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Hořava-Lifshitz gravity from dynamical Newton-Cartan geometry

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ABSTRACT: Recently it has been established that torsional Newton-Cartan (TNC) geometry is the appropriate geometrical framework to which non-relativistic field theories couple. We show that when these geometries are made dynamical they give rise to Hořava-Lifshitz (HL) gravity. Projectable HL gravity corresponds to dynamical Newton-Cartan (NC) geometry without torsion and non-projectable HL gravity corresponds to dynamical NC geometry with twistless torsion (hypersurface orthogonal foliation). We build a precise dictionary relating all fields (including the scalar khronon), their transformations and other properties in both HL gravity and dynamical TNC geometry. We use TNC invariance to construct the effective action for dynamical twistless torsional Newton-Cartan geometries in 2+1 dimensions for dynamical exponent $1 < z \leq 2$ and demonstrate that this exactly agrees with the most general forms of the HL actions constructed in the literature. Further, we identify the origin of the U(1) symmetry observed by Hořava and Melby-Thompson as coming from the Bargmann extension of the local Galilean algebra that acts on the tangent space to TNC geometries. We argue that TNC geometry, which is manifestly diffeomorphism covariant, is a natural geometrical framework underlying HL gravity and discuss some of its implications.

KEYWORDS: Classical Theories of Gravity, Space-Time Symmetries, Models of Quantum Gravity

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1 Introduction

In the search for consistent theories of quantum gravity, Hořava-Lifshitz (HL) gravity [1, 2] has appeared as a tantalizing possibility of a non-Lorentz invariant and renormalizable UV completion of gravity. While observational constraints and the matching to general relativity in the IR put severe limitations on the phenomenological viability of this proposal, HL gravity is of intrinsic theoretical interest as an example of gravity with anisotropic scaling between time and space. In particular, in the context of holography it holds the prospect of providing an alternative way [3, 4] of constructing gravity duals for strongly coupled systems with non-relativistic scaling, including those of interest to condensed matter physics. More generally, one might expect that HL gravity has a natural embedding in the larger framework of string theory [5].

In parallel to this development, and with in part similar motivations, there has been considerable effort to extend the original AdS-setup in (conventional) relativistic gravity to space-times with non-relativistic scaling [6–9]. Such space-times typically exhibit a dynamical exponent z that characterizes the anisotropy between time and space on the

boundary. This includes in particular holography for Lifshitz space-times, for which it was found that the boundary geometry is described by a novel extension of Newton-Cartan (NC) geometry¹ with a specific torsion tensor, called torsional Newton-Cartan (TNC) geometry. The aim of this paper is to construct the theory of dynamical TNC geometry and show that it exactly agrees with the most general forms of HL gravity.

TNC geometry was first observed in [17, 18] as the boundary geometry for a specific action supporting $z = 2$ Lifshitz geometries, and subsequently generalized to a large class of holographic Lifshitz models for arbitrary values of z in [19, 20]. In parallel, it was shown in detail in [21] how TNC geometry arises by gauging the Schrödinger algebra, following the earlier work [22] on obtaining NC geometry from gauging the Bargmann algebra. In this paper we will show that TNC geometry can also be obtained by generalizing directly the work of [22] to include torsion without using the Schrödinger algebra. In its broadest sense the results of [19, 20] imply that Lifshitz holography describes a dual version of field theories on TNC backgrounds. In [23] it was shown that the Lifshitz vacuum (in Poincaré type coordinates) exhibits the same symmetry properties as a flat NC space-time. In particular it was found that the conformal Killing vectors of flat NC space-time span the Lifshitz algebra. In order to understand the properties of field theories on TNC backgrounds some simple scale invariant scalar field models on flat NC space-time were studied in [23, 24]. It was shown that two scenarios can occur: i). either the theory has an internal local U(1) symmetry related to particle number or ii). it does not. In case i). there is a mechanism that enhances the global Lifshitz symmetries to include particle number and Galilean boosts (and possibly even special conformal transformations) whereas in the other case no such symmetry enhancement can take place. This means that the notion of global symmetries depends on the type of matter fields one considers on such a background. In support of this it was demonstrated in ref. [23] that one can define probe scalars on a Lifshitz background that have a global Schrödinger invariance. The field-theoretic perspective of coupling Galilean invariant field theories to TNC² was independently considered in [30].

The relevant geometric fields in TNC are a time-like vielbein τ_μ , an inverse spatial metric $h^{\mu\nu}$ and a vector field $M_\mu = m_\mu - \partial_\mu\chi$ where χ is a Stückelberg scalar whose role in TNC geometry will be elucidated in section 6. The torsion in TNC geometry is always proportional to $\partial_\mu\tau_\nu - \partial_\nu\tau_\mu$ where τ_μ defines the local flow of time. The amount of torsion depends on the properties of τ_μ and we distinguish the three cases:³

- Newton-Cartan (NC) geometry
- twistless torsional (TTNC) geometry
- torsional Newton-Cartan (TNC) geometry

¹We refer to [10–16] for earlier work on Newton-Cartan geometry.

²Ref. [25] introduced NC geometry to field theory analyses of problems with strongly correlated electrons, such as the fractional quantum Hall effect. Later torsion was added to this analysis in [26]. The type of torsion introduced there is what we call twistless torsion. See also [27–29] for a different approach to Newton-Cartan geometry.

³These three cases also naturally arise in Lifshitz holography [17, 18]. We note that TTNC geometry was already observed in [16] but in that work the torsion was eliminated using a conformal rescaling.

where the first possibility has no torsion and the latter option has general torsion with the twistless case being an important in-between situation. More specifically, in the first case the time-like vielbein of the geometry is closed and defines an absolute time. In the second case the time-like vielbein is hypersurface orthogonal and thereby allows for a foliation of equal time spatial surfaces described by Riemannian (i.e. torsion free) geometry. In the third, most general, case there is no constraint on τ_μ .

As is clear from holographic studies of the boundary energy-momentum tensor as for example in [17–19, 23, 31] the addition of torsion to the NC geometry is crucial in order to be able to calculate the energy density and energy flux of the theory. This is because they are the response to varying τ_μ (see also [30]). Hence in order to be able to compute these quantities τ_μ better be unconstrained, i.e. one should allow for arbitrary torsion. If we work with TTNC geometry one can only compute the energy density and the divergence of the energy current [20] because in that case $\tau_\mu = \psi \partial_\mu \tau$ where one has to vary ψ and τ with ψ sourcing the energy density and τ sourcing the divergence (after partial integration) of the energy current. In any case the point is that, contrary to the relativistic setting, adding torsion is a very natural thing to do in NC geometry. Moreover, as will be shown later, the torsion is not something one can freely pick and is actually fixed by the formalism.

In all of these works the TNC geometry appears as a fixed background and is hence not dynamical. The purpose of this paper is to consider what theory of gravity appears when letting the TNC geometry fluctuate. We find, perhaps not entirely unexpected,⁴ that depending on the amount of torsion the resulting theories include HL gravity and all of its known extensions.

Our focus in this paper will be mainly on the first two of the three cases listed above, leaving the details of the dynamics of the most general case (TNC gravity) for future work. In particular, we will show that:

- dynamical NC geometry = projectable HL gravity
- dynamical TTNC geometry = non-projectable HL gravity.

The khronon field introduced by [32] (to make HL gravity generally covariant whereby making manifest the presence of an extra scalar mode) naturally appears (see also [33]) in our formulation. We furthermore show that the U(1) extension of [34] (see also [35–37]) emerges as well in a natural fashion. The essential identification between the covariant⁵ NC-type geometric structures and those appearing in the ADM parametrization that forms the starting point of HL gravity is as follows

$$\tau_\mu \sim \text{lapse}, \quad \hat{h}_{\mu\nu} \sim \text{spatial metric}, \quad m_\mu \sim \text{shift} + \text{Newtonian potential},$$

where the fields $\hat{h}_{\mu\nu}$ and m_μ are defined in section 4. We will show that the effective action for the TTNC fields leads to two kinetic terms for the metric $\hat{h}_{\mu\nu}$ (giving rise to the λ parameter of HL gravity [1, 2]) including the potential terms computed in refs. [32, 36, 37, 39].

⁴A HL-type action in TNC covariant form was already observed in [18] where the anisotropic Weyl-anomaly in a specific $z = 2$ holographic four-dimensional bulk Lifshitz model was obtained via null Scherk-Schwarz reduction of the AdS₅ conformal anomaly of gravity coupled to an axion.

⁵Note that in e.g. ref. [38] there is also a type of covariantization of HL gravity (see also eq. (3.9) of [3]), but there is still inherently a Lorentzian metric structure present. This only works up to second order in derivatives so that it only captures the IR limit of HL gravity.

Furthermore the Stückelberg scalar χ entering in the TNC quantity $M_\mu = m_\mu - \partial_\mu\chi$ (see [17–19, 21, 23, 24]) will be directly related to the Newtonian prepotential introduced in [34]. The relation to TTNC geometry will, however, provide a new perspective on the nature of the U(1) symmetry studied in the context of HL gravity. As a further confirmation that TNC geometry is a natural framework for HL gravity we will demonstrate in this paper that when we include dilatation symmetry (local Schrödinger invariance) one obtains conformal HL gravity.

As we will review in this paper, the various versions of TNC geometry defined above arise by gauging non-relativistic symmetry algebras (Galilean, Bargmann, Schrödinger). In particular, in this procedure the internal symmetries are made into local symmetries, and translations are turned into diffeomorphisms. This is in the same way that Riemannian geometry comes from gauging the Poincaré algebra, thereby imposing local Lorentz symmetry and turning translations into space-time diffeomorphisms. Thus HL gravity theories (and more generally TNC gravity) can be seen as the most general gravity theories for which the Einstein equivalence principle (that locally space-time is described by flat Minkowski space-time) is applied to *local* non-relativistic (Galilean) symmetries, rather than to the local Lorentz symmetry that one has in special relativity.

We point out that in general relativity (GR) the global symmetries (Killing vectors) of Minkowski space-time (the Poincaré algebra) form the same algebra from which upon gauging (and replacing local space-time translations by diffeomorphisms as explained in appendix A) we obtain the geometrical framework of GR. On the other hand the Killing vectors of flat NC space-time only involve space and time translations and spatial rotations [23] while the local tangent space group that we gauge in order to obtain the TNC geometrical framework is the Galilean algebra (where again we also replace local time and space translations by diffeomorphisms), which also contains Galilean boosts and is thus not the same algebra as the algebra of Killing vectors of flat NC space-time. We bring this up to highlight the fact that the local tangent space symmetries and the Killing vectors of flat space-time are in general two very different concepts that are often mistakenly assumed to be the same. Basically this happens because the M_μ vector allows for the construction of a new set of vielbeins (defined in section 4) that are invariant under G transformations and that only see diffeomorphisms and local rotations which agrees with the Killing vectors of flat NC space-time. Nevertheless the fact that M_μ is one of the background fields to which we can couple a field theory can, under special circumstances, lead to additional symmetries such as G and N (and even special conformal symmetries) [23].

Our results on dynamical TNC geometry and its relation to HL gravity provide a new perspective on these theories of gravity. For one thing, the vacuum of HL gravity (without a cosmological constant) has so far been taken to be Minkowski space-time, but since the underlying geometry appears to be TNC geometry, it seems more natural to take this as flat NC space-time [23, 24]. Thus it would seem worthwhile to reexamine HL gravity and the various issues⁶ that have been raised following its introduction. As another application, we emphasize that, independent of a possible UV completion of gravity, our results on dynamical TNC geometry are of relevance to constructing IR effective field theories of

⁶There is an extensive literature on this (e.g. instabilities and strong coupling at low energies), see e.g. refs. [40–47].

non-relativistic systems following the recent developments of applying this to condensed matter systems. For these kinds of applications, the question whether HL gravity flows to a theory with local Lorentz invariance ($\lambda = 1$) in the IR is of no concern. Finally, from a broader perspective our results might be useful towards a proper description of the non-relativistic quantum gravity corner of the “ $(\hbar, G_N, 1/c)$ -cube”, perhaps aiding the formulation of a well-defined perturbative $1/c$ expansion around such a theory.

Outline of the paper. The first part of the paper (sections 2 to 7) is devoted to setting up the geometrical framework for torsional Newton-Cartan geometry, presented in such a way that the subsequent connection to HL gravity is most clearly displayed. We thus take a pedagogical approach that introduces the relevant ingredients in a step-by-step way. To this end we begin in section 2 with the geometry that is obtained by gauging the Galilean algebra, extending the original work of [22] to include torsion. We exhibit the transformation properties of the relevant geometrical fields under space-time diffeomorphisms and the internal transformations, consisting of Galilean boosts (G) and spatial rotations (J). We also discuss the vielbein postulates and curvatures entering the field strength of the gauge field. We point out that the only G, J invariants are the time-like vielbein τ_μ and the inverse spatial metric $h^{\mu\nu}$. In section 3 we then present the most general affine connection that satisfies the property that the latter quantities are covariantly conserved.

In section 4, we go one step further and add the central element (N) to the Galilean algebra, and consider the gauging of the resulting Bargmann algebra (as also considered in [22] for the case with no torsion). We show that the extra gauge field m_μ that enters in this description, does not alter the transformation properties of the objects considered in section 2, but allows for the introduction of further useful G, J invariants, namely an inverse time-like vielbein \hat{v}^μ , a spatial metric $\bar{h}_{\mu\nu}$ (or $\hat{h}_{\mu\nu}$) and a “Newtonian potential” $\tilde{\Phi}$. We then return to the construction of the affine connection in section 5 and employ the geometric quantities of section 2 and 4 to construct the most general connection that can be built out of the invariants. We discuss two special choices of affine connections with particular properties, one of them being especially convenient for the comparison with HL gravity. We point out that, in the case of non-vanishing torsion, there is no choice of affine connection that is also N -invariant, but that one can formally remedy this by introducing a Stückelberg scalar χ (defining $M_\mu = m_\mu - \partial_\mu\chi$) to the setup that cancels this non-invariance. This has the advantage that one can deal simultaneously with theories that have a local $U(1)$ symmetry and those that do not have this, and further it will prove useful when comparing to HL gravity (especially [34–37]). We also show how the TNC invariants can be used to build a non-degenerate symmetric rank 2 tensor with Lorentzian signature, which will later be used to make contact with the ADM decomposition that enters HL gravity.

In section 6 we discuss the specific form of the torsion tensor that emerges from gauging the Bargmann algebra and introduce the three relevant cases for torsion (NC, TTNC and TNC) that were already mentioned above. We also introduce a vector a_μ that describes the TTNC torsion, which will turn out to be very useful in order to make contact with the literature on non-projectable HL gravity. Further we will identify the khronon field of [32].

Then in section 7 we give some basic properties of the curvatures (extrinsic curvature and Ricci tensor for TTNC) that will be useful when constructing HL actions.

In section 8 we relate the TNC invariants introduced in the previous sections to those appearing in the corresponding ADM parameterization employed in HL gravity. This identification and the match of the properties and number of components and local symmetries in the case of NC and TTNC already strongly suggest that dynamical (TT)NC is expected to be the same as (non)-projectable HL gravity. We then proceed in section 9 by showing that the generic action that describes dynamical TTNC geometries agrees on the nose with the most general HL actions appearing in the literature. For simplicity we treat the case of 2 spatial dimensions with $1 < z \leq 2$ and organize the terms in the action according to their dilatation weight. In particular, we construct all G, J invariant terms that are relevant or marginal, using as building blocks the TNC invariants (including the torsion tensor and curvature tensor) and covariant derivatives. The resulting action is written in (9.18), (9.19) and gives the HL kinetic terms [1, 2] while the potential is exactly the same as the 3D version of the potential given in [32, 36, 37, 39].

We then proceed in section 10 to consider the extension of the action to include invariance under the central extension N , leading to HL actions with local Bargmann invariance. This can be achieved by including couplings to $\tilde{\Phi}$, which did not appear yet in section 9. Importantly, in the projectable case with the HL coupling constant $\lambda = 1$ we reproduce the U(1) invariant action of [34]. When we consider the non-projectable version or $\lambda \neq 1$ we need additional terms to make the theory U(1) invariant which is precisely achieved by adding the Stückelberg field χ that we introduced in section 5 (see also [23]). We can then write a Bargmann invariant action that precisely reproduces the actions considered in the literature, where in particular the χ -dependent pieces agree with those in [36, 37]. This comes about in part via coupling to the natural TNC Newton potential, $\tilde{\Phi}_\chi$, which is the Bargmann invariant generalization of $\tilde{\Phi}$, and the simple covariant form of the action (10.14) is one of our central results.

We emphasize that adding the χ field to the action means that we have trivialized the U(1) symmetry by Stückelberging it or in other words we have removed the U(1) transformations all together. We further expand on this fact in section 11, commenting on statements in the literature regarding the relevance of the U(1) invariance (which is not there unless we have zero torsion and $\lambda = 1$) in relation to the elimination of a scalar degree of freedom. In particular, we will present a different mechanism that accomplishes this and which involves a constraint equation obtained by varying the TNC potential $\tilde{\Phi}_\chi$.

Finally in section 12 we consider the case where we add dilatations to the Bargmann algebra, i.e. we consider the dynamics we get from a geometry that is locally Schrödinger invariant. We will show that the resulting theory is conformal HL gravity, providing further evidence for our claim that TNC geometry is the underlying geometry of HL gravity. In particular, employing the local Schrödinger algebra we will arrive at the invariant $z = d$ action (12.50) for conformal HL gravity in $d + 1$ dimensions.

We end in section 13 with our conclusions and discuss a large variety of possible open directions. For comparison to general relativity and as an introduction to the logic followed in sections 2 to 7, we have included appendix A which discusses the gauging of the Poincaré algebra leading to Riemannian geometry (possibly with torsion added).

2 Local Galilean transformations

The present section until section 7 is devoted to setting up the general geometrical framework for torsional Newton-Cartan geometry. We will follow an approach that is very similar to what in general relativity is known as the gauging of the Poincaré algebra. This provides us in a very efficient manner with all basic geometrical objects used in the formulation of general relativity (and higher curvature modifications thereof). For the interested reader unfamiliar with this method we give a short summary of it in appendix A.

To obtain torsional Newton-Cartan geometry we follow the same logic as in appendix A for the case of the Galilean algebra and its central extension known as the Bargmann algebra. This was first considered in [22] for the case without torsion. Here we generalize this interesting work to the case with torsion. Adding torsion to Newton-Cartan geometry can also be done by making it locally scale invariant, i.e. gauging the Schrödinger algebra as in [21]. However upon gauging the Schrödinger algebra the resulting geometric objects are all dilatation covariant which is useful for the construction of conformal HL gravity as we will study in section 12 but it is less useful for the study of general non-conformally invariant HL actions which is why we start our analysis by adding torsion to the analysis of [22].

Consider the Galilean algebra whose generators are denoted by H, P_a, G_a, J_{ab} and whose commutation relations are

$$\begin{aligned} [H, G_a] &= P_a, & [P_a, G_b] &= 0, \\ [J_{ab}, P_c] &= \delta_{ac}P_b - \delta_{bc}P_a, & [J_{ab}, G_c] &= \delta_{ac}G_b - \delta_{bc}G_a, \\ [J_{ab}, J_{cd}] &= \delta_{ac}J_{bd} - \delta_{ad}J_{bc} - \delta_{bc}J_{ad} + \delta_{bd}J_{ac}. \end{aligned} \quad (2.1)$$

Let us consider a connection \mathcal{A}_μ taking values in the Galilean algebra⁷

$$\mathcal{A}_\mu = H\tau_\mu + P_a e_\mu^a + G_a \Omega_\mu^a + \frac{1}{2} J_{ab} \Omega_\mu^{ab}. \quad (2.2)$$

This connection transforms in the adjoint as

$$\delta \mathcal{A}_\mu = \partial_\mu \Lambda + [\mathcal{A}_\mu, \Lambda]. \quad (2.3)$$

With this transformation we can associate another transformation denoted by $\bar{\delta}$ as follows. Write (without loss of generality)

$$\Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma, \quad (2.4)$$

where

$$\Sigma = G_a \lambda^a + \frac{1}{2} J_{ab} \lambda^{ab}, \quad (2.5)$$

is chosen to only include the internal symmetries G and J . We define $\bar{\delta} \mathcal{A}_\mu$ as

$$\bar{\delta} \mathcal{A}_\mu = \delta \mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu, \Sigma], \quad (2.6)$$

⁷Our notation is such that $\mu, \nu = 0 \dots d$ are spacetime indices and $a, b = 1 \dots d$ are spatial tangent space indices.

where $\mathcal{F}_{\mu\nu}$ is the curvature

$$\begin{aligned}\mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \\ &= HR_{\mu\nu}(H) + P_a R_{\mu\nu}{}^a(P) + G_a R_{\mu\nu}{}^a(G) + \frac{1}{2} J_{ab} R_{\mu\nu}{}^{ab}(J).\end{aligned}\quad (2.7)$$

Often in works on gauging space-time symmetry groups it is suggested that diffeomorphisms can only be obtained once specific curvature constraints are imposed.⁸ We emphasize that the transformation $\bar{\delta}\mathcal{A}_\mu$ exists no matter what we choose for the curvature $\mathcal{F}_{\mu\nu}$.

If we write in components what (2.6) states we obtain the transformation properties

$$\bar{\delta}\tau_\mu = \mathcal{L}_\xi \tau_\mu, \quad (2.8)$$

$$\bar{\delta}e_\mu^a = \mathcal{L}_\xi e_\mu^a + \lambda^a{}_b e_\mu^b + \lambda^a \tau_\mu, \quad (2.9)$$

$$\bar{\delta}\Omega_\mu{}^a = \mathcal{L}_\xi \Omega_\mu{}^a + \partial_\mu \lambda^a + \lambda^a{}_b \Omega_\mu{}^b + \lambda^b \Omega_{\mu b}{}^a, \quad (2.10)$$

$$\bar{\delta}\Omega_\mu{}^{ab} = \mathcal{L}_\xi \Omega_\mu{}^{ab} + \partial_\mu \lambda^{ab} + 2\lambda^{[a} \Omega_\mu{}^{c|b]}, \quad (2.11)$$

where \mathcal{L}_ξ is the Lie derivative along ξ^μ and λ^a , λ^{ab} the parameters of the internal G , J transformations, respectively.

We can now write down covariant derivatives that transform covariantly under these transformations. They are

$$\mathcal{D}_\mu \tau_\nu = \partial_\mu \tau_\nu - \Gamma_{\mu\nu}^\rho \tau_\rho, \quad (2.12)$$

$$\mathcal{D}_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\rho e_\rho^a - \Omega_\mu{}^a \tau_\nu - \Omega_\mu{}^a{}_b e_\nu^b, \quad (2.13)$$

where $\Gamma_{\mu\nu}^\rho$ is an affine connection transforming as

$$\bar{\delta}\Gamma_{\mu\nu}^\rho = \partial_\mu \partial_\nu \xi^\rho + \xi^\sigma \partial_\sigma \Gamma_{\mu\nu}^\rho + \Gamma_{\sigma\nu}^\rho \partial_\mu \xi^\sigma + \Gamma_{\mu\sigma}^\rho \partial_\nu \xi^\sigma - \Gamma_{\mu\nu}^\sigma \partial_\sigma \xi^\rho. \quad (2.14)$$

It is in particular inert under the G and J transformations. The form of the covariant derivatives is completely fixed by the local transformations $\bar{\delta}\mathcal{A}_\mu$. However any tensor redefinition of the connections $\Gamma_{\mu\nu}^\rho$, $\Omega_\mu{}^a$ and $\Omega_\mu{}^{ab}$ that leaves the covariant derivatives form-invariant leads to an allowed set of connections with the exact same transformation properties.

We impose the vielbein postulates

$$\mathcal{D}_\mu \tau_\nu = 0, \quad (2.15)$$

$$\mathcal{D}_\mu e_\nu^a = 0, \quad (2.16)$$

which allows us to express $\Gamma_{\mu\nu}^\rho$ in terms of $\Omega_\mu{}^a$ and $\Omega_\mu{}^{ab}$ via

$$\Gamma_{\mu\nu}^\rho = -v^\rho \partial_\mu \tau_\nu + e_a^\rho \left(\partial_\mu e_\nu^a - \Omega_\mu{}^a \tau_\nu - \Omega_\mu{}^a{}_b e_\nu^b \right), \quad (2.17)$$

⁸This is because setting to zero some of the curvatures in $\mathcal{F}_{\mu\nu}$ identifies $\bar{\delta}$ with δ in (2.6) for those fields that are not fixed by the curvature constraints. There is no need for the δ and $\bar{\delta}$ transformations to coincide. As we show in appendix A this is no longer the case in GR when there is non-vanishing torsion.

where we defined inverse vielbeins v^μ and e_a^μ via

$$v^\mu \tau_\mu = -1, \quad v^\mu e_\mu^a = 0, \quad e_a^\mu \tau_\mu = 0, \quad e_a^\mu e_\mu^b = \delta_a^b. \quad (2.18)$$

The vielbein postulates for the inverses read

$$\mathcal{D}_\mu v^\nu = \partial_\mu v^\nu + \Gamma_{\mu\rho}^\nu v^\rho - \Omega_\mu^a e_a^\nu = 0, \quad (2.19)$$

$$\mathcal{D}_\mu e_a^\nu = \partial_\mu e_a^\nu + \Gamma_{\mu\rho}^\nu e_a^\rho + \Omega_\mu^b e_b^a e_c^\nu = 0. \quad (2.20)$$

Using that Ω_μ^{ab} is antisymmetric we find that $h^{\mu\nu} = \delta^{ab} e_a^\mu e_b^\nu$ satisfies

$$\nabla_\mu h^{\nu\rho} = 0, \quad (2.21)$$

which together with equations (2.12) and (2.15), i.e.

$$\nabla_\mu \tau_\nu = 0, \quad (2.22)$$

constrain $\Gamma_{\mu\nu}^\rho$. Equations (2.21) and (2.22) are the TNC analogue of metric compatibility in GR.

The components of the field strength $\mathcal{F}_{\mu\nu}$ in (2.7) are given by

$$R_{\mu\nu}(H) = 2\partial_{[\mu} \tau_{\nu]}, \quad (2.23)$$

$$R_{\mu\nu}^a(P) = 2\partial_{[\mu} e_{\nu]}^a - 2\Omega_{[\mu}^a \tau_{\nu]} - 2\Omega_{[\mu}^a e_{\nu]}^b, \quad (2.24)$$

$$R_{\mu\nu}^a(G) = 2\partial_{[\mu} \Omega_{\nu]}^a - 2\Omega_{[\mu}^{ab} \Omega_{\nu]}^b, \quad (2.25)$$

$$R_{\mu\nu}^{ab}(J) = 2\partial_{[\mu} \Omega_{\nu]}^{ab} - 2\Omega_{[\mu}^{ca} \Omega_{\nu]}^b. \quad (2.26)$$

The first two appear in the antisymmetric part of the covariant derivatives $\mathcal{D}_\mu \tau_\nu$ and $\mathcal{D}_\mu e_\nu^a$. More precisely we have

$$R_{\mu\nu}(H) = 2\Gamma_{[\mu\nu]}^\rho \tau_\rho, \quad (2.27)$$

$$R_{\mu\nu}^a(P) = 2\Gamma_{[\mu\nu]}^\rho e_\rho^a. \quad (2.28)$$

In other words they are equal to the torsion tensor, i.e.

$$2\Gamma_{[\mu\nu]}^\rho = -v^\rho R_{\mu\nu}(H) + e_a^\rho R_{\mu\nu}^a(P). \quad (2.29)$$

The other two curvature tensors can be found by computing the Riemann tensor defined as

$$[\nabla_\mu, \nabla_\nu] X_\sigma = R_{\mu\nu\sigma}^\rho X_\rho - 2\Gamma_{[\mu\nu]}^\rho \nabla_\rho X_\sigma. \quad (2.30)$$

Using that

$$R_{\mu\nu\sigma}^\rho = -\partial_\mu \Gamma_{\nu\sigma}^\rho + \partial_\nu \Gamma_{\mu\sigma}^\rho - \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda + \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda, \quad (2.31)$$

together with (2.17) tells us that

$$R_{\mu\nu\sigma}^\rho = e_a^\rho \tau_\sigma R_{\mu\nu}^a(G) - e_{\sigma a} e_b^\rho R_{\mu\nu}^{ab}(J). \quad (2.32)$$

So far all components of \mathcal{A}_μ are independent or what is the same τ_μ , e_μ^a and $\Gamma_{\mu\nu}^\rho$ (obeying (2.21) and (2.22)) are all independent. The inverse vielbeins v^μ and e_a^μ transform as

$$\bar{\delta}v^\mu = \mathcal{L}_\xi v^\mu + e_a^\mu \lambda^a, \tag{2.33}$$

$$\bar{\delta}e_a^\mu = \mathcal{L}_\xi e_a^\mu + \lambda_a{}^b e_b^\mu. \tag{2.34}$$

There are only two invariants, i.e. tensors invariant under G and J transformations, that we can build out of the vielbeins. These are τ_μ and $h^{\mu\nu} = \delta^{ab} e_a^\mu e_b^\nu$. This is not enough to construct an affine connection that transforms as (2.14). The reason we cannot build any other invariants is because v^μ and $h_{\mu\nu} = \delta_{ab} e_\mu^a e_\nu^b$ undergo shift transformations under local Galilean boosts λ^a (also known as Milne boosts [30]).

3 The affine connection: part 1

The most general $\Gamma_{\mu\nu}^\rho$ obeying (2.21) and (2.22) is of the form

$$\Gamma_{\mu\nu}^\rho = -v^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} (\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \frac{1}{2} h^{\rho\sigma} Y_{\sigma\mu\nu} \tag{3.1}$$

where $h^{\rho\sigma} Y_{\sigma\mu\nu}$ satisfies

$$\left(h^{\lambda\sigma} h^{\rho\nu} + h^{\rho\sigma} h^{\lambda\nu} \right) Y_{\sigma\mu\nu} = 0. \tag{3.2}$$

It follows that $Y_{\sigma\mu\nu}$ can be written as

$$Y_{\sigma\mu\nu} = \tau_\sigma X_{\mu\nu}^1 + \tau_\nu X_{\sigma\mu}^2 + X_{\sigma\mu\nu}^3, \tag{3.3}$$

where $X_{\mu\nu}^1$ and $X_{\sigma\mu}^2$ and $X_{\sigma\mu\nu}^3 = -X_{\nu\mu\sigma}^3$ are arbitrary. We write $X_{\sigma\mu}^2 = K_{\sigma\mu} + X_{(\sigma\mu)}^2$ so that $K_{\sigma\mu} = -K_{\mu\sigma}$. Further we write $X_{\sigma\mu\nu}^3 = \tau_\mu K_{\sigma\nu} + \tilde{X}_{\sigma\mu\nu}^3$ so that we can write

$$Y_{\sigma\mu\nu} = \tau_\sigma \left(X_{\mu\nu}^1 + X_{(\mu\nu)}^2 \right) + \tau_\mu K_{\sigma\nu} + \tau_\nu K_{\sigma\mu} + L_{\sigma\mu\nu}, \tag{3.4}$$

where $L_{\sigma\mu\nu} = -L_{\nu\mu\sigma}$ is defined as

$$L_{\sigma\mu\nu} = \tau_\nu X_{(\sigma\mu)}^2 - \tau_\sigma X_{(\nu\mu)}^2 + \tilde{X}_{\sigma\mu\nu}^3. \tag{3.5}$$

Since $Y_{\sigma\mu\nu}$ is defined as $h^{\rho\sigma} Y_{\sigma\mu\nu}$ we can drop the part in (3.4) that is proportional to τ_σ . We thus find the following form for the connection $\Gamma_{\mu\nu}^\rho$

$$\Gamma_{\mu\nu}^\rho = -v^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} (\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \frac{1}{2} h^{\rho\sigma} (\tau_\mu K_{\sigma\nu} + \tau_\nu K_{\sigma\mu} + L_{\sigma\mu\nu}). \tag{3.6}$$

The variation of $\Gamma_{\mu\nu}^\rho$ under local Galilean boosts yields

$$\begin{aligned} \delta_G \Gamma_{\mu\nu}^\rho &= \frac{1}{2} h^{\rho\sigma} \tau_\mu (\delta_G K_{\sigma\nu} + \partial_\nu \lambda_\sigma - \partial_\sigma \lambda_\nu) + \frac{1}{2} h^{\rho\sigma} \tau_\nu (\delta_G K_{\sigma\mu} + \partial_\mu \lambda_\sigma - \partial_\sigma \lambda_\mu) \\ &\quad + \frac{1}{2} h^{\rho\sigma} (\delta_G L_{\sigma\mu\nu} - \lambda_\sigma (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu) + \lambda_\mu (\partial_\nu \tau_\sigma - \partial_\sigma \tau_\nu) + \lambda_\nu (\partial_\mu \tau_\sigma - \partial_\sigma \tau_\mu)), \end{aligned} \tag{3.7}$$

where $\lambda_\mu = \lambda_a e_\mu^a$. In section 5 we will choose $K_{\mu\nu}$ and $L_{\sigma\mu\nu}$ such that $\delta_G \Gamma_{\mu\nu}^\rho = 0$.

4 Local Bargmann transformations

It is well known that the Galilean algebra admits a central extension with central element N called the Bargmann algebra. This latter element appears via the commutator $[P_a, G_b] = \delta_{ab}N$. We denote the associated gauge connection by m_μ . Following the same recipe as in section 2 with

$$\mathcal{A}_\mu = H\tau_\mu + P_a e_\mu^a + G_a \Omega_\mu^a + \frac{1}{2} J_{ab} \Omega_\mu^{ab} + N m_\mu, \quad (4.1)$$

$$\Sigma = G_a \lambda^a + \frac{1}{2} J_{ab} \lambda^{ab} + N \sigma, \quad (4.2)$$

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \\ &= H R_{\mu\nu}(H) + P_a R_{\mu\nu}^a(P) + G_a R_{\mu\nu}^a(G) + \frac{1}{2} J_{ab} R_{\mu\nu}^{ab}(J) + N R_{\mu\nu}(N), \end{aligned} \quad (4.3)$$

we obtain

$$\bar{\delta} m_\mu = \mathcal{L}_\xi m_\mu + \partial_\mu \sigma + e_\mu^a \lambda_a, \quad (4.4)$$

where $\bar{\delta}$ is defined in the same way as in (2.6). Note that we have an extra parameter σ associated with the N transformation. Because N is central, all results of the previous section remain unaffected.

Our primary focus in this section is local Galilean boost invariance. The new field m_μ is shifted under the λ^a transformation and so in combinations such as

$$\hat{v}^\mu = v^\mu - h^{\mu\nu} m_\nu, \quad (4.5)$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tau_\mu m_\nu - \tau_\nu m_\mu, \quad (4.6)$$

the Galilean boost parameter λ^a is cancelled. However we now have two other things to worry about. First of all the new field m_μ also transforms under a local $U(1)$ transformation with parameter σ and secondly we have introduced more than is strictly necessary to have local Galilean invariance. This is because the component

$$\tilde{\Phi} = -v^\mu m_\mu + \frac{1}{2} h^{\mu\nu} m_\mu m_\nu \quad (4.7)$$

is G invariant (and of course also J invariant). In previous works we have introduced another background field χ , a Stückelberg scalar, transforming as $\bar{\delta}\chi = \mathcal{L}_\xi \chi + \sigma$ so that the combination $M_\mu = m_\mu - \partial_\mu \chi$ is invariant under the local N transformation and replaced everywhere m_μ by M_μ . Here it will prove convenient, for the sake of comparison with work on HL gravity to postpone this step until later.⁹ Hence for now we will work with m_μ as opposed to M_μ .

We introduce a new set of Galilean invariant vielbeins: τ_μ, \hat{e}_μ^a whose inverses are \hat{v}^μ and e_a^μ where $\hat{e}_\mu^a = e_\mu^a - m^a \tau_\mu$ with $m^a = e^{\mu a} m_\mu$. They satisfy the relations

$$\hat{v}^\mu \tau_\mu = -1, \quad \hat{v}^\mu \hat{e}_\mu^a = 0, \quad e_a^\mu \tau_\mu = 0, \quad e_a^\mu \hat{e}_\mu^b = \delta_a^b. \quad (4.8)$$

⁹In previous work [19, 21, 23, 24] we denoted by $\hat{v}^\mu, \bar{h}_{\mu\nu}$ and $\tilde{\Phi}$ the invariants with m_μ replaced by M_μ . Here we temporarily work with the forms (4.5)–(4.7) for reasons that will become clear as we go on. We return to our notation from previous works in section 12. We also point out that compared to [19, 21, 23, 24] we denote by m_μ here what was referred to as \tilde{m}_μ in these papers and vice versa we denote by \tilde{m}_μ here what was denoted by m_μ in these respective works.

We also have the completeness relation $e_a^\mu \hat{e}_\nu^a = \delta_\nu^\mu + \hat{v}^\mu \tau_\nu$. The introduction of m^a thus leads to the G, J invariants \hat{v}^μ and

$$\hat{h}_{\mu\nu} = \delta_{ab} \hat{e}_\mu^a \hat{e}_\nu^b = \bar{h}_{\mu\nu} + 2\tau_\mu \tau_\nu \tilde{\Phi}, \quad (4.9)$$

where $\bar{h}_{\mu\nu}$ is given in (4.6). The part of m_μ that is responsible for the Galilean boost invariance is m^a that transforms as (ignoring the σ transformation)

$$\bar{\delta} m^a = \mathcal{L}_\xi m^a + \lambda^a + \lambda^a{}_b m^b. \quad (4.10)$$

We can write

$$m_\mu = e_\mu^a m_a - \frac{1}{2} m_a m^a \tau_\mu + \tilde{\Phi} \tau_\mu, \quad (4.11)$$

where the last term is an invariant.

5 The affine connection: part 2

In section 2 we realized the Galilean algebra on the fields $\tau_\mu, e_\mu^a, \Omega_\mu^a$ and Ω_μ^{ab} or what is the same on τ_μ, e_μ^a and $\Gamma_{\mu\nu}^\rho$ where the affine connection obeys (2.21) and (2.22). Now that we have introduced a new field m_μ transforming as in (4.4) we will see that we can realize the Galilean algebra on a smaller set of fields, namely τ_μ, e_μ^a and m_μ . We can also realize the Galilean algebra on τ_μ, e_μ^a and m^a with m^a transforming as in (4.10), i.e. no dependence on $\tilde{\Phi}$ or realize it on τ_μ, e_μ^a, m^a and $\tilde{\Phi}$ which is another way of writing the dependence on τ_μ, e_μ^a and m_μ . These different options lead to different choices for the affine connection as we will now discuss.

The most straightforward way of constructing a $\Gamma_{\mu\nu}^\rho$ that is made out of vielbeins and either i). m_μ or ii). m^a , that obeys (2.21) and (2.22) and transforms as in (2.14), is to use the invariants $\tau_\mu, \bar{h}_{\mu\nu}, \hat{v}^\mu, h^{\mu\nu}$ and $\tilde{\Phi}$. The most general connection we can build out of these invariants reads [23]

$$\Gamma_{\mu\nu}^\rho = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} (\partial_\mu H_{\nu\sigma} + \partial_\nu H_{\mu\sigma} - \partial_\sigma H_{\mu\nu}), \quad (5.1)$$

where $H_{\mu\nu}$ is given by

$$H_{\mu\nu} = \bar{h}_{\mu\nu} + \alpha \tau_\mu \tau_\nu \tilde{\Phi}, \quad (5.2)$$

where α is any constant. If we want the connection to depend linearly on m_μ , which is a special case of case i). above, we should take $\alpha = 0$. If we wish that the connection is independent of $\tilde{\Phi}$ as in case ii). we should take $\alpha = 2$ because of the identity (4.9) so that $H_{\mu\nu} = \hat{h}_{\mu\nu}$ where $\hat{h}_{\mu\nu}$ only depends on m^a . For the general case i). i.e. general dependence on m^a and $\tilde{\Phi}$, we can take any α . For case i). with a linear dependence on m_μ we denote $\Gamma_{\mu\nu}^\rho$ by $\bar{\Gamma}_{\mu\nu}^\rho$ which is given by

$$\bar{\Gamma}_{\mu\nu}^\rho = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} (\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu}). \quad (5.3)$$

This form of $\Gamma_{\mu\nu}^\rho$ has been used in [19, 21, 24, 30, 48]. The form of $\bar{\Gamma}_{\mu\nu}^\rho$ corresponds to taking in (3.6) the following choices for $K_{\mu\nu}$ and $L_{\sigma\mu\nu}$, namely

$$K_{\mu\nu} = \partial_\mu m_\nu - \partial_\nu m_\mu, \quad (5.4)$$

$$L_{\sigma\mu\nu} = m_\sigma (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu) - m_\mu (\partial_\nu \tau_\sigma - \partial_\sigma \tau_\nu) - m_\nu (\partial_\mu \tau_\sigma - \partial_\sigma \tau_\mu). \quad (5.5)$$

For case ii). we denote $\Gamma_{\mu\nu}^\rho$ by $\hat{\Gamma}_{\mu\nu}^\rho$ which reads

$$\hat{\Gamma}_{\mu\nu}^\rho = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left(\partial_\mu \hat{h}_{\nu\sigma} + \partial_\nu \hat{h}_{\mu\sigma} - \partial_\sigma \hat{h}_{\mu\nu} \right). \quad (5.6)$$

The two connections $\hat{\Gamma}_{\mu\nu}^\rho$ and $\bar{\Gamma}_{\mu\nu}^\rho$ differ by a tensor as follows from

$$\hat{\Gamma}_{\mu\nu}^\rho = \bar{\Gamma}_{\mu\nu}^\rho + \tilde{\Phi} h^{\rho\sigma} \tau_\nu (\partial_\mu \tau_\sigma - \partial_\sigma \tau_\mu) + \tilde{\Phi} h^{\rho\sigma} \tau_\mu (\partial_\nu \tau_\sigma - \partial_\sigma \tau_\nu) - \tau_\mu \tau_\nu h^{\rho\sigma} \partial_\sigma \tilde{\Phi}. \quad (5.7)$$

In this work it will prove most convenient to use the connection (5.6) as this eases comparison with HL gravity. We stress though that in principle one can take any of the above choices, i.e. any value for α , and that the final form of the effective action for HL gravity will take the same form regardless which $\Gamma_{\mu\nu}^\rho$ one chooses as all dependence on α drops out when forming the scalar terms appearing in the action.¹⁰

The reader familiar with the literature on NC geometry without torsion might wonder which of these connections relates to the one of NC geometry (as written for example in [22] and references therein). The usual NC connection is obtained by taking (5.3) with $K_{\mu\nu}$ as given in (5.5) and $L_{\sigma\mu\nu} = 0$ which follows from (5.5) and the fact that for NC geometry we have $\partial_\mu \tau_\nu - \partial_\nu \tau_\mu = 0$. The possibility of modifying these connections by terms proportional to α was never considered before probably because this breaks manifest local N invariance of the NC connection which depends on m_μ only via its curl.

In the presence of torsion the fact that $L_{\sigma\mu\nu}$ is given by (5.5) tells us that we have no manifest N invariance of the connection. Further, for no value of α can we find such an invariance. This can be formally solved by adding a new field to the formalism, a Stückelberg scalar χ , that cancels the non-invariance. This will be discussed in the next section. One can also take the point of view as in [30] that we should just accept the fact that $\bar{\Gamma}_{\mu\nu}^\rho$ is not N invariant as a mere fact and organize couplings to these geometries and fields living on it in such a way that the action is N invariant. This is certainly a viable point of view and agrees with our approach in all these cases where the dependence on χ can be removed from the theory by field redefinition or simply because it drops out when one tries to make its appearance explicit.

If one includes χ there is the benefit that one can also deal with theories that do not have a local U(1) symmetry (because there is an explicit dependence on χ so that the U(1) invariance disappears in the Stückelberg coupling between m_μ and χ). This is what allows us to use fixed TNC background geometries for both Lifshitz field theories (explicit dependence on χ) as well as Schrödinger field theories (no dependence on χ) as discussed in [23, 24]. The χ field also allows us, as we will see in section 10, to construct two types of HL actions: those that have a local U(1) symmetry without any dependence on χ and those that have no local U(1) because m_μ always appears as $M_\mu = m_\mu - \partial_\mu \chi$.

¹⁰This statement can be made more precise in the following way. The Hořava-Lifshitz actions of section 9 such as (9.18) take exactly the same form when written in terms of $\bar{\Gamma}_{\mu\nu}^\rho$ as when expressed in terms of $\hat{\Gamma}_{\mu\nu}^\rho$. To show this one needs to use the fact that in section 9 it is assumed that τ_μ is hypersurface orthogonal which is something that we do not yet impose at this stage. This is because the difference between covariant derivatives using either one or the other connection involves terms proportional to τ_μ and since the scalars in the action are formed by using inverse spatial metrics $h^{\mu\nu}$ those terms drop out. The same comments apply when using the general α of (5.1), i.e. there is no dependence on α .

From now on we will work with (5.6) and simply denote it by $\Gamma_{\mu\nu}^\rho$ unless specifically stated otherwise. With this realization of $\Gamma_{\mu\nu}^\rho$ the other connections Ω_μ^{ab} and Ω_μ^a are fixed by the vielbein postulates. For an invariant such as \hat{v}^μ the covariant derivatives ∇_μ and \mathcal{D}_μ are the same so we can write

$$\nabla_\mu \hat{v}^\nu = \mathcal{D}_\mu \hat{v}^\nu = -e_a^\nu D_\mu m^a, \tag{5.8}$$

where we used (2.19) and (2.20) and where $D_\mu m^a$ is given by

$$D_\mu m^a = \partial_\mu m^a - \Omega_\mu^a{}_b m^b - \Omega_\mu^a. \tag{5.9}$$

In this section we focussed on making the affine connection G invariant (J invariance is automatic). It so far is not N invariant. This will be fixed in the next section. We could have made the connection N but not G invariant by taking $K_{\mu\nu}$ as in (5.5) and $L_{\sigma\mu\nu} = 0$. However in this case we are not achieving anything as the connection without $K_{\mu\nu}$ is also N invariant and so imposing N invariance does not constrain $\Gamma_{\mu\nu}^\rho$. Furthermore since in the transformation of m_μ the G boost parameter λ^a appears without a derivative, whereas the N transformation parameter σ appears with a derivative, it is more natural to use m_μ to make various tensors G invariant.

Using the invariants τ_μ , $h^{\mu\nu}$, \hat{v}^μ , $\hat{h}_{\mu\nu}$ we can build a non-degenerate symmetric rank 2 tensor with Lorentzian signature $g_{\mu\nu}$ that in the case of a relativistic theory we would refer to as a Lorentzian metric. The metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are given by

$$g_{\mu\nu} = -\tau_\mu \tau_\nu + \hat{h}_{\mu\nu}, \tag{5.10}$$

$$g^{\mu\nu} = -\hat{v}^\mu \hat{v}^\nu + h^{\mu\nu}, \tag{5.11}$$

for which we have

$$g_{\mu\nu} \hat{v}^\mu = \tau_\nu, \tag{5.12}$$

$$g_{\mu\nu} e_a^\mu = \hat{e}_{\nu a}. \tag{5.13}$$

However the natural Galilean metric structures are τ_μ and $h^{\mu\nu}$. For example, as we will see in section 9, $g_{\mu\nu}$ does not transform homogeneously under local scale transformations and so it is not on the same footing as the Riemannian metric in GR.

6 Torsion and the Stückelberg scalar

In the case of gauging the Poincaré algebra (appendix A) the torsion is the part of $\Gamma_{\mu\nu}^\rho$ that is not fixed by the vielbein postulates. In the case of the Bargmann algebra we see on the other hand that it is the torsion that is fixed, namely it is given by the antisymmetric part of (5.1), which reads

$$2\hat{\Gamma}_{[\mu\nu]}^\rho = -\hat{v}^\rho (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu). \tag{6.1}$$

It follows that the curvature (2.28) obeys

$$R_{\mu\nu}{}^a(P) = m^a (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu), \tag{6.2}$$

while $R_{\mu\nu}(H) = \partial_\mu\tau_\nu - \partial_\nu\tau_\mu$ is left arbitrary. Using that $R_{\mu\nu}{}^a(P)$ transforms as

$$\bar{\delta}R_{\mu\nu}{}^a(P) = \mathcal{L}_\xi R_{\mu\nu}{}^a(P) + \lambda^a R_{\mu\nu}(H) + \lambda^a{}_b R_{\mu\nu}{}^b(P), \quad (6.3)$$

we see that the right hand side of (6.2) transforms in exactly the same way as the left hand side (ignoring the central extension N). The right hand side of (6.2) can be matched to transform correctly under the N transformation by adding the Stückelberg scalar χ , i.e. by replacing m^a by $M^a = e^{\mu a}(m_\mu - \partial_\mu\chi)$. This explains why in the presence of torsion, i.e. when $\partial_\mu\tau_\nu - \partial_\nu\tau_\mu \neq 0$, we need the scalar χ . In section 10 we will see that there is a similar field in HL gravity whose couplings are precisely obtained by replacing everywhere m_μ by $M_\mu = m_\mu - \partial_\mu\chi$. From a purely geometrical point of view χ is needed whenever we have torsion, i.e. when the right hand side of (6.2) is nonzero to ensure correct transformations under the N generator.

This does not automatically mean that any field theory coupled to such a background has a nontrivial χ dependence. There are important cases where the χ field can be removed by a field redefinition or it simply drops out of the action once one tries to make its appearance explicit. We refer to [23] for field theory examples of the first possibility of removing χ by field redefinition and to section 10 for a HL action that exhibits the second property, namely that χ drops out.

The χ field also allows us to make the curvature $R_{\mu\nu}(N)$ appearing in (4.3), which so far played no role, visible. This goes via the following commutator

$$[D_\mu, D_\nu]\chi = -2\Gamma_{[\mu\nu]}^\rho D_\rho\chi - R_{\mu\nu}(N), \quad (6.4)$$

where $D_\mu\chi = \partial_\mu\chi - m_\mu$ and where $R_{\mu\nu}(N)$ is given by

$$R_{\mu\nu}(N) = \partial_\mu m_\nu - \partial_\nu m_\mu - 2\Omega_{[\mu}{}^a e_{\nu]a}. \quad (6.5)$$

We note that by covariance $D_\mu D_\nu\chi$ involves the Galilean boost connection $\Omega_\mu{}^a$. Using the general form of $\Gamma_{\mu\nu}^\rho$ given in (3.6) as well as the vielbein postulate (2.16) to express $\Omega_\mu{}^a$ in terms of $\Gamma_{\mu\nu}^\rho$ we obtain

$$R_{\mu\nu}(N) = \partial_\mu m_\nu - \partial_\nu m_\mu - K_{\mu\nu} + v^\sigma L_{\sigma[\mu\nu]}. \quad (6.6)$$

For the choice $\Gamma_{\mu\nu}^\rho = \bar{\Gamma}_{\mu\nu}^\rho$ (5.3), i.e. for $K_{\mu\nu}$ and $L_{\sigma\mu\nu}$ as in (5.5) and (5.5) we find

$$R_{\mu\nu}(N) = v^\sigma m_\sigma (\partial_\mu\tau_\nu - \partial_\nu\tau_\mu). \quad (6.7)$$

This curvature constraint is in agreement with the curvature constraint (6.2) because it obeys the transformation rule for the curvatures under Galilean boosts which according to (2.9) and (2.10) reads $\delta_G R_{\mu\nu}(N) = \lambda^a R_{\mu\nu a}(P)$. Again in order that $R_{\mu\nu}(N)$ remains inert under N transformations in the presence of torsion we need to replace in $\bar{\Gamma}_{\mu\nu}^\rho$ (more precisely in $L_{\sigma\mu\nu}$ as given in (5.5)) m_μ by $M_\mu = m_\mu - \partial_\mu\chi$. The field χ is an essential part of NC geometry with torsion.

The curvature constraints derived here by using the approach of section 2 agree with [22] where the torsionless case was studied. The analysis of sections 2–6 can thus

be viewed as adding torsion to the gauging of the Bargmann algebra (without adding dilatations as in [21]). By employing the relation (5.7) between $\bar{\Gamma}_{\mu\nu}^\rho$ and $\hat{\Gamma}_{\mu\nu}^\rho$ we can find the curvature constraint for $R_{\mu\nu}(N)$ that relates to this choice of affine connection. The curvature constraint (6.2) is the same for all affine connections (5.1).

Following [17, 18] we distinguish three cases for the torsion (6.1):

1. No torsion: $\partial_\mu\tau_\nu - \partial_\nu\tau_\mu = 0$ which is called Newton-Cartan (NC) geometry.
2. Twistless torsion: $\tau_{[\mu}\partial_\nu\tau_{\rho]} = 0$ which means that τ_μ is hypersurface orthogonal and is called twistless torsional Newton-Cartan (TTNC) geometry because it is equivalent to (6.8) which states that the twist tensor is zero.
3. No constraint on τ_μ which is a novel extension of Newton-Cartan (TNC) geometry.

TTNC geometry goes back to [16] but in that work a conformal rescaling was done to go to a frame in which there is no torsion. The benefit of adding torsion to the formalism was first considered in [17, 18] including the case with no constraint on τ_μ .

We will see below that making NC and TTNC geometries dynamical corresponds to projectable and non-projectable HL gravity. In this work we will always assume that we are dealing with TTNC geometry which contains NC geometry as a special case.

For twistless torsional Newton-Cartan (TTNC) geometry we have by definition

$$h^{\mu\rho}h^{\nu\sigma}(\partial_\rho\tau_\sigma - \partial_\sigma\tau_\rho) = 0. \tag{6.8}$$

This implies that the geometry induced on the slices to which τ_μ is hypersurface orthogonal is described by (torsion free) Riemannian geometry.

To make contact with the HL literature concerning non-projectable HL gravity it will prove convenient to define a vector a_μ as follows

$$a_\mu = \mathcal{L}_{\hat{v}}\tau_\mu. \tag{6.9}$$

In section 8 we will exhibit a coordinate parameterization of a_μ (see equations (8.15) and (8.16)) that will appear more familiar in the context of HL gravity, where this becomes the acceleration of the unit vector field orthogonal to equal time slices.

For TTNC we have the following useful identities

$$h^{\mu\rho}h^{\nu\sigma}(\partial_\rho a_\sigma - \partial_\sigma a_\rho) = h^{\mu\rho}h^{\nu\sigma}(\nabla_\rho a_\sigma - \nabla_\sigma a_\rho) = 0, \tag{6.10}$$

$$\partial_\mu\tau_\nu - \partial_\nu\tau_\mu = a_\mu\tau_\nu - a_\nu\tau_\mu. \tag{6.11}$$

The first of these two identities tells us that the twist tensor (the left hand side) vanishes which is why we refer to the geometry as twistless torsional NC geometry. The last identity tells us that a_μ describes the TTNC torsion. We will thus refer to it as the torsion vector.

7 Curvatures

We start by giving some basic properties of the Riemann tensor (2.31) with connection (5.6). Using that

$$\Gamma_{\mu\rho}^\rho = e^{-1}\partial_\mu e, \tag{7.1}$$

where $e = \det(\tau_\mu, e_\mu^a)$, we obtain

$$R_{\mu\nu\rho}{}^\rho = 0. \quad (7.2)$$

Note that because of torsion we have

$$\Gamma_{\rho\mu}^\rho = e^{-1} \partial_\mu e - \hat{v}^\rho (\partial_\rho \tau_\mu - \partial_\mu \tau_\rho). \quad (7.3)$$

From the definition of the Riemann tensor and our choice of connection we can derive the identity

$$\begin{aligned} 3R_{[\mu\nu\sigma]}{}^\rho &= (\nabla_\mu \hat{v}^\rho) (\partial_\nu \tau_\sigma - \partial_\sigma \tau_\nu) + (\nabla_\sigma \hat{v}^\rho) (\partial_\mu \tau_\nu - \partial_\nu \tau_\mu) \\ &\quad + (\nabla_\nu \hat{v}^\rho) (\partial_\sigma \tau_\mu - \partial_\mu \tau_\sigma). \end{aligned} \quad (7.4)$$

The trace of this equation gives us the antisymmetric part of the Ricci tensor $R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho$.

The covariant derivative of \hat{v}^μ is essentially the extrinsic curvature. Using the connection (5.6) we find the identity

$$\nabla_\mu \hat{v}^\rho = -e_a^\rho D_\mu m^a = -h^{\rho\sigma} K_{\mu\sigma}, \quad (7.5)$$

where the extrinsic curvature is defined as

$$K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_{\hat{v}} \hat{h}_{\mu\nu}. \quad (7.6)$$

For TTNC geometries the antisymmetric part of the Ricci tensor is given by

$$2R_{\rho[\mu\nu]}{}^\rho = (\nabla_\rho \hat{v}^\rho) (a_\mu \tau_\nu - a_\nu \tau_\mu) + \hat{v}^\rho (\tau_\nu \nabla_\mu a_\rho - \tau_\mu \nabla_\nu a_\rho), \quad (7.7)$$

using (6.11) and (7.4). We can also derive a TTNC Bianchi identity that reads

$$3\nabla_{[\lambda} R_{\mu\nu]\sigma}{}^\kappa = 2\Gamma_{[\mu\nu]}^\rho R_{\lambda\rho\sigma}{}^\kappa + 2\Gamma_{[\lambda\mu]}^\rho R_{\nu\rho\sigma}{}^\kappa + 2\Gamma_{[\nu\lambda]}^\rho R_{\mu\rho\sigma}{}^\kappa. \quad (7.8)$$

Contracting λ and κ and the remaining indices with $\hat{v}^\mu h^{\nu\sigma}$ leads to the identity

$$\begin{aligned} 0 &= e^{-1} \partial_\mu (e \hat{v}^\nu h^{\mu\sigma} R_{\nu\kappa\sigma}{}^\kappa) - \frac{1}{2} e^{-1} \partial_\mu (e \hat{v}^\mu h^{\nu\sigma} R_{\nu\kappa\sigma}{}^\kappa) + h^{\mu\rho} h^{\nu\sigma} K_{\rho\sigma} R_{\mu\kappa\nu}{}^\kappa \\ &\quad - \frac{1}{2} h^{\mu\nu} K_{\mu\nu} h^{\rho\sigma} R_{\rho\kappa\sigma}{}^\kappa, \end{aligned} \quad (7.9)$$

where we used (7.3) and (7.5). Since we will mostly work in 2+1 dimensions we focus on what happens in that case. Using (2.32) we find

$$e^{-1} \partial_\mu (e \hat{v}^\nu h^{\mu\sigma} R_{\nu\kappa\sigma}{}^\kappa) + \frac{1}{2} e^{-1} \partial_\mu (e \hat{v}^\mu \mathcal{R}) = 0, \quad (7.10)$$

where we used that in 2 spatial dimensions

$$R_{abcd}(J) = \frac{1}{2} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) \mathcal{R}. \quad (7.11)$$

8 Coordinate (ADM) parametrizations

Even though we treat the NC fields τ_μ and $\hat{h}_{\mu\nu}$ as independent we can parametrize them in such a way that $g_{\mu\nu}$ in (5.10) is written in an ADM decomposition. Writing

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt) (dx^j + N^j dt), \quad (8.1)$$

leads to

$$\hat{h}_{tt} = \gamma_{ij} N^i N^j + \tau_t^2 - N^2, \quad (8.2)$$

$$\hat{h}_{ti} = \gamma_{ij} N^j + \tau_i \tau_t, \quad (8.3)$$

$$\hat{h}_{ij} = \gamma_{ij} + \tau_i \tau_j. \quad (8.4)$$

For the inverse metric (5.11) the ADM decomposition reads

$$g^{tt} = -N^{-2}, \quad (8.5)$$

$$g^{ti} = N^i N^{-2}, \quad (8.6)$$

$$g^{ij} = \gamma^{ij} - N^i N^j N^{-2}. \quad (8.7)$$

From this we conclude that

$$h^{tt} = -N^{-2} + \hat{v}^t \hat{v}^t, \quad (8.8)$$

$$h^{ti} = N^i N^{-2} + \hat{v}^t \hat{v}^i, \quad (8.9)$$

$$h^{ij} = \gamma^{ij} - N^i N^j N^{-2} + \hat{v}^i \hat{v}^j. \quad (8.10)$$

The choice (6.8) implies that τ_μ is hypersurface orthogonal, i.e.

$$\tau_\mu = \psi \partial_\mu \tau. \quad (8.11)$$

If we fix our choice of coordinates such that $\tau = t$ we obtain

$$\tau_i = 0. \quad (8.12)$$

Using that $\tau_\mu h^{\mu\nu} = 0$ and (8.12) we obtain $h^{tt} = h^{ti} = 0$ as well as $\hat{h}_{ti} = \gamma_{ij} N^j$ and $\hat{h}_{ij} = \gamma_{ij}$. Further using that $h^{\mu\rho} \hat{h}_{\nu\rho} = \delta_\nu^\mu + \hat{v}^\mu \tau_\nu$ we find $h^{ij} = \gamma^{ij}$. This in turn tells us that $\hat{v}^i = N^i N^{-1}$, so that $h^{tt} = h^{ti} = 0$ leads to $\hat{v}^t = -N^{-1}$. Since $\hat{v}^\mu \tau_\mu = -1$ we also obtain $\tau_t = \psi = N$ so that $\hat{h}_{tt} = \gamma_{ij} N^i N^j$. Since $h^{tt} = h^{ti} = 0$ we also have $\hat{v}^t = v^t = -N^{-1}$ which in turn tells us that $h_{ti} = h_{tt} = 0$, so that we find

$$m_i = -\gamma_{ij} \frac{N^j}{N}. \quad (8.13)$$

Furthermore we have $h_{ij} = \gamma_{ij}$ and $v^i = 0$. For the time component of m_μ we obtain

$$m_t = -\frac{1}{2N} \gamma_{ij} N^i N^j + N \tilde{\Phi}, \quad (8.14)$$

where we used (4.11) or alternatively (4.9) and (4.6). In general $\tau_t = N = N(t, x)$ so that we are dealing with non-projectable HL gravity. Projectable HL gravity corresponds to $N = N(t)$ which is precisely what we get when we impose $\partial_\mu \tau_\nu - \partial_\nu \tau_\mu = 0$.

In these coordinates the torsion vector (6.9) reduces to

$$a_t = N^i a_i, \tag{8.15}$$

$$a_i = N^{-1} \partial_i N, \tag{8.16}$$

which contains no time derivatives. The determinant e in this parametrization is given by $N\sqrt{\gamma}$ where γ is the determinant of γ_{ij} so that using (7.3) we find $\Gamma_{\rho i}^\rho = \partial_i \log \sqrt{\gamma}$ making an object such as $\nabla_\mu (h^{\mu\nu} X_\nu)$ a γ -covariant spatial divergence.

The number of components in $g_{\mu\nu}$ in $d + 1$ space-time dimensions is $(d + 1)(d + 2)/2$ whereas the total number of components in τ_μ and $\hat{h}_{\mu\nu}$ is $(d + 1)(d + 2)/2 + d + 1 - 1$ where the extra $d + 1$ originate from τ_μ and the -1 comes from the fact that $\hat{h}_{\mu\nu} = \delta_{ab} \hat{e}_\mu^a \hat{e}_\nu^b$ so that it has zero determinant. If we furthermore use the fact that τ_μ is hypersurface orthogonal, i.e. $\tau_\mu = \psi \partial_\mu \tau$, we can remove another $d - 1$ components ending up with $(d + 1)(d + 2)/2 + 1$ which is one component more than we have in $g_{\mu\nu}$. If we next restrict to coordinate systems for which $\tau = t$ we obtain the same number of components in the ADM decomposition as we have for our TTNC geometry without $\tilde{\Phi}$. Later we will see what the scalars $\tilde{\Phi}$ and the Stückelberg scalar χ (mentioned below (4.7)) correspond to in the context of HL gravity.

This counting exercise also shows that in general for arbitrary τ_μ TNC gravity is much more general than HL gravity. We leave the study of this more general case for future research. Here we restrict to a hypersurface orthogonal τ_μ .

We thus see that the field τ_μ describes many properties that we are familiar with from the HL literature. For example the TTNC form of τ_μ in (8.11) agrees with the Khronon field of [32]. More precisely the Khronon field φ of [32] corresponds to what we call τ and what is called u_μ in [32] corresponds to what we call τ_μ . Further the torsion field a_i that we defined via (6.9) and that has the parametrization (8.16) agrees with the same field appearing in [32] where it is referred to as the acceleration vector. We will now show that the generic action describing dynamical TTNC geometries agrees on the nose with the most general HL actions appearing in the literature.

9 Hořava-Lifshitz actions

We will consider the dynamics of geometries described by τ_μ , e_μ^a and m^a (in the next section we will add $\tilde{\Phi}$ and χ) by ensuring manifest G and J invariance and by constructing in a systematic manner (essentially a derivative expansion) an action for these fields. Since we demand manifest G and J invariance the generic theory will be described by the independent fields τ_μ and $\hat{h}_{\mu\nu}$ and derivatives thereof.

For simplicity we will work with twistless torsion and in 2 spatial dimensions with $1 < z \leq 2$. It is straightforward to consider higher dimensions. We will do this in section 12 where we treat the conformal case. A convenient way to organize the terms in the action is according to their dilatation weight. The dilatation weights of the invariants are given

G, J invariant	τ_μ	$\hat{h}_{\mu\nu}$	\hat{v}^μ	$h^{\mu\nu}$	e	$\tilde{\Phi}$	χ
dilatation weight	$-z$	-2	z	2	$-(z+2)$	$2(z-1)$	$z-2$

Table 1. Dilatation weights of the G, J invariants.

in table 1 where e is the determinant of the matrix (τ_μ, e_μ^a) . The assignment of these dilatation weights to the TNC fields is consistent with the fact that adding dilatations to the Bargmann algebra leads to the Schrödinger algebra for general z [19, 21]. These assignments agree with [2]. If we choose the foliation as in the previous section with $\tau_i = 0$ and assign the length dimensions z and 1 to the coordinates t and x^i , respectively, we obtain that $[\tau_t] = [N] = L^0$, $[N^i] = L^{1-z}$ and $\gamma_{ij} = L^0$. Note that in table 1 we do not assign any dilatation weights to the coordinates. In the last two columns we have added the scalars $\tilde{\Phi}$ and χ that will not be used in this section but that will appear in the following sections. Even though the fields transform in representations of the Schrödinger algebra this does not mean that this is a local symmetry of the action. This case will be studied in section 12 leading to conformal HL actions.

There are three ways of building derivative terms, namely by i). employing the torsion tensor (6.1), ii). taking covariant derivatives of τ_μ and $\hat{h}_{\mu\nu}$ as well as covariant derivatives of the torsion tensor and iii). by building scalars out of the G, J (and later N) invariants and the curvature tensor $R_{\mu\nu\sigma}{}^\rho$. Option one amounts to using the combination $\partial_\mu\tau_\nu - \partial_\nu\tau_\mu$ which because of our choice (6.8) means that the only relevant component is the one obtained by contracting $\partial_\mu\tau_\nu - \partial_\nu\tau_\mu$ with \hat{v}^μ which equals the Lie derivative of τ_ν along \hat{v}^μ . In other words we can employ the vector a_μ defined in (6.9). Option two reduces to just the covariant derivative of $\hat{h}_{\mu\nu}$ and a_μ because of what was just said about the torsion tensor and the fact that $\nabla_\mu\tau_\nu = 0$. If we contract $\nabla_\rho\hat{h}_{\mu\nu}$ with $h^{\lambda\mu}h^{\kappa\nu}$ we obtain zero because of the fact that $\nabla_\rho h^{\lambda\kappa} = 0$. This means that the only relevant part of $\nabla_\rho\hat{h}_{\mu\nu}$ is obtained by contracting it with one \hat{v}^μ (two would give zero). Since we have $\hat{v}^\mu\nabla_\rho\hat{h}_{\mu\nu} = -\hat{h}_{\mu\nu}\nabla_\rho\hat{v}^\mu$ we can reduce option 2 to taking covariant derivatives of \hat{v}^μ and $h^{\mu\nu}a_\nu$ (note that $\hat{v}^\mu a_\mu = 0$). Because of the identity (7.5) or what is the same

$$\hat{h}_{\nu\rho}\nabla_\mu\hat{v}^\rho = -K_{\mu\nu}, \tag{9.1}$$

the extrinsic curvature can be viewed as the covariant derivative of \hat{v}^μ . Options 1 and 2 thus amount to taking the vectors $h^{\mu\nu}a_\nu$ and \hat{v}^μ as well as products thereof and to form scalar invariants by acting on these tensors with covariant derivatives and/or (products of) a_μ . We will now first classify these terms before discussing option 3.

We will classify all terms that are at most second order in time derivatives and that have no dilatation weights higher than $z+2$ (which is the negative of the dilatation weight of e). In other words we only consider relevant and marginal couplings. The only terms containing time derivatives are extrinsic curvature terms which as we observed are covariant derivatives of \hat{v}^μ . In the previous section we observed that a_μ does not contain any time derivatives, see equations (8.15) and (8.16). We start by writing down all products of \hat{v}^μ and $h^{\mu\nu}a_\nu$ that have dilatation weight at most $z+2$, taking into consideration that we

restrict our attention to the range $1 < z \leq 2$. The possibilities are

$$\begin{array}{ll}
 \hat{v}^\mu & z \\
 h^{\mu\nu} a_\nu & 2 \\
 \hat{v}^\mu \hat{v}^\nu & 2z \\
 \hat{v}^\mu h^{\nu\rho} a_\rho & z + 2 \\
 h^{\mu\rho} a_\rho h^{\nu\sigma} a_\sigma & 4,
 \end{array} \tag{9.2}$$

where the dilatation weights are indicated in the second column. Terms with weight 4 are only relevant for the case $z = 2$. We now hit these terms with ∇_μ and a_μ in all possible ways to form scalars. This does not change the dilatation weights because both ∇_μ and a_μ have weight zero. Keeping in mind that $\hat{v}^\mu a_\mu = 0$ the first two terms in (9.2) give rise to the following scalars

$$\begin{array}{ll}
 \nabla_\mu \hat{v}^\mu & z \\
 \nabla_\mu (h^{\mu\nu} a_\nu) & 2 \\
 h^{\mu\nu} a_\mu a_\nu & 2.
 \end{array} \tag{9.3}$$

Using (7.3) we have the identity

$$\nabla_\mu X^\mu = e^{-1} \partial_\mu (e X^\mu) - a_\mu X^\mu. \tag{9.4}$$

It follows that the first term in (9.3) is a total derivative and the second equals minus the third up to a total derivative. Nevertheless these quantities will be useful as they can be multiplied with a Ricci-type curvature scalar as we will see later. We now focus on the last three terms in (9.2). There are two free indices so we can contract them with $a_\mu a_\nu$, $a_\mu \nabla_\nu$ and $\nabla_\mu \nabla_\nu$. Using two a_μ 's only leads to one possibility which is

$$(h^{\mu\nu} a_\mu a_\nu)^2 \quad 4. \tag{9.5}$$

Contracting the term $\hat{v}^\mu \hat{v}^\nu$ with $a_\mu \nabla_\nu$ gives always zero because we have $a_\mu \hat{v}^\mu \nabla_\nu \hat{v}^\nu = 0$ and $a_\mu (\nabla_\nu \hat{v}^\mu) \hat{v}^\nu = 0$ where the last identity follows from (7.5). Doing the same with the term $\hat{v}^\mu h^{\nu\rho} a_\rho$ in the list (9.2) we obtain the following three allowed scalars

$$\begin{array}{ll}
 h^{\nu\rho} a_\nu a_\rho \nabla_\mu \hat{v}^\mu & z + 2 \\
 h^{\nu\rho} a_\rho a_\mu \nabla_\nu \hat{v}^\mu & z + 2 \\
 a_\nu \hat{v}^\mu \nabla_\mu (h^{\nu\rho} a_\rho) & z + 2.
 \end{array} \tag{9.6}$$

However, because of the identity

$$a_\nu \hat{v}^\mu \nabla_\mu (h^{\nu\rho} a_\rho) = \frac{1}{2} \hat{v}^\mu \nabla_\mu (h^{\nu\rho} a_\nu a_\rho) = -\frac{1}{2} h^{\nu\rho} a_\nu a_\rho \nabla_\mu \hat{v}^\mu + \text{tot.der.}, \tag{9.7}$$

the last of these three terms brings nothing new. Finally the last term in the list (9.2) when contracted with one a_μ and one ∇_ν provides two more scalars, namely

$$\begin{array}{ll}
 h^{\mu\rho} a_\mu a_\rho \nabla_\nu (h^{\nu\sigma} a_\sigma) & 4 \\
 h^{\mu\rho} a_\rho a_\nu \nabla_\mu (h^{\nu\sigma} a_\sigma) & 4.
 \end{array} \tag{9.8}$$

The second term however brings nothing new because of the identity

$$h^{\mu\rho}a_\rho a_\nu \nabla_\mu (h^{\nu\sigma}a_\sigma) = -\frac{1}{2}(h^{\mu\nu}a_\mu a_\nu)^2 - \frac{1}{2}h^{\mu\rho}a_\mu a_\rho \nabla_\nu (h^{\nu\sigma}a_\sigma) + \text{tot.der.} \quad (9.9)$$

Finally we can contract the last three terms in (9.2) with two ∇_μ 's leading to the following set of scalars

$$\begin{aligned} \nabla_\mu \hat{v}^\mu \nabla_\nu \hat{v}^\nu & 2z \\ \nabla_\nu \hat{v}^\mu \nabla_\mu \hat{v}^\nu & 2z \\ \nabla_\mu \hat{v}^\mu \nabla_\nu (h^{\nu\rho}a_\rho) & z+2 \\ \nabla_\nu \hat{v}^\mu \nabla_\mu (h^{\nu\rho}a_\rho) & z+2 \\ \nabla_\mu (h^{\mu\rho}a_\rho) \nabla_\nu (h^{\nu\sigma}a_\sigma) & 4 \\ \nabla_\nu (h^{\mu\rho}a_\rho) \nabla_\mu (h^{\nu\sigma}a_\sigma) & 4. \end{aligned} \quad (9.10)$$

There is one other set of scalar terms containing two covariant derivatives that follow by acting with $a_\mu \square$ where $\square = h^{\rho\sigma} \nabla_\rho \nabla_\sigma$, which is a dimension 2 operator, on the first two terms appearing in the list (9.2). This leads to

$$\begin{aligned} a_\mu \square \hat{v}^\mu & 2+z \\ a_\mu \square (h^{\mu\nu}a_\nu) & 4. \end{aligned} \quad (9.11)$$

Both of these however give nothing new as can be shown by partial integration and upon using the TTNC identity (6.10).

We are left with the possibility to add scalar curvature terms. To this end we first introduce a Ricci-type scalar curvature \mathcal{R} defined as

$$\mathcal{R} = -h^{\mu\nu} R_{\mu\rho\nu}{}^\rho, \quad (9.12)$$

which has dilatation weight 2. Using the scalars (9.3) we can thus build the following list of scalar terms

$$\begin{aligned} \mathcal{R} & 2 \\ \mathcal{R} \nabla_\mu \hat{v}^\mu & z+2 \\ \mathcal{R}^2 & 4 \\ \mathcal{R} \nabla_\mu (h^{\mu\nu}a_\nu) & 4 \\ \mathcal{R} h^{\mu\nu} a_\mu a_\nu & 4. \end{aligned} \quad (9.13)$$

The last term in (9.13) makes it possible to remove $\nabla_\mu (h^{\mu\rho}a_\rho) \nabla_\nu (h^{\nu\sigma}a_\sigma)$ from the list (9.10). This is due to the identity

$$\begin{aligned} \nabla_\mu (h^{\mu\rho}a_\rho) \nabla_\nu (h^{\nu\sigma}a_\sigma) &= \nabla_\nu (h^{\mu\rho}a_\rho) \nabla_\mu (h^{\nu\sigma}a_\sigma) - \frac{1}{2}(h^{\mu\nu}a_\mu a_\nu)^2 \\ &\quad - \frac{3}{2}h^{\mu\rho}a_\mu a_\rho \nabla_\nu (h^{\nu\sigma}a_\sigma) - \frac{1}{2}\mathcal{R}h^{\rho\sigma}a_\rho a_\sigma + \text{tot.der.}, \end{aligned} \quad (9.14)$$

where we used (2.30), (2.32), (7.3), (7.11) and partial integrations.

In $d = 2$ spatial dimensions there are no other curvature invariants other than \mathcal{R} . The reason is that all curvature invariants built out of the tensor $R_{\mu\nu\sigma}{}^\rho$ only involve the spatial Riemann tensor $R_{\mu\nu}{}^{ab}(J)$. The tensor $R_{abcd} = e_a^\mu e_b^\nu R_{\mu\nu cd}(J)$ has the same symmetry properties as the Riemann tensor of a d -dimensional Riemannian geometry. Hence since

here $d = 2$ the only component is the Ricci scalar \mathcal{R} . Any other term involving the curvature tensor contracted with \hat{v}^μ or $h^{\mu\nu}a_\nu$ can be written as a combination of terms we already classified using (2.30) and other identities.

We thus conclude that for $d = 2$ and $1 < z \leq 2$ the scalar terms that can appear in the action are

$$\begin{aligned}
 & h^{\mu\nu}a_\mu a_\nu & 2 \\
 & \mathcal{R} & 2 \\
 & \nabla_\mu \hat{v}^\mu \nabla_\nu \hat{v}^\nu & 2z \\
 & \nabla_\nu \hat{v}^\mu \nabla_\mu \hat{v}^\nu & 2z \\
 & h^{\nu\rho}a_\nu a_\rho \nabla_\mu \hat{v}^\mu & z + 2 \\
 & h^{\nu\rho}a_\rho a_\mu \nabla_\nu \hat{v}^\mu & z + 2 \\
 & \nabla_\mu \hat{v}^\mu \nabla_\nu (h^{\nu\rho}a_\rho) & z + 2 \\
 & \nabla_\nu \hat{v}^\mu \nabla_\mu (h^{\nu\rho}a_\rho) & z + 2 \\
 & \mathcal{R} \nabla_\mu \hat{v}^\mu & z + 2 \\
 & (h^{\mu\nu}a_\mu a_\nu)^2 & 4 \\
 & h^{\mu\rho}a_\mu a_\rho \nabla_\nu (h^{\nu\sigma}a_\sigma) & 4 \\
 & \nabla_\nu (h^{\mu\rho}a_\rho) \nabla_\mu (h^{\nu\sigma}a_\sigma) & 4 \\
 & \mathcal{R}^2 & 4 \\
 & \mathcal{R} \nabla_\mu (h^{\mu\nu}a_\nu) & 4 \\
 & \mathcal{R} h^{\mu\nu}a_\mu a_\nu & 4.
 \end{aligned} \tag{9.15}$$

Consequently, we arrive at the action

$$\begin{aligned}
 S = \int d^3x e & \left[c_1 h^{\mu\nu}a_\mu a_\nu + c_2 \mathcal{R} + c_3 \nabla_\mu \hat{v}^\mu \nabla_\nu \hat{v}^\nu + c_4 \nabla_\nu \hat{v}^\mu \nabla_\mu \hat{v}^\nu + c_5 h^{\nu\rho}a_\nu a_\rho \nabla_\mu \hat{v}^\mu \right. \\
 & + c_6 h^{\nu\rho}a_\rho a_\mu \nabla_\nu \hat{v}^\mu + c_7 \nabla_\mu \hat{v}^\mu \nabla_\nu (h^{\nu\rho}a_\rho) + c_8 \nabla_\nu \hat{v}^\mu \nabla_\mu (h^{\nu\rho}a_\rho) + c_9 \mathcal{R} \nabla_\mu \hat{v}^\mu \\
 & + \delta_{z,2} \left[c_{10} (h^{\mu\nu}a_\mu a_\nu)^2 + c_{11} h^{\mu\rho}a_\mu a_\rho \nabla_\nu (h^{\nu\sigma}a_\sigma) + c_{12} \nabla_\nu (h^{\mu\rho}a_\rho) \nabla_\mu (h^{\nu\sigma}a_\sigma) \right. \\
 & \left. \left. + c_{13} \mathcal{R}^2 + c_{14} \mathcal{R} \nabla_\mu (h^{\mu\nu}a_\nu) + c_{15} \mathcal{R} h^{\mu\nu}a_\mu a_\nu \right] \right]. \tag{9.16}
 \end{aligned}$$

The coefficients c_1 and c_2 have mass dimension z and the coefficients c_3 and c_4 have mass dimension $2 - z$. All the others are dimensionless. The terms with coefficients c_3 and c_4 are the kinetic terms because

$$c_3 \nabla_\mu \hat{v}^\mu \nabla_\nu \hat{v}^\nu + c_4 \nabla_\nu \hat{v}^\mu \nabla_\mu \hat{v}^\nu = C \left(h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - \lambda (h^{\mu\nu} K_{\mu\nu})^2 \right). \tag{9.17}$$

The terms with coefficients c_1 , c_2 and c_{10} to c_{15} only involve spatial derivatives and belong to the potential term \mathcal{V} . They agree with the potential terms in [32, 36, 37, 39] taking into consideration that we are in 2+1 dimensions. The terms with coefficients c_5 to c_9 involve mixed time and space derivatives and are in particular odd under time reversal. Hence in order to not to break time reversal invariance we will set these coefficients equal to zero. All other terms are time reversal and parity preserving. We thus obtain

$$S = \int d^3x e \left[C \left(h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - \lambda (h^{\mu\nu} K_{\mu\nu})^2 \right) - \mathcal{V} \right], \tag{9.18}$$

where the potential \mathcal{V} is given by

$$\begin{aligned}
 -\mathcal{V} = & 2\Lambda + c_1 h^{\mu\nu} a_\mu a_\nu + c_2 \mathcal{R} + \delta_{z,2} \left[c_{10} (h^{\mu\nu} a_\mu a_\nu)^2 + c_{11} h^{\mu\rho} a_\mu a_\rho \nabla_\nu (h^{\nu\sigma} a_\sigma) \right. \\
 & \left. + c_{12} \nabla_\nu (h^{\mu\rho} a_\rho) \nabla_\mu (h^{\nu\sigma} a_\sigma) + c_{13} \mathcal{R}^2 + c_{14} \mathcal{R} \nabla_\mu (h^{\mu\nu} a_\nu) + c_{15} \mathcal{R} h^{\mu\nu} a_\mu a_\nu \right], \quad (9.19)
 \end{aligned}$$

which also includes a cosmological constant Λ . The kinetic terms in (9.18) display the λ parameter of [1, 2]. The potential is exactly the same as the 3D version of the potential given in [32, 36, 37, 39]. We will not impose that \mathcal{V} obeys the detailed balance condition. In the ADM parametrization of section 8 the extrinsic curvature terms in (9.17) are just

$$\gamma^{ik} \gamma^{jl} K_{ij} K_{kl} - \lambda (\gamma^{ij} K_{ij})^2, \quad (9.20)$$

where K_{ij} is given by

$$K_{ij} = \frac{1}{2N} (\partial_t \gamma_{ij} - \mathcal{L}_N \gamma_{ij}) = \frac{1}{2N} \left(\partial_t \gamma_{ij} - \nabla_i^{(\gamma)} N_j - \nabla_j^{(\gamma)} N_i \right), \quad (9.21)$$

where $N_i = \gamma_{ij} N^j$ and $\nabla_i^{(\gamma)}$ is the covariant derivative that is metric compatible with respect to γ_{ij} .

10 Local Bargmann invariance of the HL action: local U(1) vs. Stückelberg coupling

The action (9.18) is by construction invariant under local Galilean transformations because it depends only on the invariants τ_μ and $\hat{h}_{\mu\nu}$. So far we did not consider the possibility of adding $\tilde{\Phi}$. The action (9.18) is not invariant under the central extension of the Galilean algebra. We will now study what happens when we vary m_μ in (9.18) as $\delta m_\mu = \partial_\mu \sigma$. We have that the connection (5.6) transforms under the central element N of the Bargmann algebra as

$$\begin{aligned}
 \delta_N \Gamma_{\mu\nu}^\rho = & \frac{1}{2} h^{\rho\lambda} [(a_\mu \tau_\nu - a_\nu \tau_\mu) \partial_\lambda \sigma + a_\lambda \tau_\nu \partial_\mu \sigma + a_\lambda \tau_\mu \partial_\nu \sigma] \\
 & + h^{\rho\lambda} \tau_\mu \tau_\nu [\partial_\lambda (\hat{v}^\kappa \partial_\kappa \sigma) + 2a_\lambda \hat{v}^\kappa \partial_\kappa \sigma]. \quad (10.1)
 \end{aligned}$$

Using that $\Omega_\mu{}^a{}_b$ is given via (2.13) and (2.16) by

$$\Omega_\mu{}^a{}_b = e_b^\nu (\partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\rho e_\rho^a), \quad (10.2)$$

we obtain

$$\delta_N \Omega_\mu{}^a{}_b = \frac{1}{2} \tau_\mu e_b^\nu e^{\lambda a} (a_\nu \partial_\lambda \sigma - a_\lambda \partial_\nu \sigma). \quad (10.3)$$

This implies that

$$\delta_N R_{abcd}(J) = 0. \quad (10.4)$$

Further $h^{\mu\nu} a_\nu$ is gauge invariant. Using the above results it can be shown that the whole potential \mathcal{V} in (9.19) is gauge invariant. What is left is to transform the kinetic terms under N . We have

$$\delta_N (\nabla_\nu \hat{v}^\mu \nabla_\mu \hat{v}^\nu - \nabla_\mu \hat{v}^\mu \nabla_\nu \hat{v}^\nu) = -\mathcal{R} \hat{v}^\mu \partial_\mu \sigma + 2 \left(h^{\mu\lambda} a_\mu \nabla_\nu \hat{v}^\nu - h^{\mu\nu} a_\mu \nabla_\nu \hat{v}^\lambda \right) \partial_\lambda \sigma, \quad (10.5)$$

where we used (7.10). The first term can be cancelled by adding $\tilde{\Phi}\mathcal{R}$ to the action using that $\tilde{\Phi}$ transforms as

$$\delta_N \tilde{\Phi} = -\hat{v}^\mu \partial_\mu \sigma = N^{-1} (\partial_t \sigma - N^i \partial_i \sigma), \quad (10.6)$$

where in the second equality we expressed the results in terms of the ADM parameterization of section 8.

In [34] the following U(1) transformation was introduced

$$\delta_\alpha N_i = N \partial_i \alpha. \quad (10.7)$$

Together with two new fields A and ν transforming as

$$\delta_\alpha A = \partial_t \alpha - N^i \partial_i \alpha, \quad (10.8)$$

$$\delta_\alpha \nu = -\alpha, \quad (10.9)$$

with ν called the Newtonian prepotential [34]. We see that the α transformation is none other than the Bargmann extension (the σ transformation here) as follows from the identification of m_i in (8.13). More precisely we have $\alpha = -\sigma$. We thus see that the A and ν fields can be identified with $\tilde{\Phi}$ and χ as follows: $A = -N\tilde{\Phi}$ and $\nu = \chi$. The term $\int d^3x e \mathcal{R} \tilde{\Phi}$ is what in [34] is denoted by $\int d^3x \sqrt{\gamma} A \mathcal{R}$. If we work in the context of projectable HL gravity for which $a_\mu = 0$ the action (9.18) with $\lambda = 1$ can be made U(1) invariant by writing

$$S = \int d^3x e \left[C \left(h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - (h^{\mu\nu} K_{\mu\nu})^2 - \tilde{\Phi} \mathcal{R} \right) - \mathcal{V} \right]. \quad (10.10)$$

However if we work with the non-projectable version or with $\lambda \neq 1$ we still need to add additional terms to make the theory U(1) invariant. To see this we use the Stückelberg scalar χ that we already mentioned under (4.7) (see also [23]). Using the field χ that transforms as $\delta\chi = \sigma$ we can construct the following gauge invariant action (the invariance is up to a total derivative) for $\lambda = 1$

$$S = \int d^3x e \left[C \left(h^{\mu\rho} h^{\nu\sigma} - h^{\mu\nu} h^{\rho\sigma} \right) \left(K_{\mu\nu} K_{\rho\sigma} - 2a_\mu \partial_\nu \chi K_{\rho\sigma} + a_\rho \partial_\sigma \chi \nabla_\mu \partial_\nu \chi + \frac{1}{2} a_\mu a_\rho \partial_\nu \chi \partial_\sigma \chi \right) - C \tilde{\Phi} \mathcal{R} - \mathcal{V} \right]. \quad (10.11)$$

The χ dependent terms agree with the result of [36, 37] (eq. (3.8) of that paper).¹¹ We thus see that when there is torsion $a_\mu \neq 0$ we need to introduce a Stückelberg scalar χ to make the action U(1) invariant. While when there is no torsion we can use (10.10).

¹¹To ease comparison it is useful to note that in the notation of [36, 37] one has the identity

$$\frac{1}{3} \hat{\mathcal{G}}^{ijkl} \left[4 (\nabla_i \nabla_j \varphi) a_{(k} \nabla_{l)} \varphi + 2 (\nabla_{(i} \varphi) a_{j)(k} \nabla_{l)} \varphi + 5 a_{(i} (\nabla_{j)} \varphi) a_{(k} \nabla_{l)} \varphi \right] = \hat{\mathcal{G}}^{ijkl} \left[(\nabla_i \nabla_j \varphi) a_k \nabla_l \varphi + \frac{1}{2} a_j a_k \nabla_i \varphi \nabla_l \varphi \right] + \text{tot.der.},$$

where in the notation of [36, 37] the field φ is what we call χ . We also note that the coefficients of these terms are dimension independent.

This nicely agrees with the comments made below (6.2). In [34] the χ field is denoted by ν . This means that we have the following invariance $\delta_N m_\mu = \partial_\mu \sigma$ and $\delta_N \chi = \sigma$. As a consequence we may simply replace everywhere m_μ by $M_\mu = m_\mu - \partial_\mu \chi$. This is consistent with the observations made in [35] (see in particular eq. (20) of said paper). Essentially adding the χ field to the action means that we have trivialized the U(1) symmetry by Stückelberging it or in other words we have removed the U(1) transformations all together (see the next section).

Let us define $K_{\mu\nu}^\chi$ as (7.6) with m_μ replaced by M_μ . It can be shown that

$$h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu}^\chi = h^{\mu\rho} h^{\nu\sigma} \left(K_{\rho\sigma} - \nabla_\rho \partial_\sigma \chi - \frac{1}{2} a_\rho \partial_\sigma \chi - \frac{1}{2} a_\sigma \partial_\rho \chi \right), \quad (10.12)$$

which is now by construction manifestly U(1) invariant. Similarly we can write a manifestly U(1) invariant $\tilde{\Phi}$ as

$$\tilde{\Phi}_\chi = \tilde{\Phi} + \hat{v}^\mu \partial_\mu \chi + \frac{1}{2} h^{\mu\nu} \partial_\mu \chi \partial_\nu \chi, \quad (10.13)$$

obtained by replacing m_μ by M_μ in $\tilde{\Phi}$. Instead of (10.11) we then write

$$S = \int d^3 x e \left[C \left(h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu}^\chi K_{\rho\sigma}^\chi - (h^{\mu\nu} K_{\mu\nu}^\chi)^2 - \tilde{\Phi}_\chi \mathcal{R} \right) - \mathcal{V} \right]. \quad (10.14)$$

It can be checked that this is up to total derivative terms the same as (10.11).

It is now straightforward to generalize this to arbitrary λ and to add for example the $\Omega \tilde{\Phi}$ coupling (with Ω being the cosmological constant) considered in [34] leading to

$$S = \int d^3 x e \left[C \left(h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu}^\chi K_{\rho\sigma}^\chi - \lambda (h^{\mu\nu} K_{\mu\nu}^\chi)^2 - \tilde{\Phi}_\chi (\mathcal{R} - 2\Omega) \right) - \mathcal{V} \right]. \quad (10.15)$$

If we isolate the part of the action that depends on χ we find precisely the same answer as in eq. (3.12) of [37] specialized to 2+1 dimensions.¹² We note that for uniformity we have chosen the coefficient of the $\tilde{\Phi}_\chi \mathcal{R}$ term in (10.14) and (10.15) such that the action has a U(1) symmetry in the absence of torsion, i.e. when the action is independent of χ . In the presence of χ one can in fact allow for an arbitrary coefficient in front of the $\tilde{\Phi}_\chi \mathcal{R}$ term.

As a final confirmation that TNC geometry is a natural framework for HL gravity we will show in section 12 that the conformal HL gravity theories can be obtained by adding dilatations to the Bargmann algebra, i.e. by considering the Schrödinger algebra.

11 A constraint equation

What we have learned is that unless the χ field drops out of the action, as in (10.10) for the case of projectable HL gravity with $\lambda = 1$, we no longer have a non-trivial local U(1) invariance. This is because we can express everything in terms of M_μ which is inert under the U(1). Essentially the fact that we had to introduce a Stückelberg scalar tells us that the U(1) was not there in the first place.

¹²This simply means that we can take in the notation of [37] $G^{ij} = 0$.

There are several statements in the literature expressing that one can remove a scalar degree of freedom from the theory by employing the U(1) invariance, but since we have just established that unless we are dealing with (10.10) there is no U(1) these statements are not clear to us. What we will show is that there is a different mechanism that essentially accomplishes the same effect, via a constraint equation obtained by varying $\tilde{\Phi}_\chi$ in (10.15), to the claims made in the literature.

Since $\tilde{\Phi}_\chi$ is a field like any other we should, in order to be fully general, allow for arbitrary couplings to $\tilde{\Phi}_\chi$ that do not lead to terms of dimension higher than $z + 2$. Put another way the most general HL action can be obtained by writing down the most general action depending on τ_μ , $\hat{h}_{\mu\nu}$ and $\tilde{\Phi}_\chi$ containing terms up to order (dilatation weight) $z + 2$. The first thing to notice is that we typically cannot write down a kinetic term for $\tilde{\Phi}_\chi$ because the dilatation weight of $(\hat{v}^\mu \partial_\mu \tilde{\Phi}_\chi)^2$ is $6z - 4$ which is larger than $z + 2$ whenever $z > 6/5$. The same is true for $K \hat{v}^\mu \partial_\mu \tilde{\Phi}_\chi$ while a term like $\hat{v}^\mu \partial_\mu \tilde{\Phi}_\chi$ or what is the same upon partial integration $K \tilde{\Phi}_\chi$ breaks time reversal invariance. Let us assume that we have a z value larger than $6/5$ so that we cannot write a kinetic term. This means that $\tilde{\Phi}_\chi$ will appear as a non-propagating scalar field.

Let us enumerate the possible allowed couplings to $\tilde{\Phi}_\chi$. Starting with the kinetic terms we can have schematically $\tilde{\Phi}_\chi^\alpha K^2$ where by K^2 we mean both ways of contracting the product of two extrinsic curvatures. In order for this term to have a dimension less than or equal to $z + 2$ we need that $\alpha \leq \frac{2-z}{2(z-1)}$. It follows that for $z > 4/3$ we need $\alpha < 1$. Consider next a term of the form $\tilde{\Phi}_\chi^\beta X$ where X is any term of dimension 2. The condition that the weight does not exceed $z + 2$ gives us $\beta \leq \frac{z}{2(z-1)}$ which means that if $z > 4/3$ we need $\beta < 2$. Finally we can have terms of the form $\tilde{\Phi}_\chi^\gamma$ where $\gamma \leq \frac{2+z}{2(z-1)}$ so that for $z > 8/5$ we need that $\gamma < 3$. In particular it is allowed for all values of $1 < z \leq 2$ to add a term of the form $\tilde{\Phi}_\chi^2$.

Since for $z > 6/5$ we are not allowed to add a kinetic term for $\tilde{\Phi}_\chi$ we can integrate it out. We demand that the resulting action after integrating out $\tilde{\Phi}_\chi$ is local. This puts constraints on what α , β and γ can be since they influence the solution for $\tilde{\Phi}_\chi$. We assume here that α , β and γ are non-negative integers. We will be interested in values of z close to $z = 2$ so we assume that $z > 8/5$. In that case we have the following allowed non-negative integer values: $\alpha = 0$, $\beta = 0, 1$ and $\gamma = 0, 1, 2$. In other words we can add the following $\tilde{\Phi}_\chi$ dependent terms

$$\tilde{\Phi}_\chi \left(d_1 + d_2 \mathcal{R} + d_3 \nabla_\mu (h^{\mu\nu} a_\nu) + d_4 h^{\mu\nu} a_\mu a_\nu + d_5 \tilde{\Phi}_\chi \right). \tag{11.1}$$

There are now two cases of interest: either $d_5 \neq 0$ or $d_5 = 0$. When $d_5 \neq 0$ we can solve for $\tilde{\Phi}_\chi$ and substitute the result back into the action. The resulting action will be of the same form as (9.18) where all the terms originating from solving for $\tilde{\Phi}_\chi$ and substituting the result back into the action can be absorbed into the potential terms by renaming the coefficients in \mathcal{V} . The other possibility that $d_5 = 0$ leads to a rather different situation. In that case the equation of motion of $\tilde{\Phi}_\chi$ leads to the constraint equation

$$d_1 + d_2 \mathcal{R} + d_3 \nabla_\mu (h^{\mu\nu} a_\nu) + d_4 h^{\mu\nu} a_\mu a_\nu = 0. \tag{11.2}$$

The remaining equations of motion for τ_μ etc. will depend on $\tilde{\Phi}_\chi$ because there is no local symmetry (in particular no U(1)) that allows us to gauge fix this field to zero. Since there is no kinetic term for $\tilde{\Phi}_\chi$, and hence its value will not be determined dynamically, we fix it by adding a coupling to an auxiliary field. Recall that for any value of z in the range $1 < z \leq 2$ it is allowed by the effective action approach to add a term proportional to $\tilde{\Phi}_\chi^2$. Consider now the following action

$$S = \int d^3x e \left[\tilde{\Phi}_\chi \text{ indep. part} + \tilde{\Phi}_\chi (d_1 + d_2 \mathcal{R} + d_3 \nabla_\mu (h^{\mu\nu} a_\nu) + d_4 h^{\mu\nu} a_\mu a_\nu) + \lambda \tilde{\Phi}_\chi^2 \right], \quad (11.3)$$

where crucially now λ is an auxiliary field, which has the property that its equation of motion tells us that $\tilde{\Phi}_\chi = 0$ and further the equation of motion of $\tilde{\Phi}_\chi$ will lead to the constraint equation (11.2), which is a more general version of the constraint equation used in [34] and related works. Since $\tilde{\Phi}_\chi = 0$ the $\tilde{\Phi}_\chi$ dependent terms do not affect the remaining equations of motion. This essentially accomplishes that $\tilde{\Phi}_\chi$ is not present in the theory and that we have the constraint equation (11.2). More generally we should think of $\tilde{\Phi}_\chi$ as a background field whose value can be set to be equal to some fixed function f . This is accomplished by writing instead of (11.3) the following

$$S = \int d^3x e \left[\tilde{\Phi}_\chi \text{ indep. part} + (\tilde{\Phi}_\chi - f) (d_1 + d_2 \mathcal{R} + d_3 \nabla_\mu (h^{\mu\nu} a_\nu) + d_4 h^{\mu\nu} a_\mu a_\nu) + \lambda (\tilde{\Phi}_\chi - f)^2 \right]. \quad (11.4)$$

The λ equation of motion enforces the background value $\tilde{\Phi}_\chi = f$, the equation of motion of $\tilde{\Phi}_\chi$ leads again to (11.2) while the remaining equations of motion involve terms depending on f through the variation of terms linear in f .

12 Conformal HL gravity from the Schrödinger algebra

In this section we will work with an arbitrary number of spatial dimensions d . In order to study conformal HL actions we add dilatations to the Bargmann algebra of section 4 and study the various conformal invariants that one can build. To this end we use the connection \mathcal{A}_μ that takes values in the Schrödinger algebra (where for $z = 2$ we leave out for now the special conformal transformations that will be introduced later)¹³

$$\mathcal{A}_\mu = H\tau_\mu + P_a e_\mu^a + G_a \omega_\mu^a + \frac{1}{2} J_{ab} \omega_\mu^{ab} + N \tilde{m}_\mu + D b_\mu, \quad (12.1)$$

where the new connection b_μ is called the dilatation connection. The reason that we renamed the connections in (12.1) as compared to (4.1) is because the dilatation generator D is not central so that it modifies the transformations under local D transformations as compared to how say Ω_μ^a and ω_μ^{ab} would transform using (2.13), (2.16) and (5.1). The transfor-

¹³Compared to e.g. [19, 21] we have interchanged the field m_μ appearing in front of N in the Bargmann algebra and the field \tilde{m}_μ appearing in front of N in the Schrödinger algebra, see also footnote 9.

mation properties and curvatures of the various fields follow from the Schrödinger algebra:

$$\begin{aligned}
[D, H] &= -zH, & [D, P_a] &= -P_a, \\
[D, G_a] &= (z-1)G_a, & [D, N] &= (z-2)N, \\
[H, G_a] &= P_a, & [P_a, G_b] &= \delta_{ab}N, \\
[J_{ab}, P_c] &= \delta_{ac}P_b - \delta_{bc}P_a, & [J_{ab}, G_c] &= \delta_{ac}G_b - \delta_{bc}G_a, \\
[J_{ab}, J_{cd}] &= \delta_{ac}J_{bd} - \delta_{ad}J_{bc} - \delta_{bc}J_{ad} + \delta_{bd}J_{ac}.
\end{aligned} \tag{12.2}$$

We perform the same steps as before (see (2.3) and onwards), namely we consider the adjoint transformation of \mathcal{A}_μ , i.e.

$$\delta\mathcal{A}_\mu = \partial_\mu\Lambda + [\mathcal{A}_\mu, \Lambda], \tag{12.3}$$

where we write (without loss of generality)

$$\Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma, \tag{12.4}$$

with now

$$\Sigma = G_a \lambda^a + \frac{1}{2} J_{ab} \lambda^{ab} + N\sigma + D\Lambda_D, \tag{12.5}$$

and we define $\bar{\delta}\mathcal{A}_\mu$ as

$$\bar{\delta}\mathcal{A}_\mu = \delta\mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu, \Sigma], \tag{12.6}$$

where $\mathcal{F}_{\mu\nu}$ is the curvature

$$\begin{aligned}
\mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \\
&= H \tilde{R}_{\mu\nu}(H) + P_a \tilde{R}_{\mu\nu}{}^a(P) + G_a \tilde{R}_{\mu\nu}{}^a(G) + \frac{1}{2} J_{ab} \tilde{R}_{\mu\nu}{}^{ab}(J) + N \tilde{R}_{\mu\nu}(N) \\
&\quad + D \tilde{R}_{\mu\nu}(D),
\end{aligned} \tag{12.7}$$

where we put tildes on the curvatures to distinguish them from those given in sections 2 and 4. From this we obtain among others that the dilatation connection b_μ transforms as

$$\bar{\delta}b_\mu = \mathcal{L}_\xi b_\mu + \partial_\mu \Lambda_D. \tag{12.8}$$

The following discussion closely follows section 4 of [21]. We will use this b_μ connection to rewrite the covariant derivatives (2.12) and (2.13) in a manifestly dilatation covariant manner.

As a note on our notation we remark that, now that we have learned that we should work with $M_\mu = m_\mu - \partial_\mu \chi$ we take it for granted that we have replaced everywhere m_μ by M_μ and we from now on suppress χ labels as in (10.12) and (10.13). The Schrödinger algebra for general z tells us that the dilatation weights of the fields are as in table 1 while m_μ and χ (and thus M_μ) have dilatation weight $z-2$. This also agrees with the weights assigned to A and ν in [34].

Coming back to the introduction of b_μ , to make expressions dilatation covariant we take $\bar{\Gamma}_{\mu\nu}^\rho$ of equation (5.3) and replace ordinary derivatives by dilatation covariant ones leading

to a new connection $\tilde{\Gamma}_{\mu\nu}^\rho$ that is invariant under the G_a , J_{ab} , N and D transformations and which is given by [21]

$$\tilde{\Gamma}_{\mu\nu}^\rho = -\hat{v}^\rho (\partial_\mu - z b_\mu) \tau_\nu + \frac{1}{2} h^{\rho\sigma} ((\partial_\mu - 2b_\mu) \bar{h}_{\nu\sigma} + (\partial_\nu - 2b_\nu) \bar{h}_{\mu\sigma} - (\partial_\sigma - 2b_\sigma) \bar{h}_{\mu\nu}). \quad (12.9)$$

For the most part of this section we will work with $\tilde{\Gamma}_{\mu\nu}^\rho$ and its dilatation covariant generalization $\tilde{\Gamma}_{\mu\nu}^\rho$. The final scalars out of which we will build the HL action, i.e. for dynamical TTNC geometries, are such that it does not matter whether we use $\tilde{\Gamma}_{\mu\nu}^\rho$ or $\hat{\Gamma}_{\mu\nu}^\rho$ which are related via (5.7).

With the help of b_μ and $\tilde{\Gamma}_{\mu\nu}^\rho$ we can now rewrite the covariant derivatives (2.12) and (2.13) as follows

$$\mathcal{D}_\mu \tau_\nu = \partial_\mu \tau_\nu - \tilde{\Gamma}_{\mu\nu}^\rho \tau_\rho - z b_\mu \tau_\nu = 0, \quad (12.10)$$

$$\mathcal{D}_\mu e_\nu^a = \partial_\mu e_\nu^a - \tilde{\Gamma}_{\mu\nu}^\rho e_\rho^a - \omega_\mu^a \tau_\nu - \omega_\mu^a e_\nu^b - b_\mu e_\nu^a = 0. \quad (12.11)$$

The ω_μ^a and ω_μ^{ab} connections are such that they can be written in terms of Ω_μ^a and Ω_μ^{ab} together with b_μ dependent terms such that all the b_μ terms drop out on the right hand side of (12.10) and (12.11) when expressing it in terms of the connections $\Gamma_{\mu\nu}^\rho$, Ω_μ^a and Ω_μ^{ab} .

The field $M_\mu = m_\mu - \partial_\mu \chi$ can be expressed in terms of the Schrödinger connection \tilde{m}_μ as follows. According to (12.2) and (12.6) the Schrödinger connection \tilde{m}_μ transforms as

$$\bar{\delta} \tilde{m}_\mu = \mathcal{L}_\xi \tilde{m}_\mu + \partial_\mu \sigma + \lambda^a e_{\mu a} + (z - 2) (\sigma b_\mu - \Lambda_D \tilde{m}_\mu). \quad (12.12)$$

The Stückelberg scalar χ transforms as

$$\bar{\delta} \chi = \mathcal{L}_\xi \chi + \sigma + (2 - z) \Lambda_D \chi. \quad (12.13)$$

A Schrödinger covariant derivative $\mathcal{D}_\mu \chi$ is given by

$$\mathcal{D}_\mu \chi = \partial_\mu \chi - \tilde{m}_\mu - (2 - z) b_\mu \chi. \quad (12.14)$$

Defining $M_\mu = -\mathcal{D}_\mu \chi = m_\mu - \partial_\mu \chi$ we see that M_μ transforms as

$$\bar{\delta} M_\mu = \mathcal{L}_\xi M_\mu + e_\mu^a \lambda_a + (2 - z) \Lambda_D M_\mu, \quad (12.15)$$

and that

$$m_\mu = \tilde{m}_\mu + (2 - z) b_\mu \chi. \quad (12.16)$$

Hence the dilatation covariant derivative of M_μ reads

$$\mathcal{D}_\mu M_\nu = \partial_\mu M_\nu - \tilde{\Gamma}_{\mu\nu}^\rho M_\rho - (2 - z) b_\mu M_\nu - \omega_\mu^a e_{\nu a}. \quad (12.17)$$

The torsion $\tilde{\Gamma}_{[\mu\nu]}^\rho$ has to be a G , J , N and D invariant tensor. With our TTNC field content the only option is to take it zero, i.e. $\tilde{\Gamma}_{\mu\nu}^\rho$ becomes torsionless [21]. This means that different from the relativistic case the b_μ connection is not entirely independent, but instead reads

$$b_\mu = \frac{1}{z} \hat{v}^\rho (\partial_\rho \tau_\mu - \partial_\mu \tau_\rho) - \hat{v}^\rho b_\rho \tau_\mu = \frac{1}{z} a_\mu - \hat{v}^\rho b_\rho \tau_\mu. \quad (12.18)$$

Let X^ρ be a tensor with dilatation weight w , i.e.

$$\delta_D X^\rho = -w \Lambda_D X^\rho. \quad (12.19)$$

A dilatation covariant derivative is given by

$$\tilde{\nabla}_\nu X^\rho + w b_\nu X^\rho, \quad (12.20)$$

where $\tilde{\nabla}_\nu$ is covariant with respect to $\tilde{\Gamma}_{\nu\mu}^\rho$ as given in (12.9). Let us compute the commutator

$$\begin{aligned} & \left(\tilde{\nabla}_\mu + w b_\mu \right) \left(\tilde{\nabla}_\nu + w b_\nu \right) X^\rho - (\mu \leftrightarrow \nu) \\ & = -\tilde{R}_{\mu\nu\lambda}{}^\rho X^\lambda + w (\partial_\mu b_\nu - \partial_\nu b_\mu) X^\rho, \end{aligned} \quad (12.21)$$

where

$$\tilde{R}_{\mu\nu\lambda}{}^\rho = -\partial_\mu \tilde{\Gamma}_{\nu\lambda}^\rho + \partial_\nu \tilde{\Gamma}_{\mu\lambda}^\rho - \tilde{\Gamma}_{\mu\sigma}^\rho \tilde{\Gamma}_{\nu\lambda}^\sigma + \tilde{\Gamma}_{\nu\sigma}^\rho \tilde{\Gamma}_{\mu\lambda}^\sigma. \quad (12.22)$$

The introduction of the b_μ connection has led to a new component $\hat{v}^\mu b_\mu$ as visible in (12.18). We can introduce a special conformal transformation (denoted by K) that allows us to remove this component. Hence we assign a new transformation rule to b_μ namely

$$\delta_K b_\mu = \Lambda_K \tau_\mu. \quad (12.23)$$

Under special conformal transformations we have

$$\delta_K \tilde{\Gamma}_{\mu\nu}^\rho = \Lambda_K \left((z-2) \hat{v}^\rho \tau_\mu \tau_\nu - \delta_\mu^\rho \tau_\nu - \delta_\nu^\rho \tau_\mu \right). \quad (12.24)$$

In order that $\left(\tilde{\nabla}_\mu + w b_\mu \right) \left(\tilde{\nabla}_\nu + w b_\nu \right) X^\rho$ transforms covariantly we define the K -covariant derivative

$$\begin{aligned} \left(\tilde{\mathcal{D}}_\mu + w b_\mu \right) \left(\tilde{\nabla}_\nu + w b_\nu \right) X^\rho &= \left(\tilde{\nabla}_\mu + w b_\mu \right) \left(\tilde{\nabla}_\nu + w b_\nu \right) X^\rho \\ &\quad - w f_\mu \tau_\nu X^\rho - f_\mu \left((z-2) \hat{v}^\rho \tau_\nu \tau_\lambda - \delta_\nu^\rho \tau_\lambda - \delta_\lambda^\rho \tau_\nu \right) X^\lambda, \end{aligned} \quad (12.25)$$

where f_μ is a connection for local K transformations that transforms as [21]

$$\bar{\delta} f_\mu = \mathcal{L}_\xi f_\mu + \partial_\mu \Lambda_K - z \Lambda_D f_\mu + z \Lambda_K b_\mu. \quad (12.26)$$

In order not to introduce yet another independent field f_μ (recall that we are trying to remove $\hat{v}^\mu b_\mu$) we demand that f_μ is a completely dependent connection that transforms as in (12.26). This is in part realized by setting the curvature of the dilatation connection b_μ equal to zero, i.e. by imposing

$$\check{R}_{\mu\nu}(D) = \partial_\mu b_\nu - \partial_\nu b_\mu - f_\mu \tau_\nu + f_\nu \tau_\mu = 0. \quad (12.27)$$

This fixes all but the $\hat{v}^\mu f_\mu$ component of f_μ . This latter component will be fixed later by equation (12.42). The notation is such that a tilde refers to a curvature of the $\bar{\delta}$ transformation (12.6) without the K transformation while a curvature with a check sign refers to

a curvature that is covariant under the $\bar{\delta}$ transformations with the K transformation. We note that while for the Schrödinger algebra, i.e. with the δ transformations (12.3) we can only add special conformal transformations when $z = 2$ while for the (different) group of transformations transforming under $\bar{\delta}$ we can define K transformations for any z [21].

Taking the commutator of (12.25) we find

$$\left(\tilde{\mathcal{D}}_\mu + wb_\mu\right)\left(\tilde{\nabla}_\nu + wb_\nu\right)X^\rho - (\mu \leftrightarrow \nu) = -\check{R}_{\mu\nu\lambda}{}^\rho X^\lambda, \quad (12.28)$$

where $\check{R}_{\mu\nu\lambda}{}^\rho$ is given by

$$\begin{aligned} \check{R}_{\mu\nu\lambda}{}^\rho &= \tilde{R}_{\mu\nu\lambda}{}^\rho + (z-2)\hat{v}^\rho\tau_\lambda(f_\mu\tau_\nu - f_\nu\tau_\mu) - \delta_\nu^\rho\tau_\lambda f_\mu + \delta_\mu^\rho\tau_\lambda f_\nu \\ &\quad - \delta_\lambda^\rho(f_\mu\tau_\nu - f_\nu\tau_\mu). \end{aligned} \quad (12.29)$$

Under K transformations the curvature tensor $\check{R}_{\mu\nu\lambda}{}^\rho$ transforms as

$$\delta_K \check{R}_{\mu\nu\lambda}{}^\rho = \Lambda_K [-(z-2)\tau_\lambda\tau_\nu\mathcal{D}_\mu\hat{v}^\rho + (z-2)\tau_\lambda\tau_\mu\mathcal{D}_\nu\hat{v}^\rho]. \quad (12.30)$$

Besides this property, the tensor $\check{R}_{\mu\nu\lambda}{}^\rho$ is by construction invariant under D , G , N and J transformations.

Using the vielbein postulates (12.10) and (12.11) we can write

$$\tilde{\Gamma}_{\mu\nu}{}^\rho = -v^\rho(\partial_\mu\tau_\nu - zb_\mu\tau_\nu) + e_a^\rho\left(\partial_\mu e_\nu^a - \omega_\mu{}^a\tau_\nu - \omega_\mu{}^a{}_b e_\nu^b - b_\mu e_\nu^a\right). \quad (12.31)$$

With this result we can derive

$$\check{R}_{\mu\nu\sigma}{}^\rho = -e^{\rho d}e_\sigma^c\check{R}_{\mu\nu cd}(J) + e^{\rho c}\tau_\sigma\check{R}_{\mu\nu c}(G), \quad (12.32)$$

where $\check{R}_{\mu\nu cd}(J)$ and $\check{R}_{\mu\nu c}(G)$ are given by

$$\check{R}_{\mu\nu}{}^{ab}(J) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{ca}\omega_{\nu]}{}^b{}_c, \quad (12.33)$$

$$\begin{aligned} \check{R}_{\mu\nu}{}^a(G) &= \tilde{R}_{\mu\nu}{}^a(G) - 2f_{[\mu}(e_{\nu]}{}^a + (z-2)\tau_{\nu]}M^a) \\ &= 2\partial_{[\mu}\omega_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab}\omega_{\nu]}{}_b - 2(1-z)b_{[\mu}\omega_{\nu]}{}^a \\ &\quad - 2f_{[\mu}(e_{\nu]}{}^a + (z-2)\tau_{\nu]}M^a). \end{aligned} \quad (12.34)$$

We next present some basic properties of $\check{R}_{\mu\nu\sigma}{}^\rho$. The first is

$$\check{R}_{\mu\nu\rho}{}^\rho = 0, \quad (12.35)$$

and the second is

$$\check{R}_{[\mu\nu\sigma]}{}^\rho = 0. \quad (12.36)$$

Equations (12.32) and (12.36) together give us the Bianchi identity

$$\check{R}_{[\mu\nu}{}^{ab}(J)e_{\rho]b} + \check{R}_{[\mu\nu}{}^a(G)\tau_{\rho]} = 0. \quad (12.37)$$

By contracting this with $v^\mu e^\nu{}_c e^\rho{}_a$ we find

$$\check{R}_{ca}{}^a(G) + v^\mu\check{R}_{\mu a}{}^a{}_c(J) = 0, \quad (12.38)$$

and by contracting (12.37) with $e^\mu_b e^\nu_a e^\rho_c$ we obtain

$$\tilde{R}_{ba}{}^a{}_c(J) - \tilde{R}_{ca}{}^a{}_b(J) = 0. \quad (12.39)$$

The two identities (12.36) and (12.35) imply that

$$\check{R}_{\rho[\nu\sigma]}{}^\rho = 0. \quad (12.40)$$

We define $\check{R}_{\nu\sigma} = \check{R}_{\nu\rho\sigma}{}^\rho$.

Using the identity (12.38) we can derive

$$\begin{aligned} \hat{v}^\sigma \hat{v}^\nu \check{R}_{\sigma\nu} &= -\hat{v}^\nu (R_{\nu a}{}^a(G) + M^c R_{\nu a}{}^a{}_c) \\ &= -\hat{v}^\mu (R_{\mu a}{}^a(G) + 2M^c R_{\mu a}{}^a{}_c(J)) - M^b (R_{ba}{}^a(G) + M^c R_{ba}{}^a{}_c(J)). \end{aligned} \quad (12.41)$$

We now turn to the question what $\hat{v}^\sigma \hat{v}^\nu \check{R}_{\sigma\nu}$ should be equal to. Following [21] we will take this to be equal to

$$\hat{v}^\sigma \hat{v}^\nu \check{R}_{\sigma\nu} = \frac{1}{2d} (z - 2) (h^{\mu\nu} \mathcal{D}_\mu M_\nu)^2, \quad (12.42)$$

because the right hand side has the exact same transformation properties under all local symmetries as $\hat{v}^\sigma \hat{v}^\nu \check{R}_{\sigma\nu}$. The combination of $\check{R}_{\mu\nu}(D) = 0$ together with (12.42) fixes f_μ entirely in terms of τ_μ , e_μ^a , m_μ and χ in such a way that it transforms as in (12.26).

Using that

$$h^{\mu\rho} h^{\nu\sigma} \mathcal{D}_{(\mu} M_{\nu)} = h^{\mu\rho} h^{\nu\sigma} \left(K_{\mu\nu} + \hat{v}^\lambda b_\lambda \hat{h}_{\mu\nu} \right), \quad (12.43)$$

where $K_{\mu\nu}$ is the extrinsic curvature, we see that

$$h^{\mu\rho} h^{\nu\sigma} \mathcal{D}_{(\mu} M_{\nu)} \mathcal{D}_{(\rho} M_{\sigma)} - \frac{1}{d} (h^{\mu\nu} \mathcal{D}_\mu M_\nu)^2, \quad (12.44)$$

is invariant under the K transformation because the term $\hat{v}^\mu b_\mu$ cancels out from the above difference. Another scalar quantity of interest is

$$h^{\mu\nu} \check{R}_{\mu\nu} = -\check{R}^{ab}{}_{ab}(J), \quad (12.45)$$

which is K invariant and has dilatation weight 2. With these ingredients we can build a $z = d$ conformally invariant Lagrangian

$$\mathcal{L} = e \left[A \left(h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - \frac{1}{d} (h^{\mu\nu} K_{\mu\nu})^2 \right) + B (h^{\mu\nu} \check{R}_{\mu\nu})^d \right]. \quad (12.46)$$

This is an example of a Lagrangian for non-projectable HL gravity that is conformally invariant.

The quantity $h^{\mu\nu} \check{R}_{\mu\nu}$ can be expressed in terms of \mathcal{R} and the torsion vector a_μ defined in sections 6 and 7 as follows. Solving (12.11) for $\omega_\mu{}^{ab}$ and using the relation between $\tilde{\Gamma}_{\mu\nu}^\rho$ and $\bar{\Gamma}_{\mu\nu}^\rho$ given in (5.3) which reads

$$\tilde{\Gamma}_{\mu\nu}^\rho = \bar{\Gamma}_{\mu\nu}^\rho + z \hat{v}^\rho b_\mu \tau_\nu - h^{\rho\sigma} (b_\mu \bar{h}_{\nu\sigma} + b_\nu \bar{h}_{\mu\sigma} - b_\sigma \bar{h}_{\mu\nu}), \quad (12.47)$$

we obtain (12.11), via

$$\omega_\mu{}^{ab} = \hat{\Omega}_\mu{}^{ab} + e^{\nu b} b_\nu \hat{e}_\mu^a - e^{\nu a} b_\nu \hat{e}_\mu^b, \quad (12.48)$$

where we used that ω_μ^{ab} and Ω_μ^{ab} are related, as follows from the vielbein postulates (2.13), (2.16) and where we furthermore used that for TTNC $\bar{\Omega}_\mu^{ab} = \hat{\Omega}_\mu^{ab}$ as follows from (5.7) and the TTNC relation (6.10). In the relation $\bar{\Omega}_\mu^{ab} = \hat{\Omega}_\mu^{ab}$ the connection $\bar{\Omega}_\mu^{ab}$ is found by employing the vielbein postulate expressed in terms of $\bar{\Gamma}_{\mu\nu}^\rho$ and likewise $\hat{\Omega}_\mu^{ab}$ is obtained by using the vielbein postulate written in terms of $\hat{\Gamma}_{\mu\nu}^\rho$. Then using (12.33) and (12.48) we find

$$h^{\mu\nu} \check{R}_{\mu\nu} = -\check{R}^{cd}_{cd}(J) = -\mathcal{R} + 2(d-1)\nabla_\mu (h^{\mu\nu} a_\nu) - (d-1)(d-2)h^{\mu\nu} a_\mu a_\nu, \quad (12.49)$$

where we used (12.18) and $R^{cd}_{cd}(J) = \mathcal{R}$ which is merely a definition of \mathcal{R} .

By fully employing the local Schrödinger algebra we arrive at the conformally invariant $z = d$ action [1, 34]

$$S = \int d^{d+1}x e \left[A \left(h^{\mu\rho} h^{\nu\sigma} K_{\mu\nu} K_{\rho\sigma} - \frac{1}{d} (h^{\mu\nu} K_{\mu\nu})^2 \right) + B (\mathcal{R} - 2(d-1)\nabla_\mu (h^{\mu\nu} a_\nu) + (d-1)(d-2)h^{\mu\nu} a_\mu a_\nu)^d \right]. \quad (12.50)$$

For $z = d$ the dilatation weight of $\tilde{\Phi}$ is given by $2(d-1)$ so that the terms

$$-a\tilde{\Phi} (\mathcal{R} - 2(d-1)\nabla_\mu (h^{\mu\nu} a_\nu) + (d-1)(d-2)h^{\mu\nu} a_\mu a_\nu) + b\tilde{\Phi}^{\frac{d}{d-1}}, \quad (12.51)$$

can be added to the action in a conformally invariant manner. Assuming $b \neq 0$ we can integrate out $\tilde{\Phi}$ which leads to the action (12.50) with a different constant B . The case with $b = 0$ can be viewed as a constrained system as discussed in section 11. The integrand of (12.50) has been obtained in Lifshitz holography and field theory using different techniques and found to describe the Lifshitz scale anomaly [4, 18, 49–51] where A and B play the role of two central charges. In [18] it was shown that for $d = z = 2$ the integrand of (12.50) together with (12.51) for specific values of a and b arises from the (Scherk-Schwarz) null reduction of the AdS₅ conformal anomaly of gravity coupled to an axion.

13 Discussion

We have shown that the dynamics of TTNC geometries, for which there is a hypersurface orthogonal foliation of constant time hypersurfaces, is precisely given by non-projectable Hořava-Lifshitz gravity. The projectable case corresponds to dynamical NC geometries without torsion. One can build a precise dictionary, between properties of TNC and HL gravities, which we give below in table 2.

We conclude with some general comments about interesting future research directions.

TNC geometries have appeared so far as fixed background geometries for non-relativistic field theories and hydrodynamics [23–26, 30, 52–54] as well as in holographic setups based on Lifshitz bulk space-times [17–19, 21, 23]. In all these cases the TNC geometry is treated as non-dynamical. This is a valid perspective provided the backreaction onto the geometry can be considered small, e.g. a small amount of energy or mass density should not lead to pathological behavior of the geometry when allowing it to backreact.

TNC gravity	HL gravity
twistless torsion: $h^{\mu\rho}h^{\nu\sigma}(\partial_\mu\tau_\nu - \partial_\nu\tau_\mu) = 0$	non-projectable
no torsion: $\partial_\mu\tau_\nu - \partial_\nu\tau_\mu = 0$	projectable
$\tau_\mu = \psi\partial_\mu\tau$	scalar khronon φ in u_μ [32]
τ invariant under Galilean tangent space group	foliation breaks local Lorentz invariance
torsion vector a_μ	acceleration a_μ [32]
TNC invariant: $-\tau_\mu\tau_\nu + \hat{h}_{\mu\nu}$	metric with Lorentz signature $g_{\mu\nu}$
$\tau_i = 0$	ADM decomposition
τ_t	lapse N
$m_i = -N^{-1}N_i$	ADM shift vector N_i
\hat{h}_{ij}	metric on constant t slices γ_{ij}
scalar $\tilde{\Phi}$ in $m_t = -\frac{1}{2N}\gamma_{ij}N^iN^j + N\tilde{\Phi}$	$N^{-1}A$ [34]
Stückelberg scalar χ	Newtonian prepotential ν [34]
Bargmann central extension acting on m_μ and χ	local U(1) acting on A , N_i and ν
$\nabla_\mu\hat{v}^\nu$	extrinsic curvature
two scalar invariants $\nabla_\mu\hat{v}^\mu\nabla_\nu\hat{v}^\nu$ and $\nabla_\nu\hat{v}^\mu\nabla_\mu\hat{v}^\nu$ allowed by local Galilean symmetries	the λ parameter in the kinetic term
Effective action organized by Schrödinger representations	Dimensions: $[N] = L^0$, $[\gamma_{ij}] = L^0$, $[N^i] = L^{1-z}$, $[A] = L^{-2(z-1)}$
Local Schrödinger invariance	conformal HL actions (invariant under anisotropic Weyl rescalings)
general torsion: no constraint on τ_μ	vector khronon [3]

Table 2. Dictionary between TNC and HL terminology.

This renders the question of the consistency of HL gravity in the limit of small fluctuations around flat space-time of crucial importance for applications of TNC geometry to the realm of non-relativistic physics.

In this light we wish to point out that (in the absence of a cosmological constant) the ground state is not Minkowski space-time but flat NC space-time which has different symmetries than Minkowski space-time as worked out in detail in [23]. It would be interesting to work out the properties of perturbations of TTNC gravity around flat NC space-time. In particular we have shown that generically there is no local U(1) symmetry in the problem but that rather one can either integrate out $\tilde{\Phi}_\chi$ without modifying the effective action in an essential way or in such a way that it imposes a non-trivial constraint on the spatial

part of the geometry. It would also be interesting to study the theory from a Hamiltonian perspective and derive the first and second class constraints and compare the resulting counting of degrees of freedom with the linearized analysis.

Since it is well understood how to couple matter to TNC geometries the question of how to couple matter to HL gravity can be readily addressed in this framework. For example it would be interesting to find Bianchi identities for the TTNC curvature tensor (as studied in section 7) in such a way that they are compatible with the on-shell diffeomorphism Ward identity for the energy-momentum tensor as defined in [19, 23, 24]. We emphasize once more that matter systems coupled to TNC geometries can have but do not necessarily need to have a particle number symmetry [19, 23]. It would be important to study what the fate of particle number symmetry is once we make the geometry dynamical. In the matter sector particle number symmetry comes about as a gauge transformation acting on M_μ in such a manner that the Stückelberg scalar χ can be removed from the matter action [19, 23] making this formulation consistent with [30]. We have seen in section 10 that generically the χ field cannot be removed from the actions describing the dynamics of the TNC geometry. Hence, it seems that the dynamics of the geometry breaks particle number symmetry except when we use the model (10.10) for projectable HL gravity with $\lambda = 1$ in which case the central extension of the Bargmann algebra is a true local U(1) symmetry and the χ field does not appear in the HL action.

Another interesting extension of this work is to consider the case of unconstrained torsion, i.e. TNC gravity, in which case τ_μ is no longer restricted to be hypersurface orthogonal. In table 2 we refer to this as the vector khronon extension in the last row. The main difference with TTNC geometry is that now the geometry orthogonal to τ_μ is no longer torsion free Riemannian geometry but becomes torsionful. This extra torsion is described by an object which we call the twist tensor (see e.g. [21]) denoted by $T_{\mu\nu}$ and defined as

$$T_{\mu\nu} = \frac{1}{2} (\delta_\mu^\rho + \tau_\mu \hat{v}^\rho) (\delta_\nu^\sigma + \tau_\nu \hat{v}^\sigma) (\partial_\rho \tau_\sigma - \partial_\sigma \tau_\rho). \tag{13.1}$$

Therefore apart from the fact that now the τ_μ appearing in the actions of sections 9–12 is no longer of the form $\psi \partial_\mu \tau$ but completely free, we can also add additional terms containing the twist tensor $T_{\mu\nu}$. Another such tensor is $T_{(a)\mu\nu}$ (see again [21] where it was denoted by $T_{(b)\mu\nu}$) which is defined as

$$T_{(a)\mu\nu} = \frac{1}{2} (\delta_\mu^\rho + \tau_\mu \hat{v}^\rho) (\delta_\nu^\sigma + \tau_\nu \hat{v}^\sigma) (\partial_\rho a_\sigma - \partial_\sigma a_\rho). \tag{13.2}$$

Hence we can add for example a term such as

$$T_{\mu\nu} T_{\rho\sigma} h^{\mu\rho} h^{\nu\sigma}, \tag{13.3}$$

which has weight $4 - 2z$ so that it is relevant for $z > 1$. In fact for $z = 2$ this term has weight zero and so one can add an arbitrary function of the twist tensor squared. In the IR however the two-derivative term dominates.

Another aspect that would be worthwhile examining using our results is whether one could learn more about non-relativistic field theories at finite temperature using holography

for HL gravity [3–5, 49, 55, 56]. Independently of whether HL gravity is UV complete, assuming it makes sense as a classical theory it may be a useful tool to compute properties such as correlation functions of the (non-relativistic) boundary field theory. In particular, this implies that there must exist bulk gravity duals to thermal states of the field theory, i.e. classical solutions of HL gravity that resemble black holes as we know them in general relativity. In light of this it would be interesting to re-examine the status of black hole solutions in HL gravity (see e.g. [57–59]). Moreover, it is expected that in a long-wave length regime some version of the fluid/gravity correspondence should exist, enabling the computation of for example transport coefficients in finite temperature non-relativistic field theories on flat (or more generally curved) NC backgrounds.

TNC geometry also appears in the context of WCFTs [60] as the geometry to which these $SL(2) \times U(1)$ invariant CFTs couple to. This was called warped geometry and corresponds to TNC geometry in $1 + 1$ dimensions with $z = \infty$ (or $z = 0$ if one interchanges the two coordinates). In that case there is no spatial curvature so the entire dynamics is governed by torsion. It would be interesting to write down the map to the formulation in [60] and furthermore explicitly write the HL actions for that case.

It would also be interesting to explore the relation of TNC gravity to Einstein-aether theory. It was shown in [38] that any solution of Einstein-aether theory with hypersurface orthogonal τ_μ is a solution of the IR limit of non-projectable HL gravity. It would thus be natural to expect that any solution of Einstein-aether theory with unconstrained τ_μ is a solution to the IR limit of TNC gravity. In view of the relation [61, 62] between causal dynamical triangulations (CDT) and HL quantum gravity, both involving a global time foliation, there may also be useful applications of TNC geometry in the context of CDT [63]. Finally, since HL gravity is connected to the mathematics of Ricci flow (see e.g. [64]), examining this from the TNC perspective presented in this paper could lead to novel insights.

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A Gauging Poincaré

In this appendix we briefly discuss the gauging of the Poincaré algebra to show the power of the method in a more familiar context. Consider the Poincaré algebra whose generators are P_a and M_{ab} satisfying the commutation relations

$$[M_{ab}, P_c] = \eta_{ac}P_b - \eta_{bc}P_a, \tag{A.1}$$

$$[M_{ab}, M_{cd}] = \eta_{ac}M_{bd} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad} + \eta_{bd}M_{ac}. \tag{A.2}$$

We introduce the Lie algebra valued connection \mathcal{A}_μ given by

$$\mathcal{A}_\mu = P_a e_\mu^a + \frac{1}{2} M_{ab} \omega_\mu^{ab}. \tag{A.3}$$

This connection transforms in the adjoint as

$$\delta \mathcal{A}_\mu = \partial_\mu \Lambda + [\mathcal{A}_\mu, \Lambda], \tag{A.4}$$

where Λ is given by

$$\Lambda = P_a \zeta^a + \frac{1}{2} M_{ab} \sigma^{ab}. \tag{A.5}$$

What we have done so far is to make the Poincaré transformations local. However we would like to connect this to a set of transformations that replace local space-time translations by diffeomorphisms. This can be achieved as follows. We define a new set of local transformations that we denote by $\bar{\delta}$. The main step is to replace the parameters in Λ corresponding to local space-time translations, i.e. ζ^a by a space-time vector ξ^μ via $\zeta^a = \xi^\mu e_\mu^a$. This can be achieved by the following way of writing Λ

$$\Lambda = \xi^\mu \mathcal{A}_\mu + \Sigma, \tag{A.6}$$

where

$$\Sigma = \frac{1}{2} M_{ab} \lambda^{ab}, \tag{A.7}$$

with $\sigma^{ab} = \xi^\mu \omega_\mu^{ab} + \lambda^{ab}$. Next we define $\bar{\delta} \mathcal{A}_\mu$ as

$$\bar{\delta} \mathcal{A}_\mu = \delta \mathcal{A}_\mu - \xi^\nu \mathcal{F}_{\mu\nu} = \mathcal{L}_\xi \mathcal{A}_\mu + \partial_\mu \Sigma + [\mathcal{A}_\mu, \Sigma], \tag{A.8}$$

where the second equality is an identity and where $\mathcal{F}_{\mu\nu}$ is the curvature

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \\ &= P_a R_{\mu\nu}{}^a(P) + \frac{1}{2} M_{ab} R_{\mu\nu}{}^{ab}(M), \end{aligned} \tag{A.9}$$

in which we have

$$R_{\mu\nu}{}^a(P) = 2\partial_{[\mu} e_{\nu]}^a - 2\omega_{[\mu}{}^{ab} e_{\nu]}{}_b, \tag{A.10}$$

$$R_{\mu\nu}{}^{ab}(M) = 2\partial_{[\mu} \omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{ca} \omega_{\nu]}{}^b{}_c. \tag{A.11}$$

Under the $\bar{\delta}$ transformations, the connection e_μ^a associated with the Lorentz momenta P_a , transforms as a vielbein while the connection ω_μ^{ab} associated with the Lorentz boosts M_{ab} become the spin connection coefficients.

In order to define a covariant derivative on the space-time we first introduce a covariant derivative \mathcal{D}_μ via

$$\mathcal{D}_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\rho e_\rho^a - \omega_\mu^a{}_b e_\nu^b, \quad (\text{A.12})$$

which transforms covariantly under the $\bar{\delta}$ transformations. The affine connection $\Gamma_{\mu\nu}^\rho$ transforms under the $\bar{\delta}$ transformations as

$$\bar{\delta}\Gamma_{\mu\nu}^\rho = \partial_\mu \partial_\nu \xi^\rho + \xi^\sigma \partial_\sigma \Gamma_{\mu\nu}^\rho + \Gamma_{\sigma\nu}^\rho \partial_\mu \xi^\sigma + \Gamma_{\mu\sigma}^\rho \partial_\nu \xi^\sigma - \Gamma_{\mu\nu}^\sigma \partial_\sigma \xi^\rho, \quad (\text{A.13})$$

so that it is inert under the local Lorentz (tangent space) transformations. We will now relate the properties of the curvatures $R_{\mu\nu}{}^a(P)$ and $R_{\mu\nu}{}^{ab}(M)$ to those of $\Gamma_{\mu\nu}^\rho$. This goes via the vielbein postulate which reads

$$\mathcal{D}_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\rho e_\rho^a - \omega_\mu^a{}_b e_\nu^b = 0, \quad (\text{A.14})$$

relating $\Gamma_{\mu\nu}^\rho$ to ω_μ^{ab} . Taking the antisymmetric part of the vielbein postulate and moving $\Gamma_{[\mu\nu]}^\rho$ to the other side we obtain

$$R_{\mu\nu}{}^a(P) = 2\partial_{[\mu} e_{\nu]}^a - 2\omega_{[\mu}{}^{ab} e_{\nu]b} = 2\Gamma_{[\mu\nu]}^\rho e_\rho^a. \quad (\text{A.15})$$

From this we conclude that the curvature $R_{\mu\nu}{}^a(P)$ is the torsion tensor. To identify the other curvature tensor $R_{\mu\nu}{}^{ab}(M)$ we compute the commutator of two covariant derivatives ∇_μ (containing only the connection $\Gamma_{\mu\nu}^\rho$) leading to

$$[\nabla_\mu, \nabla_\nu]X_\rho = R_{\mu\nu\rho}{}^\sigma X_\sigma - 2\Gamma_{[\mu\nu]}^\sigma \nabla_\sigma X_\rho, \quad (\text{A.16})$$

where $R_{\mu\nu\rho}{}^\sigma$ is the Riemann curvature tensor

$$R_{\mu\nu\rho}{}^\sigma = -\partial_\mu \Gamma_{\nu\rho}^\sigma + \partial_\nu \Gamma_{\mu\rho}^\sigma - \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\rho}^\lambda + \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\rho}^\lambda, \quad (\text{A.17})$$

that is related to $R_{\mu\nu}{}^{ab}(M)$ (as follows from the vielbein postulate) via

$$R_{\mu\nu\rho}{}^\sigma = -e_{\rho a} e_b^\sigma R_{\mu\nu}{}^{ab}(M), \quad (\text{A.18})$$

so that $R_{\mu\nu}{}^{ab}(M)$ is the Riemann curvature 2-form. The vielbein postulate, because of the fact that ω_μ^{ab} is antisymmetric in a and b , also tells us that the metric $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$, which is the unique Lorentz invariant tensor we can build out of the vielbeins, is covariantly constant, i.e.

$$\nabla_\rho g_{\mu\nu} = 0. \quad (\text{A.19})$$

As is well known this fixes completely the symmetric part of the connection making it equal to the Levi-Civita connection plus torsion terms which are left unfixed. The common choice in GR to work with torsion-free connections then implies that from the gauging perspective one imposes the curvature constraint $R_{\mu\nu}{}^a(P) = 0$. This in turn makes ω_μ^{ab} a fully dependent connection expressible in terms of the vielbeins and their derivatives. Without fixing the torsion e_μ^a and ω_μ^{ab} remain independent.

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