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On bounds in Poisson approximation for distributions of independent negative-binomial distributed random variables

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Abstract

Using the Stein–Chen method some upper bounds in Poisson approximation for distributions of row-wise triangular arrays of independent negative-binomial distributed random variables are established in this note.

Keywords: Stein–Chen method, Poisson approximation, Le Cam's inequality, Negative-binomial variable

Mathematics Subject Classification: 60F05, 60G50, 41A36

Background

Let $X_{n,1}, X_{n,2}, \ldots; n = 1, 2, \ldots$ be a row-wise triangular array of independent negative-binomial distributed random variables with probabilities

$$P\left(X_{n,i} = k\right) = C^k_{r_{n,i}+k-1}\left(1 - p_{n,i}\right)^k p^{r_{n,i}}_{n,i}, \tag{1}$$

where $p_{n,i} \in (0, 1)$; $r_{n,i} = 1, 2, \ldots; i = 1, 2, \ldots; k = 0, 1, \ldots$. It is worth pointing out that if all $r_{n,1} = r_{n,2} = \cdots = 1; n = 1, 2, \ldots$, then we have the sequence of independent geometric distributed random variables with success probabilities $p_{n,1}, p_{n,2}, \ldots; n = 1, 2, \ldots$. Write $W_n = \sum_{i=1}^n X_{n,i}$ and $\lambda_n = E(W_n) = \sum_{i=1}^n r_{n,i}\left(1 - p_{n,i}\right)p^{-1}_{n,i}$. We will denote by Z_{λ_n} the Poisson random variable with positive mean λ_n.

The main aim of this paper is to establish some upper bounds in Poisson approximation for $\sum_{k=1}^{\infty} \mid P(W_n = k) - P(Z_{\lambda_n} = k) \mid$ for the sequence $X_{n,1}, X_{n,2}, \ldots; n = 1, 2, \ldots$ by the well-known Stein–Chen method.

It has long been known that the remarkable Le Cam's inequality in Poisson approximation for the row-wise triangular array of independent Bernoulli distributed random variables $Y_{n,1}, Y_{n,2}, \ldots; n = 1, 2, \ldots$ with probabilities $P(Y_{n,i} = 1) = p_{n,i} = 1 - P(Y_{n,i} = 0), i = 1, 2, \ldots$ is defined as follows:

$$\sum_{k=1}^{\infty} \mid P(S_n = k) - P(Z_{\beta_n} = k) \mid \leq 2\sum_{i=1}^{n} p^2_{n,i}, \tag{2}$$

where $S_n = \sum_{i=1}^n Y_{n,i}$ and $\beta_n = E(S_n) = \sum_{i=1}^n p_{n,i}$ [see Le Cam (1960), Neammanee (2003) for more details]. Moreover, a shape inequality has been established as follows:

$$\sum_{k=1}^{\infty} | P(S_n = k) - P(Z_{\beta_n} = k) | \leq \frac{2(1-e^{-\beta_n})}{\beta_n} \sum_{i=1}^{n} p_{n,i}^2. \quad (3)$$

[We refer the reader to Barbour et al. (1992) and Chen (1975)]. As far as we know the Stein–Chen method is the well-known method have been used in Poisson approximation problems and it can be applied to a wide class of discrete random variables as geometric distributed random variables and negative-binomial distributed random variables. In recent years, using the Stein–Chen method, many results related to Poisson approximation for various discrete random variables are established in Teerapabolarn and Wongkasem (2007), Teerapabolarn (2009, 2013). These results are included here for the sake of completeness. Let $Z_1, Z_2, \ldots$ be a sequence of independent geometric distributed random variables with probabilities $P(Z_i = k) = (1-p_i)^k p_i, k = 0, 1, 2, \ldots; i = 1, 2, \ldots$ Then, for $A \subseteq \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$,

$$\sup_A | P(V_n \in A) - \sum_{k \in A} \frac{{\gamma_n}^k e^{-\gamma_n}}{k!} | \leq \sum_{i=1}^{n} \min\left\{ \frac{{\gamma_n}^{-1}(1-e^{-\gamma_n})}{p_i}, 1 \right\} (1-p_i)^2 p_i^{-1}, \quad (4)$$

and for $A \subseteq \mathbb{Z}_+, w_0 \in \mathbb{Z}_+$

$$| P(V_n \leq w_0) - \sum_{k=0}^{w_0} \frac{{\gamma_n}^k e^{-\gamma_n}}{k!} | \leq {\gamma_n}^{-1}(e^{-\gamma_n} - 1) \sum_{i=1}^{n} \min\left\{ \frac{1}{p_i(w_0+1)}, 1 \right\} (1-p_i)^2 p_i^{-1}, \quad (5)$$

where $V_n = \sum_{i=1}^n Z_i, \gamma_n = E(V_n) = \sum_{i=1}^n (1-p_i)p_i^{-1}$ [see Teerapabolarn and Wongkasem (2007), for more details]. It should be noted that in case when the mean $\gamma_n = E(V_n)$ will be replaced by a parameter $\bar{\gamma_n} = \sum_{i=1}^n (1-p_i)$, another results will be established as follows:

$$-\bar{\gamma_n}^{-1}(e^{-\bar{\gamma_n}} - 1) \sum_{i=1}^{n} \min\left\{ \frac{1}{p_i(w_0+1)}, 1 \right\} (1-p_i)^2 \leq P(V_n \leq w_0) - \sum_{k=0}^{w_0} \frac{\bar{\gamma_n}^k e^{-\bar{\gamma_n}}}{k!} \leq 0, \quad (6)$$

and

$$\begin{aligned} &| P(V_n \leq w_0) - \sum_{k=0}^{w_0} \frac{e^{-\bar{\gamma_n}} \bar{\gamma_n}^k}{k!} | \\ &\leq \frac{\sum_{k=0}^{w_0} \frac{e^{-\bar{\gamma_n}} \bar{\gamma_n}^k}{k!} (1 - \sum_{k=0}^{w_0} \frac{e^{-\bar{\gamma_n}} \bar{\gamma_n}^k}{k!})}{\frac{e^{\bar{\gamma_n}} \bar{\gamma_n}^{w_0+1}}{(w_0+1)!}} \sum_{i=1}^{n} \min\left\{ \frac{1}{p_i(w_0+1)}, 1 \right\} (1-p_i)^2, \end{aligned} \quad (7)$$

for $A \subseteq \mathbb{Z}_+$ [results of this nature may be found in Teerapabolarn (2013)]. It is easy to check that when the values $r_{n,1} = r_{n,2} = \cdots = 1; n = 1, 2, \ldots$ the desired sequence $(X_n, n \geq 1)$ will become the sequence $Z_1, Z_2, \ldots$. Therefore, it makes sense to consider the results in (4), (5), (6), and (7) for negative-binomial random variables with probabilities in term of (1).

It should be noted that in recent years the same problem was tackled in Upadhye and Vellaisamy (2014) and Vellaisamy and Upadhye (2009) by using Kerstans method (1964) and the method of exponents [see Upadhye and Vellaisamy (2013, 2014) and Vellaisamy and Upadhye (2009), for more details]. The compound negative binomial and compound Poisson approximations to the generalized Poisson binomial distribution are studied and applications are also discussed [see Upadhye and Vellaisamy (2013, 2014), for more details]. Specifically, using Kerstans method (1964) and the method of exponents, Vellaisamy and Upadhye (2009) have established the bounds in Poisson approximation as following inequality:

$$d_{TV}(S_n, Z_\lambda) \leq \sum_{j=1}^{n} \frac{\alpha_j q_j^2}{p_j} \min\left\{1, \frac{1}{\sqrt{2\lambda e}}\right\},$$

where $\lambda = \sum_{i=1}^{n} \alpha_i q_i = \alpha q$, for $X_1, X_2, \ldots, X_n$ are independent negative binomial distributed random variables with parameters α_j and $q_j, j = 1, 2, \ldots, n$ and Z_λ is a Poisson random variable with mean λ.

It is worth pointing out that comparison of bounds in negative binomial approximation and Poisson approximation is showing that an negative binomial approximation is better than Poisson approximation in the case $X_j, j = 1, 2, \ldots$ are independent negative binomial random variables [see Theorem 2.2 and Theorem 2.4 in Vellaisamy and Upadhye (2009)].

Besides, Poisson approximation is also considered for a wide class of discrete random variables via operator method and method of probability distance [see Hung and Thao (2013) and Hung and Giang (2014), for more details].

The main purpose of this paper is to use the Stein–Chen method for providing the bounds of Le Cam-type inequality (2) and (3) in Poisson approximation for row-wise arrays of independent negative-binomial distributed random variables. The results obtained in this paper are extensions and generalizations of some results in Teerapabolarn and Wongkasem (2007), Teerapabolarn (2009, 2013).

Preliminaries

During the last several decades the Stein–Chen method has risen to become one of the most important tools available for studying in Poisson approximation problems. The Stein–Chen method has been dealt with in detail in many articles [the reader is referred to Stein (1972), Chen (1975), Chen and Röllin (2013), Barbour et al. (1992) and Barbour and Chen (2004) for fuller development]. The Stein–Chen method can be summarized as follows:

Let us denote by $F_X(A)$ the probability distribution function of a discrete random variable $X \in A$ and we will denoted by $P_{\alpha_n}(A) = \sum_{k\in A} e^{-\alpha_n} \frac{{\alpha_n}^k}{k!}$ the Poisson distribution function, defined on the set $A \subseteq \mathbb{Z}_+$. The best known method for estimating

$$\Delta = \sup_x \left|F_X(A) - P_{\alpha_n}(A)\right|$$

is basing on the following arguments [see Chen (1975) for more details]:

Assume that $h(u)$ is a real-valued bounded function and $P_{\alpha_n} h = e^{-\alpha_n} \sum_{k=0}^{\infty} h(k) \frac{{\alpha_n}^k}{k!}$. Consider the function $f(.)$ which is a solution of the differential equation

$$\alpha_n f(x+1) - x f(x) = h(x) - P_{\alpha_n} h.$$

Setting

$$h(x) = h_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Putting $x = X$ and taking the expectation of both sides of the above differential equation, we have

$$F_X(A) - P_{\alpha_n}(A) = E[\alpha_n f(X+1) - Xf(X)].$$

Thus, the problem of estimating Δ can be reduced to that of estimating the difference of the expectations

$$\left|E\alpha_n f(X+1) - EXf(X)\right|.$$

Before starting the main results in the next section we first recall the following remarkable lemmas:

Lemma 1 (Barbour et al. 1992) *Let* $Vf_A(w) = f_A(w+1) - f_A(w)$. *Then, for* $A \subseteq \mathbb{Z}_+$ *and* $k \in \mathbb{Z}_+ \setminus \{0\}$,

$$\sup_{w \geq k} \left|Vf_A(w)\right| \leq \min\left\{\alpha_n{}^{-1}\left(1 - e^{-\alpha_n}\right), \frac{1}{k}\right\}.$$

Lemma 2 (Teerapabolarn and Wongkasem 2007) *Let* $w_0 \in \mathbb{Z}_+$ *and* $k \in \mathbb{Z}_+ \setminus \{0\}$, *we have*

$$\sup_{w \geq k} \left|Vf_{C_{w_0}}(w)\right| \leq \gamma_n{}^{-1}\left(e^{\gamma_n} - 1\right) \min\left\{\frac{1}{w_0+1}, \frac{1}{k}\right\}.$$

Lemma 3 (Teerapabolarn 2009) *Let* $w_0 \in \mathbb{Z}_+$ *and* $k \in \mathbb{Z}_+ \setminus \{0, 1\}$. *Then, we have*

$$0 < \sup_{w \geq k} f(w) \leq \bar{\gamma}_n{}^{-1}\left(e^{\bar{\gamma}_n} - 1\right) \min\left\{\frac{1}{k}, \frac{1}{w_0+1}\right\}.$$

Lemma 4 (Teerapabolarn 2013) *For* $w_0 \in \mathbb{Z}_+$ *and* $k \in \mathbb{Z}_+ \setminus \{0, 1\}$, *let* $p_{\bar{\gamma}_n}(w_0) = \dfrac{e^{-\bar{\gamma}_n}\bar{\gamma}_n{}^{w_0}}{w_0!}$ *and* $P_{\bar{\gamma}_n}(w_0) = \sum_{k=0}^{w_0} \dfrac{\bar{\gamma}_n{}^k e^{-\bar{\gamma}_n}}{k!}$. *Then the following inequality is true*

$$\sup_{w \geq k} f_{C_{w_0}}(w) \leq \frac{P_{\bar{\gamma}_n}(w_0)\left(1 - P_{\bar{\gamma}_n}(w_0)\right)}{p_{\bar{\gamma}_n}(w_0+1)} \min\left\{\frac{1}{w_0+1}, \frac{1}{k}\right\}.$$

Results

Throughout the forthcoming, unless otherwise specified, we shall denote by $X_{n,1}, X_{n,2}, \ldots; n = 1, 2, \ldots$ a row-wise triangular array of independent negative-binomial distributed random variables with probabilities

$$P\left(X_{n,i} = k\right) = C^k_{r_{n,i}+k-1}\left(1 - p_{n,i}\right)^k p_{n,i}^{r_{n,i}},$$

where $p_{n,i} \in (0,1)$; $r_{n,i} = 1, 2, \ldots$; $i = 1, 2, \ldots$; $k = 0, 1, \ldots$. Let $W_n = \sum_{i=1}^{n} X_{n,i}$ and set $\lambda_n = E(W_n) = \sum_{i=1}^{n} r_{n,i}\big(1 - p_{n,i}\big)p_{n,i}^{-1}$. Then, for $r_{n,i} \in \{1, 2, \ldots\}$ we have the following theorems:

Theorem 1 *For $A \subseteq \mathbb{Z}_+$,*

$$\sup_{A} \left| P(W_n \in A) - \sum_{k\in A} \frac{\lambda_n^k e^{-\lambda_n}}{k!} \right| \leq \sum_{i=1}^{n} \min\left\{ \lambda_n^{-1}\left(1 - e^{-\lambda_n}\right) r_{n,i}\big(1 - p_{n,i}\big)p_{n,i}^{-1}, 1 - p_{n,i}^{r_{n,i}} \right\} \big(1 - p_{n,i}\big)p_{n,i}^{-1}.$$

Proof Let f and h are bounded real-valued functions defined on $\mathbb{Z}_+$. For $w = 0, 1, \ldots$ we have the Stein's equation for Poisson distribution with a mean λ_n

$$\lambda_n f(w+1) - wf(w) = h(w) - P_{\lambda_n}(h),$$

where $P_{\lambda_n}(h) = e^{-\lambda_n} \sum_{k=0}^{\infty} h(k) \dfrac{{\lambda_n}^k}{k!}$.

For $A \subseteq \mathbb{Z}_+$, let us denote by $h_A : \mathbb{Z}_+ \to \mathbb{R}$ and by $f_A(w)$ the functions defined by

$$h_A(w) = \begin{cases} 1, & \text{if } w \in A, \\ 0, & \text{if } w \notin A. \end{cases}$$

and

$$f_A(w) = \begin{cases} (w-1)!{\lambda_n}^{-w} e^{\lambda_n} \left[P_{\lambda_n}\left(h_{A\cap C_{w-1}}\right) - P_{\lambda_n}(h_A) P_{\lambda_n}\left(h_{C_{w-1}}\right) \right], & \text{if } w \geq 1, \\ 0, & \text{if } w = 0, \end{cases}$$

where $C_w = \{0, 1, \cdots, w\}$.

Given $f = f_A$ and $h = h_A$, We have the following Stein's equation:

$$\lambda_n f(w+1) - wf(w) = h_A(w) - P_{\lambda_n}(h_A),$$

where

$$P_{\lambda_n}(h_A) = e^{-\lambda_n} \sum_{k=0}^{\infty} h_A(k) \frac{\lambda_n^k}{k!} = \sum_{k\in A} e^{-\lambda_n} \frac{\lambda_n^k}{k!}.$$

Therefore, the Stein's equation can be written as follows:

$$h_A(w) - \sum_{k\in A} e^{-\lambda_n} \frac{\lambda_n^k}{k!} = \lambda_n f(w+1) - wf(w).$$

Taking expectations of both sides of above equation, we have

$$P(W_n \in A) - \sum_{k\in A} \frac{\lambda_n^k e^{-\lambda_n}}{k!} = E[\lambda_n f(W_n + 1) - W_n f(W_n)].$$

It follows that

$$
\begin{aligned}
&\mid P(W_n \in A) - \sum_{k\in A} \frac{\lambda_n^k e^{-\lambda_n}}{k!} \mid = E[\lambda_n f(W_n+1) - W_n f(W_n)] \mid \\
&\leq \sum_{i=1}^{n} \mid E[r_{n,i}(p_{n,i}^{-1}-1) f(W_n+1) - X_{n,i} f(W_n)] \mid .
\end{aligned} \tag{8}
$$

Let $W_i = W_n - X_{n,i}$. Then, for each i, we get

$$
\begin{aligned}
&E[r_{n,i}(p_{n,i}^{-1}-1) f(W_n+1) - X_{n,i} f(W_n)] \\
&= E[r_{n,i}(p_{n,i}^{-1}-1) f(W_i+X_{n,i}+1) - X_{n,i} f(W_i+X_{n,i})] \\
&= E[E[(r_{n,i}(p_{n,i}^{-1}-1) f(W_i+X_{n,i}+1) - X_{n,i} f(W_i+X_{n,i}))|X_{n,i}]] \\
&= E[(r_{n,i}(p_{n,i}^{-1}-1) f(W_i+X_{n,i}+1) - X_{n,i} f(W_i+X_{n,i}))|X_{n,i}=0] p_{n,i}^{r_{n,i}} \\
&\quad + E[(r_{n,i}(p_{n,i}^{-1}-1) f(W_i+X_{n,i}+1) - X_{n,i} f(W_i+X_{n,i}))|X_{n,i}=1] r_{n,i} p_{n,i}^{r_{n,i}}(1-p_{n,i}) \\
&\quad + \sum_{k\geq 2} E[\Big(r_{n,i}(p_{n,i}^{-1}-1) f(W_i+X_{n,i}+1) \\
&\quad - X_{n,i} f(W_i+X_{n,i})\Big)|X_{n,i}=k] C_{r_{n,i}+k-1}^{k} p_{n,i}^{r_{n,i}}(1-p_{n,i})^k \\
&= E[r_{n,i}(p_{n,i}^{-1}-1) p_{n,i}^{r_{n,i}} f(W_i+1)] \\
&\quad + E[r_{n,i}^2 \big(1-p_{n,i}\big)^2 p_{n,i}^{r_{n,i}-1} f(W_i+2) - r_{n,i} p_{n,i}^{r_{n,i}}(1-p_{n,i}) f(W_i+1)] \\
&\quad + \sum_{k\geq 2} E[C_{r_{n,i}+k-1}^{k} r_{n,i}(1-p_{n,i})^{k+1} p_{n,i}^{r_{n,i}-1} f(W_i+k+1) \\
&\quad - k C_{r_{n,i}+k-1}^{k} p_{n,i}^{r_{n,i}}(1-p_{n,i})^k f(W_i+k)] \\
&= r_{n,i}\big(1-p_{n,i}\big)^2 p_{n,i}^{r_{n,i}-1} E[f(W_i+1)] + E[r_{n,i}^2(1-p_{n,i})^2 p_{n,i}^{r_{n,i}-1} f(W_i+2)] \\
&\quad + \sum_{k\geq 2} E[C_{r_{n,i}+k-1}^{k} r_{n,i}(1-p_{n,i})^{k+1} p_{n,i}^{r_{n,i}-1} f(W_i+k+1) \\
&\quad - k C_{r_{n,i}+k-1}^{k} p_{n,i}^{r_{n,i}}(1-p_{n,i})^k f(W_i+k)] \\
&= r_{n,i}\big(1-p_{n,i}\big)^2 p_{n,i}^{r_{n,i}-1} E[f(W_i+1)] \\
&\quad + \sum_{k\geq 2} E[C_{r_{n,i}+k-2}^{k-1} r_{n,i}(1-p_{n,i})^k p_{n,i}^{r_{n,i}-1} f(W_i+k) - k C_{r_{n,i}+k-1}^{k} p_{n,i}^{r_{n,i}}(1-p_{n,i})^k f(W_i+k)] \\
&= r_{n,i}\big(1-p_{n,i}\big)^2 p_{n,i}^{r_{n,i}-1} E[f(W_i+1)] \\
&\quad + \sum_{k\geq 2} E[C_{r_{n,i}+k-2}^{k-1} r_{n,i}(1-p_{n,i})^k p_{n,i}^{r_{n,i}-1} f(W_i+k) \\
&\quad - \big(r_{n,i}+k-1\big) C_{r_{n,i}+k-2}^{k-1} p_{n,i}^{r_{n,i}}(1-p_{n,i})^k f(W_i+k)]
\end{aligned}
$$

$$
\begin{aligned}
&= r_{n,i}\left(1-p_{n,i}\right)^2 p_{n,i}^{r_{n,i}-1} E[f(W_i+1)] \\
&\quad + \sum_{k\ge 2} E\left[\frac{r_{n,i}+k-1}{r_{n,i}} C_{r_{n,i}+k-2}^{k-1} r_{n,i}\left(1-p_{n,i}\right)^k p_{n,i}^{r_{n,i}-1} f(W_i+k)\right. \\
&\quad \left. -\ \left(r_{n,i}+k-1\right) C_{r_{n,i}+k-2}^{k-1} p_i^{r_{n,i}}\left(1-p_{n,i}\right)^k f(W_i+k)\right] \\
&\quad - \sum_{k\ge 2}\left(\frac{r_{n,i}+k-1}{r_{n,i}}-1\right) C_{r_{n,i}+k-2}^{k-1} r_{n,i}\left(1-p_{n,i}\right)^k p_{n,i}^{r_{n,i}-1} E[f(W_i+k)] \\
&= r_{n,i}\left(1-p_{n,i}\right)^2 p_{n,i}^{r_{n,i}-1} E[f(W_i+1)] \\
&\quad + \sum_{k\ge 2}\left(r_{n,i}+k-1\right) C_{r_{n,i}+k-2}^{k-1}(1-p_{n,i})^{k+1} p_{n,i}^{r_{n,i}-1} E[f(W_i+k)] \\
&\quad - \sum_{k\ge 2}\left(\frac{r_{n,i}+k}{r_{n,i}}-1\right) C_{r_{n,i}+k-1}^{k} r_{n,i}(1-p_{n,i})^{k+1} p_{n,i}^{r_{n,i}-1} E[f(W_i+k+1)] \\
&\quad - r_{n,i}\left(1-p_{n,i}\right)^2 p_{n,i}^{r_{n,i}-1} E[f(W_i+2)] \\
&= r_{n,i}\left(1-p_{n,i}\right)^2 p_{n,i}^{r_{n,i}-1} E[f(W_i+1)] - r_{n,i}\left(1-p_{n,i}\right)^2 p_{n,i}^{r_{n,i}-1} E[f(W_i+2)] \\
&\quad + \sum_{k\ge 2} k C_{r_{n,i}+k-1}^{k}(1-p_{n,i})^{k+1} p_{n,i}^{r_{n,i}-1} E[f(W_i+k)] \\
&\quad - \sum_{k\ge 2} k C_{r_{n,i}+k-1}^{k}(1-p_{n,i})^{k+1} p_{n,i}^{r_{n,i}-1} E[f(W_i+k+1)] \\
&= r_{n,i}\left(1-p_{n,i}\right)^2 p_{n,i}^{r_{n,i}-1} E[f(W_i+1)-f(W_i+2)] \\
&\quad + \sum_{k\ge 2} k C_{r_{n,i}+k-1}^{k}(1-p_{n,i})^{k+1} p_{n,i}^{r_{n,i}-1} E[f(W_i+k)-f(W_i+k+1)] \\
&= \sum_{k\ge 1} k C_{r_{n,i}+k-1}^{k}(1-p_{n,i})^{k+1} p_{n,i}^{r_{n,i}-1} E[f(W_i+k)-f(W_i+k+1)].
\end{aligned}
$$

By using Lemma 1, we have

$$
\begin{aligned}
&|E[r_{n,i}(p_{n,i}^{-1}-1)f(W_n+1)-X_{n,i}f(W_n)]| \\
&\le \sum_{k\ge 1} k C_{r_{n,i}+k-1}^{k}(1-p_{n,i})^{k+1} p_{n,i}^{r_{n,i}-1} E|f(W_i+k)-f(W_i+k+1)| \\
&\le \sum_{k\ge 1} k C_{r_{n,i}+k-1}^{k}(1-p_{n,i})^{k+1} p_{n,i}^{r_{n,i}-1} \sup_{w\ge k}|Vf(w)| \\
&\le \min\left\{\lambda_n{}^{-1}(1-e^{-\lambda_n})p_{n,i}^{r_{n,i}-1}\sum_{k\ge 1} k C_{r_{n,i}+k-1}^{k}(1-p_{n,i})^{k+1},\right. \\
&\quad \left. p_{n,i}^{r_{n,i}-1}\sum_{k\ge 1} C_{r_{n,i}+k-1}^{k}(1-p_{n,i})^{k+1}\right\} \\
&= \min\left\{\lambda_n{}^{-1}(1-e^{-\lambda_n})p_{n,i}^{r_{n,i}-1}\left(1-p_{n,i}\right)\sum_{k\ge 1} k C_{r_{n,i}+k-1}^{k}(1-p_{n,i})^{k},\right. \\
&\quad \left. p_{n,i}^{r_{n,i}-1}\left(1-p_{n,i}\right)\left(p_{n,i}^{-r_{n,i}}-1\right)\right\} \\
&= \min\left\{\lambda_n^{-1}\left(1-e^{-\lambda_n}\right)p_{n,i}^{r_{n,i}-1}\left(1-p_{n,i}\right)r_{n,i}\left(1-p_{n,i}\right)p_{n,i}^{-r_{n,i}-1}, p_{n.i}^{-1}\left(1-p_{n,i}\right)\left(1-p_{n,i}^{r_{n,i}}\right)\right\} \\
&= \min\left\{\lambda_n^{-1}\left(1-e^{-\lambda_n}\right)r_{n,i}\left(1-p_{n,i}\right)p_{n,i}^{-1}, 1-p_{n,i}^{r_{n,i}}\right\}\left(1-p_{n,i}\right)p_{n,i}^{-1}. \qquad (9)
\end{aligned}
$$

To combine (8) and (9), we have

$$\sup_A \left| P(W_n \in A) - \sum_{k\in A} \frac{\lambda_n^k e^{-\lambda_n}}{k!} \right| \leq \sum_{i=1}^{n} \min\left\{ \lambda_n^{-1}\left(1-e^{-\lambda_n}\right) r_{n,i}(1-p_{n,i})p_{n,i}^{-1}, 1-p_{n,i}^{r_{n,i}} \right\}(1-p_{n,i})p_{n,i}^{-1}.$$

The proof is complete. □

Remark 1 It is easily seen that the (4) is a special case of the Theorem 1 with $r_{n,i} = 1; n = 1, 2, \ldots; i = 1, 2, \ldots n$

Theorem 2 *Let W_n and λ_n be defined as in Theorem 1. Then, for $w_0 \in \mathbb{N}$,*

$$\left| P(W_n \leq w_0) - \sum_{k\leq w_0} \frac{\lambda_n^k e^{-\lambda_n}}{k!} \right| \leq \lambda_n^{-1}\left(e^{\lambda_n}-1\right) \sum_{i=1}^{n} \min\left\{ \frac{r_{n,i}(1-p_{n,i})}{p_{n,i}(w_0+1)}, 1-p_{n,i}^{r_{n,i}} \right\}(1-p_{n,i})p_{n,i}^{-1}.$$

Proof For $C_w = \{0, \ldots, w\}$ and $w_0 \in N$, let $h_{w_0} : \mathbb{Z}_+ \to \mathbb{R}$, $f_{C_{w_0}}(w_0)$ be defined by

$$h_{C_{w_0}}(w) = \begin{cases} 1 & \text{if } w \leq w_0, \\ 0 & \text{if } w > w_0. \end{cases}$$

$$f_{C_{w_0}}(w) = \begin{cases} (w-1)!\lambda_n{}^{-w} e^{\lambda_n} \left[P_{\lambda_n}(h_{C_{w0}})P_{\lambda_n}(1-h_{C_{w-1}})\right] & \text{if } w_0 < w, \\ (w-1)!\lambda_n{}^{-w} e^{\lambda_n} \left[P_{\lambda_n}(h_{C_{w-1}})P_{\lambda_n}\left(1-h_{C_{w_0}}\right)\right] & \text{if } w_0 \geq w, \\ 0 & \text{if } w = 0. \end{cases}$$

Given $f = f_{C_{w_0}}$ and $h = h_{C_{w_0}}$. We have the Stein's equation

$$h_{C_{w_0}}(w) - \sum_{k\leq w_0} e^{-\lambda_n} \frac{\lambda_n^k}{k!} = \lambda_n f(w+1) - wf(w).$$

Taking expectations of both sides and arguing similarly to the proof of Theorem 1 we prove that

$$\left| P(W_n \leq w_0) - \sum_{k\leq w_0} \frac{\lambda_n^k e^{-\lambda_n}}{k!} \right| \leq \sum_{i=1}^{n} |E[r_{n,i}(p_{n,i}^{-1}-1)f(W_n+1) - X_{n,i}f(W_n)]|. \quad (10)$$

According to the Theorem 1, we have

$$E[r_{n,i}(p_{n,i}^{-1}-1)f(W_n+1) - X_{n,i}f(W_n)] = \sum_{k\geq 1} kC^k_{r_{n,i}+k-1}(1-p_{n,i})^{k+1} p_{n,i}^{r_{n,i}-1} E[f(W_i+k) - f(W_i+k+1)]. \quad (11)$$

Hence, by (10), (11) and Lemma 2, we have

$$\left|P(W_n \le w_0) - \sum_{k\le w_0} \frac{\lambda_n^k e^{-\lambda_n}}{k!}\right| \le \sum_{i=1}^n |E[r_{n,i}(p_{n,i}^{-1}-1)f(W_n+1) - X_{n,i}f(W_n)]|$$

$$\le \sum_{i=1}^n \left(\sum_{k\ge1} kC^k_{r_{n,i}+k-1}(1-p_{n,i})^{k+1}p_{n,i}^{r_{n,i}-1}\sup_{w\ge k}|Vf(w)|\right)$$

$$\le \sum_{i=1}^n \left(\lambda_n{}^{-1}\left(e^{\lambda_n}-1\right)\min\left\{\frac{p_{n,i}^{r_{n,i}-1}}{w_0+1}\sum_{k\ge1}kC^k_{r_{n,i}+k-1}(1-p_{n,i})^{k+1},\right.\right.$$

$$\left.\left. p_{n,i}^{r_{n,i}-1}\sum_{k\ge1}C^k_{r_{n,i}+k-1}(1-p_{n,i})^{k+1}\right\}\right)$$

$$\le \sum_{i=1}^n \left(\lambda_n{}^{-1}\left(e^{\lambda_n}-1\right)\min\left\{\frac{p_{n,i}^{r_{n,i}-1}(1-p_{n,i})}{w_0+1}\sum_{k\ge1}kC^k_{r_{n,i}+k-1}(1-p_{n,i})^{k},\right.\right.$$

$$\left.\left. p_{n,i}^{r_{n,i}-1}(1-p_{n,i})\left(p_{n,i}^{-r_{n,i}}-1\right)\right\}\right)$$

$$= \sum_{i=1}^n \left(\lambda_n^{-1}\left(e^{\lambda_n}-1\right)\min\left\{\frac{p_{n,i}^{r_{n,i}-1}(1-p_{n,i})r_{n,i}(1-p_{n,i})}{(w_0+1)p_{n,i}^{r_{n,i}+1}}, (1-p_{n,i})p_{n,i}^{-1}(1-p_{n,i}^{r_{n,i}})\right\}\right)$$

$$= \sum_{i=1}^n \left(\lambda_n^{-1}\left(e^{\lambda_n}-1\right)\min\left\{\frac{r_{n,i}(1-p_{n,i})}{p_{n,i}(w_0+1)}, 1-p_{n,i}^{r_{n,i}}\right\}(1-p_{n,i})p_{n,i}^{-1}\right).$$

Thus

$$\left|P(W_n \le w_0) - \sum_{k\le w_0} \frac{\lambda_n^k e^{-\lambda_n}}{k!}\right|$$

$$\le \lambda_n{}^{-1}\left(e^{\lambda_n}-1\right)\sum_{i=1}^n \min\left\{\frac{r_{n,i}(1-p_{n,i})}{p_{n,i}(w_0+1)}, 1-p_{n,i}^{r_{n,i}}\right\}(1-p_{n,i})p_{n,i}^{-1}.$$

This finishes the proof. □

Remark 2 It is easy to check that the (5) is a special case of Theorem 2 with $r_{n,i} = 1; n = 1, 2, \ldots; i = 1, 2, \ldots n$.

Theorem 3 *Let* $W_n = \sum_{i=1}^n X_i$ *and* $\bar{\lambda}_n = \sum_{i=1}^n r_{n,i}q_{n,i}$ *with* $q_{n,i} = 1 - p_{n,i}$. *Then, we have*

$$-\bar{\lambda}_n{}^{-1}\left(e^{\bar{\lambda}_n}-1\right)\sum_{i=1}^n \min\left\{\alpha_i, \frac{\beta_i-\alpha_i}{w_0+1}\right\} \le P(W_n \le w_0) - \sum_{k=0}^{w_0}\frac{\bar{\lambda}_n{}^k e^{-\bar{\lambda}_n}}{k!} \le 0,$$

With $\alpha_i = 1 - p_{n,i}^{r_{n,i}} - r_{n,i}q_{n,i}p_{n,i}^{r_{n,i}}$, $\beta_i = r_{n,i}\left(p_{n,i}^{-r_{n,i}} - 1 - r_{n,i}q_{n,i}p_{n,i}^{r_{n,i}}\right)$.

Proof Arguing as in theorem (3), we have the Stein's equation

$$h_{w_0}(w) - \sum_{k=0}^{w_0} e^{-\bar{\lambda}_n}\frac{\bar{\lambda}_n{}^k}{k!} = \bar{\lambda}_n f(w+1) - wf(w).$$

Taking expectations of both sides, we get

$$\begin{aligned} P(W_n \le w_0) - \sum_{k=0}^{w_0} \frac{\bar{\lambda}_n^{\,k} e^{-\bar{\lambda}_n}}{k!} &= E\left[\bar{\lambda}_n f(W_n+1) - W_n f(W_n)\right] \\ &= \sum_{i=1}^{n} E[r_{n,i} q_{n,i} f(W_n+1) - X_{n,i} f(W_n)]. \end{aligned} \tag{12}$$

Let $W_i = W_n - X_{n,i}$. Then, for each i, we deduce

$$\begin{aligned} & E[r_{n,i}q_{n,i}f(W_n+1) - X_{n,i}f(W_n)] \\ &= E[E[(r_{n,i}q_{n,i}f(W_i+X_{n,i}+1) - X_{n,i}f(W_i+X_{n,i}))|X_{n,i}]] \\ &= E\left[r_{n,i}q_{n,i}p_{n,i}^{r_{n,i}}f(W_i+1)\right] + E[r_{n,i}^2 q_{n,i}^2 p_{n,i}^{r_{n,i}} f(W_i+2) - r_{n,i}q_{n,i}p_{n,i}^{r_{n,i}} f(W_i+X_{n,i})] \\ &\quad + \sum_{k\ge 2} E[r_{n,i}C_{r_{n,i}+k-1}^{k} q_{n,i}^{k+1} p_{n,i}^{r_{n,i}} f(W_i+k+1) - kC_{r_{n,i}+k-1}^{k} q_{n,i}^{k} p_{n,i}^{r_{n,i}} f(W_i+k)] \\ &= \sum_{k\ge 2} E[r_{n,i}C_{r_{n,i}+k-2}^{k-1} q_{n,i}^{k} p_{n,i}^{r_{n,i}} f(W_i+k) - kC_{r_{n,i}+k-1}^{k} q_{n,i}^{k} p_{n,i}^{r_{n,i}} f(W_i+k)] \\ &= \sum_{k\ge 2} E[\frac{r_{n,i}k}{r_{n,i}+k-1} C_{r_{n,i}+k-1}^{k} q_{n,i}^{k} p_{n,i}^{r_{n,i}} f(W_i+k) - kC_{r_{n,i}+k-1}^{k} q_{n,i}^{k} p_{n,i}^{r_{n,i}} f(W_i+k)] \\ &= \sum_{k} \frac{k(1-k)}{r_{n,i}+k-1} C_{r_{n,i}+k-1}^{k} q_{n,i}^{k} p_{n,i}^{r_{n,i}} f(W_i+k) \\ &\ge -\sum_{k} \frac{k(k-1)}{r_{n,i}+k-1} C_{r_{n,i}+k-1}^{k} q_{n,i}^{k} p_{n,i}^{r_{n,i}} \sup_{w\ge k} f(w). \end{aligned}$$

By using Lemma 3, then we have

$$\begin{aligned} & -\sum_{k} \frac{k(k-1)}{r_{n,i}+k-1} C_{r_{n,i}+k-1}^{k} q_{n,i}^{k} p_{n,i}^{r_{n,i}} \sup_{w\ge k} f(w) \\ &\ge -\bar{\lambda}_n^{-1}\left(e^{\bar{\lambda}_n}-1\right) p_{n,i}^{r_{n,i}} \min\left\{ \sum_{k} \frac{k-1}{r_{n,i}+k-1} C_{r_{n,i}+k-1}^{k} q_{n,i}^{k}, \right. \\ &\quad \left. \frac{1}{w_0+1} \sum_{k} \frac{k(k-1)}{r_{n,i}+k-1} C_{r_{n,i}+k-1}^{k} q_{n,i}^{k} \right\}. \end{aligned} \tag{13}$$

Moreover, we have

$$p_{n,i}^{r_{n,i}} \sum_{k} \frac{k-1}{r_{n,i}+k-1} C_{r_{n,i}+k-1}^{k} q_{n,i}^{k} \le 1 - p_{n,i}^{r_{n,i}} - r_{n,i}q_{n,i}p_{n,i}^{r_{n,i}} \tag{14}$$

and

$$\begin{aligned} & p_{n,i}^{r_{n,i}} \sum_{k} \frac{k(k-1)}{r_{n,i}+k-1} C_{r_{n,i}+k-1}^{k} q_{n,i}^{k} \\ &\le r_{n,i}\left(p_{n,i}^{-r_{n,i}} - 1 - r_{n,i}q_{n,i}p_{n,i}^{r_{n,i}}\right) - \left(1 - p_{n,i}^{r_{n,i}} - r_{n,i}q_{n,i}p_{n,i}^{r_{n,i}}\right). \end{aligned} \tag{15}$$

Hence, by (12), (13), (14) and (15), we can assert that

$$-\bar{\lambda}_n^{-1}\left(e^{\bar{\lambda}_n}-1\right)\sum_{i=1}^{n}\min\left\{\alpha_i, \frac{\beta_i-\alpha_i}{w_0+1}\right\} \leq P(W_n\leq w_0)-\sum_{k=0}^{w_0}\frac{\bar{\lambda}_n^{\,k} e^{-\bar{\lambda}_n}}{k!}\leq 0.$$

The proof is complete. □

Remark 3 When $r_{n,i}=1$, we have

$$\alpha_i = 1-p_{n,i}-q_{n,i}p_{n,i} = \left(1-p_{n,i}\right)\left(1-p_{n,i}\right) = q_{n,i}^2,$$
$$\beta_i = p_{n,i}^{-1}-1-q_{n,i}p_{n,i} = \frac{1-p_{n,i}}{p_{n,i}}-\left(1-p_{n,i}\right)p_{n,i} = \frac{\left(1-p_{n,i}\right)\left(1-p_{n,i}^2\right)}{p_{n,i}} = q_{n,i}^2\frac{1+p_{n,i}}{p_{n,i}},$$
$$\beta_i-\alpha_i = q_{n,i}^2\left(\frac{1+p_{n,i}}{p_{n,i}}-1\right) = \frac{q_{n,i}^2}{p_{n,i}}.$$

It is clear that the (6) is a special case of Theorem 3 with $r_{n,i}=1$; $n=1,2,\ldots$; $i=1,2,\ldots n$.

Theorem 4 *Let* $W_n=\sum_{i=1}^{n}X_{n,i}$ *and* $\overline{\lambda_n}=\sum_{i=1}^{n}r_{n,i}q_i$ *with* $q_{n,i}=1-p_{n,i}$. *Then, for* $w_0\in\mathbb{N}$ *we have*

$$\left|P(W_n\leq w_0)-P_{\overline{\lambda_n}}(w_0)\right| \leq \frac{P_{\overline{\lambda_n}}(w_0)\left(1-P_{\overline{\lambda_n}}(w_0)\right)}{p_{\overline{\lambda_n}}(w_0+1)}\sum_{i=1}^{n}\min\left\{\alpha_i, \frac{\beta_i-\alpha_i}{w_0+1}\right\},$$

where

$$\alpha_i = 1-p_{n,i}^{r_{n,i}}-r_{n,i}q_{n,i}p_{n,i}^{r_{n,i}} \text{ and } \beta_i = r_{n,i}\left(p_{n,i}^{-r_{n,i}}-1-r_{n,i}q_{n,i}p_{n,i}^{r_{n,i}}\right).$$

Proof According to Theorem 3 we obtain the following inequality

$$\left|P(W_n\leq w_0)-P_{\bar{\lambda}_n}(w_0)\right| \leq \sum_{i=1}^{n}\sum_{k}\frac{k(k-1)}{r_{n,i}+k-1}C_{r_{n,i}+k-1}^{k}q_{n,i}^{k}p_{n,i}^{r_{n,i}}\sup_{w\geq k}f_{C_{w_0}}(w).$$

By using Lemma 4, then we have

$$\begin{aligned}
&\left|P(W_n\leq w_0)-P_{\bar{\lambda}_n}(w_0)\right| \\
&\quad\leq \frac{P_{\bar{\lambda}_n}(w_0)\left(1-P_{\bar{\lambda}_n}(w_0)\right)}{p_{\bar{\lambda}_n}(w_0+1)}\sum_{i=1}^{n}\sum_{k}\frac{k(k-1)}{r_{n,i}+k-1}C_{r_{n,i}+k-1}^{k}q_{n,i}^{k}p_{n,i}^{r_{n,i}}\min\left\{\frac{1}{w_0+1},\frac{1}{k}\right\} \\
&\quad\leq \frac{P_{\bar{\lambda}_n}(w_0)\left(1-P_{\bar{\lambda}_n}(w_0)\right)}{p_{\bar{\lambda}_n}(w_0+1)}\sum_{i=1}^{n}\min\left\{\alpha_i, \frac{\beta_i-\alpha_i}{w_0+1}\right\},
\end{aligned}$$

with $\alpha_i=1-p_{n,i}^{r_{n,i}}-r_{n,i}q_{n,i}p_{n,i}^{r_{n,i}}$, $\beta_i=r_{n,i}\left(p_{n,i}^{-r_{n,i}}-1-r_{n,i}q_{n,i}p_{n,i}^{r_{n,i}}\right)$.

Hence, the theorem is proved. □

Remark 4 In the same way as in Remarks 3, we notice that (7) is a special case of Theorem 4 with $r_{n,i} = 1$; $n = 1, 2, \ldots$; $i = 1, 2, \ldots n$.

Conclusions

We conclude this paper with the following comments. The received results in this paper are extensions and generalizations of results in Teerapabolarn and Wongkasem (2007), Teerapabolarn (2009, 2013). The results would be more interesting and valuable if the discussed negative-binomial random variables in this paper are dependent. We shall take this up in the next study.

Authors' contributions

All authors contributed equally and significantly to this work. All authors drafted the manuscript. Both authors read and approved the final manuscript.

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Competing interests

The authors declare that they have no competing interests.

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