Discrete Comput Geom (2013) 49:296–301
DOI 10.1007/s00454-012-9477-6

f-Vectors Implying Vertex Decomposability

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Received: 3 February 2012 / Revised: 27 August 2012 / Accepted: 2 November 2012 /
Published online: 4 December 2012

Abstract We prove that if a pure simplicial complex Δ of dimension d with n facets has the least possible number of $(d-1)$-dimensional faces among all complexes with n faces of dimension d, then it is vertex decomposable. This answers a question of J. Herzog and T. Hibi. In fact, we prove a generalization of their theorem using combinatorial methods.

Keywords Vertex decomposable simplicial complex · f-Vector · Kruskal–Katona theorem · Cohen–Macaulay module · Stanley–Reisner ring

Mathematics Subject Classification (2010) 05E45 · 05E40 · 13F55

1 Introduction

We call a simplicial complex *pure* if all its facets are of the same dimension.

Definition 1 A pure simplicial complex Δ of dimension d and n facets is called *extremal* if it has the least possible number of $(d-1)$-dimensional faces among all complexes with n faces of dimension d.

In particular, for $d=0$, all zero-dimensional complexes are extremal, since all of them have exactly one (-1)-dimensional face, namely the empty set.

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In this paper, we generalize and prove by only combinatorial means the following theorem of Herzog and Hibi from 1999.

Theorem 1 ([8], Theorem 2.3) *An extremal simplicial complex is Cohen–Macaulay over an arbitrary field.*

Their proof is algebraic and uses results from Refs. [1] and [6]. In fact, they asked for a combinatorial proof. We give it by proving that an extremal simplicial complex is vertex decomposable. It is well known that vertex decomposable complexes are Cohen–Macaulay. Our proof goes along the lines of the proof of Kruskal–Katona inequality. We start with a presentation of some necessary preliminaries.

1.1 Vertex Decomposable and Cohen–Macaulay Complexes

For a simplex σ in a simplicial complex Δ, the simplicial complex

$$\{\tau \in \Delta : \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}$$

is called a *link* of σ in Δ, and denoted by $\mathrm{lk}_\Delta\sigma$. For a vertex x of Δ by $\Delta \setminus x$, we mean the simplicial complex $\{\tau \in \Delta : x \notin \tau\}$.

Definition 2 A pure simplicial complex Δ is vertex decomposable if one of the following holds:

(1) Δ is empty,
(2) Δ is a single vertex,
(3) for some vertex x, both $\mathrm{lk}_\Delta\{x\}$ and $\Delta \setminus x$ are pure and vertex decomposable.

Definition 3 For a simplicial complex Δ on the set of vertices $\{1, \dots, n\}$ and a given field $\mathbb{K}$, the *Stanley–Reisner ring* (the *face ring*) is $\mathbb{K}[\Delta] := \mathbb{K}[x_1, \dots, x_n]/I$, where I is generated by all square-free monomials $x_{i_1} \dots x_{i_l}$ for which $\{i_1, \dots, i_l\}$ is not a face in Δ.

When we say that a simplicial complex is *Cohen–Macaulay*, we always mean that its Stanley–Reisner ring has this property.

The following is a folklore result (we refer the reader to, e.g., [2]).

Theorem 2 *For a simplicial complex Δ, the following implications hold:*

$$\Delta \textit{ is vertex decomposable} \Rightarrow \Delta \textit{ is shellable} \Rightarrow \Delta \textit{ is Cohen–Macaulay over any field.}$$

We recall a combinatorial description of Cohen–Macaulay complexes.

Theorem 3 ([13]) *Let $R = \mathbb{K}[\Delta]$ be the face ring of Δ. Then the following conditions are equivalent:*

(1) *R is Cohen–Macaulay ring,*
(2) *$\tilde{H}_i(lk_\Delta\sigma; \mathbb{K}) = 0$ for $i < dim(lk_\Delta\sigma)$ for all simplices $\sigma \in \Delta$.*

For classical techniques of counting homologies, we refer the reader to [7,14]. For entertaining ones, we advise Sect. 3.2 of [12].

1.2 Kruskal–Katona Theorem

One of the most natural questions concerning simplicial complexes is:
What is the minimum number of $(k-1)$-element faces in simplicial complex with n faces of size k?
This question was answered independently by Kruskal [11] and Katona [10] in 1960s. For a positive integer k, they enlisted all k-element subsets of integers in the following order, called the *squashed order*: $A < B$ if $\max(A \setminus B) < \max(B \setminus A)$. Let $S_k(n)$ be the set of first n sets in this list. For a given set $\mathfrak{U}$ of k-element sets, denote by $\Delta\mathfrak{U}$, the set of all $(k-1)$-element sets which are contained in some member of $\mathfrak{U}$. The Kruskal–Katona theorem reads as follows.

Theorem 4 *For a positive integers n, k and a set $\mathfrak{U}$ of n sets of size k, we have*

$$\left|\Delta\mathfrak{U}\right| \geq \left|\Delta S_k(n)\right|.$$

This result was further generalized by Clements and Lindström in [3]. Daykin [4,5] gave two simple proofs, and later Hilton [9] gave another one. For an algebraic proof, we refer the reader to [1]. We will work mainly with Hilton's idea.

Note that the cardinality of $\Delta S_k(n)$ may be easily determined. For a given k, each positive integer n can be uniquely expressed as

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t},$$

with $1 \leq t \leq a_t$ and $a_t < \cdots < a_k$. We have

$$\delta_{k-1}(n) := \left|\Delta S_k(n)\right| = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_t}{t-1}.$$

As a consequence of Kruskal–Katona theorem, we get

Corollary 1 *A pure simplicial complex Δ of dimension $d > 0$ with f-vector $(f_0, \ldots, f_d)$ is extremal if and only if $f_{d-1} = \delta_d(f_d)$.*

2 The Main Result

For a better understanding of the assumption that Δ is extremal, we will use Hilton's idea from his proof [9] of the Kruskal–Katona theorem. First, we define sets similar to $S_k(n)$. Let $S_k^i(n)$ denote the first n sets of k-element subsets of integers in the squashed order ($A < B$ if $\max(A \setminus B) < \max(B \setminus A)$) which do not contain i. We also denote by $\{i\}(\cup)\mathfrak{U}$ the set $\{\{i\} \cup A : A \in \mathfrak{U}\}$.

Let $\mathfrak{U}$ be a n-element set of k-element sets, let $V = \bigcup_{A\in\mathfrak{U}} A$ be an underlying set, and let v be its cardinality. For $i \in V$, let $B_i = \{A \in \mathfrak{U} : i \notin A\}$, $C_i = \{A \setminus \{i\} : i \in A \in \mathfrak{U}\}$, and let b_i, c_i be the respective cardinalities. Note that $c_i \neq 0$. We want to find an index i such that $\left|\Delta B_i\right| > \left|C_i\right|$.

Lemma 1 *Either there exists an i such that $\left|\Delta B_i\right| > \left|C_i\right|$, or $\mathfrak{U}$ consists of all possible k-element subsets of V.*

Proof We are going to count the sum of cardinalities of both sets when i runs over all elements of V. Then

$$\sum_{i \in V} \left|\Delta B_i\right| \geq kn(v-k)/(v-k) = kn = \sum_{i \in V} \left|C_i\right|,$$

since at left hand side each $A \in \mathfrak{U}$ gives k distinct sets in its boundary, and it is counted once for every $i \notin A$. Some sets in boundaries of sets from B_i can be the same, but their number is at most $(v-1)-(k-1) = v-k$. On the other hand, each $A \in \mathfrak{U}$ is counted k times at the right side. Hence, we can find a desired i, or the above bounds are tight. In the latter case, when $A \in \Delta B_i$, all $v-k$ possibilities of completing it to a k-element set has to be in $\mathfrak{U}$. This means that $\mathfrak{U}$ consists of all possible k-element subsets of V because from any set in $\mathfrak{U}$ we can delete any element and insert any other. □

Lemma 2 *If Δ is an extremal simplicial complex of positive dimension, then there exists a vertex x such that both $lk_{\Delta}\{x\}$ and $\Delta \setminus x$ are extremal.*

Proof Let Δ be of dimension $d-1 > 0$ and let $\mathfrak{U}$ be the set of all d-element sets in Δ. If $\mathfrak{U}$ consists of all possible d-element subsets of a given v-element set, then the assertion of the lemma is clearly true (we can take any vertex). Otherwise, due to Lemma 1, there exists an $i \in V$ such that $\left|\Delta B_i\right| > \left|C_i\right|$. We have that

$$\Delta\mathfrak{U} = \Delta B_i \cup C_i \cup (\{i\}(\cup)\Delta C_i).$$

Since ΔB_i and $\{i\}(\cup)\Delta C_i$ are disjoint, it follows that

$$\left|\Delta\mathfrak{U}\right| \geq \left|\Delta B_i\right| + \left|\{i\}(\cup)\Delta C_i\right| > \left|C_i\right| + \left|\{i\}(\cup)\Delta C_i\right|.$$

So, by Theorem 4,

$$\left|\Delta\mathfrak{U}\right| \geq \left|\Delta S^i_d(b_i)\right| + \left|\{i\}(\cup)\Delta S^i_{d-1}(c_i)\right|, \tag{2.1}$$

and

$$\left|\Delta\mathfrak{U}\right| > \left|S^i_{d-1}(c_i)\right| + \left|\{i\}(\cup)\Delta S^i_{d-1}(c_i)\right|. \tag{2.2}$$

Since $\Delta S^i_d(b_i) = S^i_{d-1}(e)$ for some e, there are now two possibilities:

(1) $\Delta S^i_d(b_i) \subset S^i_{d-1}(c_i)$, then by (2.2) we get $\left|\Delta\mathfrak{U}\right| > \left|S^i_{d-1}(c_i)\right| + \left|\{i\}(\cup)\Delta S^i_{d-1}(c_i)\right| = \left|\Delta(S^i_d(b_i) \cup (\{i\}(\cup)S^i_{d-1}(c_i)))\right|$, which contradicts the assumption that Δ is extremal, since a complex generated by sets $S^i_d(b_i) \cup (\{i\}(\cup)S^i_{d-1}(c_i))$ has $b_i + c_i = \left|\mathfrak{U}\right|$ maximal faces.

(2) $\Delta S^i_d(b_i) \supset S^i_{d-1}(c_i)$, then by (2.1), we obtain $|\Delta\mathfrak{U}| \geq |\Delta S^i_d(b_i)| + |\{i\}(\cup)\Delta S^i_{d-1}(c_i)| = |\Delta(S^i_d(b_i) \cup (\{i\}(\cup)S^i_{d-1}(c_i)))|$, and equality holds if and only if $C_i \subset \Delta B_i$, $|\Delta B_i| = |\Delta S^i_d(b_i)|$, and $|\Delta C_i| = |\{i\}(\cup)\Delta S^i_{d-1}(c_i)| = |\Delta S^i_{d-1}(c_i)|$.

The complex Δ is extremal, so equality holds, and we get that $C_i \subset \Delta B_i$ and $[B_i]$, $[C_i]$ are extremal, where $[A]$ means the simplicial complex generated by the set of faces A. Observe that $\mathrm{lk}_\Delta\{i\} = [C_i]$ and $\Delta \setminus i = [B_i]$. The first equality is obvious, while the second is not as clear. If $\sigma = \{v_1, \dots, v_k\}$ is a face in $\Delta \setminus i$ then it is a subface of some facet $F = \{v_1, \dots, v_d\}$. If i does not belong to F then $F \in [B_i]$ and so σ does. Otherwise, $F \setminus \{i\} \in C_i \subset \Delta B_i$, so $F \setminus \{i\} \cup \{j\} \in B_i$ for some j. Hence, $\sigma \in [B_i]$. Now $i = x$ gives the assertion. □

Finally, we are ready to prove the generalization of Theorem 1.

Theorem 5 *An extremal simplicial complex is vertex decomposable.*

Proof The proof goes by an induction on d the dimension of Δ and secondly on the number of facets. If $d = 0$, then Δ consists of points and by the definition it is vertex decomposable. When $d > 0$, then by Lemma 2 there exists a vertex x, such that both complexes $\mathrm{lk}_\Delta\{x\}$ and $\Delta \setminus x$ are extremal. The first is of lower dimension and the second either has the same dimension as Δ but fewer facets, or it has smaller dimension. By the inductive hypothesis, both $\mathrm{lk}_\Delta\{x\}$ and $\Delta \setminus x$ are vertex decomposable, and as a consequence Δ also is. □

The above result is best possible in the following sense. Let Δ be a pure simplicial complex of dimension $d > 0$ with f-vector $(f_0, \dots, f_d)$, and with $f_{d-1} = \delta_d(f_d) + c$, $c \in \mathbb{N}$. Due to Corollary 1, the meaning of Theorem 5 is that if $c = 0$, then Δ is vertex decomposable. But even for $c = 1$, complex Δ does not have to be Cohen–Macaulay, which by Theorem 2 is a weaker property then vertex decomposability. We show the following.

Example 1 We have that $\delta_d(2) = 2d + 1$. Let Δ be a pure simplicial complex of dimension d with the set of facets $\mathfrak{U}$ consisting of two disjoint ones. Then $|\Delta\mathfrak{U}| = 2d+2$, and $\tilde{H}_0(\mathrm{lk}_\Delta\emptyset; \mathbb{K}) = 1$, so due to Theorem 3, complex Δ is not Cohen–Macaulay over any field $\mathbb{K}$, and as a consequence it is also not vertex decomposable.

Acknowledgments This publication is partly supported by the Polish National Science Centre grant no. 2011/03/N/ST1/02918. The author would like to thank Ralf Fröberg and Jürgen Herzog for many inspiring conversations and introduction to this subject. The author would also like to thank Jarek Grytczuk and Piotr Micek for help in preparation of this manuscript.

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