.

260, 258]. In particular in Moncrief's work [268] the linear perturbations of a relativistic perfect fluid on an arbitrary general relativistic metric were studied, and it was shown that the wave equation for such perturbations can be expressed as a relativistic wave equation on some effective (acoustic) metric (which can admit acoustic horizons). In this sense [268] can be seen as a precursor to the later works on acoustic geometries and acoustic horizons.[18]

#### 3.2.3 Electro-mechanical analogy

The so-called "electro-mechanical analogy" has also had a long history within the engineering community. It is sometimes extended to obtain an "electro-mechanical-acoustic" analogy, or even an "electro-thermal" analogy. Unfortunately the issues of interest to the engineering community rarely resonate within the relativity community, and these engineering analogies (though powerful in their own right) have no immediate impact for our purposes.[19]

### 3.3 Motivation

The motivation for these investigations (both historical and current) is rather mixed. In modern language the reasons to investigate analogue models are:

- Partly to use condensed matter to gain insight into classical general relativity.
- Partly to use condensed matter to gain insight into curved-space quantum field theory.
- Partly to develop an observational window on curved-space quantum field theory.
- Partly to use classical general relativity to gain insight into condensed matter physics.
- Partly (much more tenuous) to gain insight into new and radically different ways of dealing with "quantum gravity".

### 3.4 Going further

To further complicate the history, there is large body of work for which analogue spacetime ideas provide part of the background gestalt, even if the specific connection may sometimes be somewhat tenuous. Among such articles we mention:

- Analogue-based "geometrical" interpretations of pseudo-momentum, Iordanskii forces, Magnus forces, and the acoustic Aharanov–Bohm effect [133, 365, 366, 367, 368, 408].
- An analogue-inspired interpretation of the Kerr spacetime [157].
- The use of analogies to clarify the Newtonian limit of general relativity [373], to provide heuristics for motivating interest in specific spacetimes [320, 395], and to discuss a simple interpretation of the notion of a horizon [287].

[18] Indeed the results of Moncrief [268] are more general than those considered in the standard acoustic gravity papers that followed because they additionally permit a general relativistic curved background.

In spite of these impressive results, we consider these papers to be part of the "historical period" for the main reason that such works are philosophically orthogonal to modern developments in analogue gravity. Indeed the main motivation for such works was the study of perfect fluid dynamics in accretion flows around black holes, or in cosmological expansion, and in this context the description via an acoustic effective background was just a tool in order to derive results concerning conservation laws and stability. This is probably why even if temporally reference [268] pre-dates Unruh's 1981 paper by one year, and [259, 260, 258] post-date Unruh's 1981 paper by a few years, there seems to have not been any cross-connection.

[19] A recent attempt at connecting the electro-mechanical analogy back to relativity can be found in [434].

- Discrete [359] and non-commutative [81] spacetimes partially influenced and flavoured by analogue ideas.
- Analogue-based hints on how to implement “double special relativity” (DSR) [215, 216, 217, 370], and a cautionary analysis of why this might be difficult [346].
- Possible black-hole phase transitions placed in an analogue context [364].
- Cosmological structure formation viewed as noise amplification [351].
- Modified inflationary scenarios [81, 83].
- Discussions of unusual topology, “acoustic wormholes”, and unusual temporal structure [270, 272, 313, 357, 358, 433].
- Analogue models based on plasmon physics [357, 358].
- Abstract quantum field theoretic considerations of the Unruh effect [428].
- Numerous suggestions regarding possible trans-Planckian physics [7, 29, 69, 74, 75, 171, 262, 322, 371].
- Numerous suggestions regarding a minimum length in quantum gravity [32, 35, 49, 85, 135, 175, 176, 215, 216, 217, 243, 245, 244, 355].
- Standard quantum field theory physics reformulated in the light of analogue models [4, 5, 117, 127, 238, 239, 240, 241, 248, 285, 286, 295, 294, 296, 301, 314, 428].
- Standard general relativity supplemented with analogue viewpoints and insights [212, 225, 248].
- The discussion of, and argument for, a possible reassessment of fundamental features of quantum physics and general relativity [6, 152, 206, 226, 241, 297, 328, 335].
- Non-standard viewpoints on quantum physics and general relativity [93, 174, 290, 324, 323, 336, 337, 338, 339].
- Soliton physics [302], defect physics [246], and the Fizeau effect [271], presented with an analogue flavour.
- Analogue-inspired models of black hole accretion [315, 316].
- Cosmological horizons from an analogue spacetime perspective [146].
- Analogue-inspired insights into renormalization group flow [60].
- An analysis of “wave catastrophes” inspired by analogue models [210].
- Improved numerical techniques for handling wave equations [426], and analytic techniques for handling wave tails [37], partially based on analogue ideas.

From the above the reader can easily appreciate the broad interest in, and wide applicability of, analogue spacetime models.

There is not much more that we can usefully say here. We have doubtless missed some articles of historical importance, but with a good library or a fast internet connection the reader will be in as good a position as we are to find any additional historical articles.

# 4 A Catalogue of Models

In this chapter, we will attempt to categorise the very many analogue models researchers have investigated. Perhaps the most basic subdivision is into classical models and quantum models, but even then many other levels of refinement are possible. Consider for instance the following list:

- Classical models:
  - Classical sound.
  - Water waves (gravity waves).
  - Classical refractive index.
  - Normal modes.
- Quantum models:
  - Bose–Einstein condensates (BECs).
  - The Heliocentric universe.
    (Helium as an exemplar for just about anything.)
  - Slow light.

We will now provide a few words on each of these topics.

## 4.1 Classical models

### 4.1.1 Classical sound

Sound in a moving fluid has already been extensively discussed in Section 2, and we will not repeat such discussion here. In contrast, sound in a solid exhibits its own distinct and interesting features, notably in the existence of a generalization of the normal notion of birefringence – longitudinal modes travel at a different speed (typically faster) than do transverse modes. This may be viewed as an example of an analogue model which breaks the "light cone" into two at the classical level; as such this model is not particularly useful if one is trying to simulate special relativistic kinematics with its universal speed of light, though it may be used to gain insight into yet another way of "breaking" Lorentz invariance.

### 4.1.2 Shallow water waves (gravity waves)

A wonderful example of the occurrence of an effective metric in nature is that provided by gravity waves in a shallow basin filled with liquid [345] (see Figure 7).[20] If one neglects the viscosity and considers an irrotational flow, $\mathbf{v} = \boldsymbol{\nabla}\phi$, one can write Bernoulli's equation in the presence of Earth's gravity as

$$\partial_t \phi + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 = -\frac{p}{\rho} - gz - V_{\|}. \tag{119}$$

Here $\rho$ is the density of the fluid, $p$ its pressure, $g$ the gravitational acceleration and $V_{\|}$ a potential associated with some external force necessary to establish an horizontal flow in the fluid.

[20] Of course we now mean "gravity wave" in the traditional fluid mechanics sense of a water wave whose restoring force is given by ordinary Newtonian gravity. Waves in the fabric of spacetime are more properly called "gravitational waves", though this usage seems to be in decline within the general relativity community. Be very careful in any situation where there is even a possibility of confusing the two concepts.

We denote that flow by $\mathbf{v}_{\mathrm{B}}^{\parallel}$. We must also impose the boundary conditions that the pressure at the surface, and the vertical velocity at the bottom, both vanish: $p(z = h_{\mathrm{B}}) = 0$, $v_{\perp}(z = 0) = 0$.

Once a horizontal background flow is established, one can see that the perturbations of the velocity potential satisfy

$$\partial_t \, \delta\phi + \mathbf{v}_{\mathrm{B}}^{\parallel} \cdot \boldsymbol{\nabla}_{\parallel} \delta\phi = -\frac{p}{\rho}. \tag{120}$$

If we now expand this perturbation potential in a Taylor series

$$\delta\phi = \sum_{n=0}^{\infty} \frac{z^n}{n!} \, \delta\phi_n(x, y), \tag{121}$$

it is not difficult to prove [345] that surface waves with long wavelengths (long compared with the depth of the basin, $\lambda \gg h_{\mathrm{B}}$), can be described to a good approximation by $\delta\phi_0(x, y)$ and that this field "sees" an effective metric of the form

$$\mathrm{d}s^2 = \frac{1}{c^2} \left[ -(c^2 - v_{\mathrm{B}}^{\parallel 2}) \, \mathrm{d}t^2 - 2\mathbf{v}_{\mathrm{B}}^{\parallel} \cdot \mathbf{dx} \, \mathrm{d}t + \mathbf{dx} \cdot \mathbf{dx} \right], \tag{122}$$

where $c \equiv \sqrt{g h_{\mathrm{B}}}$. The link between small variations of the potential field and small variations of the position of the surface is provided by the following equation

$$\delta v_{\perp} = -h_{\mathrm{B}} \, \nabla_{\parallel}^2 \delta\phi_0 = \partial_t \, \delta h + \mathbf{v}_{\mathrm{B}}^{\parallel} \cdot \boldsymbol{\nabla}_{\parallel} \delta h = \frac{\mathrm{d}}{\mathrm{d}t} \, \delta h. \tag{123}$$

The entire previous analysis can be generalised to the case in which the bottom of the basin is not flat, and the background flow not purely horizontal [345]. Therefore, one can create different effective metrics for gravity waves in a shallow fluid basin by changing (from point to point) the background flow velocity and the depth, $h_{\mathrm{B}}(x, y)$.

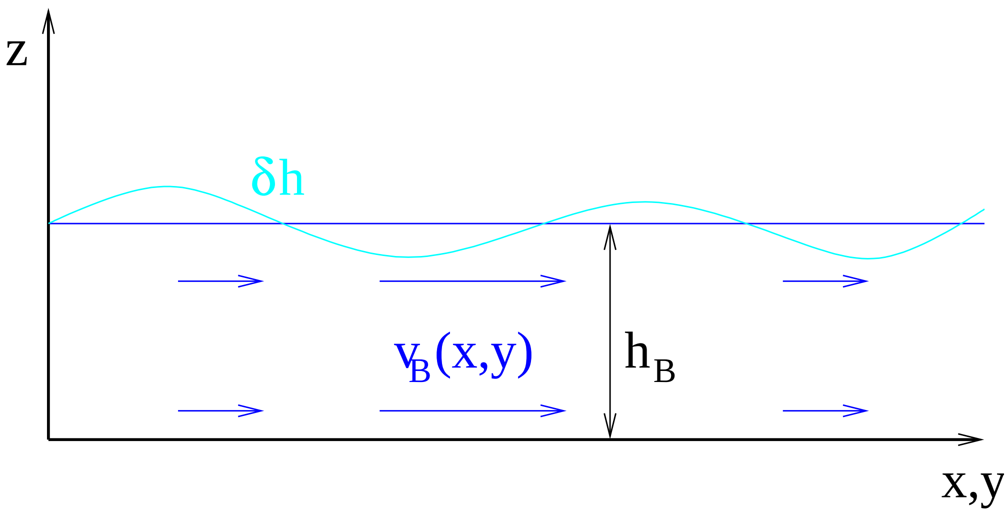


Figure 7: *Gravity waves in a shallow fluid basin with a background horizontal flow.*

The main advantage of this model is that the velocity of the surface waves can very easily be modified by changing the depth of the basin. This velocity can be made very slow, and consequently, the creation of ergoregions should be relatively easier than in other models. As described here, this model is completely classical given that the analogy requires long wavelengths and slow propagation speeds for the gravity waves. Although the latter feature is convenient for the practical realization of analogue horizons, it is a disadvantage in trying to detect analogue Hawking radiation as the relative temperature will necessarily be very low. (This is why, in order to have a possibility of experimentally observing Hawking evaporation and other quantum phenomena, one would need to use ultra cold quantum fluids.) However, the gravity wave analogue can certainly serve to investigate the classical phenomena of mode mixing that underlies the quantum processes.

#### 4.1.3 Classical refractive index

The macroscopic Maxwell equations inside a dielectric take the well-known form

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{E} + \partial_t \mathbf{B} = 0, \tag{124}$$

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 0, \quad \boldsymbol{\nabla} \times \mathbf{H} - \partial_t \mathbf{D} = 0, \tag{125}$$

with the constitutive relations $\mathbf{H} = \boldsymbol{\mu}^{-1} \cdot \mathbf{B}$ and $\mathbf{D} = \boldsymbol{\epsilon} \cdot \mathbf{E}$. Here, $\boldsymbol{\epsilon}$ is the $3 \times 3$ permittivity tensor and $\boldsymbol{\mu}$ the $3 \times 3$ permeability tensor of the medium. These equations can be written in a condensed way as

$$\partial_\alpha \left( Z^{\mu\alpha\nu\beta} \, F_{\nu\beta} \right) = 0 \tag{126}$$

where $F_{\nu\beta} = A_{[\nu,\beta]}$ is the electromagnetic tensor,

$$F_{0i} = -F_{i0} = -E_i, \quad F_{ij} = \varepsilon_{ijk} B^k, \tag{127}$$

and (assuming the medium is at rest) the non-vanishing components of the 4th rank tensor $Z$ are given by

$$Z^{0i0j} = -Z^{0ij0} = Z^{i0j0} = -Z^{i00j} = -\frac{1}{2}\epsilon^{ij}; \tag{128}$$

$$Z^{ijkl} = \frac{1}{2}\varepsilon^{ijm} \, \varepsilon^{kln} \, \mu^{-1}_{mn}; \tag{129}$$

supplemented by the conditions that $Z$ is antisymmetric on its first pair of indices and antisymmetric on its second pair of indices. Without significant loss of generality we can ask that $Z$ also be symmetric under pairwise interchange of the first pair of indices with the second pair – thus $Z$ exhibits most of the algebraic symmetries of the Riemann tensor, though this appears to merely be accidental, and not fundamental in any way.

If we compare this to the Lagrangian for electromagnetism in curved spacetime

$$\mathcal{L} = \sqrt{-g} \; g^{\mu\alpha} \; g^{\nu\beta} \; F_{\mu\nu} \; F_{\alpha\beta} \tag{130}$$

we see that in curved spacetime we can also write the electromagnetic equations of motion in the form (126) where now (for some constant $K$):

$$Z^{\mu\nu\alpha\beta} = K \, \sqrt{-g} \, \left\{ g^{\mu\alpha} \, g^{\nu\beta} - g^{\mu\beta} \, g^{\nu\alpha} \right\} \tag{131}$$

If we consider a static gravitational field we can always re-write it as a conformal factor multiplying an ultra-static metric

$$g_{\mu\nu} = \Omega^2 \, \{ -1 \oplus g_{ij} \} \tag{132}$$

then

$$Z^{0i0j} = -Z^{0ij0} = Z^{i0j0} = -Z^{i00j} = -K\,\sqrt{-g}\,g^{ij}; \tag{133}$$
$$Z^{ijkl} = K\,\sqrt{-g}\,\{g^{ik}\,g^{jl} - g^{il}\,g^{jk}\} \tag{134}$$

The fact that $Z$ is independent of the conformal factor $\Omega$ is simply the reflection of the well-known fact that the Maxwell equations are conformally invariant in (3+1) dimensions. Thus if we wish to have the analogy (between a static gravitational field and a dielectric medium at rest) hold *at the level of the wave equation* (physical optics) we must satisfy the two stringent constraints

$$K\,\sqrt{-g}\,g^{ij} = \frac{1}{2}\,\epsilon^{ij}; \tag{135}$$
$$K\,\sqrt{-g}\,\{g^{ik}\,g^{jl} - g^{il}\,g^{jk}\} = \frac{1}{2}\,\varepsilon^{ijm}\,\varepsilon^{kln}\,\mu^{-1}_{mn}. \tag{136}$$

The second of these constraints can be written as

$$K\,\sqrt{-g}\,\varepsilon_{ijm}\,\varepsilon_{kln}\,\{g^{ik}\,g^{jl}\} = \mu^{-1}_{mn}. \tag{137}$$

In view of the standard formula for $3\times 3$ determinants

$$\varepsilon_{ijm}\,\varepsilon_{kln}\,\{X^{ik}\,X^{jl}\} = 2\ \det X\ X^{-1}_{mn}, \tag{138}$$

this now implies

$$2K\,\frac{g_{ij}}{\sqrt{-g}} = \mu^{-1}_{ij}, \tag{139}$$

whence

$$\frac{1}{2K}\,\sqrt{-g}\,g^{ij} = \mu^{ij}. \tag{140}$$

Comparing this with

$$2K\,\sqrt{-g}\,g^{ij} = \epsilon^{ij}, \tag{141}$$

we now have:

$$\epsilon^{ij} = 4\,K^2\,\mu^{ij}; \tag{142}$$
$$g^{ij} = \frac{4\,K^2}{\det\boldsymbol{\epsilon}}\,\epsilon^{ij}; \tag{143}$$
$$g^{ij} = \frac{1}{4\,K^2\ \det\boldsymbol{\mu}}\,\mu^{ij}. \tag{144}$$

To rearrange this, introduce the matrix square root $[\boldsymbol{\mu}^{1/2}]^{ij}$, which always exists because $\boldsymbol{\mu}$ is real positive definite and symmetric. Then

$$g^{ij} = \left[\left\{\frac{\boldsymbol{\mu}^{1/2}\,\boldsymbol{\epsilon}\,\boldsymbol{\mu}^{1/2}}{\det(\boldsymbol{\mu}\,\boldsymbol{\epsilon})}\right\}^{1/2}\right]^{ij}. \tag{145}$$

Note that if you are given the static gravitational field (in the form $\Omega$, $g_{ij}$) one can always solve to find an equivalent analogue in terms of permittivity/permeability (albeit an analogue that satisfies the mildly unphysical constraint $\boldsymbol{\epsilon} \propto \boldsymbol{\mu}$).[21] On the other hand, if you are given permeability and

[21] The existence of this constraint has been independently re-derived several times in the literature. In contrast, other segments of the literature seem blithely unaware of this important restriction on just when permittivity and permeability are truly equivalent to an effective metric.

permittivity tensors $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$, then it is *only* for that subclass of media that satisfy $\boldsymbol{\epsilon} \propto \boldsymbol{\mu}$ that one can perfectly mimic *all* of the electromagnetic effects by an equivalent gravitational field. Of course this can be done provided one only considers wavelengths that are sufficiently long for the macroscopic description of the medium to be valid. In this respect it is interesting to note that the behaviour of the refractive medium at high frequencies has been used to introduce an effective cutoff for the modes involved in Hawking radiation [319]. We shall encounter this model later on when we shall consider the trans-Planckian problem for Hawking radiation.

**4.1.3.1 Eikonal approximation:** With a bit more work this discussion can be extended to a medium in motion, leading to an extension of the Gordon metric. Alternatively, one can agree to ask more limited questions by working at the level of geometrical optics (adopting the eikonal approximation), in which case there is no longer any restriction on the permeability and permittivity tensors. To see this construct the matrix

$$C^{\mu\nu} = Z^{\mu\alpha\nu\beta}\, k_\alpha\, k_\beta. \tag{146}$$

The dispersion relations for the propagation of photons (and therefore the sought for geometrical properties) can be obtained from the reduced determinant of $C$ (notice that the [full] determinant of $C$ is identically zero as $C^{\mu\nu} k_\nu = 0$; the reduced determinant is that associated with the three directions orthogonal to $k_\nu = 0$). By choosing the gauge $A_0 = 0$ one can see that this reduced determinant can be obtained from the determinant of the $3 \times 3$ sub-matrix $C^{ij}$. This determinant is

$$\det(C^{ij}) = \frac{1}{8} \det\left(-\omega^2 \epsilon^{ij} + \varepsilon^{ikm}\, \varepsilon^{jln}\, \mu^{-1}_{mn} k_k k_l\right). \tag{147}$$

or, after making some manipulations,

$$\det(C^{ij}) = \frac{1}{8} \det\left[-\omega^2 \epsilon^{ij} + (\det \boldsymbol{\mu})^{-1} (\mu^{ij} \mu^{kl} k_k k_l - \mu^{im} k_m \mu^{jl} k_l)\right]. \tag{148}$$

To simplify this, again introduce the matrix square roots $[\boldsymbol{\mu}^{1/2}]^{ij}$ and $[\boldsymbol{\mu}^{-1/2}]_{ij}$, which always exist because the relevant matrices are real positive definite and symmetric. Then define

$$\tilde{k}^i = [\boldsymbol{\mu}^{1/2}]^{ij}\, k_j \tag{149}$$

and

$$[\tilde{\boldsymbol{\epsilon}}]^{ij} = \det(\boldsymbol{\mu})\, [\boldsymbol{\mu}^{-1/2}\, \boldsymbol{\epsilon}\, \boldsymbol{\mu}^{-1/2}]_{ij} \tag{150}$$

so that

$$\det(C^{ij}) \propto \det\left\{-\omega^2\, [\tilde{\boldsymbol{\epsilon}}]^{ij} + \delta^{ij}\, [\delta_{mn}\, \tilde{k}^m\, \tilde{k}^n] - \tilde{k}^i\, \tilde{k}^j\right\}. \tag{151}$$

The behaviour of this dispersion relation now depends critically on the way that the eigenvalues of $\tilde{\boldsymbol{\epsilon}}$ are distributed.

**4.1.3.2 3 degenerate eigenvalues:** If all eigenvalues are degenerate then $\tilde{\boldsymbol{\epsilon}} = \tilde{\epsilon}\, \mathbf{I}$, implying $\boldsymbol{\epsilon} \propto \boldsymbol{\mu}$ but now with the possibility of a position dependent proportionality factor (in the case of physical optics the proportionality factor was constrained to be a position-independent constant). In this case we now easily evaluate

$$\boldsymbol{\epsilon} = \frac{\mathrm{tr}(\boldsymbol{\epsilon})}{\mathrm{tr}(\boldsymbol{\mu})}\, \boldsymbol{\mu} \qquad \text{and} \qquad \tilde{\epsilon} = \det \boldsymbol{\mu}\, \frac{\mathrm{tr}(\boldsymbol{\epsilon})}{\mathrm{tr}(\boldsymbol{\mu})}, \tag{152}$$

while

$$\det(C^{ij}) \propto \omega^2 \left\{\omega^2 - [\tilde{\epsilon}^{-1}\, \delta_{mn}\, \tilde{k}^m\, \tilde{k}^n]\right\}^2. \tag{153}$$

That is

$$\det(C^{ij}) \propto \omega^2 \left\{\omega^2 - [g^{ij}\, k_i\, k_j]\right\}^2, \tag{154}$$

with

$$g^{ij} = \frac{1}{\tilde{\epsilon}}\, [\boldsymbol{\mu}]^{ij} = \frac{\mathrm{tr}(\boldsymbol{\mu})\ [\boldsymbol{\mu}]^{ij}}{\mathrm{tr}(\boldsymbol{\epsilon})\ \det \boldsymbol{\mu}} = \frac{\mathrm{tr}(\boldsymbol{\epsilon})\ [\boldsymbol{\epsilon}]^{ij}}{\mathrm{tr}(\boldsymbol{\mu})\ \det \boldsymbol{\epsilon}}. \tag{155}$$

This last result is compatible with but more general than the result obtained under the more restrictive conditions of physical optics. In the situation where both permittivity and permeability are isotropic, ($\epsilon^{ij} = \epsilon\, \delta^{ij}$ and $\mu^{ij} = \mu\, \delta^{ij}$) this reduces to the perhaps more expected result

$$g^{ij} = \frac{\delta^{ij}}{\epsilon\, \mu}. \tag{156}$$

**4.1.3.3 2 degenerate eigenvalues:** If $\tilde{\boldsymbol{\epsilon}}$ has two distinct eigenvalues then the determinant $\det(C^{ij})$ factorises into a trivial factor of $\omega^2$ and two quadratics. Each quadratic corresponds to a distinct effective metric. This is the physical situation encountered in uni-axial crystals, where the *ordinary* and *extraordinary* rays each obey distinct quadratic dispersion relations [39]. From the point of view of analogue models this corresponds to a two-metric theory.

**4.1.3.4 3 distinct eigenvalues:** If $\tilde{\boldsymbol{\epsilon}}$ has three distinct eigenvalues then the determinant $\det(C^{ij})$ is the product of a trivial factor of $\omega^2$ and a *non-factorizable quartic*. This is the physical situation encountered in bi-axial crystals [39, 399], and it seems that no meaningful notion of effective Riemannian metric can be assigned to this case. (The use of Finsler geometries in this situation is an avenue that may be worth pursuing [184].)

**4.1.3.5 Abstract linear electrodynamics:** Hehl and co-workers have championed the idea of using the linear constitutive relations of electrodynamics as the primary quantities, and then treating the spacetime metric (even for flat space) as a *derived* concept. See [288, 165, 220, 166].

**4.1.3.6 Nonlinear electrodynamics:** In general, the permittivity and permeability tensors can be modified by applying strong electromagnetic fields (this produces an effectively non-linear electrodynamics). The entire previous discussion still applies if one considers the photon as the linear perturbation of the electromagnetic field over a background configuration

$$F_{\mu\nu} = F^{\mathrm{bg}}_{\mu\nu} + f^{\mathrm{ph}}_{\mu\nu}. \tag{157}$$

The background field $F^{\mathrm{bg}}_{\mu\nu}$ sets the value of $\epsilon^{ij}(F^{\mathrm{bg}})$, and $\mu^{ij}(F^{\mathrm{bg}})$. Equation (126) then becomes an equation for $f^{\mathrm{ph}}_{\mu\nu}$. This approach has been extensively investigated by Novello and co-workers [279, 283, 102, 282, 280, 281, 278, 126].

**4.1.3.7 Summary:** The propagation of photons in a dielectric medium characterised by $3 \times 3$ permeability and permittivity tensors constrained by $\boldsymbol{\epsilon} \propto \boldsymbol{\mu}$ is equivalent (at the level of geometric optics) to the propagation of photons in a curved spacetime manifold characterised by the ultrastatic metric (155), provided one only considers wavelengths that are sufficiently long for the macroscopic description of the medium to be valid. If in addition one takes a fluid dielectric, by controlling its flow one can generalise the Gordon metric and again reproduce metrics of the Painlevé–Gullstrand type, and therefore geometries with ergo-regions. If the proportionality constant relating $\boldsymbol{\epsilon} \propto \boldsymbol{\mu}$ is position independent, one can make the stronger statement (144) which holds true at the level of physical optics.

#### 4.1.4 Normal mode meta-models

We have already seen how linearizing the Euler–Lagrange equations for a single scalar field naturally leads to the notion of an effective spacetime metric. If more than one field is involved the situation becomes more complicated, in a manner similar to that of geometrical optics in uni-axial and bi-axial crystals. (This should, with hindsight, not be too surprising since electromagnetism, even in the presence of a medium, is definitely a Lagrangian system and definitely involves more than one single scalar field.) A normal mode analysis based on a general Lagrangian (many fields but still first order in derivatives of those fields) leads to a concept of *refringence*, or more specifically *multi-refringence*, a generalization of the birefringence of geometrical optics. To see how this comes about, consider a straightforward generalization of the one-field case.

We want to consider linearised fluctuations around some background solution of the equations of motion. As in the single-field case we write (here we will follow the notation and conventions of [16])

$$\phi^A(t,\vec{x}) = \phi_0^A(t,\vec{x}) + \epsilon\, \phi_1^A(t,\vec{x}) + \frac{\epsilon^2}{2}\, \phi_2^A(t,\vec{x}) + O(\epsilon^3). \tag{158}$$

Now use this to expand the Lagrangian

$$\begin{aligned}\mathcal{L}(\partial_\mu\phi^A,\phi^A) = {} & \mathcal{L}(\partial_\mu\phi_0^A,\phi_0^A) + \epsilon\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\,\partial_\mu\phi_1^A + \frac{\partial\mathcal{L}}{\partial\phi^A}\,\phi_1^A\right] \\ & + \frac{\epsilon^2}{2}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\,\partial_\mu\phi_2^A + \frac{\partial\mathcal{L}}{\partial\phi^A}\,\phi_2^A\right] \\ & + \frac{\epsilon^2}{2}\Bigg[\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi^A)\,\partial(\partial_\nu\phi^B)}\,\partial_\mu\phi_1^A\,\partial_\nu\phi_1^B \\ & \qquad +2\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi^A)\,\partial\phi^B}\,\partial_\mu\phi_1^A\,\phi_1^B + \frac{\partial^2\mathcal{L}}{\partial\phi^A\,\partial\phi^B}\,\phi_1^A\,\phi_1^B\Bigg] \\ & \qquad +O(\epsilon^3).\end{aligned} \tag{159}$$

Consider the action

$$S[\phi^A] = \int \mathrm{d}^{d+1}x\; \mathcal{L}(\partial_\mu\phi^A,\phi^A). \tag{160}$$

Doing so allows us to integrate by parts. As in the single-field case we can use the Euler–Lagrange equations to discard the linear terms (since we are linearizing around a solution of the equations of motion) and so get

$$\begin{aligned} S[\phi^A] = {} & S[\phi_0^A] \\ & + \frac{\epsilon^2}{2}\int \mathrm{d}^{d+1}x\Bigg[\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi^A)\,\partial(\partial_\nu\phi^B)}\right\}\,\partial_\mu\phi_1^A\,\partial_\nu\phi_1^B \\ & \qquad +2\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi^A)\,\partial\phi^B}\right\}\,\partial_\mu\phi_1^A\,\phi_1^B + \left\{\frac{\partial^2\mathcal{L}}{\partial\phi^A\,\partial\phi^B}\right\}\,\phi_1^A\,\phi_1^B\Bigg] \\ & + O(\epsilon^3).\end{aligned} \tag{161}$$

Because the fields now carry indices $(AB)$ we cannot cast the action into quite as simple a form as was possible in the single-field case. The equation of motion for the linearised fluctuations are now read off as

$$\partial_\mu \left( \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)} \right\} \partial_\nu \phi_1^B \right) + \partial_\mu \left( \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial\phi^B}\, \phi_1^B \right)$$
$$-\partial_\mu \phi_1^B\, \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^B)\, \partial \phi^A} - \left( \frac{\partial^2 \mathcal{L}}{\partial\phi^A\, \partial\phi^B} \right) \phi_1^B = 0. \tag{162}$$

This is a linear second-order *system* of partial differential equations with position-dependent coefficients. This system of PDEs is automatically self-adjoint (with respect to the trivial "flat" measure $\mathrm{d}^{d+1}x$).

To simplify the notation we introduce a number of definitions. First

$$f^{\mu\nu}{}_{AB} \equiv \frac{1}{2} \left( \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)} + \frac{\partial^2 \mathcal{L}}{\partial(\partial_\nu \phi^A)\, \partial(\partial_\mu \phi^B)} \right). \tag{163}$$

This quantity is independently symmetric under interchange of $\mu$, $\nu$ and $A$, $B$. We will want to interpret this as a generalization of the "densitised metric", $f^{\mu\nu}$, but the interpretation is not as straightforward as for the single-field case. Next, define

$$\Gamma^\mu{}_{AB} \equiv + \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial\phi^B} - \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^B)\, \partial\phi^A}$$
$$+ \frac{1}{2} \partial_\nu \left( \frac{\partial^2 \mathcal{L}}{\partial(\partial_\nu \phi^A)\, \partial(\partial_\mu \phi^B)} - \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)} \right). \tag{164}$$

This quantity is anti-symmetric in $A$, $B$. One might want to interpret this as some sort of "spin connexion", or possibly as some generalization of the notion of "Dirac matrices". Finally, define

$$K_{AB} = -\frac{\partial^2 \mathcal{L}}{\partial\phi^A\, \partial\phi^B} + \frac{1}{2} \partial_\mu \left( \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial\phi^B} \right) + \frac{1}{2} \partial_\mu \left( \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^B)\, \partial\phi^A} \right). \tag{165}$$

This quantity is by construction symmetric in $(AB)$. We will want to interpret this as some sort of "potential" or "mass matrix". Then the crucial point for the following discussion is to realise that Equation (162) can be written in the compact form

$$\partial_\mu \left( f^{\mu\nu}{}_{AB}\, \partial_\nu \phi_1^B \right) + \frac{1}{2} \left[ \Gamma^\mu_{AB}\, \partial_\mu \phi_1^B + \partial_\mu (\Gamma^\mu_{AB}\, \phi_1^B) \right] + K_{AB}\, \phi_1^B = 0. \tag{166}$$

Now it is more transparent that this is a formally self-adjoint second-order linear *system* of PDEs. Similar considerations can be applied to the linearization of any hyperbolic system of second-order PDEs.

Consider an eikonal approximation for an arbitrary direction in field space, that is, take

$$\phi^A(x) = \epsilon^A(x)\, \exp[-i\varphi(x)], \tag{167}$$

with $\epsilon^A(x)$ a slowly varying amplitude, and $\varphi(x)$ a rapidly varying phase. In this eikonal approximation (where we neglect gradients in the amplitude, and gradients in the coefficients of the PDEs, retaining only the gradients of the phase) the linearised system of PDEs (166) becomes

$$\{ f^{\mu\nu}{}_{AB}\, \partial_\mu \varphi(x)\, \partial_\nu \varphi(x) + \Gamma^\mu{}_{AB}\, \partial_\mu \varphi(x) + K_{AB} \}\, \epsilon^B(x) = 0. \tag{168}$$

This has a nontrivial solution if and only if $\epsilon^A(x)$ is a null eigenvector of the matrix

$$f^{\mu\nu}{}_{AB}\, k_\mu\, k_\nu + \Gamma^\mu{}_{AB}\, k_\mu + K_{AB}, \tag{169}$$

where $k_\mu = \partial_\mu \varphi(x)$. Now, the condition for such a null eigenvector to exist is that

$$F(x,k) \equiv \det\left\{ f^{\mu\nu}{}_{AB}\; k_\mu\; k_\nu + \Gamma^\mu{}_{AB}\; k_\mu + K_{AB} \right\} = 0, \tag{170}$$

with the determinant to be taken on the field space indices $AB$. This is the natural generalization to the current situation of the Fresnel equation of bi-refringent optics [39, 224]. Following the analogy with the situation in electrodynamics (either nonlinear electrodynamics, or more prosaically propagation in a bi-refringent crystal), the null eigenvector $\epsilon^A(x)$ would correspond to a specific "polarization". The Fresnel equation then describes how different polarizations can propagate at different velocities (or in more geometrical language, can see different metric structures). In particle physics language this determinant condition $F(x,k) = 0$ is the natural generalization of the "mass shell" constraint. Indeed it is useful to define the mass shell as a subset of the cotangent space by

$$\mathcal{F}(x) \equiv \left\{ k_\mu \;\middle|\; F(x,k) = 0 \right\}. \tag{171}$$

In more mathematical language we are looking at the null space of the determinant of the "symbol" of the system of PDEs. By investigating $F(x,k)$ one can recover part (not all) of the information encoded in the matrices $f^{\mu\nu}{}_{AB}$, $\Gamma^\mu{}_{AB}$, and $K_{AB}$, or equivalently in the "generalised Fresnel equation" (170). (Note that for the determinant equation to be useful it should be non-vacuous; in particular one should carefully eliminate all gauge and spurious degrees of freedom before constructing this "generalised Fresnel equation", since otherwise the determinant will be identically zero.) We now want to make this analogy with optics more precise, by carefully considering the notion of characteristics and characteristic surfaces. We will see how to extract from the the high-frequency high-momentum regime described by the eikonal approximation all the information concerning the causal structure of the theory.

One of the key structures that a Lorentzian spacetime metric provides is the notion of causal relationships. This suggests that it may be profitable to try to work backwards from the causal structure to determine a Lorentzian metric. Now the causal structure implicit in the system of second-order PDEs given in Equation (166) is described in terms of the characteristic surfaces, and it is for this reason that we now focus on characteristics as a way of encoding causal structure, and as a surrogate for some notion of Lorentzian metric. Note that via the Hadamard theory of surfaces of discontinuity the characteristics can be identified with the infinite-momentum limit of the eikonal approximation [155]. That is, when extracting the characteristic surfaces we neglect subdominant terms in the generalised Fresnel equation and focus only on the leading term in the symbol ($f^{\mu\nu}{}_{AB}$). In particle physics language going to the infinite-momentum limit puts us on the light cone instead of the mass shell; and it is the light cone that is more useful in determining causal structure. The "normal cone" at some specified point $x$, consisting of the locus of normals to the characteristic surfaces, is defined by

$$\mathcal{N}(x) \equiv \left\{ k_\mu \;\middle|\; \det\left( f^{\mu\nu}{}_{AB}\;\; k_\mu\; k_\mu \right) = 0 \right\}. \tag{172}$$

As was the case for the Fresnel Equation (170), the determinant is to be taken on the field indices $AB$. (Remember to eliminate spurious and gauge degrees of freedom so that this determinant is not identically zero.) We emphasise that the algebraic equation defining the normal cone is the leading term in the Fresnel equation encountered in discussing the eikonal approximation. If there are $N$ fields in total then this "normal cone" will generically consist of $N$ nested sheets each with the topology (not necessarily the geometry) of a cone. Often several of these cones will coincide, which is not particularly troublesome, but unfortunately it is also common for some of these cones to be degenerate, which is more problematic.

It is convenient to define a function $Q(x,k)$ on the co-tangent bundle

$$Q(x,k) \equiv \det\left(f^{\mu\nu}{}_{AB}(x)\; k_\mu\; k_\mu\right). \tag{173}$$

The function $Q(x,k)$ defines a completely-symmetric spacetime tensor (actually, a tensor density) with $2N$ indices

$$Q(x,k) = Q^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_N\nu_N}(x)\;\; k_{\mu_1}\; k_{\nu_1}\; k_{\mu_2}\; k_{\nu_2}\cdots\; k_{\mu_N}\; k_{\nu_N}. \tag{174}$$

(Remember that $f^{\mu\nu}{}_{AB}$ is symmetric in both $\mu\nu$ and $AB$ independently.) Explicitly, using the expansion of the determinant in terms of completely antisymmetric field-space Levi–Civita tensors

$$Q^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_N\nu_N} = \frac{1}{N!}\,\epsilon^{A_1A_2\cdots A_N}\;\epsilon^{B_1B_2\cdots B_N}\; f^{\mu_1\nu_1}{}_{A_1B_1}\; f^{\mu_2\nu_2}{}_{A_2B_2}\;\cdots f^{\mu_N\nu_N}{}_{A_NB_N}. \tag{175}$$

In terms of this $Q(x,k)$ function the normal cone is

$$\mathcal{N}(x) \equiv \left\{ k_\mu \;\middle|\; Q(x,k)=0 \right\}. \tag{176}$$

In contrast, the "Monge cone" (*aka* "ray cone", *aka* "characteristic cone", *aka* "null cone") is the envelope of the set of characteristic surfaces through the point $x$. Thus the "Monge cone" is dual to the "normal cone", its explicit construction is given by (Courant and Hilbert [92, volume 2, page 583]):

$$\mathcal{M}(x) = \left\{ t^\mu = \frac{\partial Q(x,k)}{\partial k_\mu} \;\middle|\; k_\mu \in \mathcal{N}(x) \right\}. \tag{177}$$

The structure of the normal and Monge cones encode all the information related with the causal propagation of signals associated with the system of PDEs. We will now see how to relate this causal structure with the existence of effective spacetime metrics, from the experimentally favoured single-metric theory compatible with the Einstein equivalence principle to the most complicated case of pseudo-Finsler geometries [184].

- Suppose that $f^{\mu\nu}{}_{AB}$ factorises

  $$f^{\mu\nu}{}_{AB} = h_{AB}\; f^{\mu\nu}. \tag{178}$$

  Then

  $$Q(x,k) = \det(h_{AB})\;[f^{\mu\nu}\; k_\mu\; k_\nu]^N \tag{179}$$

  The Monge cones and normal cones are then true geometrical cones (with the $N$ sheets lying directly on top of one another). The normal modes all see the same spacetime metric, defined up to an unspecified conformal factor by $g^{\mu\nu} \propto f^{\mu\nu}$. This situation is the most interesting from the point of view of general relativity. Physically it corresponds to a single-metric theory, and mathematically it corresponds to a strict algebraic condition on the $f^{\mu\nu}{}_{AB}$.

- The next most useful situation corresponds to the commutativity condition:

  $$f^{\mu\nu}{}_{AB}\; f^{\alpha\beta}{}_{BC} = f^{\alpha\beta}{}_{AB}\; f^{\mu\nu}{}_{BC}; \qquad \text{that is} \qquad [\,\boldsymbol{f}^{\mu\nu}, \boldsymbol{f}^{\alpha\beta}\,] = 0. \tag{180}$$

  If this algebraic condition is satisfied, then for all spacetime indices $\mu\nu$ and $\alpha\beta$ the $f^{\mu\nu}{}_{AB}$ can be simultaneously diagonalised in field space leading to

  $$\bar{f}^{\mu\nu}{}_{AB} = \mathrm{diag}\{\bar{f}^{\mu\nu}_1, \bar{f}^{\mu\nu}_2, \bar{f}^{\mu\nu}_3, \ldots, \bar{f}^{\mu\nu}_N\} \tag{181}$$

  and then

  $$Q(x,k) = \prod_{A=1}^{N}\,[\bar{f}^{\mu\nu}_A\; k_\mu\; k_\nu]. \tag{182}$$

  This case corresponds to an $N$-metric theory, where up to an unspecified conformal factor $g^{\mu\nu}_A \propto \bar{f}^{\mu\nu}_A$. This is the natural generalization of the two metric situation in bi-axial crystals.

- If $f^{\mu\nu}{}_{AB}$ is completely general, satisfying no special algebraic condition, then $Q(x,k)$ does not factorise and is in general a polynomial of degree $2N$ in the wave vector $k_\mu$. This is the natural generalization of the situation in bi-axial crystals. (And for any deeper analysis of this situation one will almost certainly need to adopt pseudo-Finsler techniques [184].)

The message to be extracted from this rather formal discussion is that effective metrics are rather general and mathematically robust objects that can arise in quite abstract settings – in the abstract setting discussed here it is the algebraic properties of the object $f^{\mu\nu}{}_{AB}$ that eventually leads to mono-metricity, multi-metricity, or worse. The current abstract discussion also serves to illustrate, yet again,

1. that there is a significant difference between the levels of physical normal modes (wave equations), and geometrical normal modes (dispersion relations), and
2. that the densitised inverse metric is in many ways more fundamental than the metric itself.

## 4.2 Quantum models

### 4.2.1 Bose–Einstein condensates

We have seen that one of the main aims of research in analogue models of gravity is the possibility of simulating semiclassical gravity phenomena, such as the Hawking radiation effect or cosmological particle production. In this sense systems characterised by a high degree of quantum coherence, very cold temperatures, and low speeds of sound offer the best test field. Hence it is not surprising that in recent years Bose–Einstein condensates (BECs) have become the subject of extensive study as possible analogue models of general relativity [136, 137, 16, 19, 18, 115, 114].

Let us start by very briefly reviewing the derivation of the acoustic metric for a BEC system, and show that the equations for the phonons of the condensate closely mimic the dynamics of a scalar field in a curved spacetime. In the dilute gas approximation, one can describe a Bose gas through a quantum field $\widehat{\Psi}$ satisfying

$$\mathrm{i}\hbar\,\frac{\partial}{\partial t}\widehat{\Psi} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\mathbf{x}) + \kappa(a)\;\widehat{\Psi}^\dagger\widehat{\Psi}\right)\widehat{\Psi}. \tag{183}$$

Here $\kappa$ parameterises the strength of the interactions between the different bosons in the gas. It can be re-expressed in terms of the scattering length as

$$\kappa(a) = \frac{4\pi a\hbar^2}{m}. \tag{184}$$

As usual, the quantum field can be separated into a macroscopic (classical) condensate and a fluctuation: $\widehat{\Psi} = \psi + \widehat{\varphi}$, with $\langle\widehat{\Psi}\rangle = \psi$. Then, by adopting the self-consistent mean field approximation (see for example [153])

$$\widehat{\varphi}^\dagger\widehat{\varphi}\widehat{\varphi} \simeq 2\langle\widehat{\varphi}^\dagger\widehat{\varphi}\rangle\;\widehat{\varphi} + \langle\widehat{\varphi}\widehat{\varphi}\rangle\;\widehat{\varphi}^\dagger, \tag{185}$$

one can arrive at the set of coupled equations:

$$\begin{aligned}\mathrm{i}\hbar\,\frac{\partial}{\partial t}\psi(t,\mathbf{x}) &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\mathbf{x}) + \kappa\;n_c\right)\psi(t,\mathbf{x})\\ &\quad +\kappa\left\{2\tilde{n}\psi(t,\mathbf{x}) + \tilde{m}\psi^*(t,\mathbf{x})\right\};\end{aligned} \tag{186}$$

$$\begin{aligned}\mathrm{i}\hbar\,\frac{\partial}{\partial t}\widehat{\varphi}(t,\mathbf{x}) &= \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\mathbf{x}) + \kappa\;2n_T\right)\widehat{\varphi}(t,\mathbf{x})\\ &\quad +\kappa\;m_T\;\widehat{\varphi}^\dagger(t,\mathbf{x}).\end{aligned} \tag{187}$$

Here

$$n_c \equiv |\psi(t,\mathbf{x})|^2; \quad m_c \equiv \psi^2(t,\mathbf{x}); \tag{188}$$
$$\tilde{n} \equiv \langle \widehat{\varphi}^\dagger\, \widehat{\varphi}\rangle; \quad \tilde{m} \equiv \langle \widehat{\varphi}\, \widehat{\varphi}\rangle; \tag{189}$$
$$n_T = n_c + \tilde{n}; \quad m_T = m_c + \tilde{m}. \tag{190}$$

The equation for the classical wave function of the condensate is closed only when the back-reaction effect due to the fluctuations are neglected. (This back-reaction is hiding in the parameters $\tilde{m}$ and $\tilde{n}$.) This is the approximation contemplated by the Gross–Pitaevskii equation. In general one will have to solve both equations simultaneously. Adopting the Madelung representation for the wave function of the condensate

$$\psi(t,\mathbf{x}) = \sqrt{n_c(t,\mathbf{x})}\; \exp[-\mathrm{i}\theta(t,\mathbf{x})/\hbar], \tag{191}$$

and defining an irrotational "velocity field" by $\mathbf{v} \equiv \boldsymbol{\nabla}\theta/m$, the Gross–Pitaevskii equation can be rewritten as a continuity equation plus an Euler equation:

$$\frac{\partial}{\partial t} n_c + \boldsymbol{\nabla}\cdot(n_c \mathbf{v}) = 0, \tag{192}$$
$$m\frac{\partial}{\partial t}\mathbf{v} + \boldsymbol{\nabla}\left(\frac{mv^2}{2} + V_{\mathrm{ext}}(t,\mathbf{x}) + \kappa n_c - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}}\right) = 0. \tag{193}$$

These equations are completely equivalent to those of an irrotational and inviscid fluid apart from the existence of the so-called quantum potential

$$V_{\mathrm{quantum}} = -\hbar^2\nabla^2\sqrt{n_c}/(2m\sqrt{n_c}), \tag{194}$$

which has the dimensions of an energy. Note that

$$n_c\; \nabla_i V_{\mathrm{quantum}} \equiv n_c\; \nabla_i \left[-\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}}\right] = \nabla_j\left[-\frac{\hbar^2}{4m}\; n_c\; \nabla_i\nabla_j \ln n_c\right], \tag{195}$$

which justifies the introduction of the so-called quantum stress tensor

$$\sigma_{ij}^{\mathrm{quantum}} = -\frac{\hbar^2}{4m}\; n_c\; \nabla_i\nabla_j \ln n_c. \tag{196}$$

This tensor has the dimensions of pressure, and may be viewed as an intrinsically quantum anisotropic pressure contributing to the Euler equation. If we write the mass density of the Madelung fluid as $\rho = m\, n_c$, and use the fact that the flow is irrotational then the Euler equation takes the form

$$\rho\left[\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right] + \rho\; \boldsymbol{\nabla}\left[\frac{V_{\mathrm{ext}}(t,\mathbf{x})}{m}\right] + \boldsymbol{\nabla}\left[\frac{\kappa\rho^2}{2m^2}\right] + \boldsymbol{\nabla}\cdot\sigma^{\mathrm{quantum}} = 0. \tag{197}$$

Note that the term $V_{\mathrm{ext}}/m$ has the dimensions of specific enthalpy, while $\kappa\rho^2/(2m)$ represents a bulk pressure. When the gradients in the density of the condensate are small one can neglect the quantum stress term leading to the standard hydrodynamic approximation. Because the flow is irrotational, the Euler equation is often more conveniently written in Hamilton–Jacobi form:

$$m\frac{\partial}{\partial t}\theta + \left(\frac{[\boldsymbol{\nabla}\theta]^2}{2m} + V_{\mathrm{ext}}(t,\mathbf{x}) + \kappa n_c - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}}\right) = 0. \tag{198}$$

Apart from the wave function of the condensate itself, we also have to account for the (typically small) quantum perturbations of the system (187). These quantum perturbations can be described in several different ways, here we are interested in the "quantum acoustic representation"

$$\widehat{\varphi}(t,\mathbf{x}) = e^{-\mathrm{i}\theta/\hbar}\left(\frac{1}{2\sqrt{n_c}}\,\widehat{n}_1 - \mathrm{i}\,\frac{\sqrt{n_c}}{\hbar}\,\widehat{\theta}_1\right), \tag{199}$$

where $\widehat{n}_1, \widehat{\theta}_1$ are real quantum fields. By using this representation Equation (187) can be rewritten as

$$\partial_t \widehat{n}_1 + \frac{1}{m}\boldsymbol{\nabla}\cdot\left(n_1\,\boldsymbol{\nabla}\theta + n_c\,\boldsymbol{\nabla}\widehat{\theta}_1\right) = 0, \tag{200}$$

$$\partial_t \widehat{\theta}_1 + \frac{1}{m}\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\widehat{\theta}_1 + \kappa(a)\,n_1 - \frac{\hbar^2}{2m}\,D_2 \widehat{n}_1 = 0. \tag{201}$$

Here $D_2$ represents a second-order differential operator obtained from linearizing the quantum potential. Explicitly:

$$D_2\,\widehat{n}_1 \equiv -\frac{1}{2}n_c^{-3/2}\,[\nabla^2(n_c^{+1/2})]\,\widehat{n}_1 + \frac{1}{2}n_c^{-1/2}\,\nabla^2(n_c^{-1/2}\,\widehat{n}_1). \tag{202}$$

The equations we have just written can be obtained easily by linearizing the Gross–Pitaevskii equation around a classical solution: $n_c \to n_c + \widehat{n}_1$, $\phi \to \phi + \widehat{\phi}_1$. It is important to realise that in those equations the back-reaction of the quantum fluctuations on the background solution has been assumed negligible. We also see in Equations (200, 201), that time variations of $V_{\mathrm{ext}}$ and time variations of the scattering length $a$ appear to act in very different ways. Whereas the external potential only influences the background Equation (198) (and hence the acoustic metric in the analogue description), the scattering length directly influences both the perturbation and background equations. From the previous equations for the linearised perturbations it is possible to derive a wave equation for $\widehat{\theta}_1$ (or alternatively, for $\widehat{n}_1$). All we need is to substitute in Equation (200) the $\widehat{n}_1$ obtained from Equation (201). This leads to a PDE that is second-order in time derivatives but infinite order in space derivatives – to simplify things we can construct the symmetric $4\times 4$ matrix

$$f^{\mu\nu}(t,\mathbf{x}) \equiv \left[\begin{array}{c|c} f^{00} & f^{0j} \\ \hline f^{i0} & f^{ij} \end{array}\right]. \tag{203}$$

(Greek indices run from 0–3, while Roman indices run from 1–3.) Then, introducing (3+1)-dimensional space-time coordinates

$$x^\mu \equiv (t;\, x^i) \tag{204}$$

the wave equation for $\theta_1$ is easily rewritten as

$$\partial_\mu(f^{\mu\nu}\,\partial_\nu \widehat{\theta}_1) = 0. \tag{205}$$

Where the $f^{\mu\nu}$ are *differential operators* acting on space only:

$$f^{00} = -\left[\kappa(a) - \frac{\hbar^2}{2m}\,D_2\right]^{-1} \tag{206}$$

$$f^{0j} = -\left[\kappa(a) - \frac{\hbar^2}{2m}\,D_2\right]^{-1}\frac{\nabla^j\theta_0}{m} \tag{207}$$

$$f^{i0} = -\frac{\nabla^i\theta_0}{m}\left[\kappa(a) - \frac{\hbar^2}{2m}\,D_2\right]^{-1} \tag{208}$$

$$f^{ij} = \frac{n_c\,\delta^{ij}}{m} - \frac{\nabla^i\theta_0}{m}\left[\kappa(a) - \frac{\hbar^2}{2m}\,D_2\right]^{-1}\frac{\nabla^j\theta_0}{m}. \tag{209}$$

Now, if we make a spectral decomposition of the field $\widehat{\theta}_1$ we can see that for wavelengths larger than $\hbar/mc_{\rm s}$ (this corresponds to the "healing length", as we will explain below), the terms coming from the linearization of the quantum potential (the $D_2$) can be neglected in the previous expressions, in which case the $f^{\mu\nu}$ can be approximated by numbers, instead of differential operators. (This is the heart of the acoustic approximation.) Then, by identifying

$$\sqrt{-g}\, g^{\mu\nu} = f^{\mu\nu}, \tag{210}$$

the equation for the field $\widehat{\theta}_1$ becomes that of a (massless minimally coupled) quantum scalar field over a curved background

$$\Delta\theta_1 \equiv \frac{1}{\sqrt{-g}}\, \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu\right) \widehat{\theta}_1 = 0, \tag{211}$$

with an effective metric of the form

$$g_{\mu\nu}(t,\mathbf{x}) \equiv \frac{n_c}{m\, c_{\rm s}(a,n_c)} \left[ \begin{array}{ccc} -\{c_{\rm s}(a,n_c)^2 - v^2\} & \vdots & -v_j \\ \cdots\cdots\cdots\cdots & & \cdots\cdots \\ -v_i & \vdots & \delta_{ij} \end{array} \right]. \tag{212}$$

Here the magnitude $c_{\rm s}(n_c,a)$ represents the speed of the phonons in the medium:

$$c_{\rm s}(a,n_c)^2 = \frac{\kappa(a)\, n_c}{m}. \tag{213}$$

With this effective metric now in hand, the analogy is fully established, and one is now in a position to start asking more specific physics questions.

### 4.2.2 BEC models in the eikonal approximation

It is interesting to consider the case in which the above "hydrodynamical" approximation for BECs does not hold. In order to explore a regime where the contribution of the quantum potential cannot be neglected we can use the so called *eikonal* approximation, a high-momentum approximation where the phase fluctuation $\widehat{\theta}_1$ is itself treated as a slowly-varying amplitude times a rapidly varying phase. This phase will be taken to be the same for both $\widehat{n}_1$ and $\widehat{\theta}_1$ fluctuations. In fact, if one discards the unphysical possibility that the respective phases differ by a time varying quantity, any time-constant difference can be safely reabsorbed in the definition of the (complex) amplitudes. Specifically, we shall write

$$\widehat{\theta}_1(t,\mathbf{x}) = {\rm Re}\left\{\mathcal{A}_\theta\ \exp(-i\phi)\right\}, \tag{214}$$

$$\widehat{n}_1(t,\mathbf{x}) = {\rm Re}\left\{\mathcal{A}_\rho\ \exp(-i\phi)\right\}. \tag{215}$$

As a consequence of our starting assumptions, gradients of the amplitude, and gradients of the background fields, are systematically ignored relative to gradients of $\phi$. (Warning: What we are doing here is not quite a "standard" eikonal approximation, in the sense that it is not applied directly on the fluctuations of the field $\psi(t,\mathbf{x})$ but separately on their amplitudes and phases $\rho_1$ and $\phi_1$.) We adopt the notation

$$\omega = \frac{\partial\phi}{\partial t}; \qquad k_i = \nabla_i\phi. \tag{216}$$

Then the operator $D_2$ can be approximated as

$$D_2\,\widehat{n}_1 \equiv -\frac{1}{2} n_c^{-3/2}\,[\Delta(n_c^{+1/2})]\,\widehat{n}_1 + \frac{1}{2} n_c^{-1/2}\,\Delta(n_c^{-1/2}\widehat{n}_1) \tag{217}$$

$$\approx +\frac{1}{2} n_c^{-1}\,[\Delta\widehat{n}_1] \tag{218}$$

$$= -\frac{1}{2} n_c^{-1}\,k^2\,\widehat{n}_1. \tag{219}$$

A similar result holds for $D_2$ acting on $\widehat{\theta}_1$. That is, under the eikonal approximation we effectively replace the *operator* $D_2$ by the *function*

$$D_2 \to -\frac{1}{2} n_c^{-1} k^2. \tag{220}$$

For the matrix $f^{\mu\nu}$ this effectively results in the replacement

$$f^{00} \to -\left[\kappa(a) + \frac{\hbar^2\,k^2}{4m\,n_c}\right]^{-1} \tag{221}$$

$$f^{0j} \to -\left[\kappa(a) + \frac{\hbar^2\,k^2}{4m\,n_c}\right]^{-1}\,\frac{\nabla^j\theta_0}{m} \tag{222}$$

$$f^{i0} \to -\frac{\nabla^i\theta_0}{m}\,\left[\kappa(a) + \frac{\hbar^2\,k^2}{4m\,n_c}\right]^{-1} \tag{223}$$

$$f^{ij} \to \frac{n_c\,\delta^{ij}}{m} - \frac{\nabla^i\theta_0}{m}\,\left[\kappa(a) + \frac{\hbar^2\,k^2}{4m\,n_c}\right]^{-1}\,\frac{\nabla^j\theta_0}{m}\,. \tag{224}$$

(As desired, this has the net effect of making $f^{\mu\nu}$ a matrix of numbers, not operators.) The physical wave equation (205) now becomes a nonlinear dispersion relation

$$f^{00}\,\omega^2 + (f^{0i} + f^{i0})\,\omega\,k_i + f^{ij}\,k_i\,k_j = 0. \tag{225}$$

After substituting the approximate $D_2$ into this dispersion relation and rearranging, we see (remember: $k^2 = ||k||^2 = \delta^{ij}\,k_i\,k_j$)

$$-\omega^2 + 2\,v_0^i\,\omega k_i + \frac{n_c k^2}{m}\left[\kappa(a) + \frac{\hbar^2}{4mn_c}k^2\right] - (v_0^i\,k_i)^2 = 0. \tag{226}$$

That is:

$$\left(\omega - v_0^i\,k_i\right)^2 = \frac{n_c k^2}{m}\left[\kappa(a) + \frac{\hbar^2}{4mn_c}k^2\right]. \tag{227}$$

Introducing the speed of sound $c_\mathrm{s}$ this takes the form:

$$\omega = v_0^i\,k_i \pm \sqrt{c_\mathrm{s}^2 k^2 + \left(\frac{\hbar}{2m}\,k^2\right)^2}. \tag{228}$$

At this stage some observations are in order:

1. It is interesting to recognise that the dispersion relation (228) is exactly in agreement with that found in 1947 by Bogoliubov [36] (reprinted in [310]; see also [223]) for the collective excitations of a homogeneous Bose gas in the limit $T \to 0$ (almost complete condensation). In his derivation Bogoliubov applied a diagonalization procedure for the Hamiltonian describing the system of bosons.

2. It is easy to see that (228) actually interpolates between two different regimes depending on the value of the wavelength $\lambda = 2\pi/||k||$ with respect to the "acoustic Compton wavelength" $\lambda_c = h/(mc_s)$. (Remember that $c_s$ is the speed of sound; this is not a standard particle physics Compton wavelength.) In particular, if we assume $v_0 = 0$ (no background velocity), then for large wavelengths $\lambda \gg \lambda_c$ one gets a standard phonon dispersion relation $\omega \approx c||k||$. For wavelengths $\lambda \ll \lambda_c$ the quasi-particle energy tends to the kinetic energy of an individual gas particle and in fact $\omega \approx \hbar^2 k^2/(2m)$.

   We would also like to highlight that in relative terms, the approximation by which one neglects the quartic terms in the dispersion relation gets worse as one moves closer to a horizon where $v_0 = -c_s$. The non-dimensional parameter that provides this information is defined by

$$\delta \equiv \frac{\sqrt{1+\frac{\lambda_c^2}{4\lambda^2}}-1}{(1-v_0/c_s)} \simeq \frac{1}{(1-v_0/c_s)}\frac{\lambda_c^2}{8\lambda^2}. \tag{229}$$

   As we will discuss in Section 5.1.3, this is the reason why sonic horizons in a BEC can exhibit different features from those in standard general relativity.

3. The dispersion relation (228) exhibits a contribution due to the background flow $v_0^i\, k_i$, plus a quartic dispersion at high momenta. The group velocity is

$$v_g^i = \frac{\partial \omega}{\partial k_i} = v_0^i \pm \frac{\left(c^2 + \frac{\hbar^2}{2m^2}k^2\right)}{\sqrt{c^2k^2 + \left(\frac{\hbar}{2m}\,k^2\right)^2}}\; k^i\,. \tag{230}$$

   Dispersion relations of this type (but in most cases with the sign of the quartic term reversed) have been used by Corley and Jacobson in analysing the issue of trans-Planckian modes in the Hawking radiation from general relativistic black holes [185, 186, 88]. The existence of modified dispersion relations (MDR), that is, dispersion relations that break Lorentz invariance, can be taken as a manifestation of new physics showing up at high energies/short wavelengths. In their analysis, the group velocity reverses its sign for large momenta. (Unruh's analysis of this problem used a slightly different toy model in which the dispersion relation saturated at high momentum [377].) In our case, however, the group velocity grows without bound allowing high-momentum modes to escape from behind the horizon. Thus the acoustic horizon is not absolute in these models, but is instead frequency dependent, a phenomenon that is common once non-trivial dispersions are included.

   Indeed, with hindsight the fact that the group velocity goes to infinity for large $k$ was preordained: After all, we started from the generalised nonlinear Schrödinger equation, and we know what its characteristic curves are. Like the diffusion equation the characteristic curves of the Schrödinger equation (linear or nonlinear) move at infinite speed. If we then approximate this generalised nonlinear Schrödinger equation in any manner, for instance by linearization, we cannot change the characteristic curves: For any well behaved approximation technique, at high frequency and momentum we should recover the characteristic curves of the system we started with. However, what we certainly do see in this analysis is a suitably large region of momentum space for which the concept of the effective metric both makes sense, and leads to finite propagation speed for medium-frequency oscillations.

   This type of superluminal dispersion relation has also been analysed by Corley and Jacobson [90]. They found that this escape of modes from behind the horizon often leads to self-amplified instabilities in systems possessing both an inner horizon as well as an outer horizon, possibly causing them to disappear in an explosion of phonons. This is also in partial agreement with the stability analysis performed by Garay *et al.* [136, 137] using the whole

Bogoliubov equations. Let us however leave further discussion regarding these developments to the Section 5.1.3 on horizon stability.

#### 4.2.3 The Heliocentric universe

Helium is one of the most fascinating elements provided by nature. Its structural richness confers on helium a paradigmatic character regarding the emergence of many and varied macroscopic properties from the microscopic world (see [418] and references therein). Here, we are interested in the emergence of effective geometries in helium, and their potential use in testing aspects of semiclassical gravity.

Helium four, a bosonic system, becomes superfluid at low temperatures (2.17 K at vapour pressure). This superfluid behaviour is associated with the condensation in the vacuum state of a macroscopically large number of atoms. A superfluid is automatically an irrotational and inviscid fluid, so in particular one can apply to it the ideas worked out in Section 2. The propagation of classical acoustic waves (scalar waves) over a background fluid flow can be described in terms of an effective Lorentzian geometry: the acoustic geometry. However, in this system one can naturally go considerably further, into the quantum domain. For long wavelengths, the quasiparticles in this system are quantum phonons. One can separate the classical behaviour of a background flow (the effective geometry) from the behaviour of the quantum phonons over this background. In this way one can reproduce, in laboratory settings, different aspects of quantum field theory over curved backgrounds. The speed of sound in the superfluid phase is typically of the order of cm/sec. Therefore, at least in principle, it should not be too difficult to establish configurations with supersonic flows and their associated ergoregions.

Helium three, the fermionic isotope of helium, in contrast becomes superfluid at very much lower temperatures (below 2.5 milli-K). The reason behind this rather different behaviour is the pairing of fermions to form effective bosons (Cooper pairing), which are then able to condense. In the so-called $^3\mathrm{He}-\mathrm{A}$ phase, the structure of the fermionic vacuum is such that it possesses two Fermi points, instead of the more typical Fermi surface. In an equilibrium configuration one can choose the two Fermi points to be located at $\{p_x = 0, p_y = 0, p_z = \pm p_F\}$ (in this way, the z-axis signals the direction of the angular momentum of the pairs). Close to either Fermi point the spectrum of quasiparticles becomes equivalent to that of Weyl fermions. From the point of view of the laboratory, the system is not isotropic, it is axisymmetric. There is a speed for the propagation of quasiparticles along the z-axis, $c_\parallel \simeq$ cm/sec, and a different speed, $c_\perp \simeq 10^{-5}\, c_\parallel$, for propagation perpendicular to the symmetry axis. However, from an internal observer's point of view this anisotropy is not "real", but can be made to disappear by an appropriate rescaling of the coordinates. Therefore, in the equilibrium case, we are reproducing the behaviour of Weyl fermions over Minkowski spacetime. Additionally, the vacuum can suffer collective excitations. These collective excitations will be experienced by the Weyl quasiparticles as the introduction of an effective electromagnetic field and a curved Lorentzian geometry. The control of the form of this geometry provides the sought for gravitational analogy.

Apart from the standard way to provide a curved geometry based on producing non-trivial flows, there is also the possibility of creating topologically non-trivial configurations with a built-in non-trivial geometry. For example, it is possible to create a domain-wall configuration [200, 199] (the wall contains the z-axis) such that the transverse velocity $c_\perp$ acquires a profile in the perpendicular direction (say along the x-axis) with $c_\perp$ passing through zero at the wall (see Figure 8). This particular arrangement could be used to reproduce a black hole–white hole configuration only if the soliton is set up to move with a certain velocity along the x-axis. This configuration has the advantage than it is dynamically stable, for topological reasons, even when some supersonic regions are created.

A third way in which superfluid Helium can be used to create analogues of gravitational config-

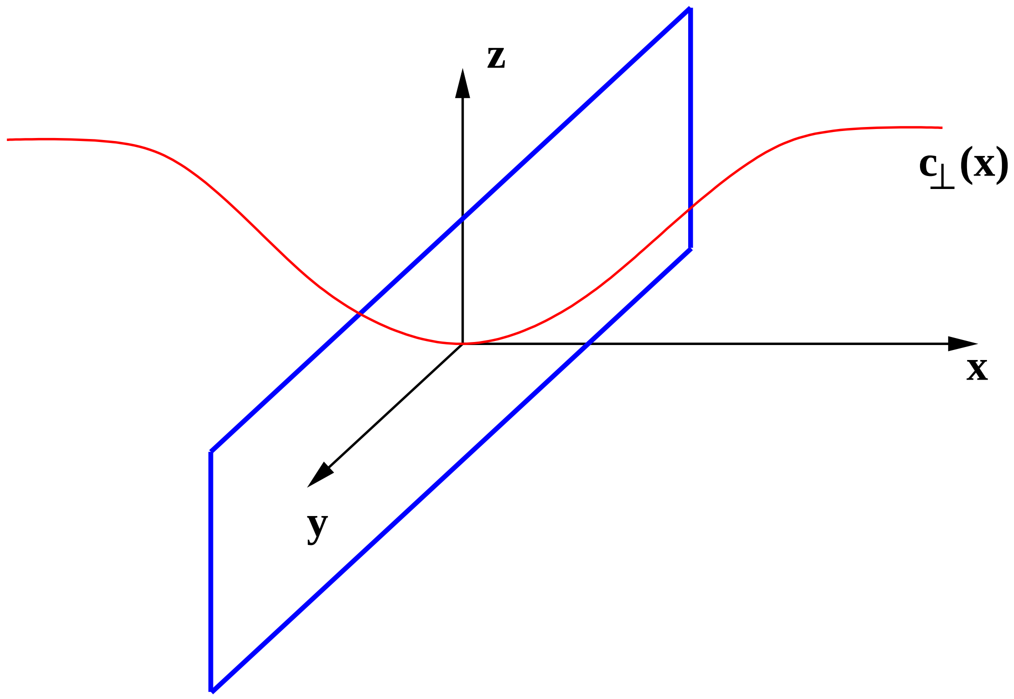


Figure 8: *Domain wall configuration in* $^3$He.

urations is the study of surface waves (or ripplons) on the interface between two different phases of $^3$He [415, 417]. In particular, if we have a thin layer of $^3\mathrm{He}-\mathrm{A}$ in contact with another thin layer of $^3\mathrm{He}-\mathrm{B}$, the oscillations of the contact surface "see" an effective metric of the form [415, 417]

$$\mathrm{d}s^2 = \frac{1}{(1-\alpha_A\alpha_B U^2)}\left[-\left(1-W^2-\alpha_A\alpha_B U^2\right)\,\mathrm{d}t^2 - 2\mathbf{W}\cdot\mathbf{dx}\,\mathrm{d}t + \mathbf{dx}\cdot\mathbf{dx}\right], \tag{231}$$

where

$$\mathbf{W} \equiv \alpha_A \mathbf{v_A} + \alpha_B \mathbf{v_B}; \qquad \mathbf{U} \equiv \mathbf{v_A} - \mathbf{v_B}; \tag{232}$$

and

$$\alpha_A \equiv \frac{h_B\rho_A}{h_A\rho_B + h_B\rho_A}; \qquad \alpha_B \equiv \frac{h_A\rho_B}{h_A\rho_B + h_B\rho_A}. \tag{233}$$

(All of this provided that we are looking at wavelengths larger than the layer thickness, $k\,h_A \ll 1$ and $k\,h_B \ll 1$.)

The advantage of using surface waves instead of bulk waves in superfluids is that one could create horizons without reaching supersonic speeds in the bulk fluid. This could alleviate the appearance of dynamical instabilities in the system, that in this case are controlled by the strength of the interaction of the ripplons with bulk degrees of freedom [415, 417].

#### 4.2.4 Slow light

The geometrical interpretation of the motion of light in dielectric media leads naturally to conjecture that the use of flowing dielectrics might be useful for simulating general relativity metrics with

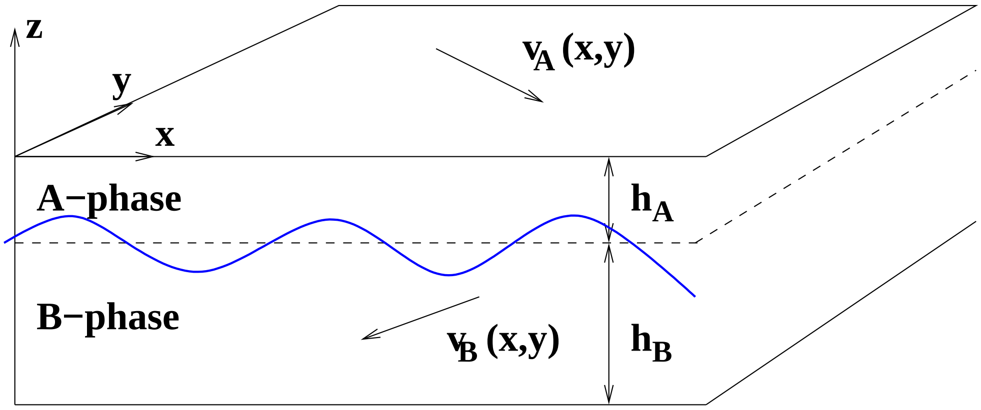


Figure 9: *Ripplons in the interface between two sliding superfluids.*

ergoregions and black holes. Unfortunately, these types of geometry require flow speeds comparable to the group velocity of the light. Since typical refractive indexes in non-dispersive media are quite close to unity, it is then clear that it is practically impossible to use them to simulate such general relativistic phenomena. However recent technological advances have radically changed this state of affairs. In particular the achievement of controlled slowdown of light, down to velocities of a few meters per second (or even down to complete rest) [383, 204, 52, 211, 309, 374, 348], has opened a whole new set of possibilities regarding the simulation of curved-space metrics via flowing dielectrics.

But how can light be slowed down to these "snail-like" velocities? The key effect used to achieve this takes the name of Electromagnetically Induced Transparency (EIT). A laser beam is coupled to the excited levels of some atom and used to strongly modify its optical properties. In particular one generally chooses an atom with two long-lived metastable (or stable) states, plus a higher energy state that has some decay channels into these two lower states. The coupling of the excited states induced by the laser light can affect the transition from a lower energy state to the higher one, and hence the capability of the atom to absorb light with the required transition energy. The system can then be driven into a state where the transitions between each of the lower energy states and the higher energy state exactly cancel out, due to quantum interference, at some specific resonant frequency. In this way the higher-energy level has null averaged occupation number. This state is hence called a "dark state". EIT is characterised by a transparency window, centred around the resonance frequency, where the medium is *both* almost transparent *and* extremely dispersive (strong dependence on frequency of the refractive index). This in turn implies that the group velocity of any light probe would be characterised by very low real group velocities (with almost vanishing imaginary part) in proximity to the resonant frequency.

Let us review the most common setup envisaged for this kind of analogue model. A more detailed analysis can be found in [232]. One can start by considering a medium in which an EIT window is opened via some control laser beam which is oriented perpendicular to the direction of the flow. One then illuminates this medium, now along the flow direction, with some probe light (which is hence perpendicular to the control beam). This probe beam is usually chosen to be weak with respect to the control beam, so that it does not modify the optical properties of the medium. In the case in which the optical properties of the medium do not vary significantly over several wavelengths of the probe light, one can neglect the polarization and can hence describe the propagation of the latter with a simple scalar dispersion relation [235, 124]

$$k^2 - \frac{\omega^2}{c^2}\left[1+\chi(\omega)\right], \qquad (234)$$

where $\chi$ is the susceptibility of the medium, related to the refractive index $n$ via the simple relation $n = \sqrt{1+\chi}$.

It is easy to see that in this case the group and phase velocities differ

$$v_{\rm g} = \frac{\partial \omega}{\partial k} = \frac{c}{\sqrt{1+\chi} + \frac{\omega}{2n}\frac{\partial \chi}{\partial \omega}}; \qquad v_{\rm ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1+\chi}}. \tag{235}$$

So even for small refractive indexes one can get very low group velocities, due to the large dispersion in the transparency window, and in spite of the fact that the phase velocity remains very near to $c$. (The phase velocity is exactly $c$ at the resonance frequency $\omega_0$). In an ideal EIT regime the probe light experiences a vanishing susceptibility $\chi$ near the the critical frequency $\omega_0$, this allows us to express the susceptibility near the critical frequency via the expansion

$$\chi(\omega) = \frac{2\alpha}{\omega_0}(\omega - \omega_0) + O\left[(\omega-\omega_0)^3\right], \tag{236}$$

where $\alpha$ is sometimes called the "group refractive index". The parameter $\alpha$ depends on the dipole moments for the transition from the metastable states to the high energy one, and most importantly depends on the ratio between the probe-light energy per photon, $\hbar\omega_0$, and the control-light energy per atom [232]. This might appear paradoxical because it seems to suggest that for a dimmer control light the probe light would be further slowed down. However this is just an artificial feature due to the extension of the EIT regime beyond its range of applicability. In particular in order to be effective the EIT requires the control beam energy to dominate all processes and hence it cannot be dimmed at will.

At resonance we have

$$v_{\rm g} = \frac{\partial \omega}{\partial k} \to \frac{c}{1+\alpha} \approx \frac{c}{\alpha}; \qquad v_{\rm ph} = \frac{\omega}{k} \to c. \tag{237}$$

We can now generalise the above discussion to the case in which our highly dispersive medium flows with a characteristic velocity profile $\mathbf{u}(\mathbf{x}, t)$. In order to find the dispersion relation of the probe light in this case we just need to transform the dispersion relation (234) from the comoving frame of the medium to the laboratory frame. Let us consider for simplicity a monochromatic probe light (more realistically a pulse with a very narrow range of frequencies $\omega$ near $\omega_0$). The motion of the dielectric medium creates a local Doppler shift of the frequency

$$\omega \to \gamma\,(\omega_0 - \mathbf{u}\cdot\mathbf{k})\,, \tag{238}$$

where $\gamma$ is the usual relativistic factor. Given that $k^2 - \omega^2/c^2$ is a Lorentz invariant, it is then easy to see that this Doppler detuning affects the dispersion relation (234) only via the susceptibility dependent term. Given further that in any realistic case one would deal with non-relativistic fluid velocities $\mathbf{u} \ll c$ we can then perform an expansion of the dispersion relation up to second order in $u/c$. Expressing the susceptibility via (236) we can then rewrite the dispersion relation in the form [235]

$$g^{\mu\nu} k_\mu k_\nu = 0, \tag{239}$$

where

$$k_\nu = \left(\frac{\omega_0}{c}, -\mathbf{k}\right), \tag{240}$$

and (most of the relevant articles adopt the signature $(+---)$, as we also do for this particular section)

$$g^{\mu\nu} = \left[\begin{array}{c|c} 1+\alpha u^2/c^2 & \alpha \mathbf{u}^T/c^2 \\ \hline \alpha\mathbf{u}/c^2 & -\mathbf{I}_{3\times 3} + 4\alpha\mathbf{u}\otimes\mathbf{u}^T/c^2 \end{array}\right]. \tag{241}$$

The inverse of this tensor will be the covariant effective metric experienced by the probe light, whose rays would then be null geodesics of the line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. In this sense the probe light will propagate as in a curved background. Explicitly one finds the covariant metric to be

$$g_{\mu\nu} = \left[\begin{array}{c|c} A & B\mathbf{u}^T \\ \hline B\mathbf{u} & -\mathbf{I}_{3\times 3} + C\mathbf{u}\otimes\mathbf{u}^T \end{array}\right], \tag{242}$$

where

$$A = \frac{1 - 4\alpha u^2/c^2}{1 + (\alpha^2 - 3\alpha)u^2/c^2 - 4\alpha^2 u^4/c^4}; \tag{243}$$

$$B = \frac{1}{1 + (\alpha^2 - 3\alpha)u^2/c^2 - 4\alpha^2 u^4/c^4}; \tag{244}$$

$$C = \frac{1 - (4/\alpha + 4u^2/c^2)}{1 + (\alpha^2 - 3\alpha)u^2/c^2 - 4\alpha^2 u^4/c^4}. \tag{245}$$

Several comments are in order concerning the metric (242). First of all it is clear that although more complicated than an acoustic metric it will be still possible to cast it into the Arnowitt–Deser–Misner-like form [392]

$$g_{\mu\nu} = \left[\begin{array}{c|c} -[c_{\mathrm{eff}}^2 - g_{ab}u_{\mathrm{eff}}^a u_{\mathrm{eff}}^b] & [u_{\mathrm{eff}}]_i \\ \hline [u_{\mathrm{eff}}]_j & [g_{\mathrm{eff}}]_{ij} \end{array}\right], \tag{246}$$

where the effective speed $\mathbf{u}_{\mathrm{eff}}$ is proportional to the fluid flow speed $\mathbf{u}$ and the three space effective metric $g_{\mathrm{eff}}$ is (rather differently from the acoustic case) non-trivial.

In any case, the existence of this ADM form already tells us that an ergoregion will always appear once the norm of the effective velocity exceeds the effective speed of light (which for slow light is approximately $c/\alpha$ where $\alpha$ can be extremely large due to the huge dispersion in the transparency window around the resonance frequency $\omega_0$). However a trapped surface (and hence an optical black hole) will form only if the *inward normal component* of the effective flow velocity exceeds the group velocity of light. In the slow light setup so far considered such a velocity turns out to be $u = c/(2\sqrt{\alpha})$.

The realization that ergoregions and event horizons can be simulated via slow light may lead one to the (erroneous) conclusion that this is an optimal system for simulating particle creation by gravitational fields. However, as pointed out by Unruh in [284, 379], such a conclusion would turn out to be over-enthusiastic. In order to obtain particle creation an inescapable requirement is to have so-called "mode mixing", that is, mixing between the positive and negative frequency modes of the incoming and outgoing states. This is tantamount to saying that there must be regions where the frequency of the quanta as seen by a stationary observer at infinity (laboratory frame) becomes negative beyond the ergosphere at $g_{00} = 0$.

In a flowing medium this can in principle occur thanks to the tilting of the dispersion relation due to the Doppler effect caused by the velocity of the flow Equation (238), but this also tells us that the condition $\omega_0 - \mathbf{u}\cdot\mathbf{k} < 0$ can be satisfied only if the velocity of the medium exceeds $|\omega_0/k|$ which is the *phase* velocity of the probe light, not its group velocity. Since the phase velocity in the slow light setup we are considering is very close to $c$, the physical speed of light in vacuum, not very much hope is left for realizing analogue particle creation in this particular laboratory setting.

However it was also noticed by Unruh and Schützhold [379] that a different setup for slow light might deal with this and other issues (see [379] for a detailed summary). In the setup suggested by these authors there are two strong background counter-propagating control beams illuminating the atoms. The field describing the beat fluctuations of this electromagnetic background can be shown to satisfy, once the dielectric medium is in motion, the same wave equation as that on a curved background. In this particular situation the phase velocity and the group velocity are approximately the same, and both can be made small, so that the previously discussed obstruction to mode mixing is removed. So in this new setup it *is* concretely possible to simulate classical particle creation such as, e.g., super-radiance in the presence of ergoregions.

Nonetheless the same authors showed that this does not open the possibility for a simulation of quantum particle production (e.g., Hawking radiation). This is because that effect also requires the commutation relations of the field to generate the appropriate zero-point energy fluctuations (the vacuum structure) according to the Heisenberg uncertainty principle. This is not the case for the effective field describing the beat fluctuations of the system we have just described, which is equivalent to saying that it does not have a proper vacuum state (i.e., analogue to any physical field). Hence one has to conclude that any simulation of quantum particle production is precluded.

## 4.3 Going further

We feel that the catalogue we have just presented is reasonably complete and covers the key items. For additional background on any of these topics, we would suggest sources such as the books "Artificial Black Holes" [284] and "The universe in a Helium droplet" [418]. For more specific detail, check this review's bibliography, and use Spires to check for recent developments.

# 5 Lessons from Analogue Models

Of course, the entire motivation for looking at analogue models is to be able to learn more physics. One could start studying analogue models with the idea of seeing whether it is possible (either theoretically or in practice) to reproduce in the laboratory various gravitational phenomena whose real existence in nature cannot be currently checked. These are phenomena that surpass our present (and foreseeable) observational capabilities, but yet, we believe in their existence because it follows from seemingly strong theoretical arguments within the standard frameworks of general relativity and field theory in curved space. However, the interest of this approach is not merely to reproduce these gravitational phenomena in some formal analogue model, but to see which departures from the ideal case show up in real analogue models, and to analyse whether similar deviations are likely to appear in real gravitational systems.

When one thinks about emergent gravitational features in condensed matter systems, one immediately realises that these features only appear in the low-energy regime of the analogue systems. When the systems are probed at high energies (short length scales) the effective geometrical description of the analogue models break down, as one starts to be aware that the systems are actually composed of discrete pieces (atoms and molecules). This scenario is quite similar to what one expects to happen with our geometrical description of the Universe, when explored with microscopic detail at the Planck scale.

That is, the study of analogue models of general relativity is giving us insights as to how the standard theoretical picture of different gravitational phenomena could change when taking into account additional physical knowledge coming from the existence of an underlying microphysical structure. Quite robustly, these studies are telling us already that the first deviations from the standard general relativity picture can be encoded in the form of high-energy violations of Lorentz invariance in particle dispersion relations. Beyond these first deviations, the analogue models of general relativity provide well understood examples (the underlying physics is well known) in which a description in terms of fields in curved spacetimes shows up as a low-energy-regime emergent phenomena.

The analogue models are being used to shed light on these general questions through a number of specific routes. Let us now turn to discussing several specific physics issues that are being analysed from this perspective.

## 5.1 Hawking radiation

### 5.1.1 Basics

As is well known, in 1974 Stephen Hawking announced that quantum mechanically even a spherical distribution of matter collapsing to form a black hole should emit radiation; with a spectrum approximately that of a black body [159, 160]. A black hole will tend to evaporate by emitting particles from its horizon toward infinity. Hawking radiation is a quantum-field-in-curved-space effect. The existence of radiation emission is a kinematic effect that does not rely on Einstein equations. Therefore, one can aim to reproduce it in a condensed matter system. Within standard field theory, a minimal requirement for having Hawking radiation is the existence in the background configuration of an apparent horizon [394]. So, in principle, to be able to reproduce Hawking radiation in a laboratory one would have to fulfil at least two requirements:

1. To choose an adequate analogue system; it has to be a quantum analogue model (see Section 4) such that its description could be separated into a classical effective background spacetime plus some standard relativistic quantum fields living on it (it can happen that the quantum fields do not satisfy the appropriate commutation or anti-commutation relations [379]).

2. To configure the analogue geometry such that it includes an apparent horizon. That is, within an appropriate quantum analogue model, the formation of an apparent horizon for the propagation of the quantum fields should excite the fields as to result in the emission of a thermal distribution of field particles. [22].

This is a straight and quite naive translation of the standard Hawking effect derivation to the condensed matter realm. However, in reality, this translation process has to take into account additional issues that, as we are trying to convey, instead of problems, are where the interesting physics lies.

1. The effective description of the quantum analogue systems as fields over a background geometry breaks down when probed at sufficiently short length scales. This could badly influence the main features of Hawking radiation. In fact, immediately after the inception of the idea that black holes radiate, it was realised that there was a potential problem with the calculation [375]. It strongly relies on the validity of quantum field theory on curved backgrounds up to arbitrary high energies. Following a wave packet with a certain frequency at future infinity backwards in time, we can see that it had to contain arbitrarily large frequency components with respect to a local free fall observer (well beyond the Planck scale) when it was close to the horizon. In principle any unknown physics at the Planck scale could strongly influence the Hawking process so that one should view it with suspicion. This is the so-called trans-Planckian problem of Hawking radiation. To create an analogue model exhibiting Hawking radiation will be, therefore, equivalent to giving a solution to the trans-Planckian problem.

2. In order to clearly observe Hawking radiation, one should first be sure that there is no other source of instabilities in the system that could mask the effect. In analogue models such as liquid Helium or BECs the interaction of a radial flow (with speed of the order of the critical Landau speed, which in these cases coincides with the sound speed [213]) with the surface of the container (an electromagnetic potential in the BECs case) might cause the production of rotons and quantised vortices, respectively. Thus, in order to produce an analogue model of Hawking radiation, one has to be somewhat ingenious. For example in the liquid Helium case, instead of taking acoustic waves in a supersonic flow stream as the analogue model, it is preferable to use as analogue model ripplons in the interface between two different phases, A and B phases, of Helium-3 [415]. Another option is to start from a moving domain wall configuration. Then, the topological stability of the configuration prevents its destruction when creating a horizon [199, 200]. In the case of BECs a way to suppress the formation of quantised vortices is to take effectively one-dimensional configurations. If the transverse dimension of the flow is smaller than the healing length then there is no space for the existence of a vortex [19]. In either liquid Helium or BECs, there is also the possibility of creating an apparent horizon by rapidly approaching a critical velocity profile (see Figure 10), but without actually crossing into the supersonic regime [13], softening in this way the appearance of dynamical instabilities.

3. Real analogue models cannot, strictly speaking, reproduce eternal black-hole configurations. An analogue model of a black hole has always to be created at some finite laboratory time. Therefore, one is forced to carefully analyse the creation process, as it can greatly influence the

[22] One could also imagine systems in which the effective metric fails to exist on one side of the horizon (or even more radically, on both sides). The existence of particle production in this kind of system will then depend on the specific interactions between the sub-systems characterizing each side of the horizon. For example, in stationary configurations it will be necessary that these interactions allow negative energy modes to dissappear beyond the horizon, propagating forward in time (as happens in an ergoregion). Whether these systems will provide adequate analogue models of Hawking radiation or not is an interesting question that deserve future analysis. Certainly systems of this type lie well outside the class of usual analogue models.

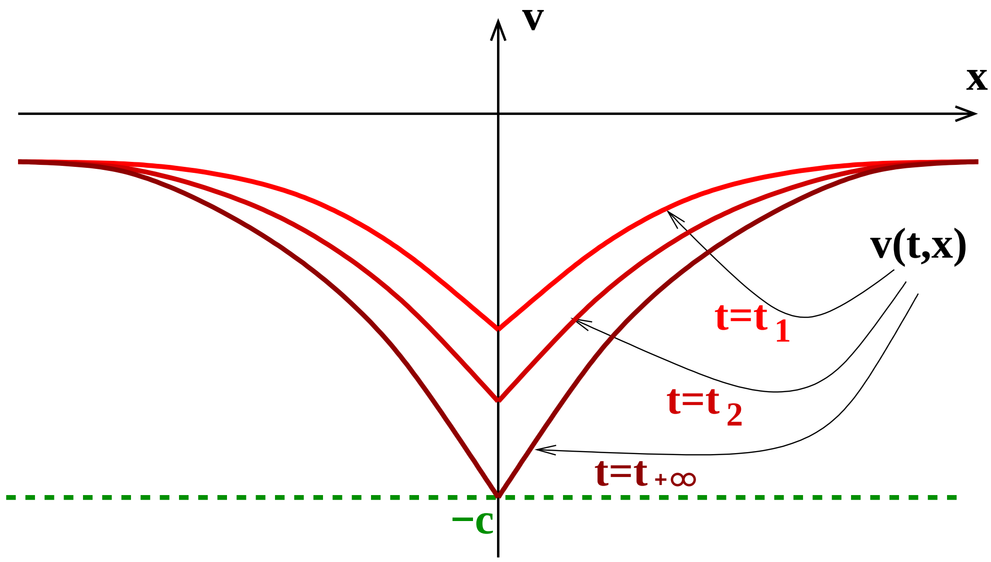


Figure 10: *Velocity profile for a left going flow; the profile is dynamically modified with time so that it reaches the profile with a sonic point at the asymptotic future.*

Hawking effect. Depending on the procedure of creation, one could end up in quite different quantum states for the field and only some of them might exhibit Hawking radiation. This becomes more important when considering that the analogue models incorporate modified dispersion relations. An inappropriate preparation, together with modified dispersion relation effects, could completely eliminate Hawking radiation [380].

4. Another important issue is the need to characterise "how quantum" a specific analogue model is. Even though, strictly speaking, one could say that any system undergoes quantum fluctuations, the point is how important they are in its description. In trying to build an analogue model of Hawking's quantum effect, the relative value of Hawking temperature with respect to the environment is going to tell us whether the system can be really thought as a quantum analogue model or as effectively classical. For example, in our standard cosmological scenario, for a black hole to radiate at temperatures higher than that of the Cosmic Microwave Background, $\approx 3$ K, the black hole should have a diameter of the order of micrometers or less. We would have to say that such black holes are no longer classical, but semiclassical. The black holes for which we have some observational evidence are of much higher mass and size, so their behaviour can be thought of as completely classical. Estimates of the Hawking temperature reachable in BECs yield $T \sim 100\,\mathrm{nK}$ [19]. This has the same order of magnitude of the temperature as the BECs themselves. This is telling us that regarding the Hawking process, BECs can be considered to be highly-quantum analogue models.

5. There is also the very real question of whether one should trust semiclassical calculations *at all* when it comes to dealing with back-reaction in the Hawking effect. See for instance the arguments presented by Helfer ([167, 168, 169], and references therein).

Because of its importance, let us now review what we know about the effects of high-energy dispersion relations on the Hawking process.

#### 5.1.2 Trans-Planckian problem

We saw in the introduction to this section that the trans-Planckian problem of Hawking radiation was one of the strongest motivations for the modern research into analogue models of gravity. In fact it was soon realised that such models could provide a physical framework within which a viable solution of the problem could be found. Let us explain why and how.

As we have said, the requirement of a reservoir of ultra-high frequency modes nearby the horizon seems to indicate a possible (and worrisome) sensitivity of the black hole radiation to the microscopic structure of spacetime. Actually by assuming exact Lorentz invariance one could in principle always locally transform the problematic ultra high frequency modes to low energy ones via some appropriate Lorentz transformation [185]. However in doing so it would have to rely on the physics of reference frames moving ultra fast with respect to us, as the reference frame needed would move arbitrarily close to the speed of light. Hence we would have to apply Lorentz invariance in a regime of arbitrary large boosts yet untested and in principle never completely testable given the non-compactness of the boost subgroup. The assumption of an exact boost symmetry is linked to a scale-free nature of spacetime given that unbounded boosts expose ultra-short distances. Hence the assumption of *exact* Lorentz invariance needs, in the end, to rely on some idea on the nature of spacetime at ultra-short distances.

It was this type of reasoning that led in the nineties to a careful reconsideration of the crucial ingredients required for the derivation of Hawking radiation [185, 186, 377]. In particular investigators explored the possibility that spacetime microphysics could provide a short distance, Lorentz-breaking cutoff, but at the same time leave Hawking's results unaffected at energy scales well below that set by the cutoff.

Of course ideas about a possible cutoff imposed by the discreteness of spacetime at the Planck scale had already been discussed in the literature well before Unruh's seminal paper [376]. However such ideas were running into serious difficulties given that a naive short distance cutoff posed on the available modes of a free field theory results in a complete removal of the evaporation process (see e.g., Jacobson's article [185] and references therein, and the comments in [167, 168, 169]). Indeed there are alternative ways through which the effect of the short scales physics could be taken into account, and analogue models provide a physical framework where these ideas could be put to the test. In fact analogue models provide explicit examples of emergent spacetime symmetries, they can be used to simulate black hole backgrounds, they may be endowed with quantizable perturbations and, in most of the cases, they have a well known microscopic structure. Given that Hawking radiation can be, at least in principle, simulated in such systems one might ask how and if the trans-Planckian problem is resolved in these cases.

**5.1.2.1 Modified dispersion relations:** The general feature that most of the work on this subject has focussed on is that in analogue models the quasi-particles propagating on the effective geometry are actually collective excitations of atoms. This generically implies that their dispersion relation will be a relativistic one only at low energies (large scales),[23] and in each case there will be some short length scale (e.g., intermolecular distance for a fluid, coherence length for a superfluid, healing length for a BEC) beyond which deviations will be non-negligible. In general such microphysics induced corrections to the dispersion relation take the form

$$E^2 = c^2 \left(m^2 c^2 + k^2 + \Delta(k, K)\right) \tag{247}$$

where $K$ is the scale that describes the transition to the full microscopic system (what we might call the "analogue Planck scale").

[23] Actually, even relativistic behaviour at low energy can be non-generic, but we assume in this discussion that an analogue model by definition is a system for which all the linearised perturbations do propagate on the same Lorentzian background at low energies.

In general the best one can do is to expand $\Delta(k, K)$ around $k = 0$, obtaining an infinite power series (of which it will be safe to retain only the lowest order terms), although in some special models (like BEC) the series is automatically finite due to intrinsic properties of the system. (In any case one can see that most of the analogue models so far considered lead to modifications of the form $\pm k^3/K^2$ or $\pm k^4/K^2$.) Depending on the sign in front of the modification the group velocity at high energy can be larger (+) or smaller (−) than the low energy speed of light $c$. These cases are usually referred in the literature as "superluminal" and "subluminal" dispersion relations.

Most of the work on the trans-Planckian problem in the nineties focussed on studying the effect on Hawking radiation due to such modifications of the dispersion relations at high energies in the case of acoustic analogues [185, 186, 377, 378, 88], and the question of whether such phenomenology could be applied to the case of real black holes (see e.g., [50, 188, 88, 299]).[24] In all the aforementioned works Hawking radiation can be recovered under some suitable assumptions, as long as neither the black hole temperature nor the frequency at which the spectrum is considered are too close to the scale of microphysics $K$. However, the applicability of these assumptions to the real case of black hole evaporation is an open question. Also, in the case of the analogue models the mechanism by which the Hawking radiation is recovered is not always the same. We concisely summarise here the main results (but see e.g., [380] for further details).

**5.1.2.2 Subluminal dispersion relations:** This was the case originally considered by Unruh [377, 378],

$$\omega = K\left(\tanh(k/K)^n\right)^{1/n}, \tag{248}$$

and by Corley and Jacobson [88]

$$\omega^2 = k^2 - k^4/K^2, \tag{249}$$

where both dispersion relations are given in the co-moving frame.

The key feature is that in the presence of a subluminal modification the group velocity of the modes increases with $k$ only up to some turning point (which is equivalent to saying that the group velocity does not asymptote to $c$, which could be the speed of sound, but instead is upper bounded). For values of $k$ beyond the turning point the group velocity decreases to zero (for (248)) or becomes imaginary (for (249)). In the latter case this can be interpreted as signifying the breakdown of the regime where the dispersion relation (249) can be trusted. The picture that emerged from these analyses concerning the origin of the outgoing Hawking modes at infinity is quite surprising. In fact, if one traces back in time an outgoing mode, as it approaches the horizon it decreases its group velocity below the speed of sound. At some point before reaching the horizon, the outgoing mode will appear as an ingoing one dragged into the black hole by the flow. Stepping further back in time it is seen that such a mode was located at larger and larger distances from the horizon, and tends to a very high energy mode far away at early times. In this case one finds what might be called a "mode conversion", where the origin of the outgoing Hawking quanta seems to originate from ingoing modes which have "bounced off" the horizon without reaching trans-Planckian frequencies in its vicinities. Several detailed analytical and numerical calculations have shown that such a conversion indeed happens [378, 50, 88, 87, 170, 330, 380] and that the Hawking result can be recovered for $\kappa \ll K$ where $\kappa$ is the black hole surface gravity.

**5.1.2.3 Superluminal dispersion relations:** The case of a superluminal dispersion relation is quite different and, as we have seen, has some experimental interest given that this is the kind of dispersion relation that arises in some promising analogue models (e.g., BECs). In this situation, the outgoing modes are actually seen as originating from behind the horizon. This implies that

[24] However see also [318, 322] for a radically different alternative approach based on the idea of "superoscillations" where ultrahigh frequency modes near the horizon can be mimicked (to arbitrary accuracy) by the exponential tail of an exponentially large amplitude mostly hidden behind the horizon.

these modes somehow originate from the singularity (which can be a region of high turbulence in acoustic black hole analogues), and hence it would seem that not much can be said in this case. However it is possible to show that if one imposes vacuum boundary conditions on these modes near the singularity, then it is still possible to recover the Hawking result, i.e., thermal radiation outside the hole [87]. It is particularly interesting to note that this recovering of the standard result is not always guaranteed in the presence of superluminal dispersion relations. Corley and Jacobson [90] in fact discovered a very peculiar type of instability due to such superluminal dispersion in the presence of black holes with inner horizons. The net result of the investigation carried out in [90] is that the compact ergo-region characterizing such configurations is unstable to self-amplifying Hawking radiation. The presence of such an instability seems to be confirmed by the analysis carried on in [136, 137] where a Bose–Einstein condensate analogue black hole was considered.

**5.1.2.4 General conditions for Hawking radiation:** Is it possible to reduce the rather complex phenomenology just described to a few basic assumptions that must be satisfied in order to recover Hawking radiation in the presence of Lorentz violating dispersion relations? A tentative answer is given in [380], where the robustness of the Hawking result is considered for general modified (subluminal as well as superluminal) dispersion relations. The authors of [380] assume that the geometrical optics approximation breaks down only in the proximity of the event horizon (which is equivalent to saying that the particle production happens only in such a region). Here, the would-be trans-Planckian modes are converted in sub-Planckian ones. Then, they try to identify the minimal set of assumptions that guarantees that such "converted modes" are generated in their ground states (with respect to a freely falling observer), as this is a well known condition in order to recover Hawking's result. They end up identifying three basic assumptions that guarantee such emergence of modes in the ground state at the horizon. First, the preferred frame selected by the breakdown of Lorentz invariance must be the freely falling one instead of the rest frame of the static observer at infinity (which coincides in this limit with the laboratory observer). Second, the Planckian excitations must start off in the ground state with respect to freely falling observers. Finally, they must evolve in an adiabatic way (i.e., the Planck dynamics must be much faster than the external sub-Planckian dynamics). Of course, although several systems can be found in which such conditions hold, it is also possible to show [380] that realistic situations in which at least one of these assumptions is violated can be imagined. It is hence still an open question whether real black hole physics does indeed satisfy such conditions, and whether it is hence robust against modifications induced by the violation of Lorentz invariance.

**5.1.2.5 Open issues:** In spite of the remarkable insight given by the models discussed above (based on modified dispersion relations) it is not possible to consider them fully satisfactory in addressing the trans-Planckian problem. In particular it was soon recognised [89, 189] that in this framework it is not possible to explain the origin of the short wavelength incoming modes which are "progenitors" of the outgoing modes after bouncing off in the proximity of the horizon. For example, in the Unruh model (248), one can see that if one keeps tracking a "progenitor" incoming mode back in time, then its group velocity (in the co-moving frame) drops to zero as its frequency becomes more and more blue shifted (up to arbitrarily large values), just the situation one was trying to avoid. This is tantamount to saying that the trans-Planckian problem has been moved from the region near the horizon out to the region near infinity. In the Corley–Jacobson model (249) this unphysical behaviour is removed thanks to the presence of the physical cutoff $K$. However it is still true that in tracking the incoming modes back in time one finally sees a wave packet so blue shifted that $|k| = K$. At this point one can no longer trust the dispersion relation (249) (which anyway in realistic analogue models is emergent and not fundamental), and hence the model has no predictive power regarding the ultimate origin of the relevant incoming modes.

These conclusions regarding the impossibility of clearly predicting the origin at early times of the modes ultimately to be converted into Hawking radiation are not specific to the particular dispersion relations (248) or (249) one is using. The Killing frequency is in fact conserved on a static background, thus the incoming modes must have the same frequency as the outgoing ones, hence there can be no mode-mixing and particle creation. This is why one has actually to assume that the WKB approximation fails in the proximity of the horizon and that the modes are there in the vacuum state for the co-moving observer. In this sense the need for these assumptions can be interpreted as evidence that these models are not fully capable of solving the trans-Planckian problem.

**5.1.2.6 Solid state and lattice models:** It was to overcome this type of issue that alternative ways of introducing an ultra-violet cutoff due to the microphysics were considered [318, 319, 89]. In particular in [319] the transparency of the refractive medium at high frequencies has been used to introduce an effective cutoff for the modes involved in Hawking radiation in a classical refractive index analogue model (see Section 4.1.3 of this review). In this model an event horizon for the electromagnetic field modes can be simulated by a surface of singular electric and magnetic permeabilities. This would be enough to recover Hawking radiation but it would imply the unphysical assumption of a refractive index which is valid at any frequency. However it was shown in [319] that the Hawking result can be recovered even in the case of a dispersive medium which becomes transparent above some fixed frequency $K$ (which we can imagine as the plasma frequency of the medium), the only (crucial) assumption being again that the "trans-Planckian" modes with $k > K$ are in their ground state near the horizon.

An alternative avenue was considered in [89]. There a lattice description of the background was used for imposing a cutoff in a more physical way with respect to the continuum dispersive models previously considered. In such a discretised spacetime the field takes values only at the lattice points, and wavevectors are identified modulo $2\pi/\ell$ where $\ell$ is the lattice characteristic spacing, correspondingly one obtains a sinusoidal dispersion relation for the propagating modes. Hence the problem of recovering a smooth evolution of incoming modes to outgoing ones is resolved by the intrinsically regularised behaviour of the wave vectors field. In [89] the authors explicitly considered the Hawking process for a discretised version of a scalar field, where the lattice is associated to the free-fall coordinate system (taken as the preferred system). With such a choice it is possible to preserve a discrete lattice spacing. Furthermore the requirement of a fixed short distance cutoff leads to the choice of a lattice spacing constant at infinity, and that the lattice points are at rest at infinity and fall freely into the black hole.[25] In this case the lattice spacing grows in time and the lattice points spread in space as they fall toward the horizon. However this time dependence of the lattice points is found to be of order $1/\kappa$, and hence unnoticeable to long wavelength modes and relevant only for those with wavelengths of the order of the lattice spacing. The net result is that on such a lattice long wavelength outgoing modes are seen to originate from short wavelength incoming modes via a process analogous to the Bloch oscillations of accelerated electrons in crystals [89, 189].

### 5.1.3 Horizon stability

Although closely related, as we will soon see, we have to distinguish carefully between the mode analysis of a linear field theory (with or without modified dispersion relations – MDR) over a fixed background and the stability analysis of the background itself.

Let us consider a three-dimensional irrotational and inviscid fluid system with a stationary sink-type of flow (see Figures 1, 2). The details of the configuration are not important for the following

[25] Reference [89] also considered the case of a lattice with proper distance spacing constant in time but this has the problem that the proper spacing of the lattice goes to zero at spatial infinity, and hence there is no fixed short distance cutoff.

discussion, only the fact that there is a spherically symmetric fluid flow accelerating towards a central sink, that sink being surrounded by a sphere acting as a sonic horizon. Then, as we have discussed in Section 2, linearizing the Euler and continuity equations leads to a massless scalar field theory over a black-hole-like spacetime. (We are assuming that the hydrodynamic regime remains valid up to arbitrarily short length scales; for instance, we are neglecting the existence of MDR). To be specific, let us choose the geometry of the canonical acoustic black-hole spacetime described in [389]:

$$\mathrm{d}s^2 = -c^2 \left(1 - \frac{r_0^4}{r^4}\right) \mathrm{d}\tau^2 + \left(1 - \frac{r_0^4}{r^4}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2\right). \tag{250}$$

In this expression we have used the Schwarzschild time coordinate $\tau$ instead of the lab time $t$; $c$ is constant. If we expand the field in spherical harmonics,

$$\phi_{lm}(\tau, r, \theta, \varphi) \equiv e^{-i\omega\tau} \, \frac{\chi_{lm}(r)}{r} \, Y_{lm}(\theta, \varphi), \tag{251}$$

we obtain the following equation for the radial part of the field:

$$\frac{\omega^2}{c^2} \, \chi = \left( -\frac{\mathrm{d}^2}{\mathrm{d}r^{*2}} + V_l(r) \right) \chi; \tag{252}$$

where

$$V_l(r) = \left(1 - \frac{r_0^4}{r^4}\right) \left[ \frac{l(l+1)}{r^2} + \frac{4r_0^4}{r^6} \right]. \tag{253}$$

Here

$$r^* \equiv r - (r_0/4)\{\ln(r+r_0)/(r-r_0) + 2 \arctan r/r_0\} \tag{254}$$

is a “tortoise” coordinate.

In a normal mode analysis one requires boundary conditions such that the field is regular everywhere, even at infinity. However, if one is analysing the solutions of the linear field theory as a way of probing the stability of the background configuration, one can consider less restrictive boundary conditions. For instance, one can consider the typical boundary conditions that lead to quasinormal modes: These modes have to be purely out-going at infinity and purely in-going at the horizon; but one does not require, for example, the modes to be normalizable. The quasinormal modes associated to this sink configuration have been analysed in [31]. The results found are qualitatively similar to those in the classical linear stability analysis of the Schwarzschild black hole in general relativity [384, 385, 317, 431, 267]. Of course, the gravitational field in general relativity has two dynamical degrees of freedom – those associated with gravitational waves – that have to be analysed separately; these are the “axial” and “polar” perturbations. In contrast, in the present situation we only have scalar perturbations. Nevertheless, the potentials associated with “axial” and “polar” perturbations of Schwarzschild spacetime, and that associated with scalar perturbations of the canonical acoustic black hole, produce qualitatively the same behaviour: There is a series of *damped* quasinormal modes – proving the linear stability of the system – with higher and higher damping rates.

An important point we have to highlight in here is that although in the linear regime the dynamical behaviour of the acoustic system is similar to general relativity, this is no longer true once one enters the non-linear regime. The underlying nonlinear equations in the two cases are very different. The differences are so profound, that in the general case of acoustic geometries constructed from compressible fluids, there exist sets of perturbations that, independently of how small they are initially, can lead to the development of shocks, a situation completely absent in vacuum general relativity.

Now, given an approximately stationary, and at the very least metastable, classical black-hole-like configuration, a standard quantum mode analysis leads to the existence of Hawking radiation in the form of phonon emission. This shows, among other things, that quantum corrections to the classical behaviour of the system must make the configuration with a sonic horizon dynamically unstable against Hawking emission. Moreover, in an analogue system with quantum fluctuations that maintain strict adherence to the equivalence principle (no MDR) it must then be impossible to create an isolated truly stationary horizon by external means – any truly stationary horizon must be provided with an external power source to stabilise it against Hawking emission. That is, in an analogue system one could in principle, by manipulating external forces, compensate for the backreaction effects that in a physical general relativity scenario cause the horizon to shrink (or evaporate) and thus become non-stationary.

Let us describe what happens when one takes into account the existence of MDR. A wonderful physical system that has MDR explicitly incorporated in its description is the Bose–Einstein condensate. The macroscopic wave function of the BEC behaves as a classical irrotational fluid but with some deviations when short length scales become involved. (For length scales of the order of or shorter than the healing length.) What are the effects of the MDR on the dynamical stability of a black-hole-like configuration in a BEC? The stability of a sink configuration in a BEC has been analysed in [136, 137] but taking the flow to be effectively one-dimensional. What they found is that these configurations are dynamically unstable: There are modes satisfying the appropriate boundary conditions such that the imaginary parts of their associated frequencies are positive. These instabilities are associated basically with the bound states inside the black hole. The dynamical tendency of the system to evolve is suggestively similar to that in the standard evaporation process of a black hole in semiclassical general relativity. This observation alone could make us question whether the first signatures of a quantum theory underlying general relativity might show up in precisely this manner. Interest in this question is reinforced by a specific analysis in the "loop quantum gravity" approach to quantizing gravity that points towards the existence of fundamental MDR at high energies [134]. The formulation of effective gravitational theories that incorporate some sort of MDR at high energies is currently under investigation (see for example [247, 2]); these exciting developments are however beyond the scope of this review.

Before continuing with the discussion on the stability of configurations with horizons, and in order not to cause confusion between the different wording used when talking about the physics of BECs and the emergent gravitational notions on them, let us write down a quite loose but useful translation dictionary:

- The "classical" or macroscopic wave function of the BEC represents the classical spacetime of GR, but only when probed at long enough wavelengths.

- The "classical" long-wavelength perturbations to a background solution of the Gross–Pitaevskii equation correspond to classical gravitational waves in GR. Of course this analogy does not imply that these are spin 2 waves, it only points out that the perturbations are made from the same "substance" as the background configuration itself.

- The macroscopic wave function of the BEC, without the restriction of being probed only at long wavelengths, corresponds to some sort of semiclassical vacuum gravity. Its "classical" behaviour (in the sense that does not involve any probability notion) is already taking into account its underlying quantum origin.

- The Bogoliubov quantum quasiparticles over the "classical" wave function correspond to a further step away from semiclassical gravity in that they are analogous to the existence of quantum gravitons over a (semiclassical) background spacetime.

At this point we would like to remark, once again, that the analysis based on the evolution of a BEC has to be used with care. For example, they cannot directly serve to shed light on what happens in the final stages of the evaporation of a black hole, as the BEC does not fulfil, at any regime, the Einstein equations.

Now continuing the discussion, what happens when treating the perturbations to the background BEC flow as quantum excitations (Bogoliubov quasiparticles)? What we certainly know is that the analysis of modes in a collapsing-to-form-a-black-hole background spacetime leads to the existence of radiation emission very much like Hawking emission [50, 87, 91]. (This is why it is said that Hawking's process is robust against modifications of the physics at high energies.) The comparison of these calculations with that of Hawking (without MDR), tells us that the main modification to Hawking's result is that now the Hawking flux of particles would not last forever but would vanish after a long enough time (this is why, in principle, we can dynamically create a configuration with a sonic horizon in a BEC). The emission of quantum particles reinforces the idea that the supersonic sink configurations are unstable.

Summarizing:

- If the perturbation to the BEC background configuration have "classical seeds" (that is, are describable by the linearised Gross–Pitaevskii equation alone), then, one will have "classical" instabilities.

- If the perturbations have "quantum seeds" (that is, are described by the Bogoliubov equations), then, one will have "quantum" instabilities.

In the light of the acoustic analogies it is natural to ask whether there are other geometric configurations with horizons of interest, besides the sink type of configurations (these are the most similar to the standard description of black holes in general relativity, but probably not the simplest in terms of realizability in a real laboratory; for an entire catalogue of them see [13]). Here, we are going to specifically mention two effectively one-dimensional black hole–white hole configurations, one in a straight line and one in a ring (see Figures 11 and 12, respectively).

A quantum mode analysis of the black hole–white hole configuration in a straight line, taking into account the existence of superluminal dispersion relations (similar to those in a BEC), led to the conclusion that the emission of particles in this configuration proceeds in a self-amplified (or runaway) manner [90]. We can understand this effect as follows: At the black horizon a virtual pair of phonons are converted into real phonons, the positive energy phonon goes towards infinity while the negative energy pair falls beyond the black horizon. However, the white horizon makes this negative energy phonon bounce back towards the black horizon (thanks to superluminal motion) stimulating the emission of additional phonons. Although related to Hawking's process this phenomenon has a quite different nature: For example, there is no temperature associated with it. A stability analysis of a configuration like this in a BEC would lead to strong instabilities. This same configuration, but compactified into a ring configuration, has been dynamically analysed in [136, 137]. What they found is that there are regions of stability and instability depending on the parameters characterizing the configuration. We suspect that the stability regions appear because of specific periodic arrangements of the modes around the ring. Among other reasons, these arrangements are interesting because they could be easier to create in laboratory with current technology and their instabilities easier to detect than Hawking radiation itself.

To conclude this subsection, we would like to highlight that there is still much to be learned by studying the different levels of description of an analogue system, and how they influence the stability or instability of configurations with horizons.

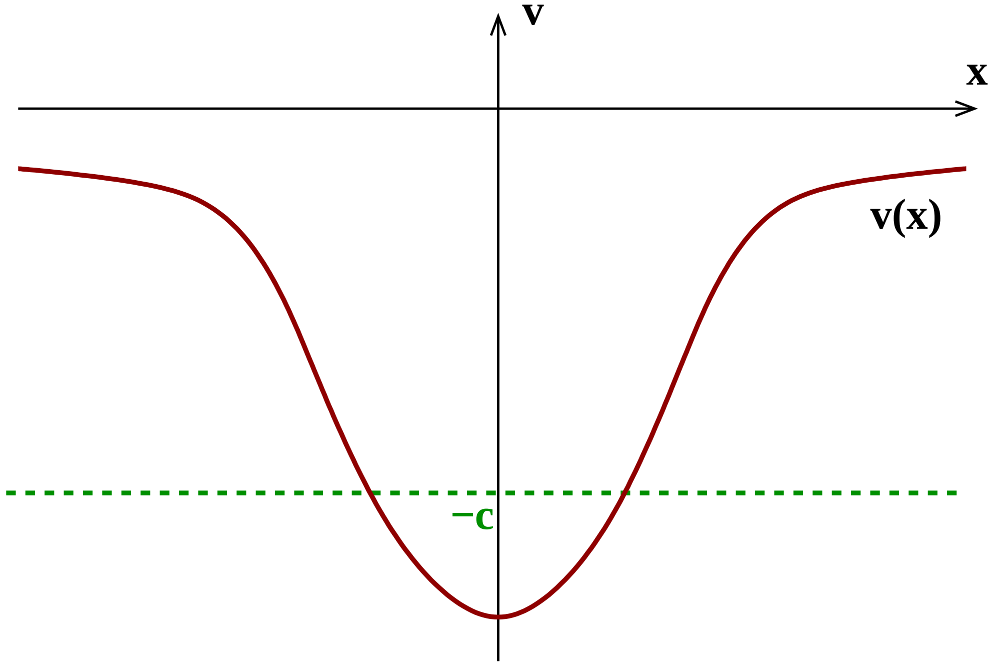


Figure 11: *One-dimensional velocity profile with a black hole horizon and a white hole horizon.*

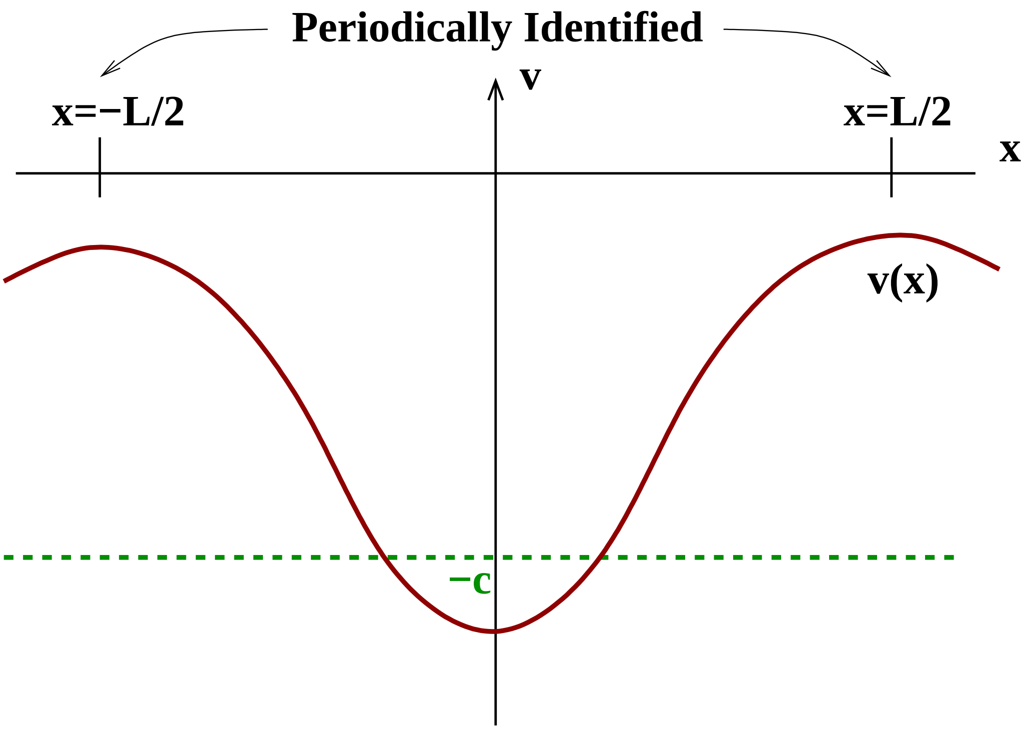


Figure 12: *One-dimensional velocity profile in a ring; the fluid flow exhibits two sonic horizons, one of black hole type and the other of white hole type.*

#### 5.1.4 Analogue spacetimes as background gestalt

In addition, among the many papers using analogue spacetimes as part of their background mindset when addressing these issues we mention:

- "Top down" calculations of Hawking radiation starting from some idealised model of quantum gravity [1, 201, 276, 277].
- "Bottom up" calculations of Hawking radiation starting from curved space quantum field theory [22, 23, 61, 62, 71, 111, 112, 128, 253, 254, 255, 256, 257, 298, 321].
- Trans-Hawking versions of Hawking radiation, either as reformulations of the physics, or as alternative scenarios [30, 68, 130, 131, 150, 156, 167, 168, 169, 233, 289, 291, 325, 329, 333, 334, 340, 341, 350, 364].
- Black hole entropy viewed in the light of analogue spacetimes [94].
- Hawking radiation interpreted as a statement about particles travelling along complex spacetime trajectories [292, 350, 360].

### 5.2 Super-radiance

Another phenomenon that has been (and is being) analysed from the analogue gravity perspective is super-radiance. The rotational kinetic energy accumulated in a rotating black hole can be extracted from it by scattering into it waves of sufficiently low frequency and high angular momentum. In general, in order that the wave can extract rotational energy from the system it has to satisfy the condition

$$\omega < m\,\Omega \tag{255}$$

where $\Omega$ is the angular speed of the black hole at the event horizon and $m$ is the harmonic azimuthal number of the wave. This is a purely classical process first considered by Penrose [305]. When dealing with quantum fields, as opposed to classical fields, this process can proceed spontaneously: Quantum mechanically, a rotating black hole will tend to radiate away all of its angular momentum, eventually approaching a non-rotating Schwarzschild black hole [429, 430]. This process is known as super-radiance. (The term super-radiance was already used in condensed matter to describe processes in which there was some coherent emission of radiation from an otherwise disordered system.)

Again, these processes have a purely kinematical origin so they are perfectly suitable for being reproduced in an analogue model. Regarding these processes, the simplest geometry that one can reproduce, thinking of analogue models based on fluid flows, is that of the draining bathtub of Section 2. Of course, this metric does not exactly correspond to Kerr geometry, nor even to a section of it [401]. However, it is *qualitatively* similar. It can be used to simulate both Penrose's classical process and quantum super-radiance as these effects do not depend on the specific multipole decomposition of Kerr's geometry, but only on its rotating character. A specific experimental setup has been put forward by Schützhold and Unruh using gravity waves in a shallow basin mimicking an ideal draining bathtub [345]. Equivalently to what happens with Kerr black holes, this configuration is classically stable (in the linear regime) [31]. A word of caution is in order here: Interactions of the gravity surface waves with bulk waves (neglected in the analysis) could cause the system to become unstable [415]. This instability has no counterpart in standard general relativity (though it might have one in braneworld theories). Super-resonant scattering of waves in this rotating sink configuration, or in a simple purely rotating vortex, could in principle be observed in this and other analogue models. There are already several articles dealing with this problem [25, 27, 26, 64, 113, 237].

A related phenomenon one can consider is the black-hole bomb mechanism [312]. One would only have to surround the rotating configuration by a mirror for it to become grossly unstable. What causes the instability is that those in-going waves that are amplified when reflected in the ergosphere would then in turn be reflected back toward the ergoregion, due to the exterior mirror, thus being amplified again, and so on.

An interesting phenomenon that appears in many condensed matter systems is the existence of quantised vortices. The angular momentum of these vortices comes in multiples of some fundamental unit (typically $\hbar$ or something proportional to $\hbar$). The extraction of rotational energy by a Penrose process in these cases could only proceed via finite-energy transitions. This would supply an additional specific signature to the process. In such a highly quantum configuration it is also important to look for the effect of having high-energy dispersion relations. For example, in BECs, the radius of the ergoregion of a single quantised vortex is of the order of the healing length, so one cannot directly associate an effective Lorentzian geometry to this portion of the configuration. Any analysis that neglects the high energy terms is not going to give any sensible result in these cases.

## 5.3 Cosmological geometries

Analogue model techniques have also been applied to cosmology. In a cosmological framework the key items of interest are the Friedmann–Robertson–Walker (FRW) geometries, more properly called the Friedmann–Lemaître–Robertson–Walker (FLRW) geometries. The simulation of such geometries has been considered in various works such as [17, 18, 116, 115, 114, 242, 58, 59, 424] with a specific view to enhancing our understanding of "cosmological particle production" driven by the expansion of the universe.

The acoustic metric can be written as

$$\mathrm{d}s^2 = \frac{\rho}{c_\mathrm{s}} \left[ -(c_\mathrm{s}^2 - v^2)\, \mathrm{d}t^2 - 2\mathbf{v}\cdot \mathbf{dx}\, \mathrm{d}t + \mathbf{dx}^2 \right]. \tag{256}$$

Essentially there are two ways to use this metric to reproduce cosmological spacetimes: One is based on physical explosion, the other on rapid variations in the "effective speed of light".

Following [18, 115, 202, 73, 203] one can take a homogeneous system $\rho(t), c_\mathrm{s}(t)$ and a radial profile for the velocity $\mathbf{v} = (\dot{b}/b)\mathbf{r}$, with $b$ a scale factor depending only on $t$. Then, defining a new radial coordinate as $r_b = r/b$ the metric can be expressed as

$$\mathrm{d}s^2 = \frac{\rho}{c_\mathrm{s}} \left[ -c_\mathrm{s}^2\, \mathrm{d}t^2 + b^2(\mathrm{d}r_b^2 + r_b^2\, \mathrm{d}\Omega_2^2) \right]. \tag{257}$$

Introducing a Hubble-like parameter,

$$H_b(t) = \frac{\dot{b}(t)}{b(t)}, \tag{258}$$

the equation of continuity can be written as

$$\dot{\rho} + 3H_b(t)\,\rho = 0; \qquad \Rightarrow \qquad \rho(t) = \frac{\rho_0}{b^3(t)}, \tag{259}$$

with $\rho_0$ constant. Finally we arrive at the metric of a flat FLRW geometry

$$\mathrm{d}s^2 = -T^2(t)\, \mathrm{d}t^2 + a_s^2(t)\, (\mathrm{d}r_b^2 + r_b^2\, \mathrm{d}\Omega_2^2), \tag{260}$$

with

$$T(t) \equiv \sqrt{\rho\, c_{\mathrm{s}}}; \qquad a_s(t) \equiv \sqrt{\frac{\rho}{c_{\mathrm{s}}}}\; b. \tag{261}$$

The proper Friedmann time, $\tau$, is related to the laboratory time, $t$, by

$$\tau = \int T(t)\; \mathrm{d}t. \tag{262}$$

The other avenue starts from a fluid at rest $v = 0$ with respect to the laboratory at all times:

$$\mathrm{d}s^2 = -\rho\, c_{\mathrm{s}}\; \mathrm{d}t^2 + \frac{\rho}{c_{\mathrm{s}}}\; \mathbf{dx}^2. \tag{263}$$

Now it is not difficult to imagine a situation in which $\rho$ remains constant, in a sufficiently large region of space, while the speed of sound decreases with time (this can be made in BECs for example by changing with time the value of the scattering length [17, 18]). This again reproduces an expanding flat FLRW Universe.

Models considered to date focus on variants of the BEC-inspired analogues:

- Fedichev and Fischer [115, 114] have investigated WKB estimates of the cosmological particle production rate and (1+1) dimensional cosmologies, both in expanding BECs.
- Lidsey [242], and Fedichev and Fischer [116] have focussed on the behaviour of cigar-like condensates in grossly asymmetric traps.
- Barcelo *et al.* [17, 18] have focussed on the central region of the BEC and thereby tried (at least locally) to mimic FLRW behaviour as closely as possible.
- Fischer and Schützhold [119] propose the use of two-component BECs to simulate cosmic inflation.
- Weinfurtner [425, 424] has concentrated on the approximate simulation of de Sitter spacetimes.

In all of these models the general expectations of the relativity community have been borne out – theory definitely predicts particle production, and the very interesting question is the extent to which the formal predictions are going to be modified when working with real systems experimentally [18]. We expect that these analogue models provide us with new insights as to how their inherent modified dispersion relations affect cosmological processes such as the generation of a primordial spectrum of perturbations (see for example [42, 41, 43, 44, 45, 46, 47, 70, 107, 158, 177, 178, 207], [229, 230, 244, 252, 249, 250, 251, 274, 275, 296, 349, 361, 362, 363, 371] where analogue-like ideas are applied to cosmological inflation).

An interesting side-effect of the original investigation, is that birefringence can now be used to model "variable speed of light" (VSL) geometries [28, 108]. Since analogue models quite often lead to two or more "excitation cones", rather than one, it is quite easy to obtain a bimetric or multi-metric model. If one of these metrics is interpreted as the "gravitational" metric and the other as the "photon" metric, then VSL cosmologies can be given a mathematically well-defined and precise meaning [28, 108].

### 5.4 Bose novae: an example of the reverse flow of information?

As we have seen in the previous sections, analogue models have in the past been very useful in providing new, condensed matter physics inspired, ideas about how to solve longstanding problems of semiclassical gravity. In closing this section, it is interesting to briefly discuss what perhaps represents, so far, the only attempt to use analogue models in the reverse direction; that is to import well known concepts of semiclassical gravity into condensed matter frameworks.

The phenomenon we are referring to is the so called "Bose nova" [104]. This is an experiment dealing with a gas of a few million $^{85}$Rb atoms at a temperature of about 3 nK. The condensate is rendered unstable by exploiting the possibility of tuning the interaction (more precisely the scattering length) between the atoms via a magnetic field. Reversing the sign of the interaction, making it attractive, destabilises the condensate. After a brief waiting time (generally called $t_{\mathrm{collapse}}$), the condensate implodes and loses a sizeable fraction of its atoms in the form of a "nova burst". If left to evolve undisturbed, the number of atoms in the burst stabilises and a remnant condensate is left. However if the condensate interaction is again made repulsive after some time $t_{\mathrm{evolve}}$, before the condensate has sufficient time to stabilise, then the formation of "jets" of atoms is observed, these jets being characterised by lower kinetic energy and a distinct shape with respect to the burst emission.

Interestingly, an elegant explanation of such a phenomenology was proposed in [58, 59], based on the well known semiclassical gravity analysis of particle creation in an expanding universe. In fact the dynamics of quantum excitations over the collapsing BEC was shown to closely mimic that for quantum excitations in a time-reversed (collapsing instead of expanding) scenario for cosmological particle creation. This is not so surprising as the quantum excitations above the BEC ground state feel a time-varying background during the collapse, and as a consequence one then expects squeezing of the vacuum state and mode mixing which are characteristic of quantum field theory in variable external fields.

However the analogy is even deeper than this. In fact in [58, 59] a key role in explaining the observed burst and jets is played by the concepts of "frozen" versus "oscillating" modes – borrowed from cosmology – (although with a reverse dynamics with respect to the standard (expanding) cosmological case). In the case of Bose novae the modes which are amplified are those for which the physical frequency is smaller than the collapse rate, while modes with higher frequencies remain basically unaffected and their amplitudes obey a harmonic oscillator equation. As the collapse rate decreases, more and more modes stop growing and start oscillating, which is equivalent to a creation of particles from the quantum vacuum. In the case of a sudden stop of the collapse by a new reversal of the sign of the interaction all of the previously growing modes are suddenly converted into particles, explaining in this way the generation of jets and their lower energy (they correspond to modes with lower frequencies with respect to those generating the bursts).

Although this simple model cannot explain all the details of the Bose novae phenomenology, we think it is remarkable how far it can go in explaining several observed features by exploiting the language and techniques so familiar to quantum cosmology. In this sense the analysis presented in [58, 59] primarily shows a possible new application of analogue models where they could be used to lend ideas and techniques developed in the context of gravitational physics to the explanation of condensed matter phenomena.

### 5.5 Going further

For more details on the trans-Planckian problem some of the key papers are the relatively early papers of Unruh [377, 378], and Jacobson [185, 186]. For super-radiance and cosmological issues there seems to be considerable ongoing interest, and one should carefully check Spires for the most recent articles.

# 6 Future Directions

The key question one should ask at this stage is: "Where can we go from here?" Apart from continuing with the analysis of the issues described in the previous section, there are additional interesting routes, both theoretical and experimental, worth of future exploration. In particular, the following topics come to mind:

- Back reaction,
- Equivalence principle,
- Emergent gravity,
- Quantum gravity phenomenology,
- Quantum gravity,
- Experimental analogue gravity.

Some work has already been done dealing with these topics in the context of analogue gravity. Let us now expand on these issues a little.

## 6.1 Back reaction

There are important phenomena in gravitational physics whose understanding needs analysis well beyond classical general relativity and field theory on curved background spacetimes. The black hole evaporation process can be considered as paradigmatic among these phenomena. Here, we particularise our discussion to this case. Since we are currently unable to analyse the entire process of black hole evaporation within a complete quantum theory of gravity, a way of proceeding is to analyse the simpler (but still extremely difficult) problem of semiclassical back reaction (see for example [99, 79, 34, 132, 51, 257]). One takes a background black hole spacetime, calculates the expectation value of the quantum energy-momentum tensor of matter fields in the appropriate quantum state (the Unruh vacuum state for a radiating black hole), and then takes this expectation value as a source for the perturbed Einstein equations. This calculation gives us information about the tendency of spacetime to evolve under vacuum polarization effects.

A nice feature of analogue models of general relativity is that although the underlying classical equations of motion have nothing to do with Einstein equations, the tendency of the analogue geometry to evolve due to quantum effects is formally equivalent (approximately, of course) to that in semiclassical general relativity. Therefore, the onset of the back reaction effects (if not their precise details) can be simulated within the class of analogue models. An example of the type of back-reaction calculations one can perform are those in [9, 10]. These authors started from an effectively one-dimensional acoustic analogue model, configured to have an acoustic horizon by using a Laval nozzle to control the flow's speed. They then considered the effect of quantizing the acoustic waves over the background flow. To calculate the appropriate back reaction terms they took advantage of the classical conformal invariance of the (1+1)-dimensional reduction of the system. In this case, we know explicitly the form of the expectation value of the energy-momentum tensor trace (via the trace anomaly). The other two independent components of the energy-momentum tensor were approximated by the Polyakov stress tensor. In this way, what they found is that the tendency of a left-moving flow with one horizon is for it to evolve in such a manner as to push the horizon down-stream at the same time that its surface gravity is decreased. This is a behaviour similar to what is found for near-extremal Reissner–Nordström black holes. (However, we should not conclude that acoustic black holes are in general closely related to near-extremal Reissner–Nordström black holes, rather than to Schwarzschild black holes. This result is quite specific to the particular one-dimensional configuration analysed.)

Can we expect to learn something new about gravitational physics by analysing the problem of back reaction in different analogue models? As we have repeatedly commented, the analyses based on analogue models force us to consider the effects of modified high-energy dispersion relations. For example, in BECs, they affect the "classical" behaviour of the background geometry as much as the behaviour of the quantum fields living on the background. In seeking a semiclassical description for the evolution of the geometry, one would have to compare the effects caused by the modified dispersion relations to those caused by pure semiclassical back reaction (which incorporates deviations from standard general relativity as well). In other words, one would have to understand the differences between the standard back reaction scheme in general relativity, and that based on Equations (186) and (187).

To end this subsection, we would like to comment that one can go beyond the semiclassical back-reaction scheme by using the so-called stochastic semiclassical gravity programme [179, 180, 181]. This programme aims to pave the way from semiclassical gravity toward a complete quantum-gravitational description of gravitational phenomena. This stochastic gravity approach not only considers the expectation value of the energy-momentum tensor but also its fluctuations, encoded in the semiclassical Einstein–Langevin equation. In a very interesting paper [299], Parentani showed that the effects of the fluctuations of the metric (due to the in-going flux of energy at the horizon) on the out-going radiation led to a description of Hawking radiation similar to that obtained with analogue models. It would be interesting to develop the equivalent formalism for quantum analogue models and to investigate the different emerging approximate regimes.

## 6.2 Equivalence principle

Analogue models are of particular interest in that the analogue spacetimes that emerge often violate, to some extent, the Einstein equivalence principle [16, 399]. Since the Einstein equivalence principle (or more precisely the universality of free fall) is experimentally tested to the accuracy of about 1 part in $10^{13}$, it is important to build this principle into realistic models of gravitation – most likely at a fundamental level.

One way of interpreting the Einstein equivalence principle is as a "principle of universality" for the geometrical structure of spacetime. Whatever the spacetime geometrical structure is, if all excitations "see" the same geometry one is well on the way to satisfying the observational and experimental constraints. In a metric theory, this amounts to the demand of mono-metricity: A *single* universal metric must govern the propagation of all excitations.

Now it is this feature that is relatively difficult to arrange in analogue models. If one is dealing with a single degree of freedom then mono-metricity is no great constraint. But with multiple degrees of freedom, analogue spacetimes generally lead to refringence – that is the occurrence of Fresnel equations that often imply multiple propagation speeds for distinct normal modes. To even obtain a bi-metric model (or more generally, a multi-metric model), requires an algebraic constraint on the Fresnel equation that it completely factorises into a product of quadratics in frequency and wavenumber. Only if this algebraic constraint is satisfied can one assign a "metric" to each of the quadratic factors. If one further wishes to impose mono-metricity then the Fresnel equation must be some integer power of some single quadratic expression, an even stronger algebraic statement [16, 399].

Faced with this situation, there are two ways in which the analogue gravity community might proceed:

1. Try to find a broad class of analogue models (either physically based or mathematically idealised) that naturally lead to mono-metricity. Little work along these lines has yet been done; at least partially because it is not clear what features such a model should have in order to be "clean" and "compelling".

2. Accept refringence as a common feature of the analogue models and attempt to use refringence to ones benefit in one or more ways:

   - There are real physical phenomena in non-gravitational settings that definitely do exhibit refringence and sometimes multi-metricity. Though situations of this type are not directly relevant to the gravity community, there is significant hope that the mathematical and geometrical tools used by the general relativity community might in these situations shed light on other branches of physics.
   - Use the refringence that occurs in many analogue models as a way of "breaking" the Einstein equivalence principle, and indeed as a way of "breaking" even more fundamental symmetries and features of standard general relativity, with a view to exploring possible extensions of general relativity. While the analogue models are not themselves primary physics, they can nevertheless be used as a way of providing *hints* as to how more fundamental physics might work.

## 6.3 Emergent gravity

One of the more fascinating approaches to "quantum gravity" is the suggestion, typically attributed to Sakharov [332, 393], that gravity itself may not be "fundamental physics". Indeed it is now a relatively common opinion, maybe not mainstream but definitely a strong minority opinion, that gravity (and in particular the whole notion of spacetime and spacetime geometry) might be no more "fundamental" than is fluid dynamics. The word "fundamental" is here used in a rather technical sense – fluid mechanics is not fundamental because there is a known underlying microphysics, that of molecular dynamics, of which fluid mechanics is only the low-energy low-momentum limit. Indeed the very concepts of density and velocity field, which are so central to the Euler and continuity equations, make no sense at the microphysical level and emerge only as one averages over timescales and distance-scales larger than the mean free time and mean free path.

In the same way, it is plausible (even though no specific and compelling model of the relevant microphysics has yet emerged) that the spacetime manifold and spacetime metric might arise only once one averages over suitable microphysical degrees of freedom. Sakharov had in mind a specific model in which gravity could be viewed as an "elasticity" of the spacetime medium, and was "induced" via one-loop physics in the matter sector [332, 393]. In this way Sakharov had hoped to relate the observed value of Newton's constant (and the cosmological constant) to the spectrum of particle masses.

More generally the phrase "emergent gravity" is now used to describe the whole class of theories in which the spacetime metric arises as a low-energy approximation, and in which the microphysical degrees of freedom might be radically different. Analogue models, and in particular analogue models based on fluid mechanics or the fluid dynamic approximation to BECs, are specific examples of "emergent physics" in which the microphysics is well understood. As such they are useful for providing *hints* as to how such a procedure might work in a more fundamental theory of quantum gravity.

## 6.4 Quantum gravity – phenomenology

Over the last few years a widespread consensus has emerged that observational tests of quantum gravity are for the foreseeable future likely to be limited to precision tests of dispersion relations and their possible deviations from Lorentz invariance [261]. The key point is that at low energies (well below the Planck energy) one expects the locally Minkowskian structure of the spacetime manifold to guarantee that one sees only special relativistic effects; general relativistic effects are negligible at short distances. However as ultra high energies are approached (although still below Planck scale energies) several quantum gravity models seem to predict that the locally Euclidean geometry of the spacetime manifold will break down. There are several scenarios for the origin of this breakdown ranging from string theory [214, 109] to brane worlds [54] and loop quantum gravity [134]. Common to all such scenarios is that the microscopic structure of spacetime is likely to show up in the form of a violation of Lorentz invariance leading to modified dispersion relations for elementary particles. Such dispersion relations are characterised by extra terms (with respect to the standard relativistic form) which are generally expected to be suppressed by powers of the Planck energy. Remarkably the last years have seen a large wealth of work in testing the effects of such dispersion relations and in particular strong constraints have been cast by making use of high energy astrophysics observations (see for example [3, 82, 195, 194, 196, 197, 261, 355] and references therein).

Several of the analogue models are known to exhibit similar behaviour, with a low-momentum effective Lorentz invariance eventually breaking down at high momentum once the microphysics is explored.[26] Thus some of the analogue models provide controlled theoretical laboratories in which at least some forms of subtle high-momentum breakdown of Lorentz invariance can be explored. As such the analogue models provide us with *hints* as to what sort of modified dispersion relation might be natural to expect given some general characteristics of the microscopic physics. Hopefully investigation of appropriate analogue models might be able to illuminate possible mechanisms leading to this kind of quantum gravity phenomenology, and so might be able to provide us new ideas about other effects of physical quantum gravity that might be observable at sub-Planckian energies.

## 6.5 Quantum gravity – fundamental models

When it comes to dealing with "fundamental" theories of quantum gravity, the analogue models play an interesting role which is complementary to the more standard approaches. The search for a quantum theory of gravity is fundamentally a search for an appropriate mathematical structure in which to simultaneously phrase both quantum questions and gravitational questions. More precisely, one is searching for a mathematical framework in which to develop an abstract quantum theory which then itself encompasses classical Einstein gravity (the general relativity), and reduces to it in an appropriate limit [65, 356, 129].

The three main approaches to quantum gravity currently in vogue, "string models" (also known as "M-models"), "loop space" (and the related "spin foams"), and "lattice models" (Euclidean or Lorentzian) all share one feature: They attempt to develop a "pre-geometry" as a replacement for classical differential geometry (which is the natural and very successful mathematical language used to describe Einstein gravity) [65, 356, 129, 327, 326, 40]. The basic idea is that the mooted replacement for differential geometry would be relevant at extremely small distances (where the quantum aspects of the theory would be expected to dominate), while at larger distances (where the classical aspects dominate) one would hope to recover both ordinary differential geometry and specifically Einstein gravity or possibly some generalization of it. The "string", "loop", and "lattice" approaches to quantum gravity differ in detail in that they emphasise different features of the long-distance model, and so obtain rather different short-distance replacements for classical differential geometry. Because the relevant mathematics is extremely difficult, and by and large not particularly well understood, it is far from clear which if any of these three standard approaches will be preferable in the long run [356].

We feel it likely that analogue models can shed new light on this very confusing field by

[26] It is however important to keep in mind that not all the above cited quantum gravity models violate the Lorentz symmetry in the same manner. The discreteness of spacetime at short scales is not the only way of breaking Lorentz invariance.

providing a concrete specific situation in which the transition from the short-distance "discrete" or "quantum" theory to the long-distance "continuum" theory is both well understood and non-controversial. Here we are specifically referring to fluid mechanics, where at short distances the system must be treated using discrete atoms or molecules as the basic building blocks, while at large distances there is a well-defined continuum limit that leads to the Euler and continuity equations. Furthermore once one is in the continuum limit, there is a well-defined manner in which a notion of "Lorentzian differential geometry", and in particular a "Lorentzian effective spacetime metric" can be assigned to any particular fluid flow [376, 389, 284]. Indeed, the "analogue gravity programme" is extremely successful in this regard, providing a specific and explicit example of a "discrete" → "continuum" → "differential geometry" chain of development. What the "analogue gravity programme" does not seem to do as easily is to provide a natural direct route to the Einstein equations of general relativity, but that merely indicates that current analogies have their limits and therefore, one should not take them too literally [389, 284]. Fluid mechanics is a guide to the mathematical possibilities, not an end in itself. The parts of the analogy that do work well are precisely the steps where the standard approaches to quantum gravity have the most difficulty, and so it would seem useful to develop an abstract mathematical theory of the "discrete" → "continuum" → "differential geometry" chain using this fluid mechanical analogy (and related analogies) as inspiration.

## 6.6 Going further

Beyond the various theoretical issues we have discussed above there is the important question of "experimental analogue gravity" – to what extent can all these ideas be tested in direct laboratory experiments? Currently several experimental groups are interested, but to the best of our knowledge no actual experiments are currently underway. Broadly speaking, for any experimental group interested in analogue spacetimes the two key issues to address are:

- Identify a particular analogue model easily amenable to laboratory investigation, and double check the extent to which the model provides a theoretically robust and clean analogue to general relativistic curved spacetime.
- Identify the technical issues involved in actually setting up a laboratory experiment.

While the consensus in the theoretical community is that Bose–Einstein condensates are likely to provide the best working model for analogue gravity, it is possible that we might still be surprised by experimental developments. We leave this as an open challenge to the experimental community.

# 7 Conclusions

In this review article we have seen the interplay between standard general relativity and various analogies that can be used to capture aspects of its behaviour. These analogies have ranged from rather general but very physical analogue models based on fluid-acoustics, geometrical optics, and wave optics, to highly specific models based on BECs, liquid helium, slow light, etc. Additionally, we have seen several rather abstract mathematical toy models that bring us to such exotic structures and ideas as birefringence, bimetricity, Finsler spaces, and Sakharov's induced gravity.

The primary reason that these analogies were developed within the general relativity community was to help in the understanding of general relativity by providing very down-to-earth models of otherwise subtle behaviour in general relativity. Secondary reasons include the rather speculative suggestion that there may be more going on than just analogy – it is conceivable (though perhaps unlikely) that one or more of these analogue models could suggest a relatively simple and useful way of quantizing gravity that side-steps much of the technical machinery currently employed in such efforts. A tertiary concern (at least as far as the general relativity community is concerned) is the use of relativity and differential geometric techniques to improve understanding of various aspects of condensed matter physics.

The authors expect interest in analogue models to continue unabated, and suspect that there are several key but unexpected issues whose resolution would be greatly aided by the analysis of appropriate analogue models

## 7.1 Going further

Though every practicing scientist already knows this, for the sake of any student reading this we mention the following resources:

- http://www.slac.stanford.edu/spires/ – the bibliographic database for keeping track of (almost all of) the relevant literature.
- http://www.arXiv.org – the electronic-preprint (e-print) database for accessing the text of (almost all, post 1992) relevant articles.
- http://www.livingreviews.org/ – the *Living Reviews* portal.

Those three access points should allow you to keep abreast of what is going on in the field.

# 8 Acknowledgements

The work of Matt Visser was supported by the Marsden fund administered by the Royal Society of New Zealand. Carlos Barceló has been supported by a Marie Curie European Reintegration Grant, and by the Education and Science Council of Junta de Andalucía (Spain). We also wish to thank Enrique Arilla for Figures 1 and 2 of the art work.

# References


[1] Amati, D., and Russo, J.G., "Black holes by analytic continuation", *Phys. Rev. D*, **56**, 974–982, (1997). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9602125. 5.1.4

[2] Amelino-Camelia, G., "Doubly-Special Relativity: First Results and Key Open Problems", *Int. J. Mod. Phys. D*, **11**, 1643–1669, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0210063. 5.1.3

[3] Amelino-Camelia, G., Ellis, J.R., Mavromatos, N.E., Nanopoulos, D.V., and Sarkar, S., "Potential Sensitivity of Gamma-Ray Burster Observations to Wave Dispersion in Vacuo", *Nature*, **393**, 763–765, (1998). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/9712103. 6.4

[4] Anglin, J.R., "Influence functionals and the accelerating detector", *Phys. Rev. D*, **47**, 4525–4537, (1993). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9210035. 3.4

[5] Antunes, N.D., "Numerical simulation of vacuum particle production: applications to cosmology, dynamical Casimir effect and time-dependent non-homogeneous dielectrics", (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/hep-ph/0310131. 3.4

[6] Arbona, A., "Is a classical Euclidean TOE reasonable?", (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0310007. 3.4

[7] Arteaga, D., Parentani, R., and Verdaguer, E., "Propagation in a thermal graviton background", *Phys. Rev. D*, **70**, 044019, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0311065. 3.4

[8] Balazs, N.L., "Effect of a gravitational field, due to a rotating body, on the plane of polarization of an electromagnetic wave", *Phys. Rev.*, **110**, 236–239, (1958). 3.2.1

[9] Balbinot, R., Fagnocchi, S., and Fabbri, A., "Quantum effects in acoustic black holes: The backreaction", *Phys. Rev. D*, **71**, 064019–1–11, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0405098. 3.1.6, 6.1

[10] Balbinot, R., Fagnocchi, S., Fabbri, A., and Procopio, G.P., "Backreaction in Acoustic Black Holes", *Phys. Rev. Lett.*, **95**, 161302–1–4, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0405096. 3.1.6, 6.1

[11] Baldovin, F., Novello, M., Perez Bergliaffa, S.E., and Salim, J.M., "A nongravitational wormhole", *Class. Quantum Grav.*, **17**, 3265–3276, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0003075. 3.1.2

[12] Barceló, C., and Campos, A., "Braneworld physics from the analog-gravity perspective", *Phys. Lett. B*, **563**, 217–223, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0206217. 3.1.4

[13] Barceló, C., Liberati, S., Sonego, S., and Visser, M., "Causal structure of analogue spacetimes", *New J. Phys.*, **6**, 186, (2004). URL (cited on 31 May 2005): http://stacks.iop.org/NJP/6/186. 6, 3.1.6, 2, 5.1.3

[14] Barceló, C., Liberati, S., and Visser, M., "Analog gravity from Bose–Einstein condensates", *Class. Quantum Grav.*, **18**, 1137–1156, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0011026. 3.1.2

[15] Barceló, C., Liberati, S., and Visser, M., "Analog gravity from field theory normal modes?", *Class. Quantum Grav.*, **18**, 3595–3610, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0104001. 2.8, 2.8, 3.1.3

[16] Barceló, C., Liberati, S., and Visser, M., "Refringence, field theory, and normal modes", *Class. Quantum Grav.*, **19**, 2961–2982, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0111059. 3.1.3, 4.1.4, 4.2.1, 6.2

[17] Barceló, C., Liberati, S., and Visser, M., "Analogue models for FRW cosmologies", *Int. J. Mod. Phys. D*, **12**, 1641–1650, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0305061. 3.1.5, 5.3, 5.3

[18] Barceló, C., Liberati, S., and Visser, M., "Probing semiclassical analogue gravity in Bose–Einstein condensates with widely tunable interactions", *Phys. Rev. A*, **68**, 053613, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/0307491. 3.1.5, 4.2.1, 5.3, 5.3, 5.3

[19] Barceló, C., Liberati, S., and Visser, M., "Towards the Observation of Hawking Radiation in Bose–Einstein Condensates", *Int. J. Mod. Phys. A*, **18**, 3735–1–11, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0110036. 3.1.3, 4.2.1, 2, 4

[20] Barceló, C., Visser, M., and Liberati, S., "Einstein gravity as an emergent phenomenon?", *Int. J. Mod. Phys. D*, **10**, 799–806, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0106002. 3.1.3

[21] Bardeen, J.M., Carter, B., and Hawking, S.W., "The four laws of black hole mechanics", *Commun. Math. Phys.*, **31**, 161–170, (1973). 6

[22] Barrabès, C., Frolov, Valeri P., and Parentani, R., "Metric fluctuation corrections to Hawking radiation", *Phys. Rev. D*, **59**, 124010–1–14, (1999). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9812076. 5.1.4

[23] Barrabès, C., Frolov, V.P., and Parentani, R., "Stochastically fluctuating black-hole geometry, Hawking radiation and the trans-Planckian problem", *Phys. Rev. D*, **62**, 044020–1–19, (2000). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0001102. 5.1.4

[24] Basak, S., "Sound wave in vortex with sink", (2003). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0310105. 3.1.5

[25] Basak, S., "Analog of Superradiance effect in BEC", (2005). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0501097. 3.1.7, 5.2

[26] Basak, S., and Majumdar, P., "Reflection coefficient for superresonant scattering", *Class. Quantum Grav.*, **20**, 2929–2936, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0303012. 3.1.5, 5.2

[27] Basak, S., and Majumdar, P., "'Superresonance' from a rotating acoustic black hole", *Class. Quantum Grav.*, **20**, 3907–3913, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0203059. 3.1.4, 5.2

[28] Bassett, B.A., Liberati, S., Molina-París, C., and Visser, M., "Geometrodynamics of variable-speed-of-light cosmologies", *Phys. Rev. D*, **62**, 103518–1–18, (2000). Related online version (cited on 31 May 2005): http://arXiv.org/abs/astro-ph/0001441. 3.1.2, 5.3

[29] Bastero-Gil, M., "What can we learn by probing trans-Planckian physics", in Khalil, S., Shafi, Q., and Tallat, H., eds., *International Conference on High Energy Physics*, Proceedings of the International Conference on High Energy Physics, January 9–14, 2001, Cairo, Egypt, 283–288, (Rinton Press, Princeton, U.S.A., 2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-ph/0106133. 3.4

[30] Belgiorno, F., "Black Hole Thermodynamics in Carathéodory's Approach", *Phys. Lett. A*, **312**, 324–330, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0210020. 5.1.4

[31] Berti, E., Cardoso, V., and Lemos, J.P.S., "Quasinormal modes and classical wave propagation in analogue black holes", *Phys. Rev. D*, **70**, 124006, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0408099. 3.1.6, 5.1.3, 5.2

[32] Bhattacharyya, G., Mathews, P., Rao, K., and Sridhar, K., "Searching for signals of minimal length in extra dimensional models using dilepton production at hadron colliders", *Phys. Lett. B*, **603**, 46–50, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-ph/0408295. 3.4

[33] Bilic, N., "Relativistic Acoustic Geometry", *Class. Quantum Grav.*, **16**, 3953–3964, (1999). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9908002. 16

[34] Birrell, N.D., and Davies, P.C.W., *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics, (Cambridge University Press, Cambridge, U.K., 1982). 6.1

[35] Błaut, A., Kowalski-Glikman, J., and Nowak-Szczepaniak, D., "$\kappa$-Poincaré dispersion relations and the black hole radiation", *Phys. Lett. B*, **521**, 364–370, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0108069. 3.4

[36] Bogoliubov, N., "On the theory of superfluidity", *J. Phys. (Moscow)*, **11**, 23, (1947). 1

[37] Bombelli, L., and Sonego, S., "Relationships between various characterizations of wave tails", *J. Phys. A*, **27**, 7177–7199, (1994). 3.4

[38] Boonserm, P., Cattoen, C., Faber, T., Visser, M., and Weinfurtner, S., "Effective refractive index tensor for weak field gravity", *Class. Quantum Grav.*, **22**, 1905–1915, (2005). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0411034. 3.1.6

[39] Born, M., and Wolf, E., *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*, (Pergamon, Oxford, U.K.; New York, U.S.A., 1980), 6th edition. 4.1.3.3, 4.1.3.4, 4.1.4

[40] Bousso, R., and Polchinski, J., "The string theory landscape", *Sci. Am.*, **291**, 60–69, (2004). 6.5

[41] Brandenberger, R.H., "Frontiers of inflationary cosmology", *Braz. J. Phys.*, **31**, 131–146, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-ph/0102183. 5.3

[42] Brandenberger, R.H., "A status review of inflationary cosmology", (2001). URL (cited on 31 May 2005): http://arXiv.org/abs/hep-ph/0101119. 5.3

[43] Brandenberger, R.H., "Trans-Planckian Physics and Inflationary Cosmology", (2002). URL (cited on 31 May 2005): http://arXiv.org/abs/hep-th/0210186. 5.3

[44] Brandenberger, R.H., "Lectures on the theory of cosmological perturbations", *Lect. Notes Phys.*, **646**, 127–167, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/0306071. 5.3

[45] Brandenberger, R.H., Joras, S.E., and Martin, J., "Trans-Planckian physics and the spectrum of fluctuations in a bouncing universe", *Phys. Rev. D*, **66**, 083514–1–9, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/0112122. 5.3

[46] Brandenberger, R.H., and Martin, J., "The robustness of inflation to changes in super-Planck-scale physics", *Mod. Phys. Lett. A*, **16**, 999–1006, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/astro-ph/0005432. 5.3

[47] Brandenberger, R.H., and Martin, J., "On signatures of short distance physics in the cosmic microwave background", *Int. J. Mod. Phys. A*, **17**, 3663–3680, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/0202142. 5.3

[48] Brevik, I., and Halnes, G., "Light rays at optical black holes in moving media", *Phys. Rev. D*, **65**, 024005–1–12, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0106045. 3.1.3

[49] Brout, R., Gabriel, C., Lubo, M., and Spindel, P., "Minimal length uncertainty principle and the trans-Planckian problem of black hole physics", *Phys. Rev. D*, **59**, 044005–1–6, (1999). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/9807063. 3.4

[50] Brout, R., Massar, S., Parentani, R., and Spindel, P., "Hawking radiation without trans-Planckian frequencies", *Phys. Rev. D*, **52**, 4559–4568, (1995). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/9506121. 23, 5.1.2.2, 5.1.3

[51] Brout, R., Massar, S., Parentani, R., and Spindel, P., "A Primer for black hole quantum physics", *Phys. Rep.*, **260**, 329–454, (1995). 6.1

[52] Budker, D., Kimball, D.F., Rochester, S.M., and Yashchuk, V.V., "Nonlinear Magneto-optics and Reduced Group Velocity of Light in Atomic Vapor with Slow Ground State Relaxation", *Phys. Rev. Lett.*, **83**, 1767–1770, (1999). 4.2.4

[53] Burgess, C.P., "Quantum Gravity in Everyday Life: General Relativity as an Effective Field Theory", *Living Rev. Relativity*, **7**, lrr-2004-5, (2004). URL (cited on 31 May 2005): http://www.livingreviews.org/lrr-2004-5. 3.1.5

[54] Burgess, C.P., Cline, J.M., Filotas, E., Matias, J., and Moore, G.D., "Loop-generated bounds on changes to the graviton dispersion relation", *J. High Energy Phys.*, **2002**(03), 043, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-ph/0201082. 6.4

[55] Cadoni, M., "Acoustic analogues of two-dimensional black holes", *Class. Quantum Grav.*, **22**, 409–419, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0410138. 3.1.6

[56] Cadoni, M., and Mignemi, S., "Acoustic analogues of black hole singularities", (2005). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0504143. 3.1.7

[57] Calogeracos, A., and Volovik, G.E., "Rotational quantum friction in superfluids: Radiation from object rotating in superfluid vacuum", *J. Exp. Theor. Phys. Lett.*, **69**, 281–287, (1999). Related online version (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/9901163. 3.1.3

[58] Calzetta, E.A., and Hu, B.L., "Bose-Novae as Squeezing of the Vacuum by Condensate Dynamics", (2002). URL (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/0208569. 5.3, 5.4

[59] Calzetta, E.A., and Hu, B.L., "Bose-Novae as Squeezing of Vacuum Fluctuations by Condensate Dynamics", (2002). URL (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/0207289. 5.3, 5.4

[60] Calzetta, E.A., Hu, B.L., and Mazzitelli, F.D., "Coarse-grained effective action and renormalization group theory in semiclassical gravity and cosmology", *Phys. Rep.*, **352**, 459–520, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/0102199. 3.4

[61] Canfora, F., and Vilasi, G., "Back Reaction from Trace Anomaly in RN-blackholes Evaporation", *J. High Energy Phys.*, **2003**(12), 055, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0402017. 5.1.4

[62] Canfora, F., and Vilasi, G., "Trace anomaly and black holes evaporation", (2003). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0302036. 5.1.4

[63] Cardoso, V., "Acoustic black holes", (2005). URL (cited on 31 May 2005): http://arXiv.org/abs/physics/0503042. 3.1.7

[64] Cardoso, V., Lemos, J.P.S., and Yoshida, S., "Quasinormal modes and stability of the rotating acoustic black hole: Numerical analysis", *Phys. Rev. D*, **70**, 124032–1–7, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0410107. 3.1.6, 5.2

[65] Carlip, S., "Quantum gravity: A progress report", *Rep. Prog. Phys.*, **64**, 885–942, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0108040. 6.5

[66] Carter, B., "Relativistic superfluid models for rotating neutron stars", in Blaschke, D., Glendenning, N.K., and Sedrakian, A., eds., *Physics of Neutron Star Interiors*, vol. 578 of Lecture Notes in Physics, 54, (Springer, Berlin, Germany; New York, U.S.A., 2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0101257. 3.1.3

[67] Carter, B., and Chamel, N., "Covariant analysis of Newtonian multi-fluid models for neutron stars: I Milne–Cartan structure and variational formulation", *Int. J. Mod. Phys. D*, **13**, 291–326, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0305186. 3.1.5

[68] Casadio, R., "On dispersion relations and the statistical mechanics of Hawking radiation", *Class. Quantum Grav.*, **19**, 2453–2462, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0111287. 5.1.4

[69] Casadio, R., "On brane-world black holes and short scale physics", *Ann. Phys. (N.Y.)*, **307**, 195–208, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-ph/0304099. 3.4

[70] Casadio, R., and Mersini, L., "Short distance signatures in cosmology: Why not in black holes?", *Int. J. Mod. Phys. A*, **19**, 1395–1412, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0208050. 5.3

[71] Casher, A., Englert, F., Itzhaki, N., Massar, S., and Parentani, R., "Black hole horizon fluctuations", *Nucl. Phys. B*, **484**, 419–434, (1997). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9606106. 5.1.4

[72] Cassidy, M.J., and Hawking, S.W., "Models for chronology selection", *Phys. Rev. D*, **57**, 2372–2380, (1998). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9709066. 2.4

[73] Castin, Y., and Dum, R., "Bose–Einstein Condensates in Time Dependent Traps", *Phys. Rev. Lett.*, **77**, 5315–5319, (1996). 5.3

[74] Chang, D., Chu, C.-S., and Lin, F.-L., "Transplanckian dispersion relation and entanglement entropy of black hole", *Fortschr. Phys.*, **52**, 477–482, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0312136. 3.4

[75] Chang, D., Chu, C.-S., and Lin, F.-L., "Transplanckian entanglement entropy", *Phys. Lett. B*, **583**, 192–198, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0306055. 3.4

[76] Chapline, G., Hohlfeld, E., Laughlin, R.B., and Santiago, D.I., "Quantum phase transitions and the breakdown of classical general relativity", *Int. J. Mod. Phys. A*, **18**, 3587–3590, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0012094. 3.1.2

[77] Chapline, G., and Mazur, P.O., "Superfluid picture for rotating space-times", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0407033. 3.1.6

[78] Cherubini, C., Federici, F., Succi, S., and Tosi, M.P., "Excised acoustic black holes: The scattering problem in the time domain", *Phys. Rev. D*, **72**, 084016–1–9, (2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0504048. 3.1.7

[79] Christensen, S.M., and Fulling, S.A., "Trace anomalies and the Hawking effect", *Phys. Rev. D*, **15**, 2088–2104, (1977). 6.1

[80] Chruściel, P.T., "Black holes", in Frauendiener, J., and Friedrich, H., eds., *The Conformal Structure of Space-Time: Geometry, Analysis, Numerics*, Proceedings of the internationl workshop, Tübingen, Germany, April 2001, vol. 604 of Lecture Notes in Physics, 61–102, (Springer, Berlin, Germany; New York, U.S.A., 2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0201053. 3

[81] Chu, C.-S., Greene, B.R., and Shiu, G., "Remarks on inflation and noncommutative geometry", *Mod. Phys. Lett. A*, **16**, 2231–2240, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0011241. 3.4

[82] Coleman, S.R., and Glashow, S.L., "High-energy tests of Lorentz invariance", *Phys. Rev. D*, **59**, 116008–1–14, (1999). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-ph/9812418. 6.4

[83] Collins, H., and Martin, M.R., "The enhancement of inflaton loops in an $\alpha$-vacuum", *Phys. Rev. D*, **70**, 084021–1–9, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-ph/0309265. 3.4

[84] Comer, G.L., "Superfluid analog of the Davies–Unruh effect", (1992). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0505005. 2.7.3, 15

[85] Consoli, M., "Approximate Lorentz invariance of the vacuum: A physical solution of the 'hierarchy problem'?", (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/hep-ph/0306070. 3.4

[86] Corley, S., "Particle creation via high frequency dispersion", *Phys. Rev. D*, **55**, 6155–6161, (1997). 2.5.2

[87] Corley, S., "Computing the spectrum of black hole radiation in the presence of high frequency dispersion: An analytical approach", *Phys. Rev. D*, **57**, 6280–6291, (1998). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9710075. 5.1.2.2, 5.1.2.3, 5.1.3

[88] Corley, S., and Jacobson, T.A., "Hawking Spectrum and High Frequency Dispersion", *Phys. Rev. D*, **54**, 1568–1586, (1996). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9601073. 2.5.2, 15, 3, 23, 5.1.2.2, 5.1.2.2

[89] Corley, S., and Jacobson, T.A., "Lattice black holes", *Phys. Rev. D*, **57**, 6269–6279, (1998). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9709166. 5.1.2.5, 5.1.2.6, 25

[90] Corley, S., and Jacobson, T.A., "Black hole lasers", *Phys. Rev. D*, **59**, 124011–1–12, (1999). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9806203. 15, 3, 5.1.2.3, 5.1.3

[91] Corley, S.R., *The role of short distance physics in the Hawking effect*, Ph.D. Thesis, (University of Maryland, College Park, U.S.A., 1997). 5.1.3

[92] Courant, R., and Hilbert, D., *Methods of Mathematical Physics*, vol. 2 of Wiley Classics Library, (Interscience, New York, U.S.A., 1989). 4.1.4

[93] Czerniawski, J., "What is wrong with Schwarzschild's coordinates?", (2002). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0201037. 3.4

[94] Damour, T., "The entropy of black holes: A primer", in Dalibard, J., Duplantier, B., and Rivasseau, V., eds., *Poincaré Seminar 2003: Bose–Einstein Condensation – Entropy*, Proceedings of the third and fourth Poincaré Seminars, vol. 38 of Progress in Mathematical Physics, (Birkhäuser, Basel, Switzerland; Boston, U.S.A., 2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0401160. 5.1.4

[95] Das, S., "Black hole thermodynamics: Entropy, information and beyond", *Pramana*, **63**, 797–816, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0403202. 3.1.6

[96] Das, T.K., "Analogous Hawking Radiation from Astrophysical Black Hole Accretion", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0404482. 3.1.6

[97] Das, T.K., "Analogue Hawking radiation from astrophysical black hole accretion", *Class. Quantum Grav.*, **21**, 5253–5260, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0408081. 3.1.6

[98] Das, T.K., "Transonic Black Hole Accretion as Analogue System", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0411006. 3.1.6

[99] Davies, P.C.W., Fulling, S.A., and Unruh, W.G., "Energy momentum tensor near an evaporating black hole", *Phys. Rev. D*, **13**, 2720–2723, (1976). 6.1

[100] de Felice, F., "On the gravitational field acting as an optical medium", *Gen. Relativ. Gravit.*, **2**, 347–357, (1971). 3.2.1

[101] De Lorenci, V.A., and Klippert, R., "Analogue gravity from electrodynamics in nonlinear media", *Phys. Rev. D*, **65**, 064027–1–6, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0107008. 3.1.3

[102] De Lorenci, V.A., Klippert, R., Novello, M., and Salim, J.M., "Nonlinear electrodynamics and FRW cosmology", *Phys. Rev. D*, **65**, 063501–1–5, (2002). 3.1.4, 4.1.3.6

[103] De Lorenci, V.A., Klippert, R., and Obukhov, Y.N., "On optical black holes in moving dielectrics", *Phys. Rev. D*, **68**, 061502–1–4, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0210104. 3.1.4

[104] Donley, E.A., Claussen, N.R., Cornish, S.L., Roberts, J.L., Cornell, E.A., and Wieman, C.E., "Dynamics of collapsing and exploding Bose-Einstein condensates", *Nature*, **412**, 295–299, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/0105019. 5.4

[105] Dumin, Y.V., "Topological defect density in 1D FRW cosmological model: Corrections inferred from the MJJL experiment", (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/hep-ph/0308184. 3.1.5

[106] Dziarmaga, J., "Analog electromagnetism in a symmetrized 3He-A", (2001). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0112041. 3.1.3

[107] Easther, R., Greene, B.R., Kinney, W.H., and Shiu, G., "Inflation as a probe of short distance physics", *Phys. Rev. D*, **64**, 103502–1–8, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0104102. 3.1.3, 5.3

[108] Ellis, G.F.R., and Uzan, J.-P., " '*c*' is the speed of light, isn't it?", *Am. J. Phys.*, **73**, 240–247, (2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0305099. 5.3

[109] Ellis, J.R., Mavromatos, N.E., Nanopoulos, D.V., and Volkov, G., "Gravitational-recoil effects on fermion propagation in space-time foam", *Gen. Relativ. Gravit.*, **32**, 1777–1798, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9911055. 6.4

[110] Eltsov, V.B., Krusius, M., and Volovik, G.E., "Superfluid 3He: A Laboratory model system of quantum field theory", (1998). URL (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/9809125v1. 16

[111] Englert, F., "The Black hole history in tamed vacuum", (1994). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9408005. 5.1.4

[112] Englert, F., Massar, S., and Parentani, R., "Source vacuum fluctuations of black hole radiance", *Class. Quantum Grav.*, **11**, 2919–2938, (1994). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9404026. 5.1.4

[113] Federici, F., Cherubini, C., Succi, S., and Tosi, M.P., "Superradiance from BEC vortices: a numerical study", (2005). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0503089. 3.1.7, 5.2

[114] Fedichev, P.O., and Fischer, U.R., "Gibbons–Hawking Effect in the Sonic de Sitter Space-Time of an Expanding Bose–Einstein-Condensed Gas", *Phys. Rev. Lett.*, **91**, 240407, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/0304342. 3.1.5, 4.2.1, 5.3, 5.3

[115] Fedichev, P.O., and Fischer, U.R., " "Cosmological" quasiparticle production in harmonically trapped superfluid gases", *Phys. Rev. A*, **69**, 033602, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/0303063. 3.1.5, 4.2.1, 5.3, 5.3, 5.3

[116] Fedichev, P.O., and Fischer, U.R., "Observer dependence for the phonon content of the sound field living on the effective curved space-time background of a Bose–Einstein condensate", *Phys. Rev. D*, **69**, 064021, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/0307200. 3.1.5, 5.3, 5.3

[117] Fischer, U.R., "Motion of quantized vortices as elementary objects", *Ann. Phys. (N.Y.)*, **278**, 62–85, (1999). Related online version (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/9907457. 3.4

[118] Fischer, U.R., "Quasiparticle universes in Bose–Einstein condensates", *Mod. Phys. Lett. A*, **19**, 1789–1812, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/0406086. 3.1.6

[119] Fischer, U.R., and Schützhold, R., "Quantum simulation of cosmic inflation in two-component Bose–Einstein condensates", *Phys. Rev. A*, **70**, 063615, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/0406470. 3.1.6, 5.3

[120] Fischer, U.R., and Visser, M., "Riemannian geometry of irrotational vortex acoustics", *Phys. Rev. Lett.*, **88**, 110201–1–4, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/0110211. 3.1.3

[121] Fischer, U.R., and Visser, M., "On the space-time curvature experienced by quasiparticle excitations in the Painleve–Gullstrand effective geometry", *Ann. Phys. (N.Y.)*, **304**, 22–39, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/0205139. 3.1.4

[122] Fischer, U.R., and Visser, M., "Warped space-time for phonons moving in a perfect nonrelativistic fluid", *Europhys. Lett.*, **62**, 1–7, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0211029. 3.1.4

[123] Fischer, U.R., and Volovik, G.E., "Thermal quasi-equilibrium states across Landau horizons in the effective gravity of superfluids", *Int. J. Mod. Phys. D*, **10**, 57–88, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0003017. 3.1.2

[124] Fiurásek, J., Leonhardt, U., and Parentani, R., "Slow-light pulses in moving media", *Phys. Rev. A*, **65**, 011802–1–4, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/quant-ph/0011100. 3.1.4, 4.2.4

[125] Fock, V.A., *The Theory of Space, Time, and Gravitation*, (Pergamon, New York, U.S.A., 1964), 2nd edition. 4

[126] Fonseca-Barbatti, C., Novello, M., Salim, J.M., and Arcuri, R.C., "Creation of a wormhole due to nonlinear electrodynamics", *Mod. Phys. Lett. A*, **17**, 1305–1314, (2002). 3.1.4, 4.1.3.6

[127] Ford, L.H., "Quantum field theory in curved spacetime", (1997). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9707062. 3.4

[128] Ford, L.H., and Svaiter, N.F., "Cosmological and black hole horizon fluctuations", *Phys. Rev. D*, **56**, 2226–2235, (1997). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9704050. 5.1.4

[129] Friedan, D., "A tentative theory of large distance physics", *J. High Energy Phys.*, **2003**(10), 063, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0204131. 6.5

[130] Frolov, V.P., "Black Hole Entropy and Physics at Planckian Scales", in Sánchez, N., and Zichichi, A., eds., *String gravity and physics at the Planck energy scale*, Proceedings of the NATO Advanced Study Institute on String Gravity and Physics at the Planck Energy Scale, Erice, Italy, September 18–19, 1995, vol. 476 of NATO ASI series. Series C, Mathematical and physical sciences, (Kluwer Academic Publishers, Dordrecht, Netherlands; Boston, U.S.A., 1996). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9510156. 5.1.4

[131] Frolov, V.P., and Larsen, A.L., "Stationary strings and 2-D black holes", *Nucl. Phys. B*, **449**, 149–158, (1995). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9503060. 5.1.4

[132] Fulling, S.A., *Aspects of Quantum Field Theory in Curved Space-Time*, (Cambridge University Press, Cambridge, U.K.; New York, U.S.A., 1989). 6.1

[133] Furtado, C., M de Carvalho, A.M., Garcia de Andrade, L.C., and Moraes, F., "Holonomy, Aharonov–Bohm effect and phonon scattering in superfluids", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0401025. 3.4

[134] Gambini, R., and Pullin, J., "Nonstandard optics from quantum space-time", *Phys. Rev. D*, **59**, 124021–1–4, (1999). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9809038. 5.1.3, 6.4

[135] Garay, L.J., "Quantum gravity and minimum length", *Int. J. Mod. Phys. A*, **10**, 145–166, (1995). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9403008. 3.4

[136] Garay, L.J., Anglin, J.R., Cirac, J.I., and Zoller, P., "Sonic Analog of Gravitational Black Holes in Bose–Einstein Condensates", *Phys. Rev. Lett.*, **85**, 4643–1–5, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0002015. 3.1.2, 4.2.1, 3, 5.1.2.3, 5.1.3, 5.1.3

[137] Garay, L.J., Anglin, J.R., Cirac, J.I., and Zoller, P., "Sonic black holes in dilute Bose–Einstein condensates", *Phys. Rev. A*, **63**, 023611–1–13, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0005131. 3.1.2, 4.2.1, 3, 5.1.2.3, 5.1.3, 5.1.3

[138] Garcia de Andrade, L.C., "Irrotational vortex geometry of torsion loops", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0409115. 14

[139] Garcia de Andrade, L.C., "Non-Riemannian acoustic black holes: Hawking radiation and Lorentz symmetry breaking", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0411103. 14

[140] Garcia de Andrade, L.C., "Non-Riemannian geometry of turbulent acoustic flows and analog gravity", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0410036. 14

[141] Garcia de Andrade, L.C., "Non-Riemannian geometry of vortex acoustics", *Phys. Rev. D*, **70**, 064004, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0405062. 14

[142] Garcia de Andrade, L.C., "Non-Riemannian vortex geometry of rotational viscous fluids and breaking of the acoustic Lorentz invariance", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0409116. 14

[143] Garcia de Andrade, L.C., "On the necessity of non-Riemannian acoustic spacetime in fluids with vorticity", *Phys. Lett. A*, **346**, 327–329, (2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0502106. 14

[144] Garcia de Andrade, L.C., "Relativistic superfluid hydrodynamics", (2005). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0503088. 14

[145] Garcia de Andrade, L.C., de M Carvalho, A.M., and Furtado, C., "Geometric phase for fermionic quasiparticles scattering by disgyration in superfluids", *Europhys. Lett.*, **67**, 538–544, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0406057. 14

[146] Ghafarnejad, H., and Salehi, H., "Hadamard renormalization, conformal anomaly and cosmological event horizons", *Phys. Rev. D*, **56**, 4633–4639, (1997). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9709158. 3.4

[147] Gibbons, G.W., and Hawking, S.W., "Action integrals and partition functions in quantum gravity", *Phys. Rev. D*, **15**, 2752–2756, (1977). 7

[148] Giovanazzi, S., "Hawking Radiation in Sonic Black Holes", *Phys. Rev. Lett.*, **94**, 061302–1–4, (2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/physics/0411064. 3.1.6

[149] Giovanazzi, S., Farrell, C., Kiss, T., and Leonhardt, U., "Conditions for one-dimensional supersonic flow of quantum gases", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/0405007. 2.7.1, 3.1.6

[150] Glass, E.N., and Krisch, J.P., "Schwarzschild atmospheric processes: A classical path to the quantum", *Gen. Relativ. Gravit.*, **32**, 735–741, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9910080. 5.1.4

[151] Gordon, W., "Zur Lichtfortpflanzung nach der Relativitätstheorie", *Ann. Phys. (Leipzig)*, **72**, 421–456, (1923). 3.2.1

[152] Górski, A.Z., and Szmigielski, J., "On Pairs of Difference Operators Satisfying: [D,X]=Id", *J. Math. Phys.*, **39**, 545–568, (1998). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9703015. 3.4

[153] Griffin, A., *Excitations in a Bose-condensed Liquid*, vol. 4 of Cambridge Studies in Low Temperature Physics, (Cambridge University Press, Cambridge, U.K.; New York, U.S.A., 1993). 4.2.1

[154] Gullstrand, A., "Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie", *Ark. Mat. Astron. Fys.*, **16**(8), 1–15, (1922). 11

[155] Hadamard, J., *Leçons sur la propagation des ondes et les équations de l'hydrodynamique (Lectures on the propagation of waves and the equations of hydrodynamics)*, (Hermann, Paris, France, 1903). 4.1.4

[156] Hambli, N., and Burgess, C.P., "Hawking radiation and ultraviolet regulators", *Phys. Rev. D*, **53**, 5717–5722, (1996). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/9510159. 5.1.4

[157] Hamilton, A.J.S., and Lisle, J.P., "The river model of black holes", (2004). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0411060. 3.4

[158] Hassan, S.F., and Sloth, M.S., "Trans-Planckian effects in inflationary cosmology and the modified uncertainty principle", *Nucl. Phys. B*, **674**, 434–458, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/0204110. 5.3

[159] Hawking, S.W., "Black hole explosions?", *Nature*, **248**, 30–31, (1974). 7, 5.1.1

[160] Hawking, S.W., "Particle creation by black holes", *Commun. Math. Phys.*, **43**, 199–220, (1975). Related online version (cited on 31 May 2005): http://projecteuclid.org/getRecord?id=euclid.cmp/1103899181. 7, 5.1.1

[161] Hawking, S.W., "Chronology protection conjecture", *Phys. Rev. D*, **46**, 603–611, (1992). 2.4

[162] Hawking, S.W., "The Chronology Protection Conjecture", in Sato, H., and Nakamura, T., eds., *The Sixth Marcel Grossmann Meeting: on recent developments in theoretical and experimental general relativity, gravitation and relativistic field theories*, Proceedings of the meeting held at Kyoto International Conference Hall, Kyoto, Japan, 23–29 June 1991, 3–16, (World Scientific, Singapore, 1992). 2.4

[163] Hawking, S.W., "Chronology protection: Making the world safe for historians", in Hawking, S.W. et al., ed., *The Future of Spacetime*, 87–108, (Norton, New York, U.S.A., 2002). 2.4

[164] Hawking, S.W., and Ellis, G.F.R., *The Large Scale Structure of Space-Time*, Cambridge Monographs on Mathematical Physics, (Cambridge University Press, Cambridge, U.K., 1973). 4, 2.4, 2.4, 5, 2.6

[165] Hehl, F.W., and Obukhov, Y.N., "To consider the electromagnetic field as fundamental, and the metric only as a subsidiary field", *Found. Phys.*, submitted, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/physics/0404101. 4.1.3.5

[166] Hehl, F.W., and Obukhov, Y.N., "Linear media in classical electrodynamics and the Post constraint", *Phys. Lett. A*, **334**, 249–259, (2005). Related online version (cited on 31 May 2005): http://arXiv.org/abs/physics/0411038. 4.1.3.5

[167] Helfer, A.D., "Trans-Planckian modes, back-reaction, and the Hawking process", (2000). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0008016. 5, 5.1.2, 5.1.4

[168] Helfer, A.D., "Do black holes radiate?", *Rep. Prog. Phys.*, **66**, 943–1008, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0304042. 5, 5.1.2, 5.1.4

[169] Helfer, A.D., "State reduction and energy extraction from black holes", *Phys. Lett. A*, **329**, 277–283, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0407055. 5, 5.1.2, 5.1.4

[170] Himemoto, Y., and Tanaka, T., "A generalization of the model of Hawking radiation with modified high frequency dispersion relation", *Phys. Rev. D*, **61**, 064004–1–18, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9904076. 5.1.2.2

[171] Ho, P.-M., "Regularization of Newton constant, trans-Planckian dispersion relation, and symmetry of particle spectrum", *Class. Quantum Grav.*, **21**, 2641–2650, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0308103. 3.4

[172] Hochberg, D., "Evaporating black holes and collapsing bubbles in fluids", unknown status, (1997). 16

[173] Hochberg, D., and Pérez-Mercader, J., "A Liquid Model Analogue for Black Hole Thermodynamics", *Phys. Rev. D*, **55**, 4880–4888, (1997). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9609043. 16

[174] Horwitz, L.P., and Oron, O., "Classical Gravity as an Eikonal Approximation to a Manifestly Lorentz Covariant Quantum Theory with Brownian Interpretation", in Reimer, A., ed., *Quantum Gravity Research Trends*, vol. 250 of Horizons in World Physics, (Nova Science, New York, U.S.A., 2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0407076. 3.4

[175] Hossenfelder, S., "The minimal length and large extra dimensions", *Mod. Phys. Lett. A*, **19**, 2727–2744, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-ph/0410122. 3.4

[176] Hossenfelder, S., "Running coupling with minimal length", *Phys. Rev. D*, **70**, 105003, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-ph/0405127. 3.4

[177] Hu, B.L., "Dynamical finite size effect, inflationary cosmology and thermal particle production", in Lee, H.C., ed., *CAP–NSERC Summer Institute in Theoretical Physics, 2 vols.*, Edmonton, Alberta, Jul 10–25, 1987, (World Scientific, Singapore; Teaneck, U.S.A., 1988). 5.3

[178] Hu, B.L., "Nonequilibrium quantum fields in cosmology: Comments on selected current topics", in De Vega, H.J., and Sánchez, N., eds., *Second Paris Cosmology Colloquium*, Proceedings of the 2nd Paris Cosmology Colloquium Within the Framework of the International School of Astrophysics, 2–4 June, 1994, Observatoire de Paris, France, 111, (World Scientific, Singapore; River Edge, U.S.A., 1995). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9409053. 5.3

[179] Hu, B.L., "Stochastic gravity", *Int. J. Theor. Phys.*, **38**, 2987–3037, (1999). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9902064. 6.1

[180] Hu, B.L., and Verdaguer, E., "Stochastic gravity: A primer with applications", *Class. Quantum Grav.*, **20**, R1–R42, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0211090. 6.1

[181] Hu, B.L., and Verdaguer, E., "Stochastic Gravity: Theory and Applications", *Living Rev. Relativity*, **7**, lrr-2004-3, (2004). URL (cited on 31 May 2005): http://www.livingreviews.org/lrr-2004-3. 6.1

[182] Huhtala, P., and Volovik, G.E., "Fermionic microstates within Painleve–Gullstrand black hole", *J. Exp. Theor. Phys.*, **94**, 853–861, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0111055. 3.1.3

[183] Israel, W., "Dark stars: the evolution of an idea", in Hawking, S.W., and Israel, W., eds., *Three Hundred Years of Gravitation*, 199–276, (Cambridge University Press, Cambridge, U.K.; New York, U.S.A., 1987). see especially the discussion on page 234. 11

[184] Ito, K., and Ugakkai, N.S., eds., *Encyclopedic Dictionary of Mathematics*, (MIT, Cambridge, U.S.A., 1987), 2nd edition. 4.1.3.4, 4.1.4, 4.1.4

[185] Jacobson, T.A., "Black hole evaporation and ultrashort distances", *Phys. Rev. D*, **44**, 1731–1739, (1991). 15, 3, 5.1.2, 23, 5.5

[186] Jacobson, T.A., "Black hole radiation in the presence of a short distance cutoff", *Phys. Rev. D*, **48**, 728–741, (1993). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/9303103. 15, 3, 5.1.2, 23, 5.5

[187] Jacobson, T.A., "Introduction to Black Hole Microscopy", in Macías, A., Quevedo, H., Obregón, O., and Matos, T., eds., *Recent Developments in Gravitation and Mathematical Physics*, Proceedings of the First Mexican School on Gravitation and Mathematical Physics, Guanajuato, Mexico, 12-16 December 1994, (World Scientific, Singapore; River Edge, U.S.A., 1996). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/9510026. 15

[188] Jacobson, T.A., "On the origin of the outgoing black hole modes", *Phys. Rev. D*, **53**, 7082–7088, (1996). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/9601064. 2.5.2, 15, 23

[189] Jacobson, T.A., "Trans-Planckian redshifts and the substance of the space-time river", *Prog. Theor. Phys. Suppl.*, **136**, 1–17, (1999). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/0001085. 5.1.2.5, 25

[190] Jacobson, T.A., "Lorentz violation and Hawking radiation", (2001). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0110079. 3.1.3

[191] Jacobson, T.A., "Introduction to quantum fields in curved spacetime and the Hawking effect", (2003). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0308048. 3.1.5

[192] Jacobson, T.A., and Kang, G., "Conformal invariance of black hole temperature", *Class. Quantum Grav.*, **10**, L201–L206, (1993). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9307002. 7, 2.5.2, 12

[193] Jacobson, T.A., and Koike, T., "Black hole and baby universe in a thin film of 3He-A", in Novello, M., Visser, M., and Volovik, G., eds., *Artificial Black Holes*, 87–108, (World Scientific, Singapore; River Edge, U.S.A., 2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/0205174. 3.1.2

[194] Jacobson, T.A., Liberati, S., and Mattingly, D., "Lorentz violation and Crab synchrotron emission: A new constraint far beyond the Planck scale", *Nature*, **424**, 1019–1021, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0212190. 3.1.4, 6.4

[195] Jacobson, T.A., Liberati, S., and Mattingly, D., "Threshold effects and Planck scale Lorentz violation: Combined constraints from high energy astrophysics", *Phys. Rev. D*, **67**, 124011–1–26, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-ph/0209264. 3.1.4, 6.4

[196] Jacobson, T.A., Liberati, S., and Mattingly, D., "Astrophysical bounds on Planck suppressed Lorentz violation", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/hep-ph/0407370. 3.1.6, 6.4

[197] Jacobson, T.A., Liberati, S., and Mattingly, D., "Quantum gravity phenomenology and Lorentz violation", in Trampetić, J., and Wess, J., eds., *Particle Physics and the Universe*, Proceedings of the 9th Adriatic Meeting, September 2003, Dubrovnik, vol. 98 of Springer Proceedings in Physics, (Springer, Berlin, Germany; New York, U.S.A., 2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0404067. 3.1.6, 6.4

[198] Jacobson, T.A., and Mattingly, D., "Hawking radiation on a falling lattice", *Phys. Rev. D*, **61**, 024017–1–10, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9908099. 15

[199] Jacobson, T.A., and Volovik, G.E., "Effective spacetime and Hawking radiation from moving domain wall in thin film of 3He-A", *J. Exp. Theor. Phys. Lett.*, **68**, 874–880, (1998). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9811014. 16, 4.2.3, 2

[200] Jacobson, T.A., and Volovik, G.E., "Event horizons and ergoregions in 3He", *Phys. Rev. D*, **58**, 064021–1–7, (1998). 4.2.3, 2

[201] Jevicki, A., and Thaler, J., "Dynamics of black hole formation in an exactly solvable model", *Phys. Rev. D*, **66**, 024041–1–6, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0203172. 5.1.4

[202] Kagan, Y., Surkov, E.L., and Shlyapnikov, G.V., "Evolution of a Bose-condensed gas under variations of the confining potential", *Phys. Rev. A*, **54**, R1753–R1756, (1996). 5.3

[203] Kagan, Y., Surkov, E.L., and Shlyapnikov, G.V., "Evolution and global collapse of trapped Bose condensates under variations of the scattering length", *Phys. Rev. Lett.*, **79**, 2604–2607, (1997). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/physics/9705005. 5.3

[204] Kash, M.M., Sautenkov, V.A., Zibrov, A.S., Hollberg, L., Welch, G.R., Lukin, M.D., Rostovtsev, Y., Fry, E.S., and Scully, M.O., "Ultraslow Group Velocity and Enhanced Nonlinear Optical Effects in a Coherently Driven Hot Atomic Gas", *Phys. Rev. Lett.*, **82**, 5229–5232, (1999). 4.2.4

[205] Kempf, A., “Mode generating mechanism in inflation with a cutoff”, *Phys. Rev. D*, **63**, 083514–1–5, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0009209. 3.1.2

[206] Kempf, A., “A covariant information-density cutoff in curved space-time”, *Phys. Rev. Lett.*, **92**, 221301, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0310035. 3.4

[207] Kempf, A., and Niemeyer, J.C., “Perturbation spectrum in inflation with cutoff”, *Phys. Rev. D*, **64**, 103501–1–6, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0103225. 3.1.3, 5.3

[208] Kim, S.-W., and Oh, John J., “Decay rate and low-energy near-horizon dynamics of acoustic black holes”, *Phys. Lett. B*, **608**, 10–16, (2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0409003. 3.1.6

[209] Kim, W.T., Son, E.J., and Yoon, M.S., “Statistical entropy and superradiance in 2+1 dimensional acoustic black holes”, (2005). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0504127. 3.1.7

[210] Kiss, T., and Leonhardt, U., “Towards a classification of wave catastrophes”, (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/physics/0309036. 3.4

[211] Kocharovskaya, O., Rostovtsev, Y., and Scully, M.O., “Stopping Light via Hot Atoms”, *Phys. Rev. Lett.*, **86**, 628–631, (2001). 4.2.4

[212] Kokkotas, K.D., and Schmidt, B.G., “Quasi-Normal Modes of Stars and Black Holes”, *Living Rev. Relativity*, **2**, lrr-1999-2, (1999). URL (cited on 31 May 2005):
http://www.livingreviews.org/lrr-1999-2. 3.4

[213] Kopnin, N.B., and Volovik, G.E., “Critical velocity and event horizon in pair-correlated systems with relativistic fermionic quasiparticles”, *J. Exp. Theor. Phys. Lett.*, **67**, 528–532, (1998). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/9712187. 16, 2

[214] Kostelecký, V.A., and Samuel, S., “Spontaneous breaking of Lorentz symmetry in string theory”, *Phys. Rev. D*, **39**, 683–685, (1989). 6.4

[215] Kowalski-Glikman, J., “Testing dispersion relations of quantum kappa-Poincare algebra on cosmological ground”, *Phys. Lett. B*, **499**, 1–8, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0006250. 3.4

[216] Kowalski-Glikman, J., “De Sitter space as an arena for doubly special relativity”, *Phys. Lett. B*, **547**, 291–296, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0207279. 3.4

[217] Kowalski-Glikman, J., “Doubly special relativity: A kinematics of quantum gravity?”, in Semikhatov, A. et al., ed., *Proceedings of the 3rd International Sakharov Conference on Physics, 2 vols.*, Moscow, Russia, June 24–29, 2002, (Scientific World, Moscow, Russia, 2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0209264. 3.4

[218] Kraus, P., and Wilczek, F., "A Simple stationary line element for the Schwarzschild Geometry, and some applications", (June, 1994). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9406042. 11

[219] Lamb, H., *Hydrodynamics*, (Dover, Mineola, U.S.A., 1932), 6th edition. Reissue of 1932 ed., first edition publ. 1879. 2.3, 2.3, 4

[220] Lämmerzahl, C., and Hehl, F.W., "Riemannian light cone from vanishing birefringence in premetric vacuum electrodynamics", *Phys. Rev. D*, **70**, 105022–1–10, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0409072. 4.1.3.5

[221] Landau, L.D., and Lifshitz, E.M., *Fluid Mechanics*, (Pergamon; Addison-Wesley, London, U.K.; Reading, U.S.A., 1959). 2.3, 2.3, 4, 2.6

[222] Landau, L.D., and Lifshitz, E.M., *The classical theory of fields*, (Pergamon Press, Oxford, U.K.; New York, U.S.A., 1971), 3rd edition. 3.2.1

[223] Landau, L.D., Lifshitz, E.M., and Pitaevskii, L.P., *Statistical Physics, Part 2*, vol. 9 of Course of Theoretical Physics, (Pergamon Press, Oxford, U.K.; New York, U.S.A., 1980). 1

[224] Landau, L.D., Lifshitz, E.M., and Pitaevskii, L.P., *Electrodynamics of continuous media*, vol. 8 of Course of Theoretical Physics, (Pergamon Press, Oxford, U.K.; New York, U.S.A., 1984), 2nd edition. 4.1.4

[225] Larsen, A.L., "Cosmic strings and black holes", (1996). URL (cited on 31 May 2005): http://arXiv.org/abs/hep-th/9610063. 3.4

[226] Laschkarew, W., "Zur Theorie der Gravitation", *Z. Phys.*, **35**, 473–476, (1926). 3.4

[227] Laughlin, R.B., "Emergent relativity", *Int. J. Mod. Phys. A*, **18**, 831–854, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0302028. 3.1.5

[228] Lemaître, G., "L'univers en expansion", *Ann. Soc. Sci. Bruxelles, Ser. A*, **53**, 51–85, (1933). 11

[229] Lemoine, M., Lubo, M., Martin, J., and Uzan, J.-P., "Stress-energy tensor for trans-Planckian cosmology", *Phys. Rev. D*, **65**, 023510–1–14, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/0109128. 5.3

[230] Lemoine, M., Martin, J., and Uzan, J.-P., "Trans-Planckian dark energy?", *Phys. Rev. D*, **67**, 103520–1–13, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/0212027. 5.3

[231] Leonhardt, U., "Space-time geometry of quantum dielectrics", *Phys. Rev. A*, **62**, 012111–1–8, (2000). Related online version (cited on 31 May 2005): http://arXiv.org/abs/physics/0001064. 3.1.2

[232] Leonhardt, U., "Slow Light", in Novello, M., Visser, M., and Volovik, G., eds., *Artificial Black Holes*, 61–85, (World Scientific, Singapore; River Edge, U.S.A., 2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0108085. 4.2.4, 4.2.4

[233] Leonhardt, U., "Quantum physics of simple optical instruments", *Rep. Prog. Phys.*, **66**, 1207–1250, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/quant-ph/0305007. 5.1.4

[234] Leonhardt, U., Kiss, T., and Öhberg, P., "Intrinsic instability of sonic white holes", (2002). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0211069. 3.1.4

[235] Leonhardt, U., and Piwnicki, P., "Relativistic Effects of Light in Moving Media with Extremely Low Group Velocity", *Phys. Rev. Lett.*, **84**, 822–825, (2000). Related online version (cited on 31 May 2005):
http://arxiv.org/abs/cond-mat/9906332. 3.1.2, 4.2.4, 4.2.4

[236] Leonhardt, U., and Piwnicki, P., "Reply to Comment on "Relativistic Effects of Light in Moving Media with Extremely Low Group Velocity" by M. Visser", *Phys. Rev. Lett.*, **85**, 5253, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0003016. 3.1.2

[237] Lepe, S., and Saavedra, J., "Quasinormal modes, superradiance and area spectrum for 2+1 acoustic black holes", *Phys. Lett. B*, **617**, 174–181, (2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0410074. 3.1.6, 5.2

[238] Liberati, S., *Quantum vacuum effects in gravitational fields: Theory and detectability*, Ph.D. Thesis, (International School for Advanced Studies, Trieste, Italy, 2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0009050. 3.4

[239] Liberati, S., Sonego, S., and Visser, M., "Unexpectedly large surface gravities for acoustic horizons?", *Class. Quantum Grav.*, **17**, 2903–2923, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0003105. 10, 12, 3.4

[240] Liberati, S., Sonego, S., and Visser, M., "Scharnhorst effect at oblique incidence", *Phys. Rev. D*, **63**, 085003–1–10, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/quant-ph/0010055. 3.4

[241] Liberati, S., Sonego, S., and Visser, M., "Faster-than-$c$ signals, special relativity, and causality", *Ann. Phys. (N.Y.)*, **298**, 167–185, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0107091. 3.4

[242] Lidsey, J.E., "Cosmic dynamics of Bose–Einstein condensates", *Class. Quantum Grav.*, **21**, 777–786, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0307037. 3.1.5, 5.3, 5.3

[243] Lubo, M., "Quantum minimal length and trans-Planckian photons", *Phys. Rev. D*, **61**, 124009–1–1, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9911191. 3.4

[244] Lubo, M., "Ultraviolet cut off, black hole-radiation equilibrium and big bang", *Phys. Rev. D*, **68**, 125005, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0306187. 3.4, 5.3

[245] Lubo, M., "Ultraviolet cutoff and bosonic dominance", *Phys. Rev. D*, **68**, 125004–1–9, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0305216. 3.4

[246] M de Carvalho, A.M., Moraes, F., and Furtado, C., "The self-energy of a charged particle in the presence of a topological defect distribution", *Int. J. Mod. Phys. A*, **19**, 2113–2122, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0401030. 3.4

[247] Magueijo, J., and Smolin, L., "Lorentz Invariance with an Invariant Energy Scale", *Phys. Rev. Lett.*, **88**, 190403–1–4, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0112090. 5.1.3

[248] Martin, J., "Inflationary cosmological perturbations of quantum-mechanical origin", in Amelino-Camelia, G., and Kowalski-Glikman, J., eds., *Planck Scale Effects in Astrophysics and Cosmology*, 40th Karpacz Winter School of Theoretical Physics, Ladek Zdrój, Poland, 4–14 February 2004, vol. 669 of Lecture Notes in Physics, (Springer, Berlin, Germany; New York, U.S.A., 2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0406011. 3.4

[249] Martin, J., and Brandenberger, R.H., "A Cosmological Window on Trans-Planckian Physics", (2001). URL (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0012031. 5.3

[250] Martin, J., and Brandenberger, R.H., "Trans-Planckian problem of inflationary cosmology", *Phys. Rev. D*, **63**, 123501–1–16, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0005209. 5.3

[251] Martin, J., and Brandenberger, R.H., "Corley–Jacobson dispersion relation and trans-Planckian inflation", *Phys. Rev. D*, **65**, 103514–1–5, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0201189. 5.3

[252] Martin, J., and Brandenberger, R.H., "Dependence of the spectra of fluctuations in inflationary cosmology on trans-Planckian physics", *Phys. Rev. D*, **68**, 063513–1–16, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0305161. 5.3

[253] Massar, S., "The semiclassical back reaction to black hole evaporation", *Phys. Rev. D*, **52**, 5857–5864, (1995). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9411039. 5.1.4

[254] Massar, S., and Parentani, R., "From Vacuum Fluctuations to Radiation: Accelerated Detectors and Black Holes", (1994). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9404057. 5.1.4

[255] Massar, S., and Parentani, R., "From vacuum fluctuations to radiation. 2. Black holes", *Phys. Rev. D*, **54**, 7444–7458, (1996). 5.1.4

[256] Massar, S., and Parentani, R., "From vacuum fluctuations to radiation: Accelerated detectors and black holes", *Phys. Rev. D*, **54**, 7426–7443, (1996). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9502024. 5.1.4

[257] Massar, S., and Parentani, R., "How the change in horizon area drives black hole evaporation", *Nucl. Phys. B*, **575**, 333–356, (2000). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9903027. 5.1.4, 6.1

[258] Matarrese, S., "On the classical and quantum irrotational motions of a relativistic perfect fluid: I. Classical Theory", *Proc. R. Soc. London, Ser. A*, **401**, 53–66, (1985). 3.2.2, 18

[259] Matarrese, S., "Perturbations of an irrotational perfect fluid", in Fabbri, R., and Modugno, M., eds., *Atti del VI Convegno Nazionale di Relatività Generale e Fisica della Gravitazione*, Proceedings of the 4th Italian conference on general relativity and the physics of gravitation, Florence, Italy, 10–13 October 1984, 283–287, (Pitagora Editrice, Bologna, Italy, 1986). 3.2.2, 18

[260] Matarrese, S., "Phonons in a relativistic perfect fluid", in Ruffini, R, ed., *The Fourth Marcel Grossmann Meeting on recent developments in theoretical and experimental general relativity, gravitation and relativistic field theories*, Proceedings of the meeting held at the University of Rome "La Sapienza", 17–21 June, 1985, 1591–1595, (North-Holland; Elsevier, Amsterdam, Netherlands; New York, U.S.A., 1986). 3.2.2, 18

[261] Mattingly, D., "Modern Tests of Lorentz Invariance", *Living Rev. Relativity*, **8**, lrr-2005-5, (2005). URL (cited on 31 May 2005): http://www.livingreviews.org/lrr-2005-5. 6.4

[262] Mersini, L., "Dark energy from the trans-Planckian regime", in Khalil, S., Shafi, Q., and Tallat, H., eds., *International Conference on High Energy Physics*, Proceedings of the International Conference on High Energy Physics, January 9–14, 2001, Cairo, Egypt, 289–294, (Rinton Press, Princeton, U.S.A., 2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-ph/0106134. 3.1.3, 3.4

[263] Mersini, L., Bastero-Gil, M., and Kanti, P., "Relic dark energy from trans-Planckian regime", *Phys. Rev. D*, **64**, 043508–1–9, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-ph/0101210. 3.1.3

[264] Milne-Thomson, L.M., *Theoretical Hydrodynamics*, (Macmillan, London, U.K., New York, U.S.A., 1968), 5th edition. 2.3, 2.3, 4

[265] Misner, C.W., Thorne, K.S., and Wheeler, J.A., *Gravitation*, (W.H. Freeman, San Francisco, U.S.A., 1973). 4, 4, 5, 2.6

[266] Møller, C., *The Theory of Relativity*, (Clarendon, Oxford, U.K., 1972), 2nd edition. 4

[267] Moncrief, V., "Gravitational perturbations of spherically symmetric systems. I. The exterior problem", *Ann. Phys. (N.Y.)*, **88**, 323–342, (1974). 5.1.3

[268] Moncrief, V., "Stability of stationary, spherical accretion onto a Schwarzschild black hole", *Astrophys. J.*, **235**, 1038–1046, (1980). Related online version (cited on 02 December 2005): http://adsabs.harvard.edu/cgi-bin/bib_query?1980ApJ...235.1038M. 2.4, 3.2.2, 18

[269] Nakano, H., Kurita, Y., Ogawa, K., and Yoo, C.-M., "Quasinormal ringing for acoustic black holes at low temperature", *Phys. Rev. D*, **71**, 084006–1–7, (2005). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0411041. 3.1.6

[270] Nandi, K.K., and Xu, D.H., "Unruh model for the Einstein–Rosen charge: Squealing wormholes?", (2004). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0410052. 3.4

[271] Nandi, K.K., Zhang, Y.-Z., Alsing, P.M., Evans, J.C., and Bhadra, A., "Analogue of the Fizeau effect in an effective optical medium", *Phys. Rev. D*, **67**, 025002–1–11, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0208035. 3.4

[272] Nandi, K.K., Zhang, Y.-Z., and Cai, R.-G., "Acoustic wormholes", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0409085. 3.4

[273] Niemeyer, J.C., "Inflation with a Planck-scale frequency cutoff", *Phys. Rev. D*, **63**, 123502–1–7, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0005533. 3.1.2

[274] Niemeyer, J.C., "Cosmological consequences of short distance physics", (2002). URL (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0201511. 5.3

[275] Niemeyer, J.C., and Parentani, R., "Trans-Planckian dispersion and scale invariance of inflationary perturbations", *Phys. Rev. D*, **64**, 101301–1–4, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0101451. 3.1.3, 5.3

[276] Nikolic, H., "Time in quantum gravity by weakening the Hamiltonian constraint", (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0312063. 5.1.4

[277] Nikolic, H., "Black holes radiate but do not evaporate", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0402145. 5.1.4

[278] Novello, M., "Effective geometry in nonlinear electrodynamics", *Int. J. Mod. Phys. A*, **17**, 4187–4196, (2002). 3.1.4, 4.1.3.6

[279] Novello, M., De Lorenci, V.A., Salim, J.M., and Klippert, R., "Geometrical aspects of light propagation in nonlinear electrodynamics", *Phys. Rev. D*, **61**, 045001–1–10, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9911085. 4.1.3.6

[280] Novello, M., and Perez Bergliaffa, S.E., "Effective Geometry", in Novello Santiago, M., and Perez Bergliaffa, E., eds., *Cosmology and Gravitation*, Xth Brazilian School of Cosmology and Gravitation, 25th Anniversary (1977–2002), Mangaratiba, Rio de Janeiro, Brazil, 29 July - 9 August 2002, vol. 668 of AIP Conference Proceedings, 288–300, (American Institute of Physics, Melville, U.S.A., 2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0302052. 3.1.5, 4.1.3.6

[281] Novello, M., Perez Bergliaffa, S.E., and Salim, J.M., "Nonlinear electrodynamics and the acceleration of the Universe", *Phys. Rev. D*, **69**, 127301–1–4, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0312093. 3.1.5, 4.1.3.6

[282] Novello, M., Perez Bergliaffa, S.E., Salim, J.M., De Lorenci, V.A., and Klippert, R., "Analog black holes in flowing dielectrics", *Class. Quantum Grav.*, **20**, 859–871, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0201061. 3.1.4, 4.1.3.6

[283] Novello, M., and Salim, J.M., "Effective electromagnetic geometry", *Phys. Rev. D*, **63**, 083511–1–4, (2001). 4.1.3.6

[284] Novello, M., Visser, M., and Volovik, G., eds., *Artificial Black Holes*, (World Scientific, Singapore; River Edge, U.S.A., 2002). 1.1, 2.2, 3.1.2, 4.2.4, 4.3, 6.5

[285] Obadia, N., and Parentani, R., "Notes on moving mirrors", *Phys. Rev. D*, **64**, 044019–1–17, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0103061. 3.4

[286] Obadia, N., and Parentani, R., "Uniformly accelerated mirrors. II: Quantum correlations", *Phys. Rev. D*, **67**, 024022–1–18, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0209057. 3.4

[287] Obukhov, Y.N., "Black hole hydrodynamics", (2003). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0309114. 3.4

[288] Obukhov, Y.N., and Hehl, F.W., "Spacetime metric from linear electrodynamics", *Phys. Lett. B*, **458**, 466–470, (1999). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9904067. 4.1.3.5

[289] Oppenheim, J., "Thermodynamics with long-range interactions: From Ising models to black holes", *Phys. Rev. E*, **68**, 016108–1–17, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0212066. 5.1.4

[290] Oron, O., and Horwitz, L.P., "Eikonal approximation to 5D wave equations as geodesic motion in a curved 4D spacetime", *Gen. Relativ. Gravit.*, **37**, 491–506, (2005). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-ph/0205018. 3.4

[291] Padmanabhan, T., "Gravity and the thermodynamics of horizons", *Phys. Rep.*, **406**, 49–125, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0311036. 5.1.4

[292] Padmanabhan, T., "Entropy of Horizons, Complex Paths and Quantum Tunneling", *Mod. Phys. Lett. A*, **19**, 2637–2643, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0405072. 5.1.4

[293] Painlevé, P., "La mécanique classique et la theorie de la relativité", *C. R. Acad. Sci.*, **173**, 677–680, (1921). 11

[294] Parentani, R., "The Recoils of the accelerated detector and the decoherence of its fluxes", *Nucl. Phys. B*, **454**, 227–249, (1995). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9502030. 3.4

[295] Parentani, R., "The Recoils of a Dynamical Mirror and the Decoherence of its Fluxes", *Nucl. Phys. B*, **465**, 175–214, (1996). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/9509104. 3.4

[296] Parentani, R., "Time dependent perturbation theory in quantum cosmology", *Nucl. Phys. B*, **492**, 501–525, (1997). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9610045. 3.4, 5.3

[297] Parentani, R., "The validity of the background field approximation", in Burko, L.M., and Ori, A., eds., *Internal Sructure of Black Holes and Space Time Singularities*, Workshop on The Internal Sructure of Black Holes and Space Time Singularities, June 29–July 3, 1997, Haifa, Israel, vol. 13 of Annals of the Israel Physical Society, (Institute of Physics Publishing, Bristol, U.K.; Philadelphia, U.S.A., 1997). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9710059. 3.4

[298] Parentani, R., "Hawking radiation from Feynman diagrams", *Phys. Rev. D*, **61**, 027501–1–4, (2000). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9904024. 5.1.4

[299] Parentani, R., "Quantum metric fluctuations and Hawking radiation", *Phys. Rev. D*, **63**, 041503–1–4, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0009011. 23, 6.1

[300] Parentani, R., "What did we learn from studying acoustic black holes?", *Int. J. Mod. Phys. A*, **17**, 2721–2726, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0204079. 3.1.4

[301] Parentani, R., "The inflationary paradigm: Predictions for CMB", (2004). URL (cited on 31 May 2005): http://arXiv.org/abs/astro-ph/0404022. 3.4

[302] Pashaev, O.K., and Lee, J.-H., "Resonance Solitons as Black Holes in Madelung Fluid", *Mod. Phys. Lett. A*, **17**, 1601–1619, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/9810139. 3.4

[303] Peblanski, J., "Electromagnetic waves in gravitational fields", *Phys. Rev.*, **118**, 1396–1408, (1960). 3.2.1

[304] Peblanski, J., *Lectures on Nonlinear Electrodynamics*, (Nordita, Copenhagen, Denmark, 1970). 3.2.1

[305] Penrose, R., "Gravitational collapse: The role of general relativity", *Riv. Nuovo Cimento*, **1**, 252–276, (1969). 5.2

[306] Perez Bergliaffa, S.E., "Effective geometry in Astrophysics", *Int. J. Mod. Phys. D*, **13**, 1469–1476, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/astro-ph/0401577. 3.1.6

[307] Perez Bergliaffa, S.E., Hibberd, K., Stone, M., and Visser, M., "Wave Equation for Sound in Fluids with Vorticity", *Physica D*, **191**, 121–136, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/0106255. 13, 3.1.3

[308] Pham, Q.M., "Sur les équations de l'electromagné dans la materie", *C. R. Hebd. Seanc. Acad. Sci.*, **242**, 465–467, (1956). 3.2.1

[309] Philips, D.F., Fleischhauer, A., Mair, A., and Walsworth, R.L., "Storage of Light in Atomic Vapor", *Phys. Rev. Lett.*, **86**, 783–786, (2001). 4.2.4

[310] Pines, D., *The Many-Body Problem: A Lecture Note and Reprint Volume*, vol. 6 of Frontiers in Physics, (W.A. Benjamin, New York, U.S.A., 1962), 2nd edition. 1

[311] Piwnicki, P., "Geometrical approach to light in inhomogeneous media", *Int. J. Mod. Phys. A*, **17**, 1543–1558, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0201007. 3.1.4

[312] Press, W.H., and Teukolsky, S.A., "Floating Orbits, Superradiant Scattering and the Black-Hole Bomb", *Nature*, **238**, 211–212, (1972). 5.2

[313] Radu, E., "On the Euclidean approach to quantum field theory in Gödel space-time", *Phys. Lett. A*, **247**, 207–210, (1998). 3.4

[314] Raval, A., Hu, B.L., and Koks, D., "Near-thermal radiation in detectors, mirrors and black holes: A stochastic approach", *Phys. Rev. D*, **55**, 4795–4812, (1997). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9606074. 3.4

[315] Ray, A.K., "Linearized perturbation on stationary inflow solutions in an inviscid and thin accretion disc", *Mon. Not. R. Astron. Soc.*, **344**, 83–88, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/astro-ph/0212515. 3.4

[316] Ray, A.K., and Bhattacharjee, J.K., "A Dynamical Systems Approach to an Inviscid and Thin Accretion Disc", (2003). URL (cited on 31 May 2005): http://arXiv.org/abs/astro-ph/0307447. 3.4

[317] Regge, T., and Wheeler, J.A., "Stability of a Schwarzschild Singularity", *Phys. Rev.*, **108**, 1063–1069, (1957). 5.1.3

[318] Reznik, B., "Trans-Planckian tail in a theory with a cutoff", *Phys. Rev. D*, **55**, 2152–2158, (1997). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9606083. 15, 24, 5.1.2.6

[319] Reznik, B., "Origin of the thermal radiation in a solid-state analogue of a black hole", *Phys. Rev. D*, **62**, 044044–1–7, (2000). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9703076. 7, 9, 15, 21, 5.1.2.6

[320] Rosquist, K., "Letter: A Moving Medium Simulation of Schwarzschild Black Hole Optics", *Gen. Relativ. Gravit.*, **36**, 1977–1982, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0309104. 3.4

[321] Rosu, H., "Towards measuring Hawking like effects in the laboratory. 1", unknown status, (1989). Related online version (cited on 7 Dec 2005): http://www.slac.stanford.edu/spires/find/hep/www?key=2056828. 5.1.4

[322] Rosu, H., "Superoscillations and trans-Planckian frequencies", *Nuovo Cimento B*, **112**, 131–132, (1997). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9606070. 3.4, 24

[323] Rosu, Haret C., "On the circular vacuum noise in electron storage rings", *Nuovo Cimento B*, **109**, 423–430, (1994). Related online version (cited on 31 May 2005): http://arXiv.org/abs/physics/9711015. 3.4

[324] Rosu, H.C., "Classical and quantum inertia: A matter of principles", *Grav. and Cosmol.*, **5**, 81–91, (1999). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9412012. 3.4

[325] Rosu, H.C., "Relativistic quantum field inertia and vacuum field noise spectra: By quest of the lost universality to high energy radiometric standards", *Int. J. Theor. Phys.*, **39**, 285–295, (2000). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9905049. 5.1.4

[326] Rovelli, C., "Loop quantum gravity", *Phys. World*, **16**, 37–41, (November, 2003). 6.5

[327] Rovelli, C., *Quantum Gravity*, Cambridge Monographs on Mathematical Physics, (Cambridge University Press, Cambridge, U.K.; New York, U.S.A., 2004). 6.5

[328] Rovelli, C., and Smolin, L., "Discreteness of area and volume in quantum gravity", *Nucl. Phys. B*, **442**, 593–619, (1995). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9411005. 3.4

[329] Russo, J.G., "Model of black hole evolution", *Phys. Rev. D*, **55**, 871–877, (1997). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/9602124. 5.1.4

[330] Saida, H., and Sakagami, M., "Black hole radiation with high frequency dispersion", *Phys. Rev. D*, **61**, 084023–1–8, (2000). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9905034. 5.1.2.2

[331] Sakagami, M., and Ohashi, A., "Hawking Radiation in Laboratories", *Prog. Theor. Phys.*, **107**, 1267–1272, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0108072. 3.1.3

[332] Sakharov, A.D., "Vacuum quantum fluctuations in curved space and the theory of gravitation", *Sov. Phys. Dokl.*, **12**, 1040–1041, (1968). 6.3

[333] Salehi, H., "Evaporating Black Holes And An Entropic Scale-Hierarchy", (1994). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9409023. 5.1.4

[334] Salehi, H., "Evaporating black holes and long range scaling", *Gen. Relativ. Gravit.*, **35**, 1679–1690, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/0302178. 5.1.4

[335] Saul, L.A., "The Dynamic Space of General Relativity in Second Atomization", in Reimer, A., ed., *General Relativity Research Trends*, vol. 249 of Horizons in World Physics, 153–172, (Nova Science, New York, U.S.A., 2005). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0405132. 3.4

[336] Schmelzer, I., "General Ether Theory", (2000). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0001101. 3.4

[337] Schmelzer, I., "A metric theory of gravity with condensed matter interpretation", (2000). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0001096. 3.4

[338] Schmelzer, I., "Derivation of the Einstein equivalence principle in a class of condensed matter theories", (2001). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0104013. 3.4

[339] Schmelzer, I., "A generalization of the Lorentz ether to gravity with general-relativistic limit", (2002). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0205035. 3.4

[340] Schützhold, R., "On the Hawking effect", *Phys. Rev. D*, **64**, 024029–1–14, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0011047. 5.1.4

[341] Schützhold, R., "Particle definition in the presence of black holes", *Phys. Rev. D*, **63**, 024014–1–20, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0003020. 5.1.4

[342] Schützhold, R., Plunien, G., and Soff, G., "Dielectric Black Hole Analogs", *Phys. Rev. Lett.*, **88**, 061101–1–4, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/quant-ph/0104121. 3.1.3

[343] Schützhold, R., Uhlmann, M., Xu, Y., and Fischer, U. R., "Quantum backreaction in dilute Bose–Einstein condensates", *Phys. Rev. D*, **72**, 105005–1–8, (2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/0503581. 13, 3.1.7

[344] Schützhold, R., and Unruh, W.G., "Gravity wave analogues of black holes", *Phys. Rev. D*, **66**, 044019–1–13, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0205099. 3.1.4

[345] Schützhold, R., and Unruh, W.G., "Gravity wave analogues of black holes", *Phys. Rev. D*, **66**, 044019–1–13, (2002). 3.1.4, 4.1.2, 20, 20, 5.2

[346] Schützhold, R., and Unruh, W.G., "Problems of doubly special relativity with variable speed of light", *J. Exp. Theor. Phys. Lett.*, **78**, 431, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0308049. 3.4

[347] Schützhold, R., and Unruh, W.G., "Hawking Radiation in an Electromagnetic Waveguide?", *Phys. Rev. Lett.*, **95**, 031301–1–4, (2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/quant-ph/0408145. 3.1.6

[348] Scully, M.O., and Zubairy, M.S., *Quantum Optics*, (Cambridge University Press, Cambridge, U.K.; New York, U.S.A., 1997). 4.2.4

[349] Shankaranarayanan, S., "Is there an imprint of Planck-scale physics on inflationary cosmology?", *Class. Quantum Grav.*, **20**, 75–83, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0203060. 5.3

[350] Shankaranarayanan, S., Padmanabhan, T., and Srinivasan, K., "Hawking radiation in different coordinate settings: complex paths approach", *Class. Quantum Grav.*, **19**, 2671–2687, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0010042. 5.1.4

[351] Siemieniec-Ozieblo, G., and Woszczyna, A., "Acoustic instabilities at the transition from the radiation-dominated to the matter-dominated universe", *Astron. Astrophys.*, **419**, 801–810, (2004). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0106562. 3.4

[352] Skrotskii, G.V., "The influence of gravitation on the propagation of light", *Sov. Phys. Dokl.*, **2**, 226–229, (1957). 3.2.1

[353] Skudrzyk, E., *The Foundations of Acoustics*, (Springer, New York, U.S.A., 1971). 2.3, 2.3, 4

[354] Slatyer, T.R., and Savage, C.M., "Superradiant scattering from a hydrodynamic vortex", *Class. Quantum Grav.*, **22**, 3833–3839, (2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/0501182. 3.1.7

[355] Smolin, L., "Experimental signatures of quantum gravity", (1995). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9503027. 3.4, 6.4

[356] Smolin, L., "How far are we from the quantum theory of gravity?", (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0303185. 6.5

[357] Smolyaninov, I.I., "Linear and nonlinear optics of surface plasmon toy-models of black holes and wormholes", (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0311062. 3.4

[358] Smolyaninov, I.I., and Davis, C.C., "Surface plasmon toy-models of black holes and wormholes", (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0306089. 3.4

[359] Sorkin, R.D., "Causal sets: Discrete gravity (Notes for the Valdivia Summer School)", (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0309009. 3.4

[360] Srinivasan, K., and Padmanabhan, T., "Particle production and complex path analysis", *Phys. Rev. D*, **60**, 24007–1–20, (1999). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9812028. 5.1.4

[361] Sriramkumar, L., and Padmanabhan, T., "Initial state of matter fields and trans-Planckian physics: Can CMB observations disentangle the two?", *Phys. Rev. D*, **71**, 103512–1–11, (2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0408034. 5.3

[362] Starobinsky, A.A., "Robustness of the inflationary perturbation spectrum to trans-Planckian physics", *J. Exp. Theor. Phys. Lett.*, **73**, 415–418, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0104043. 5.3

[363] Starobinsky, A.A., and Tkachev, I.I., "Trans-Planckian Particle Creation in Cosmology and Ultrahigh Energy Cosmic Rays", *J. Exp. Theor. Phys. Lett.*, **76**, 235–239, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0207572. 5.3

[364] Stephens, G.J., and Hu, B.L., "Notes on black hole phase transitions", *Int. J. Theor. Phys.*, **40**, 2183–2200, (2001). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0102052. 3.4, 5.1.4

[365] Stone, M., "Magnus and other forces on vortices in superfluids and superconductors", (1997). URL (cited on 31 May 2005):
http://arxiv.org/abs/cond-mat/9708017. 3.4

[366] Stone, M., "Acoustic energy and momentum in a moving medium", *Phys. Rev. E*, **62**, 1341–1350, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/9909315. 3.4

[367] Stone, M., "Iordanskii force and the gravitational Aharonov–Bohm effect for a moving vortex", *Phys. Rev. B*, **61**, 11780–11786, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/9909313. 3.4

[368] Stone, M., "Phonons and Forces: Momentum versus Pseudomomentum in Moving Fluids", (2000). URL (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/0012316. 3.4

[369] Synge, J.L., *Relativity: The General Theory*, (North-Holland, Amsterdam, Netherlands, 1960). 4

[370] Tamaki, T., Harada, T., Miyamoto, U., and Torii, T., "Particle velocity in noncommutative space-time", *Phys. Rev. D*, **66**, 105003–1–6, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0208002. 3.4

[371] Tanaka, T., "A comment on trans-Planckian physics in inflationary universe", (2001). URL (cited on 31 May 2005):
http://arXiv.org/abs/astro-ph/0012431. 3.4, 5.3

[372] Thorne, K.S., Price, R.H., and Macdonald, D.A., eds., *Black Holes: The Membrane Paradigm*, (Yale University Press, New Haven, U.S.A.; London, U.K., 1986). 6

[373] Trautman, A., "Comparison of Newtonian and relativistic theories of space-time", in Hoffman, B., ed., *Perspectives in Geometry and Relativity: Essays in honor of Václav Hlavatý*, 413–425, (Indiana University Press, Bloomington, U.S.A., 1966). 3.4

[374] Turukhin, A.V., Sudarshanam, V.S., Shahriar, M.S., Musser, J.A., Ham, B.S., and Hemmer, P.R., "Observation of Ultraslow and Stored Light Pulses in a Solid", *Phys. Rev. Lett.*, **88**, 023602–1–4, (2002). 4.2.4

[375] Unruh, W.G., "Notes on black-hole evaporation", *Phys. Rev. D*, **14**, 870–892, (1976). 1

[376] Unruh, W.G., "Experimental Black-Hole Evaporation?", *Phys. Rev. Lett.*, **46**, 1351–1353, (1981). 2, 2.1, 2.2, 7, 9, 2.7.2, 3.1.1, 5.1.2, 6.5

[377] Unruh, W.G., "Sonic analogue of black holes and the effects of high frequencies on black hole evaporation", *Phys. Rev. D*, **51**, 2827–2838, (1994). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9409008. 7, 9, 15, 3, 5.1.2, 23, 5.1.2.2, 5.5

[378] Unruh, W.G., "Sonic analog of black holes and the effects of high frequencies on black hole evaporation", *Phys. Rev. D*, **51**, 2827–2838, (1995). preprinted as gr-qc/9409008. 7, 9, 2.5.2, 15, 23, 5.1.2.2, 5.1.2.2, 5.5

[379] Unruh, W.G., and Schützhold, R., "On slow light as a black hole analogue", *Phys. Rev. D*, **68**, 024008–1–14, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0303028. 3.1.5, 4.2.4, 1

[380] Unruh, W.G., and Schützhold, R., "Universality of the Hawking effect", *Phys. Rev. D*, **71**, 024028–1–11, (2005). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0408009. 3.1.6, 3, 24, 5.1.2.2, 5.1.2.4

[381] Vachaspati, T., "Propagating phase boundaries as sonic horizons", (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0312069. 3.1.5

[382] Vachaspati, T., "Cosmic Problems for Condensed Matter Experiment", (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/0404480. 3.1.6

[383] Vestergaard Hau, L., Harris, S.E., Dutton, Z., and Behroozi, C.H., "Light speed reduction to 17 metres per second in ultracold atomic gas", *Nature*, **397**, 594–598, (1999). 4.2.4

[384] Vishveshwara, C.V., "Scattering of gravitational radiation by a Schwarzschild black-hole", *Nature*, **227**, 936–938, (1970). 5.1.3

[385] Vishveshwara, C.V., "Stability of the Schwarzschild metric", *Phys. Rev. D*, **1**, 2870–2879, (1970). 5.1.3

[386] Visser, M., "Dirty black holes: Thermodynamics and horizon structure", *Phys. Rev. D*, **46**, 2445–2451, (1992). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/9203057. 7, 9

[387] Visser, M., "Acoustic propagation in fluids: An unexpected example of Lorentzian geometry", (1993). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9311028. 2, 2.1, 2.2, 2.7.2, 15

[388] Visser, M., *Lorentzian Wormholes: From Einstein to Hawking*, AIP Series in Computational and Applied Mathematical Physics, (American Institute of Physics, Woodbury, U.S.A., 1995). 2.5.1

[389] Visser, M., "Acoustic black holes: Horizons, ergospheres, and Hawking radiation", *Class. Quantum Grav.*, **15**, 1767–1791, (1998). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9712010. 2, 2.1, 2.2, 2.3, 2.7.1, 2.7.2, 16, 5.1.3, 6.5

[390] Visser, M., "Hawking radiation without black hole entropy", *Phys. Rev. Lett.*, **80**, 3436–3439, (1998). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9712016. 16

[391] Visser, M., "Acoustic black holes", (1999). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9901047. 2, 2.1, 2.2, 2.7.2, 16

[392] Visser, M., "Comment on "Relativistic Effects of Light in Moving Media with Extremely Low Group Velocity"", *Phys. Rev. Lett.*, **85**, 5252, (2000). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0002011. 4.2.4

[393] Visser, M., "Sakharov's induced gravity: A modern perspective", *Mod. Phys. Lett. A*, **17**, 977–992, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0204062. 6.3

[394] Visser, M., "Essential and inessential features of Hawking radiation", *Int. J. Mod. Phys. D*, **12**, 649–661, (2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0106111. 5.1.1

[395] Visser, M., "Heuristic approach to the Schwarzschild geometry", (2003). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0309072. 3.4

[396] Visser, M., “The quantum physics of chronology protection”, in Gibbons, G.W., Shellard, E.P.S., and Rankin, S.J., eds., *The Future of Theoretical Physics and Cosmology: Celebrating Stephen Hawking’s 60th Birthday*, 161–175, (Cambridge University Press, Cambridge, U.K.; New York, U.S.A., 2003). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0204022. 2.4

[397] Visser, M., Barceló, C., and Liberati, S., “Acoustics in Bose–Einstein condensates as an example of Lorentz symmetry breaking”, (2001). URL (cited on 31 May 2005):
http://arXiv.org/abs/hep-th/0109033. 3.1.3

[398] Visser, M., Barceló, C., and Liberati, S., “Analogue models of and for gravity”, *Gen. Relativ. Gravit.*, **34**, 1719–1734, (2002). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0111111. 3.1.3

[399] Visser, M., Barceló, C., and Liberati, S., “Bi-refringence versus bi-metricity”, (2002). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0204017. 4.1.3.4, 6.2

[400] Visser, M., and Weinfurtner, S., “Massive phonon modes from a BEC-based analog model”, (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/0409639. 3.1.6

[401] Visser, M., and Weinfurtner, S.E.C., “Vortex analogue for the equatorial geometry of the Kerr black hole”, *Class. Quantum Grav.*, **22**, 2493–2510, (2004). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/0409014. 2.5.1, 12, 2.7.1, 3.1.6, 5.2

[402] Volovik, G.E., “Is there analogy between quantized vortex and black hole?”, (1995). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9510001. 16

[403] Volovik, G.E., “AB interface in superfluid 3He and Casimir effect”, *J. Exp. Theor. Phys. Lett.*, **63**, 483–489, (1996). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/9602129. 16

[404] Volovik, G.E., “Cosmology, particle physics and superfluid 3He”, *Czech. J. Phys.*, **46**, 3048, (1996). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/9607212. 2.5.1, 2.5.1

[405] Volovik, G.E., “Induced gravity in superfluid 3He”, *J. Low Temp. Phys.*, **113**, 667–680, (1997). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/9806010. 16

[406] Volovik, G.E., “Energy-momentum tensor of quasiparticles in the effective gravity in superfluids”, (1998). URL (cited on 31 May 2005):
http://arXiv.org/abs/gr-qc/9809081. 16

[407] Volovik, G.E., “Gravity of monopole and string and gravitational constant in 3He-A”, *J. Exp. Theor. Phys. Lett.*, **67**, 698–704, (1998). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/9804078. 16

[408] Volovik, G.E., “Vortex vs. spinning string: Iordanskii force and gravitational Aharonov–Bohm effect”, *J. Exp. Theor. Phys. Lett.*, **67**, 841–846, (1998). Related online version (cited on 31 May 2005):
http://arXiv.org/abs/cond-mat/9804308. 3.4

[409] Volovik, G.E., "Simulation of Painleve–Gullstrand black hole in thin 3He-A film", *J. Exp. Theor. Phys. Lett.*, **69**, 662–668, (1999). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9901077. 16

[410] Volovik, G.E., "3He and Universe parallelism", in Bunkov, Y.M., and Godfrin, H., eds., *Topological defects and the non-equilibrium dynamics of symmetry breaking phase transitions*, 353–387, (Kluwer Academic, Dordrecht, Netherlands; Boston, U.S.A., 2000). Related online version (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/9902171. 16

[411] Volovik, G.E., "Links between gravity and dynamics of quantum liquids", *Grav. and Cosmol.*, **6**, 187–203, (2000). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0004049. 3.1.2

[412] Volovik, G.E., "Fermion zero modes in Painleve–Gullstrand black hole", *J. Exp. Theor. Phys. Lett.*, **73**, 721–725, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0104088. 3.1.3

[413] Volovik, G.E., "Superfluid analogies of cosmological phenomena", *Phys. Rep.*, **351**, 195–348, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0005091. 1.1, 3.1.2

[414] Volovik, G.E., "Vacuum Energy and Cosmological Constant: View from Condensed Matter", *J. Low Temp. Phys.*, **124**, 25–39, (2001). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0101111. 3.1.3

[415] Volovik, G.E., "Black-hole horizon and metric singularity at the brane separating two sliding superfluids", *J. Exp. Theor. Phys. Lett.*, **76**, 296–300, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0208020. 4.2.3, 4.2.3, 2, 5.2

[416] Volovik, G.E., "Effective gravity and quantum vacuum in superfluids", in Novello, M., Visser, M., and Volovik, G., eds., *Artificial Black Holes*, 127–177, (World Scientific, Singapore; River Edge, U.S.A., 2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0104046. 3.1.3

[417] Volovik, G.E., "Momentum space topology of fermion zero modes on brane", *J. Exp. Theor. Phys. Lett.*, **75**, 63–66, (2002). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0112016. 4.2.3, 4.2.3

[418] Volovik, G.E., *The Universe in a Helium Droplet*, vol. 117 of International Series of Monographs on Physics, (Clarendon Press; Oxford University Press, Oxford, U.K.; New York, U.S.A., 2003). 1.1, 4.2.3, 4.3

[419] Volovik, G.E., "What can the quantum liquid say on the brane black hole, the entropy of extremal black hole and the vacuum energy?", *Found. Phys.*, **33**, 349–368, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0301043. 3.1.5

[420] Volovik, G.E., and Vachaspati, T., "Aspects of 3He and the Standard Electroweak Model", *Int. J. Mod. Phys. B*, **10**, 471–521, (1996). Related online version (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/9510065. 2.5.1

[421] Volovik, G.E., and Zelnikov, A.I., "On temperature corrections to Einstein equations", *J. Exp. Theor. Phys. Lett.*, **78**, 751–756, (2003). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0309066. 3.1.5

[422] Wald, R.M., *General Relativity*, (University of Chicago Press, Chicago, U.S.A., 1984). 4, 2.4, 5, 8, 2.6

[423] Wald, R.M., "The Thermodynamics of Black Holes", *Living Rev. Relativity*, **4**, lrr-2001-6, (2001). URL (cited on 31 May 2005): http://www.livingreviews.org/lrr-2001-6. 6

[424] Weinfurtner, S.E.C., "Analog model for an expanding universe", (2004). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0404063. 3.1.6, 5.3, 5.3

[425] Weinfurtner, S.E.C., *Simulation of gravitational objects in Bose–Einstein condensates*, (Technical University of Munich and Max Planck Institute of Quantum Optics, Garching, Munich, Germany, 2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0404022. in German. 5.3

[426] Winicour, J., "A New Way to Make Waves", (2000). URL (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/0003029. 3.4

[427] Winterberg, F., "Detection of gravitational waves by stellar scintillation in space", *Nuovo Cimento*, **53B**, 264–279, (1968). 3.2.1

[428] Zaslavsky, O.B., "Geometry and thermodynamics of quantum-corrected acceleration horizons", *Class. Quantum Grav.*, **17**, 497–512, (2000). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9812052. 3.4

[429] Zeldovich, Y.B., "The Generation of Waves by a Rotating Body", *J. Exp. Theor. Phys. Lett.*, **14**, 180, (1971). 5.2

[430] Zeldovich, Y.B., "Amplification of cylindrical electromagnetic waves reflected from a rotating body", *Sov. Phys. JETP*, **35**, 1085, (1972). 5.2

[431] Zerilli, F.J., "Effective potential for even parity Regge–Wheeler gravitational perturbation equations", *Phys. Rev. Lett.*, **24**, 737–738, (1970). 5.1.3

[432] Zhang, P.-M., Cao, L.-M., Duan, Y.-S., and Zhong, C.-K., "Transverse force on a moving vortex with the acoustic geometry", *Phys. Lett. A*, **326**, 375, (2004). Related online version (cited on 31 May 2005): http://arXiv.org/abs/hep-th/0501073. 3.1.7

[433] Zloshchastiev, K.G., "Acoustic phase lenses in superfluid He as models of composite spacetimes in general relativity: Classical and quantum properties with provision for spatial topology", *Acta Phys. Pol. B*, **30**, 897–905, (1999). Related online version (cited on 31 May 2005): http://arXiv.org/abs/gr-qc/9802060. 3.4

[434] Zloshchastiev, K.G., "Non-linear phenomena in electrical circuits: Simulation of non-linear relativistic field theory and possible applications", (1999). URL (cited on 31 May 2005): http://arXiv.org/abs/cond-mat/9912149. 19