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Properties of higher-order half-linear functional differential equations with noncanonical operators

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Abstract
Some new results are presented for the oscillatory and asymptotic behavior of higher-order half-linear differential equations with a noncanonical operator. We study the delayed and advanced equations subject to various conditions.
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1 Introduction

Over the past few years, there has been much research activity concerning the oscillation and asymptotic behavior of various classes of differential equations; we refer the reader to [1–36] and the references cited therein. Half-linear differential equations occur in a variety of real world problems such as in the study of p-Laplace equations, non-Newtonian fluid theory, and the turbulent flow of a polytrophic gas in a porous medium; see the related background details reported in [5]. Many authors have studied the properties of solutions of the higher-order differential equation

$$Lx + q(t)x^{\beta}\big(\tau(t)\big) = 0, \qquad Lx := \big(r\big(x^{(n-1)}\big)^{\alpha}\big)'(t). \tag{1.1}$$

The operator Lx is said to be in canonical form if $\int_{t_0}^{\infty} r^{-1/\alpha}(t)\,\mathrm{d}t = \infty$; otherwise, it is called noncanonical. Throughout the paper, we assume that α and β are ratios of odd positive integers, $r \in \mathrm{C}^1[t_0,\infty)$, $r(t) > 0$, $r'(t) \geq 0$, $q, \tau \in \mathrm{C}[t_0,\infty)$, $q(t) > 0$, and $\lim_{t\to\infty} \tau(t) = \infty$.

Agarwal *et al.* [6] established a criterion for the existence of bounded solutions of (1.1) under the assumptions that n is even, $\int_{t_0}^{\infty} q(t)\,\mathrm{d}t = \infty$, and

$$\int_{t_0}^{\infty} r^{-1/\alpha}(t)\,\mathrm{d}t < \infty. \tag{1.2}$$

Zhang *et al.* [34, 36] obtained some results on asymptotic behavior of (1.1) in the case where (1.2) holds, $\tau(t) < t$, and $\beta \leq \alpha$. In [34, 36], an unsolved problem can be formulated as follows.

(P) Is it possible to establish asymptotic criteria for equation (1.1) in the case where $\beta \geq \alpha$?

As a special case when $\alpha = 1$ and $n = 2$, equation (1.1) becomes

$$\big(rx'\big)'(t) + q(t)x^{\beta}\big(\tau(t)\big) = 0. \tag{1.3}$$

Li *et al.* [24] established the following criterion for (1.3).

Theorem 1.1 (See [24, Theorem 2.1]) *Let* (1.2) *hold with* $\alpha = 1$, $\beta \geq 1$, $\tau(t) \leq t$, *and* $\tau'(t) > 0$ *for all* $t \geq t_0$. *Assume that there exists a function* $\rho \in \mathrm{C}^1([t_0,\infty),\mathbb{R})$ *with* $\rho(t) \geq t$ *and* $\rho'(t) > 0$ *such that, for all sufficiently large* t_1 *and for all positive constants* M *and* L,

$$\int^{\infty}\left[q(t)R^{\beta}\big(\tau(t)\big) - \frac{\beta M^{1-\beta}\tau'(t)R^{\beta-1}(\tau(t))}{r(\tau(t))\int_{t_1}^{t}\frac{\tau'(s)}{r(\tau(s))}\,\mathrm{d}s}\right]\mathrm{d}t = \infty$$

and

$$\int^{\infty}\left[q(t)\xi^{\beta}(t) - \frac{\beta\rho'(t)}{L^{\beta-1}\xi(t)r(\rho(t))}\right]\mathrm{d}t = \infty, \tag{1.4}$$

where $R(t) := \int_{t_0}^{t} r^{-1}(s)\,\mathrm{d}s$ *and* $\xi(t) := \int_{\rho(t)}^{\infty} r^{-1}(s)\,\mathrm{d}s$. *Then* (1.3) *is oscillatory.*

The purpose of this paper is to solve question (P) and to improve Theorem 1.1. By a solution of equation (1.1) we mean a function $x \in \mathrm{C}^{n-1}[T_x,\infty)$, $T_x \geq t_0$, which has the property $r(x^{(n-1)})^{\alpha} \in \mathrm{C}^1[T_x,\infty)$ and satisfies (1.1) on $[T_x,\infty)$. We consider only the solutions satisfying $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$ and tacitly assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

2 Main results

In the sequel, all functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough. We use the notation $\delta(t) := \int_{t}^{\infty} r^{-1/\alpha}(s)\,\mathrm{d}s$ and $(\rho'(t))_+ := \max\{0,\rho'(t)\}$.

Theorem 2.1 *Assume* (1.2) *and let* $n \geq 2$, $\beta \geq \alpha$, *and* $\tau(t) < t$ *for all* $t \geq t_0$. *Further, assume that the differential equation*

$$y'(t) + q(t)\left(\frac{\lambda_0\tau^{n-1}(t)}{(n-1)!r^{1/\alpha}(\tau(t))}\right)^{\beta} y^{\beta/\alpha}\big(\tau(t)\big) = 0 \tag{2.1}$$

is oscillatory for some constant $\lambda_0 \in (0,1)$. *If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\big(M\delta\big(\tau(s)\big)\big)^{\beta-\alpha}q(s)\left(\frac{\lambda_1}{(n-2)!}\tau^{n-2}(s)\right)^{\beta}\delta^{\alpha}(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{\delta(s)r^{1/\alpha}(s)}\right]\mathrm{d}s = \infty \tag{2.2}$$

holds for some constant $\lambda_1 \in (0,1)$ *and for all constants* $M > 0$, *then every solution of* (1.1) *is oscillatory or tends to zero as* $t \to \infty$.

Proof Assume that (1.1) has a nonoscillatory solution x. Without loss of generality, we may assume that x is eventually positive. Moreover, suppose that $\lim_{t\to\infty} x(t) \neq 0$. It follows from (1.1) that there exist two possible cases:

(1) $x(t) > 0$, $x^{(n-1)}(t) > 0$, $x^{(n)}(t) < 0$, $(r(x^{(n-1)})^\alpha)'(t) < 0$;

(2) $x(t) > 0$, $x^{(n-2)}(t) > 0$, $x^{(n-1)}(t) < 0$, $(r(x^{(n-1)})^\alpha)'(t) < 0$

for $t \geq t_1$, where $t_1 \geq t_0$ is large enough.

Assume that case (1) holds. From [36, Lemma 2.1], we have

$$x(t) \geq \frac{\lambda t^{n-1}}{(n-1)! r^{1/\alpha}(t)} \bigl(r^{1/\alpha} x^{(n-1)}\bigr)(t) \tag{2.3}$$

for every $\lambda \in (0,1)$ and for all sufficiently large t. Hence by (1.1), we see that $y := r(x^{(n-1)})^\alpha$ is a positive solution of the differential inequality

$$y'(t) + q(t)\left(\frac{\lambda \tau^{n-1}(t)}{(n-1)! r^{1/\alpha}(\tau(t))}\right)^\beta y^{\beta/\alpha}\bigl(\tau(t)\bigr) \leq 0.$$

Using [28, Theorem 1], we see that equation (2.1) also has a positive solution, which is a contradiction.

Assume that case (2) holds. Define the function w by

$$w(t) := \frac{r(t)(x^{(n-1)})^\alpha(t)}{(x^{(n-2)})^\alpha(t)}, \quad t \geq t_1. \tag{2.4}$$

Then $w(t) < 0$ for $t \geq t_1$. Noting that $r(x^{(n-1)})^\alpha$ is decreasing, we have

$$r^{1/\alpha}(s)x^{(n-1)}(s) \leq r^{1/\alpha}(t)x^{(n-1)}(t), \quad s \geq t \geq t_1.$$

Dividing the above inequality by $r^{1/\alpha}(s)$ and integrating the resulting inequality from t to l, we obtain

$$x^{(n-2)}(l) \leq x^{(n-2)}(t) + r^{1/\alpha}(t)x^{(n-1)}(t)\int_t^l \frac{\mathrm{d}s}{r^{1/\alpha}(s)}.$$

Letting $l \to \infty$, we get

$$x^{(n-2)}(t) \geq -r^{1/\alpha}(t)x^{(n-1)}(t)\delta(t), \tag{2.5}$$

which yields

$$-\frac{r^{1/\alpha}(t)x^{(n-1)}(t)}{x^{(n-2)}(t)}\delta(t) \leq 1.$$

Thus, by (2.4), we see that

$$-w(t)\delta^\alpha(t) \leq 1. \tag{2.6}$$

Differentiating (2.4), we have

$$w'(t) = \frac{(r(x^{(n-1)})^\alpha)'(t)}{(x^{(n-2)})^\alpha(t)} - \alpha\frac{r(t)(x^{(n-1)})^{\alpha+1}(t)}{(x^{(n-2)})^{\alpha+1}(t)}.$$

It follows from (1.1) and (2.4) that

$$w'(t) = -q(t)\frac{x^{\beta}(\tau(t))}{(x^{(n-2)})^{\alpha}(t)} - \alpha\frac{w^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}(t)}. \tag{2.7}$$

By virtue of (2.5), we have

$$\left(\frac{x^{(n-2)}}{\delta}\right)'(t) \geq 0. \tag{2.8}$$

On the other hand, by [36, Lemma 2.1], we get

$$x(t) \geq \frac{\lambda}{(n-2)!}t^{n-2}x^{(n-2)}(t) \tag{2.9}$$

for every $\lambda \in (0,1)$ and for all sufficiently large t. Then from (2.7), (2.8), and (2.9), there exists a constant $M > 0$ such that

$$\begin{aligned} w'(t) &= -q(t)\frac{x^{\beta}(\tau(t))}{(x^{(n-2)}(\tau(t)))^{\beta}}\left(x^{(n-2)}\left(\tau(t)\right)\right)^{\beta-\alpha}\frac{(x^{(n-2)}(\tau(t)))^{\alpha}}{(x^{(n-2)}(t))^{\alpha}} - \alpha\frac{w^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}(t)} \\ &\leq -\left(M\delta\left(\tau(t)\right)\right)^{\beta-\alpha}q(t)\left(\frac{\lambda}{(n-2)!}\tau^{n-2}(t)\right)^{\beta} - \alpha\frac{w^{(\alpha+1)/\alpha}(t)}{r^{1/\alpha}(t)}. \end{aligned} \tag{2.10}$$

Multiplying (2.10) by $\delta^{\alpha}(t)$ and integrating the resulting inequality from t_1 to t, we have

$$\begin{aligned} &\delta^{\alpha}(t)w(t) - \delta^{\alpha}(t_1)w(t_1) + \alpha\int_{t_1}^{t} r^{-1/\alpha}(s)\delta^{\alpha-1}(s)w(s)\,\mathrm{d}s \\ &\quad + \int_{t_1}^{t}\left(M\delta\left(\tau(s)\right)\right)^{\beta-\alpha}q(s)\left(\frac{\lambda}{(n-2)!}\tau^{n-2}(s)\right)^{\beta}\delta^{\alpha}(s)\,\mathrm{d}s \\ &\quad + \alpha\int_{t_1}^{t}\frac{w^{(\alpha+1)/\alpha}(s)}{r^{1/\alpha}(s)}\delta^{\alpha}(s)\,\mathrm{d}s \leq 0. \end{aligned}$$

Set $B := r^{-1/\alpha}(s)\delta^{\alpha-1}(s)$, $A := \delta^{\alpha}(s)/r^{1/\alpha}(s)$, and $v := -w(s)$. Using (2.6) and the inequality

$$Av^{(\alpha+1)/\alpha} - Bv \geq -\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \quad A > 0, \tag{2.11}$$

we have

$$\begin{aligned} &\int_{t_1}^{t}\left[\left(M\delta\left(\tau(s)\right)\right)^{\beta-\alpha}q(s)\left(\frac{\lambda}{(n-2)!}\tau^{n-2}(s)\right)^{\beta}\delta^{\alpha}(s) \right. \\ &\quad \left. - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{1}{\delta(s)r^{1/\alpha}(s)}\right]\mathrm{d}s \leq \delta^{\alpha}(t_1)w(t_1) + 1, \end{aligned}$$

which contradicts (2.2). This completes the proof. □

Applying the result of [30] to equation (2.1), we have the following result due to Theorem 2.1.

Corollary 2.2 *Assume* (1.2) *and let* $n \geq 2$, $\beta > \alpha$, $\tau(t) < t$, *and* $\tau'(t) > 0$ *for all* $t \geq t_0$. *Moreover, assume that there exists a continuously differentiable function* φ *such that*

$$\varphi'(t) > 0 \quad \textit{and} \quad \lim_{t\to\infty} \varphi(t) = \infty, \tag{2.12}$$

$$\limsup_{t\to\infty} \frac{\varphi'(\tau(t))\tau'(t)}{\varphi'(t)} < \frac{\alpha}{\beta}, \tag{2.13}$$

and

$$\liminf_{t\to\infty} \frac{q(t)(\frac{\tau^{n-1}(t)}{r^{1/\alpha}(\tau(t))})^{\beta} \mathrm{e}^{-\varphi(t)}}{\varphi'(t)} > 0. \tag{2.14}$$

If (2.2) *holds for some constant* $\lambda_1 \in (0,1)$ *and for all constants* $M > 0$, *then every solution of* (1.1) *is oscillatory or tends to zero as* $t \to \infty$.

In the following, we establish some results for (1.1) when $n \geq 2$ is even.

Theorem 2.3 *Assume* (1.2) *and let* $n \geq 2$ *be even*, $\beta \geq \alpha$, $\tau'(t) > 0$, *and* $\tau(t) \leq t$ *for all* $t \geq t_0$. *Further, assume that there exists a function* $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[K^{\beta-\alpha} q(s)\rho(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(2(n-2)!)^{\alpha} r(s)((\rho'(s))_+)^{\alpha+1}}{(\theta_1 \tau'(s)\tau^{n-2}(s)\rho(s))^{\alpha}} \right] \mathrm{d}s = \infty \tag{2.15}$$

holds for all constants $\theta_1 \in (0,1)$ *and* $K > 0$. *If* (2.2) *holds for some constant* $\lambda_1 \in (0,1)$ *and for all constants* $M > 0$, *then every solution of* (1.1) *is oscillatory or tends to zero as* $t \to \infty$.

Proof Assume that (1.1) has a nonoscillatory solution x. Without loss of generality, we may assume that x is eventually positive. Moreover, suppose that $\lim_{t\to\infty} x(t) \neq 0$. It follows from (1.1) that there exist two possible cases (1) and (2) (as those of the proof of Theorem 2.1).

Assume that case (1) holds. From [4, Lemma 2.1], we see that $x'(t) > 0$ for $t \geq t_1$. Define the function u by

$$u(t) := \rho(t) \frac{r(t)(x^{(n-1)})^{\alpha}(t)}{x^{\alpha}(\tau(t)/2)}, \quad t \geq t_1. \tag{2.16}$$

Then $u(t) > 0$ for $t \geq t_1$ and

$$u'(t) = \frac{\rho'(t)}{\rho(t)} u(t) + \rho(t) \frac{(r(x^{(n-1)})^{\alpha})'(t)}{x^{\alpha}(\tau(t)/2)} - \alpha \frac{\rho(t)\tau'(t)}{2} \frac{r(t)(x^{(n-1)})^{\alpha}(t) x'(\tau(t)/2)}{x^{\alpha+1}(\tau(t)/2)}. \tag{2.17}$$

From [4, Lemma 2.2], there exist a $t_2 \geq t_1$ and a constant θ_1 with $0 < \theta_1 < 1$ such that

$$x'\bigl(\tau(t)/2\bigr) \geq \frac{\theta_1}{(n-2)!} \tau^{n-2}(t) x^{(n-1)}(t) \tag{2.18}$$

for all $t \geq t_2$. It follows from (1.1), (2.16), (2.17), and (2.18) that

$$u'(t) \leq \frac{\rho'(t)}{\rho(t)} u(t) - q(t)\rho(t) \frac{x^{\beta}(\tau(t))}{x^{\alpha}(\tau(t)/2)} - \frac{\alpha\tau'(t)}{2} \frac{\theta_1}{(n-2)!} \tau^{n-2}(t) \frac{u^{(\alpha+1)/\alpha}(t)}{(\rho(t)r(t))^{1/\alpha}}. \tag{2.19}$$

Using $x' > 0$ and (2.19), we get

$$u'(t) \le \frac{(\rho'(t))_+}{\rho(t)} u(t) - K^{\beta-\alpha} q(t)\rho(t) - \frac{\alpha\tau'(t)}{2} \frac{\theta_1}{(n-2)!} \tau^{n-2}(t) \frac{u^{(\alpha+1)/\alpha}(t)}{(\rho(t)r(t))^{1/\alpha}} \tag{2.20}$$

for some constant $K > 0$. Set

$$A := \frac{\alpha\tau'(t)}{2} \frac{\theta_1}{(n-2)!} \frac{\tau^{n-2}(t)}{(\rho(t)r(t))^{1/\alpha}}, \qquad B := \frac{(\rho'(t))_+}{\rho(t)}, \quad \text{and} \quad v := u(t).$$

Using inequality (2.11), we obtain

$$\frac{(\rho'(t))_+}{\rho(t)} u(t) - \frac{\alpha\tau'(t)}{2} \frac{\theta_1}{(n-2)!} \tau^{n-2}(t) \frac{u^{(\alpha+1)/\alpha}(t)}{(\rho(t)r(t))^{1/\alpha}} \le \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(2(n-2)!)^\alpha r(t)((\rho'(t))_+)^{\alpha+1}}{(\theta_1\tau'(t)\tau^{n-2}(t)\rho(t))^\alpha}.$$

Substituting the last inequality into (2.20), we get

$$u'(t) \le -K^{\beta-\alpha} q(t)\rho(t) + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(2(n-2)!)^\alpha r(t)((\rho'(t))_+)^{\alpha+1}}{(\theta_1\tau'(t)\tau^{n-2}(t)\rho(t))^\alpha}. \tag{2.21}$$

Integrating (2.21) from t_2 to t, we have

$$\int_{t_2}^{t} \left[K^{\beta-\alpha} q(s)\rho(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(2(n-2)!)^\alpha r(s)((\rho'(s))_+)^{\alpha+1}}{(\theta_1\tau'(s)\tau^{n-2}(s)\rho(s))^\alpha} \right] \mathrm{d}s \le u(t_2),$$

which contradicts (2.15). Assume that case (2) holds. Proceeding as in the proof of Theorem 2.1, we can obtain a contradiction to (2.2). This completes the proof. □

Next we establish a result for (1.1) when $n = 2$.

Theorem 2.4 *Assume* (1.2) *and let* $n = 2$, $\beta \ge \alpha$, $\tau'(t) > 0$, *and* $\tau(t) \le t$ *for* $t \ge t_0$. *Further, assume that there exists a function* $\rho \in \mathrm{C}^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[K^{\beta-\alpha} q(s)\rho(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(s)((\rho'(s))_+)^{\alpha+1}}{(\tau'(s)\rho(s))^\alpha} \right] \mathrm{d}s = \infty \tag{2.22}$$

for all constants $K > 0$. *If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\bigl(M\delta(\tau(s))\bigr)^{\beta-\alpha} q(s)\delta^\alpha(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{1}{\delta(s)r^{1/\alpha}(s)} \right] \mathrm{d}s = \infty \tag{2.23}$$

holds for all constants $M > 0$, *then* (1.1) *is oscillatory.*

Proof Assume that (1.1) has a nonoscillatory solution x. Without loss of generality, we may assume that x is eventually positive. It follows from (1.1) that there exist two possible cases

(1) and (2) with $n = 2$ (as those of the proof of Theorem 2.1). Assume that case (1) holds. Define

$$u(t) := \rho(t)\frac{r(t)(x'(t))^{\alpha}}{x^{\alpha}(\tau(t))}, \quad t \geq t_1.$$

The rest of the proof is similar to that of Theorem 2.3, and so is omitted. Assume that case (2) holds. Similar as in the proof of Theorem 2.1, we can obtain a contradiction to (2.23). This completes the proof. □

Next we establish some oscillation criteria for (1.1) when $n \geq 2$ is even and $\tau(t) > t$ for all $t \geq t_0$.

Theorem 2.5 *Assume* (1.2) *and let* $n \geq 2$ *be even*, $\beta > \alpha$, *and* $\tau(t) > t$ *for all* $t \geq t_0$. *Further, assume that there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[K^{\beta-\alpha}q(s)\rho(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(2(n-2)!)^{\alpha}r(s)((\rho'(s))_+)^{\alpha+1}}{(\theta_1 s^{n-2}\rho(s))^{\alpha}}\right]\mathrm{d}s = \infty \tag{2.24}$$

holds for all constants $\theta_1 \in (0,1)$ *and* $K > 0$. *If*

$$\int^{\infty} q(t)\delta^{\beta}\bigl(\tau(t)\bigr)\bigl(\tau^{n-2}(t)\bigr)^{\beta}\,\mathrm{d}t = \infty, \tag{2.25}$$

then every solution of (1.1) *is oscillatory or tends to zero as* $t \to \infty$.

Proof Assume that (1.1) has a nonoscillatory solution x. Without loss of generality, we may assume that x is eventually positive. Moreover, suppose that $\lim_{t\to\infty} x(t) \neq 0$. It follows from (1.1) that there exist two possible cases (1) and (2) (as those of the proof of Theorem 2.1).

Assume that case (1) holds. From [4, Lemma 2.1], we see that $x'(t) > 0$ for $t \geq t_1$. Define the function u by

$$u(t) := \rho(t)\frac{r(t)(x^{(n-1)})^{\alpha}(t)}{x^{\alpha}(t/2)}, \quad t \geq t_1.$$

Then $u(t) > 0$ for $t \geq t_1$ and

$$u'(t) = \frac{\rho'(t)}{\rho(t)}u(t) + \rho(t)\frac{(r(x^{(n-1)})^{\alpha})'(t)}{x^{\alpha}(t/2)} - \alpha\frac{\rho(t)}{2}\frac{r(t)(x^{(n-1)})^{\alpha}(t)x'(t/2)}{x^{\alpha+1}(t/2)}.$$

From [4, Lemma 2.2], there exist a $t_2 \geq t_1$ and a constant θ_1 with $0 < \theta_1 < 1$ such that

$$x'(t/2) \geq \frac{\theta_1}{(n-2)!}t^{n-2}x^{(n-1)}(t)$$

for all $t \geq t_2$. Thus

$$u'(t) \leq \frac{\rho'(t)}{\rho(t)}u(t) - q(t)\rho(t)\frac{x^{\beta}(\tau(t))}{x^{\alpha}(t/2)} - \frac{\alpha}{2}\frac{\theta_1}{(n-2)!}t^{n-2}\frac{u^{(\alpha+1)/\alpha}(t)}{(\rho(t)r(t))^{1/\alpha}}.$$

Similar as in the proof of Theorem 2.3, we can get a contradiction to (2.24). Assume that case (2) holds. We have (2.5) and (2.9) for every $\lambda \in (0,1)$ and for all sufficiently large t. Thus, we get by (1.1), (2.5), and (2.9) that

$$\left(r\left(x^{(n-1)}\right)^{\alpha}\right)'(t) - q(t)\left(\frac{\lambda}{(n-2)!}\tau^{n-2}(t)\delta\left(\tau(t)\right)\right)^{\beta}\left(r^{1/\alpha}x^{(n-1)}\right)^{\beta}\left(\tau(t)\right) \le 0.$$

Let $u := r(x^{(n-1)})^{\alpha}$. Then $y := -u > 0$ is a solution of the advanced inequality

$$y'(t) - q(t)\left(\frac{\lambda}{(n-2)!}\tau^{n-2}(t)\delta\left(\tau(t)\right)\right)^{\beta} y^{\beta/\alpha}\left(\tau(t)\right) \ge 0.$$

It follows from [8, Lemma 2.3] that the corresponding advanced differential equation

$$y'(t) - q(t)\left(\frac{\lambda}{(n-2)!}\tau^{n-2}(t)\delta\left(\tau(t)\right)\right)^{\beta} y^{\beta/\alpha}\left(\tau(t)\right) = 0$$

has an eventually positive solution. Using condition (2.25) and [22, Theorem 1], one can obtain a contradiction. This completes the proof. □

Finally, we establish a result for (1.1) when $n = 2$.

Theorem 2.6 *Assume* (1.2) *and let* $n = 2$, $\beta > \alpha$, *and* $\tau(t) > t$ *for* $t \ge t_0$. *Further, assume that there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[K^{\beta-\alpha}q(s)\rho(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)((\rho'(s))_+)^{\alpha+1}}{\rho^{\alpha}(s)}\right]\mathrm{d}s = \infty \tag{2.26}$$

for all constants $K > 0$. *If*

$$\int^{\infty} q(t)\delta^{\beta}\left(\tau(t)\right)\mathrm{d}t = \infty, \tag{2.27}$$

then (1.1) *is oscillatory.*

Proof Assume that (1.1) has a nonoscillatory solution x. Without loss of generality, we may assume that x is eventually positive. It follows from (1.1) that there exist two possible cases (1) and (2) with $n = 2$ (as those of the proof of Theorem 2.1). Assume that case (1) holds. Define

$$u(t) := \rho(t)\frac{r(t)(x'(t))^{\alpha}}{x^{\alpha}(t)}, \quad t \ge t_1.$$

The rest of the proof is similar to that of Theorem 2.3, and so is omitted. Assume that case (2) holds. Similar as in the proof of Theorem 2.5, we can obtain a contradiction to (2.27). This completes the proof. □

3 Examples and discussions

In the following, we illustrate possible applications with two examples.

Example 3.1 For $t \geq 1$, consider the second-order delay differential equation

$$\bigl(e^{t}x'(t)\bigr)' + e^{5t/2}x^{3}\left(\frac{t}{2}\right) = 0. \tag{3.1}$$

Let $\alpha = 1$, $\beta = 3$, and $\rho(t) = 1$. Note that $\delta(t) = e^{-t}$. Using Theorem 2.4, equation (3.1) is oscillatory. It is not difficult to see that Theorem 1.1 fails to apply due to condition (1.4).

Example 3.2 For $t \geq 1$, consider the second-order advanced differential equation

$$\bigl(e^{t}x'(t)\bigr)' + \frac{e^{6t}}{t}x^{3}(2t) = 0. \tag{3.2}$$

Let $\alpha = 1$, $\beta = 3$, and $\rho(t) = 1$. Note that $\delta(t) = e^{-t}$. Using Theorem 2.6, equation (3.2) is oscillatory.

In this paper, we suggested some new results on the oscillation and asymptotic behavior of differential equation (1.1). Theorem 2.1 can be applied in the odd-order and even-order equations.

We stress that the study of equation (1.1) in the case (1.2) brings additional difficulties. Since the sign of $x^{(n-1)}$ is not known, our criteria include a pair of assumptions; see, *e.g.*, (2.2) and (2.15). We utilized two different methods (Riccati substitution and comparison method) to deal with the cases $\tau(t) \leq t$ and $\tau(t) > t$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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