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The homogeneity theorem for supergravity backgrounds II: the six-dimensional theories

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Abstract: We prove that supersymmetric backgrounds of (1,0) and (2,0) six-dimensional supergravity theories preserving more than one half of the supersymmetry are locally homogeneous. As a byproduct we also establish that the Killing spinors of such a background generate a Lie superalgebra.

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1 Introduction

The study of supergravity backgrounds, which had seen its first period of intense activity in the 1980s in the context of Kaluza-Klein supergravity (see [1] for a then timely review), was retaken in earnest in the mid-to-late 1990s ushered in by the dualities-and-branes paradigm in string theory and by the gauge/gravity correspondence. Although as a result of this continuing second effort a huge number of backgrounds are now known, it is fair to say that we know very little about all but a few small corners of the landscape of supergravity backgrounds. This is perhaps not surprising given our still incomplete knowledge about solutions to the much more venerable four-dimensional Einstein-Maxwell equations.

The emphasis during the 1980s was on Freund-Rubin backgrounds, in which the geometry (if not necessarily the fluxes) decomposes into the metric product of a four-dimensional spacetime and some internal (typically compact) manifold, but the interest nowadays has widened to backgrounds with an intricate and truly higher-dimensional geometry. One particularly interesting class of backgrounds, due to the crucial role they play in the gauge/gravity correspondence, comprises those backgrounds preserving a substantial amount of the supersymmetry of the theory. In fact, the fraction of supersymmetry preserved by a background has proved to be a very useful organising principle in our efforts to tame the zoo of supergravity backgrounds, despite being a rather coarse invariant. Finer invariants, such as the holonomy group of the gravitino connection, are much harder to calculate and hence have yet to play a decisive role in the various classification efforts underway.

One particularly attractive classification problem is that of backgrounds which preserve a large fraction of the supersymmetry. In higher dimensions it is possible to make some limited progress by working one's way down from the top; i.e., classifying maximally or near-maximally supersymmetric backgrounds, but in order to make real progress in that classification, new ideas seem to be required. Based on the increasing number of known backgrounds, Patrick Meessen (in a private communication to the senior author in 2004) observed that $> \frac{1}{2}$-BPS backgrounds — i.e., those preserving more than half of the supersymmetry — were homogeneous; that is, that the Lie group of flux-preserving isometries of such a background acts transitively on the underlying manifold. He conjectured that this was always the case and after some initial partial results [2, 3], a local version of the conjecture was recently demonstrated for ten- and eleven-dimensional supergravity theories in [4]. In principle this "reduces" the classification problem of $> \frac{1}{2}$-BPS backgrounds to those backgrounds which are homogeneous. Alas, this is still a daunting task; although progress can be made.

The purpose of this paper is to extend the homogeneity theorem in [4] to two six-dimensional supergravity theories: $(1,0)$ and $(2,0)$. Other possible supergravity theories in six dimensions are either not yet constructed or can be obtained by dimensional reduction (without truncation) from higher-dimensional theories, in which case the homogeneity theorem follows by general arguments, as will be explained in a forthcoming paper.

For the $(1,0)$ theory, it was shown in [5] that backgrounds preserve either all, half or none of the supersymmetry. Therefore if a background preserves more than half of the supersymmetry, it must be maximally supersymmetric and they are known to be homogeneous. Indeed, as shown in [6], such backgrounds are lorentzian Lie groups with bi-invariant metrics and invariant 3-form. In this paper we give a different proof of this result which has the virtue of not requiring the classification (hence we prove a "Theorem" instead of a "theorem", in the nomenclature of Victor Kac) and which additional results in the construction of the Killing superalgebra of the background. We are not aware of similar results for the $(2,0)$ theory beyond the classification of maximally supersymmetric backgrounds in [6].

The proof of the homogeneity theorem in [4] consists of two steps. The first step is to show that the natural squaring map from spinor fields to vector fields, when applied to the Killing spinors of the background, yields Killing vectors which also preserve the fluxes. With a little extra effort, and because it is an interesting result in its own right, one also shows that the Killing spinors generate a Lie superalgebra, called the Killing superalgebra of the background — a more refined invariant than the fraction of supersymmetry, which only measures the dimension of the odd subspace.

The second step is purely algebraic and consists in proving the surjectivity of the squaring map restricted to any subspace of dimension greater than one half the rank of the relevant spinor bundle. This uses the fact that the vector obtained by squaring a spinor is causal. Applied to the space of Killing spinors of $> \frac{1}{2}$-BPS backgrounds, it guarantees that the tangent space at every point can be spanned by vector fields in the image of the squaring map, which by the first step are known to be infinitesimal symmetries of the background. This proves the *local* homogeneity of the background.

The paper is organised as follows. In section 2 we treat the $(1,0)$ theory, with eight real supercharges. We introduce the relevant notion of Killing spinor and show that they generate a Lie superalgebra. We then show that if the space of Killing spinors has dimension greater than four, then the background is locally homogeneous. In section 3 we do the same for the $(2,0)$ theory, with sixteen real supercharges. The calculations here are more complicated due to the presence of R-symmetry generators in the definition of Killing spinors. Again we find that the Killing spinors generate a Lie superalgebra and when its odd subspace has dimension greater than eight, the background is locally homogeneous. Finally, in appendix A, we collect the basic facts about spinors in six dimensions which are used in the bulk of the paper: pinor, spinor and R-symmetry representations, the invariant inner products, explicit matrix realisations and some useful consequences of the Clifford relations, including the relevant Fierz identities.

2 Six-dimensional (1,0) supergravity

Let (M, g, H) be a bosonic background of six-dimensional $(1,0)$ supergravity [7]. This means that (M, g) is a connected six-dimensional lorentzian spin manifold and $H \in \Omega^3_-(M)$ a closed anti-selfdual three-form and they satisfy the field equations of the theory with fermions equal to zero. We shall not need the equations in the following.

Let S_+ denote the positive-chirality spinor representation of $\mathrm{Spin}(5,1)$. It is a two-dimensional quaternionic representation, but we prefer to work with complex representations, whence we will think of S_+ as a four-dimensional complex representation with an invariant quaternionic structure; that is, with a complex antilinear map $J : S_+ \to S_+$ which obeys $J^2 = -1$.

Similarly, the fundamental representation S_1 of the R-symmetry group of $(1,0)$ supergravity, which is isomorphic to $\mathrm{Sp}(1)$, is a two-dimensional complex representation with an invariant quaternionic structure $j : S_1 \to S_1$.

The tensor product $S_+ \otimes_{\mathbb{C}} S_1$ of these two representations is an 8-dimensional complex representation with an invariant conjugation given by $J \otimes j$, whence it is a complex representation of real type. In other words, it is the complexification of a real representation $\mathcal{S}_+$, defined by

$$S_+ \otimes_{\mathbb{C}} S_1 \cong \mathcal{S}_+ \otimes_{\mathbb{R}} \mathbb{C} \; . \tag{2.1}$$

The real representation $\mathcal{S}_+$, which is the real subspace of $S_+ \otimes_{\mathbb{C}} S_1$ fixed under the conjugation, is eight-dimensional and is the relevant spinorial representation for this supergravity theory. With some abuse of language we will also denote by $\mathcal{S}_+$ the spinor bundle on M associated to this representation.

Let $\mathcal{S}_-$ be the real eight-dimensional representation defined as $\mathcal{S}_+$ but starting from the negative-chirality spinor representation S_- of $\mathrm{Spin}(5,1)$. As shown in section A.4 in

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the appendix, there is a $(\mathrm{Spin}(5,1)\times\mathrm{Sp}(1))$-invariant symplectic structure on $\mathcal{S}=\mathcal{S}_+\oplus\mathcal{S}_-$, denoted by $\langle -,-\rangle$, and satisfying

$$\langle\Gamma_a\varepsilon_1,\varepsilon_2\rangle=-\langle\varepsilon_1,\Gamma_a\varepsilon_2\rangle\ . \tag{2.2}$$

2.1 The Killing superalgebra

The supersymmetry variation of the gravitino defines a connection D on the bundle $\mathcal{S}_+$, defined for a spinor field $\varepsilon\in C^\infty(M;\mathcal{S}_+)$ and a vector field $X\in C^\infty(M;TM)$ by

$$D_X\varepsilon=\nabla_X\varepsilon+\frac{1}{4}\iota_X H\cdot\varepsilon\,, \tag{2.3}$$

where ∇ is the spin connection on $\mathcal{S}_+$ induced by the Levi-Civita connection, $\cdot$ is the Clifford action and ι_X denotes the contraction by the vector field X. Spinor fields which are parallel with respect to D are called *Killing spinors*. They form a real vector space $\mathfrak{g}_1$ whose dimension is at most the rank of $\mathcal{S}_+$, since a parallel section of a bundle over a connected manifold is uniquely determined by its value at any one point. For the theory in question, $\dim\mathfrak{g}_1\leq 8$. Once fixing a point $p\in M$, we will freely identify $\mathfrak{g}_1$ with a subspace of the fibre of $\mathcal{S}_+$ at p which we will in turn identify with the representation $\mathcal{S}_+$ itself. In other words, we will often think of $\mathfrak{g}_1$ as a subspace of $\mathcal{S}_+$.

Let $\mathfrak{g}_0$ denote the Lie algebra of Killing vector fields on (M,g) which preserve H. We will show that on $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ we can define the structure of a Lie superalgebra. This is by now a standard construction for supergravity theories [2, 3, 8].

The Lie superalgebra structure on $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ is a graded skew bilinear map $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, which unpacks into three bilinear maps:

1. a skewsymmetric bilinear map $[-,-]:\mathfrak{g}_0\times\mathfrak{g}_0\to\mathfrak{g}_0$, which is simply the Lie bracket of vector fields;

2. the action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$, which is a bilinear map $[-,-]:\mathfrak{g}_0\times\mathfrak{g}_1\to\mathfrak{g}_1$, defined by the spinorial Lie derivative (see, e.g., [9])

$$[X,\varepsilon]:=\mathscr{L}_X\varepsilon=\nabla_X\varepsilon-\rho(\nabla X)\varepsilon \tag{2.4}$$

   where ∇X is the skewsymmetric endomorphism of TM defined by $Y\mapsto\nabla_Y X$ and $\rho:\mathfrak{so}(TM)\to\mathrm{End}(\mathcal{S}_+)$ is the spin representation; and

3. a symmetric bilinear map $[-,-]:\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{g}_0$, whose restriction to the diagonal is called the *squaring map*: it is essentially the transpose of the Clifford action of vectors on spinors and will be defined presently.

These maps are then subject to the Jacobi identity, which unpacks into four components. As we will review presently, three of these components are automatically zero, but the fourth needs proof, which we provide.

The transpose of the Clifford action of TM on $\mathcal{S}$ under the metric g on TM and the symplectic structure on $\mathcal{S}$ defines a symmetric bilinear map $\mathcal{S}_+\times\mathcal{S}_+\to TM$. Being symmetric it is uniquely determined by its restriction to the diagonal, a quadratic map $\mathcal{S}_+ \to TM$, sending a spinor field ε to its *Dirac current* V_ε, defined by

$$g(V_\varepsilon, X) = \langle \varepsilon, X \cdot \varepsilon \rangle \ . \tag{2.5}$$

By the usual polarisation identity, we define the vector field $[\varepsilon_1, \varepsilon_2]$ corresponding to two spinor fields $\varepsilon_1, \varepsilon_2 \in C^\infty(M; \mathcal{S}_+)$ by

$$2[\varepsilon_1, \varepsilon_2] = V_{\varepsilon_1+\varepsilon_2} - V_{\varepsilon_1} - V_{\varepsilon_2} \ . \tag{2.6}$$

The first result is that when $\varepsilon_1, \varepsilon_2$ are Killing spinors, $[\varepsilon_1, \varepsilon_2]$ is a Killing vector which preserves the 3-form H. Clearly, it is enough to show this for the Dirac current of a Killing spinor ε. To see this, we first observe that a Killing spinor ε is parallel relative to a connection D defined by

$$D_\mu = \nabla_\mu - \frac{1}{2}\rho(H_\mu) \, , \tag{2.7}$$

where ρ is the spin representation applied to the skewsymmetric endomorphism H_μ of TM defined, relative to a pseudo-orthonormal basis e_a, by

$$H_\mu(e_a) = H_\mu{}^b{}_a e_b \ . \tag{2.8}$$

In other words, D is the spin connection corresponding to an affine connection also denoted D and defined by $D_\mu = \nabla_\mu - \frac{1}{2}H_\mu$. By covariance, if $D_\mu\varepsilon = 0$, then $D_\mu V_\varepsilon = 0$ as well. In other words, writing V for V_ε,

$$D_\mu V_\nu = \nabla_\mu V_\nu - \frac{1}{2} H_{\mu\nu\rho} V^\rho = 0 \, , \tag{2.9}$$

whence $\nabla_\mu V_\nu = \frac{1}{2} V^\rho H_{\rho\mu\nu}$. First of all, we see that $\nabla_\mu V_\nu = -\nabla_\nu V_\mu$, whence V is a Killing vector field. We also see that

$$dV^\flat = \frac{1}{2}\iota_V H \tag{2.10}$$

whence $d\iota_V H = 0$. Since $dH = 0$, this shows that $\mathscr{L}_V H = 0$, whence V preserves H. Then after polarisation we obtain a symmetric bilinear map $[-,-] : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$.

If $K \in \mathfrak{g}_0$ is any Killing vector field which preserves H, then the Lie derivative $\mathscr{L}_K$ leaves invariant the connection D:

$$\mathscr{L}_K D_X - D_X \mathscr{L}_K = D_{[K,X]} \ . \tag{2.11}$$

In turn, this means that $\mathscr{L}_K$ acting on spinors also leaves invariant the spin connection D, whence it sends Killing spinors to Killing spinors, defining a map $[-,-] = \mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ by $[K, \varepsilon] = \mathscr{L}_K \varepsilon$.

It now remains to prove the Jacobi identity for the bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ just defined on $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The only component of the Jacobi identity which needs to be checked is the $(\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1)$-component. This is given by a symmetric trilinear map $\mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ which again is determined uniquely via polarisation by the restriction to the diagonal: the map sending a Killing spinor ε to the Killing spinor $\mathscr{L}_{V_\varepsilon} \varepsilon$. We need to show that this is zero for all $\varepsilon \in \mathfrak{g}_1$.

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A quick calculation shows that

$$\mathscr{L}_{V_\varepsilon}\varepsilon = \iota_{V_\varepsilon}H \cdot \varepsilon\,, \tag{2.12}$$

whose vanishing, using the first equation (A.22), becomes

$$V_\varepsilon \cdot H \cdot \varepsilon + H \cdot V_\varepsilon \cdot \varepsilon = 0\,. \tag{2.13}$$

From Lemma 2, we know that $H \cdot \varepsilon = 0$, whence the first term vanishes. The second term will also vanish as a consequence of the following

Proposition 1. *Let $\varepsilon \in \mathcal{S}_+$ and V_ε its Dirac current. Then $V_\varepsilon \cdot \varepsilon = 0$.*

Proof. We have

$$V_\varepsilon \cdot \varepsilon = \langle \varepsilon, \Gamma^a \varepsilon\rangle\, \Gamma_a \varepsilon = \epsilon_{AB}\left(\varepsilon^A, \Gamma^a \varepsilon^B\right)\Gamma_a \varepsilon\,, \tag{2.14}$$

where we have expanded ε in terms of a symplectic basis for the fundamental representation S_1 of the R-symmetry group USp(2), as described in section A.7.1 of the appendix. But then by Lemma 3, it vanishes. □

In summary, on $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ we have the structure of a Lie superalgebra, called the *symmetry superalgebra* of the supersymmetric $(1,0)$ background (M, g, H). The ideal $\mathfrak{k} = [\mathfrak{g}_1, \mathfrak{g}_1] \oplus \mathfrak{g}_1$ generated by $\mathfrak{g}_1$ is called the *Killing superalgebra* of the background.

2.2 Homogeneity

We will now prove the strong version of the (local) homogeneity conjecture: that the even part of the Killing superalgebra already acts locally transitively on the background.

It follows from Proposition 1 that the Dirac current V_ε of a chiral spinor $\varepsilon \in \mathcal{S}_+$ is null:

$$g(V_\varepsilon, V_\varepsilon) = \langle \varepsilon, V_\varepsilon \cdot \varepsilon\rangle = 0\,. \tag{2.15}$$

The proof of homogeneity follows the same steps in [4], which we briefly review for the sake of completeness.

Let $\dim \mathfrak{g}_1 > 4 = \frac{1}{2}\dim \mathcal{S}_+$. We want to show that for each $p \in M$, the symmetric bilinear map $\varphi : \mathfrak{g}_1 \times \mathfrak{g}_1 \to T_pM$, obtained by sending the pair $(\varepsilon_1, \varepsilon_2)$ of Killing spinors to the tangent vector $[\varepsilon_1, \varepsilon_2](p)$ to M at p is surjective. Let $v \in T_pM$ be perpendicular to the image of φ; that is, to $[\varepsilon_1, \varepsilon_2](p)$ for all $\varepsilon_{1,2} \in \mathfrak{g}_1$. This means that for all $\varepsilon_{1,2} \in \mathfrak{g}_1$,

$$\langle \varepsilon_1, v \cdot \varepsilon_2\rangle = 0\,, \tag{2.16}$$

or that Clifford product by v maps $\mathfrak{g}_1$ to $\mathfrak{g}_1^\perp \subset \mathcal{S}_-$. Since $\dim \mathfrak{g}_1 > \dim \mathfrak{g}_1^\perp$, it follows that the Clifford product by v has nontrivial kernel and hence that v is null, since by the Clifford relation $v^2 \cdot \varepsilon = -g(v,v)\varepsilon$. Every vector which is perpendicular to the image of φ is therefore null and hence $(\operatorname{im}\varphi)^\perp \subset T_pM$ is an isotropic subspace. Since the isotropic subspaces of T_pM are at most one-dimensional, we have two possibilities: either φ is surjective or else $(\operatorname{im}\varphi)^\perp$ is one-dimensional and spanned by a null vector n, say. In this latter case, the Dirac current V_ε of every Killing spinor $\varepsilon \in \mathfrak{g}_1$ is a null vector perpendicular to n, whence

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it has to be proportional to n, otherwise they would span a two-dimensional isotropic subspace. But then by polarisation, every vector in the image of φ would be proportional to n, contradicting the fact that $\operatorname{im}\varphi$ has codimension one.

In summary, we have shown that the tangent space to M at any point p is spanned by the values at p of Killing vectors in $[\mathfrak{g}_1, \mathfrak{g}_1]$. This shows that the $(1,0)$ background (M, g, H) is locally homogeneous.

3 Six-dimensional (2,0) supergravity

A bosonic background (M, g, H) of $(2,0)$ supergravity [10, 11] consists of a connected 6-dimensional lorentzian spin manifold (M, g) and a closed anti-selfdual $\mathbb{V}$-valued three-form $H \in \Omega^3_-(M; \mathbb{V})$, where $\mathbb{V}$ is the real 5-dimensional orthogonal representation of the $\mathrm{USp}(4) \cong \mathrm{Spin}(5)$ R-symmetry group of the theory. We may choose an orthonormal basis e_i for $\mathbb{V}$ and hence think of $H = H^i e_i$ as five anti-selfdual three-forms $H^i \in \Omega^3_-(M)$. As in the $(1,0)$ theory, these fields are subject to the field equations of the theory with fermions put to zero, but we shall not need their explicit form in what follows.

As before, $S_\pm$ are the complex 8-dimensional irreducible spinor representations of $\mathrm{Spin}(5,1)$ and now S_2 denotes the fundamental representation of $\mathrm{USp}(4)$, which is complex and 4-dimensional. Both $S_\pm$ and S_2 have invariant quaternionic structures, whence their tensor product is a complex representation of $\mathrm{Spin}(5,1) \times \mathrm{USp}(4)$ of real type, whence the complexification of a sixteen-dimensional real representation $\mathscr{S}_\pm$. We will let $\mathscr{S} = \mathscr{S}_+ \oplus \mathscr{S}_-$, on which we have an action of $C\ell(5,1) \otimes C\ell(0,5)$ with generators Γ_a for $C\ell(5,1)$ and γ_i for $C\ell(0,5)$. As discussed in the appendix, $\mathscr{S}$ has a symplectic inner product $\langle -, - \rangle$ relative to which $\mathscr{S}_\pm$ are lagrangian subspaces and such that

$$\langle \varepsilon_1, \Gamma_a \varepsilon_2 \rangle = - \langle \Gamma_a \varepsilon_1, \varepsilon_2 \rangle \qquad \text{and} \qquad \langle \varepsilon_1, \gamma_i \varepsilon_2 \rangle = + \langle \gamma_i \varepsilon_1, \varepsilon_2 \rangle \ . \tag{3.1}$$

3.1 The Killing superalgebra

A *Killing spinor* of $(2,0)$ supergravity is a section ε of $\mathscr{S}_+$ which is parallel relative to a connection $\mathcal{D}$ defined by

$$\mathcal{D}_\mu \varepsilon = \nabla_\mu \varepsilon + \frac{1}{8} H^i_{\mu ab} \Gamma^{ab} \gamma_i \varepsilon \ . \tag{3.2}$$

The *Dirac current* V_ε of a spinor $\varepsilon \in C^\infty(M; \mathscr{S}_+)$ is the vector field defined by

$$g(V_\varepsilon, X) = \langle \varepsilon, X \cdot \varepsilon \rangle \ , \tag{3.3}$$

for all vector fields X. Its coefficients relative to an orthonormal frame are then given by $V^a_\varepsilon = \langle \varepsilon, \Gamma^a \varepsilon \rangle$.

As before, the Dirac current of a Killing spinor is a Killing vector which preserves H. Indeed, let ε be a Killing spinor and let $V = V_\varepsilon$ denote its Dirac current. Its covariant derivative is given by

$$
\begin{aligned}
\nabla_\mu V^\nu &= \nabla_\mu \langle \varepsilon, \Gamma^\nu \varepsilon\rangle \\
&= \langle \nabla_\mu \varepsilon, \Gamma^\nu \varepsilon\rangle + \langle \varepsilon, \Gamma^\nu \nabla_\mu \varepsilon\rangle \\
&= -\frac{1}{8} H^i_{\mu\rho\sigma} \left(\langle \Gamma^{\rho\sigma}\gamma_i \varepsilon, \Gamma^\nu \varepsilon\rangle + \langle \varepsilon, \Gamma^\nu \Gamma^{\rho\sigma}\gamma_i\rangle\right) \\
&= \frac{1}{8} H^i_{\mu\rho\sigma} \langle \gamma_i \varepsilon, [\Gamma^{\rho\sigma}, \Gamma^\nu]\varepsilon\rangle \\
&= \frac{1}{2} H^{i\,\nu}_{\mu\ \ \sigma} \langle \gamma_i \varepsilon, \Gamma^\sigma \varepsilon\rangle \ .
\end{aligned}
\tag{3.4}
$$

This can be rewritten as

$$
\nabla_\mu V_\nu = \frac{1}{2} H^i_{\mu\nu\rho} \langle \varepsilon, \Gamma^\rho \gamma_i \varepsilon\rangle \ , \tag{3.5}
$$

which shows that V is a Killing vector.

Let $\theta \in \Omega^1(M; \mathbb{V})$ be the $\mathbb{V}$-valued one-form defined by $\theta^i_\mu = \langle \varepsilon, \Gamma_\mu \gamma^i \varepsilon\rangle$. Its covariant derivative is given by

$$
\begin{aligned}
\nabla_\mu \theta^i_\nu &= \left\langle \nabla_\mu \varepsilon, \Gamma_\nu \gamma^i \varepsilon\right\rangle + \left\langle \varepsilon, \Gamma_\nu \gamma^i \nabla_\mu \varepsilon\right\rangle \\
&= -\frac{1}{8} H^j_{\mu\rho\sigma} \left(\left\langle \Gamma^{\rho\sigma}\gamma_j \varepsilon, \Gamma_\nu \gamma^i \varepsilon\right\rangle + \left\langle \varepsilon, \Gamma_\nu \Gamma^{\rho\sigma}\gamma^i \gamma_j \varepsilon\right\rangle\right) \\
&= \frac{1}{8} H^j_{\mu\rho\sigma} \left(\left\langle \gamma^i\gamma_j \varepsilon, \Gamma^{\rho\sigma}\Gamma_\nu \varepsilon\right\rangle - \left\langle \gamma_j\gamma^i \varepsilon, \Gamma_\nu \Gamma^{\rho\sigma} \varepsilon\right\rangle\right) \ .
\end{aligned}
\tag{3.6}
$$

Using the Clifford relations $\gamma_j \gamma^i = \delta^i_j + \gamma_j{}^i$, we can rewrite this as

$$
\begin{aligned}
\nabla_\mu \theta^i_\nu &= \frac{1}{8} H^i_{\mu\rho\sigma} \langle \varepsilon, [\Gamma^{\rho\sigma}, \Gamma_\nu]\varepsilon\rangle - \frac{1}{8} H^j_{\mu\rho\sigma} \left\langle \gamma_j{}^i \varepsilon, (\Gamma^{\rho\sigma}\Gamma_\nu + \Gamma_\nu \Gamma^{\rho\sigma})\varepsilon\right\rangle \\
&= \frac{1}{2} H^i_{\mu\nu\rho} \langle \varepsilon, \Gamma^\rho \varepsilon\rangle + \frac{1}{8} H^{j\rho\sigma}_\mu \left\langle \gamma^i{}_j \varepsilon, (\Gamma_{\rho\sigma}\Gamma_\nu + \Gamma_\nu \Gamma_{\rho\sigma})\varepsilon\right\rangle \\
&= \frac{1}{2} H^i_{\mu\nu\rho} \langle \varepsilon, \Gamma^\rho \varepsilon\rangle + \frac{1}{4} H^{j\rho\sigma}_\mu \left\langle \gamma^i{}_j \varepsilon, \Gamma_{\nu\rho\sigma} \varepsilon\right\rangle \ .
\end{aligned}
\tag{3.7}
$$

It follows from this that its exterior derivative $d\theta \in \Omega^2(M, \mathbb{V})$, with components $(d\theta)^i_{\mu\nu} = \nabla_\mu \theta^i_\nu - \nabla_\nu \theta^i_\mu$, is given by

$$
(d\theta)^i_{\mu\nu} = H^i_{\mu\nu\rho} \langle \varepsilon, \Gamma^\rho \varepsilon\rangle + \frac{1}{4} H^{j\rho\sigma}_\mu \left\langle \gamma^i{}_j \varepsilon, \Gamma_{\nu\rho\sigma}\varepsilon\right\rangle - \frac{1}{4} H^{j\rho\sigma}_\nu \left\langle \gamma^i{}_j \varepsilon, \Gamma_{\mu\rho\sigma}\varepsilon\right\rangle \ . \tag{3.8}
$$

Notice that the last two terms can be written in terms of a Clifford commutator, so that

$$
(d\theta)^i_{\mu\nu} = H^i_{\mu\nu\rho} \langle \varepsilon, \Gamma^\rho \varepsilon\rangle + \frac{1}{24} H^j_{\rho\sigma\tau} \left\langle \gamma^i{}_j \varepsilon, [\Gamma_{\mu\nu}, \Gamma^{\rho\sigma\tau}]\varepsilon\right\rangle \ . \tag{3.9}
$$

The second term in the r.h.s. is seen to vanish, since $[\Gamma_{\mu\nu}, H^j]\varepsilon = 0$, because H^j and hence also its infinitesimal Lorentz transformation $[\Gamma_{\mu\nu}, H^j]$ are anti-selfdual and hence annihilate ε by Lemma 2 in the appendix. The remaining term in the r.h.s. is precisely the contraction of H by V. This shows that $\iota_V H = d\theta$ is closed and, since so is H, that $\mathscr{L}_V H = d\iota_V H + \iota_V dH = 0$, showing that V leaves H invariant.

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We therefore have all the ingredients for a Lie superalgebra on the vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ is the Lie algebra of Killing vector fields which in addition preserve H and $\mathfrak{g}_1$ is the space of Killing spinors, which for the $(2,0)$ theory has dimension at most 16. The bracket $[-,-] : \mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ is given by the spinorial Lie derivative $[K, \varepsilon] = \mathscr{L}_K \varepsilon$, which since $K \in \mathfrak{g}_0$ leaves $\mathcal{D}$ invariant and hence takes Killing spinors to Killing spinors. The bracket $[-,-] : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ is obtained as before by polarising the construction of the Dirac current, as in equation (2.6).

Three of the four components of the Jacobi identity vanish by construction, whence only the $(\mathfrak{g}_1, \mathfrak{g}_1, \mathfrak{g}_1)$ component needs to be checked. This is a symmetric trilinear map $\mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_1$, whence it vanishes if and only if it vanishes when restricted to the diagonal, which is the map sending a Killing spinor ε to its Lie derivative along its Dirac current: $\mathscr{L}_{V_\varepsilon} \varepsilon$.

Letting $V = V_\varepsilon$, we have

$$\begin{aligned}
\mathscr{L}_V \varepsilon &= \nabla_V \varepsilon - \rho(\nabla V) \varepsilon \\
&= V^\mu \nabla_\mu \varepsilon - \frac{1}{4} \nabla_\mu V_\nu \Gamma^{\mu\nu} \varepsilon \\
&= -\frac{1}{8} H^i_{\mu\nu\rho} \left(\Gamma^{\mu\nu} \theta_i^\rho + V^\mu \Gamma^{\nu\rho} \gamma_i \right) \varepsilon \\
&= -\frac{1}{8} H^i_{\mu\nu\rho} \Gamma^{\mu\nu} \left(\theta_i^\rho + V^\rho \gamma_i \right) \varepsilon \\
&= \frac{1}{48} H^i_{\mu\nu\rho} \left(\Gamma^{\mu\nu\rho} \Gamma_\sigma + \Gamma_\sigma \Gamma^{\mu\nu\rho} \right) \left(V^\sigma \gamma_i + \theta_i^\sigma \right) \varepsilon \; .
\end{aligned} \tag{3.10}$$

We now use that H^i Clifford annihilates ε (Lemma 2 in the appendix) to arrive at

$$\begin{aligned}
\mathscr{L}_V \varepsilon &= \frac{1}{48} H^i_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \Gamma_\sigma \left(V^\sigma \gamma_i + \theta_i^\sigma \right) \varepsilon \\
&= \frac{1}{48} H^i_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \Gamma_\sigma \left(\langle \varepsilon, \Gamma^\sigma \varepsilon \rangle \gamma_i + \langle \varepsilon, \Gamma^\sigma \gamma_i \varepsilon \rangle \right) \varepsilon \; .
\end{aligned} \tag{3.11}$$

This can be rewritten in a way that allows us to use the Fierz identity (A.32), namely

$$\mathscr{L}_V \varepsilon = \frac{1}{48} H^i_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \Gamma_\sigma \left(\gamma_i (\varepsilon \otimes \varepsilon^\flat) + (\varepsilon \otimes \varepsilon^\flat) \gamma_i \right) \Gamma^\sigma \varepsilon \; . \tag{3.12}$$

Using that Fierz identity and also equation (A.23), we may rewrite this finally as

$$\begin{aligned}
\mathscr{L}_V \varepsilon &= -\frac{1}{24} H^i_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \Gamma_\sigma \left(\langle \varepsilon, \Gamma^\sigma \varepsilon \rangle \gamma_i + \langle \varepsilon, \Gamma^\sigma \gamma_i \varepsilon \rangle \right) \varepsilon \\
&= -\frac{1}{24} H^i_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \Gamma_\sigma \left(V^\sigma \gamma_i + \theta_i^\sigma \right) \varepsilon \; .
\end{aligned} \tag{3.13}$$

Comparing with equation (3.11), we see that it must vanish.

This shows that the brackets thus defined on $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ turn it into a Lie superalgebra. The ideal $\mathfrak{k} = [\mathfrak{g}_1, \mathfrak{g}_1] \oplus \mathfrak{g}_1$ is the *Killing superalgebra* of the $(2,0)$ background (M, g, H).

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3.2 Homogeneity

The proof of homogeneity follows the same steps as in the $(1,0)$ theory. An essential ingredient is that the Dirac current of a spinor is a causal vector; that is, either null or timelike. In the $(1,0)$ theory we showed that the Dirac current is always null, but in the $(2,0)$ theory this is not the case. Nevertheless we can still show that it is causal. We have only managed to do this by an explicit computation using the realisation in section A.5 in the appendix.

The Dirac current of $\varepsilon \in \mathscr{S}_+$ has components

$$K^a = \langle \varepsilon, \Gamma^a \varepsilon \rangle \ . \tag{3.14}$$

For nonzero $\varepsilon \in \mathscr{S}_+$, it follows that $\Gamma^a\varepsilon \in \mathscr{S}_-$ and hence we may express the inner product in either bilinear or sesquilinear forms. We choose the sesquilinear form and compute the 0th component of the Dirac current. In the notation of section A.5, we find

$$\begin{aligned} K^0 &= \varepsilon^\dagger (B \otimes b)(\Gamma^0 \otimes \mathbb{1}_4)\varepsilon \\ &= \varepsilon^\dagger(-\Gamma_{12345}\Gamma^0 \otimes \mathbb{1}_4)\varepsilon \\ &= \varepsilon^\dagger(\Gamma_7 \otimes \mathbb{1}_4)\varepsilon \\ &= \varepsilon^\dagger\varepsilon > 0\,, \end{aligned} \tag{3.15}$$

where we have used that $\Gamma_7\varepsilon = \varepsilon$. This shows that K^0 never vanishes and thus K cannot be spacelike, otherwise we could Lorentz transform to a frame where $K^0 = 0$.

The proof now follows *mutatis mutandis* the same steps as those outlined in section 2.2 for the $(1,0)$ case and will not be repeated here. In summary, if the dimension of the space of Killing spinors is greater than 8, then the $(2,0)$ background (M, g, H) is locally homogeneous.

In summary, we have established the existence of Killing superalgebras for supersymmetric backgrounds of six-dimensional $(1,0)$ and $(2,0)$ supergravities and used that to show that $> \frac{1}{2}$-BPS backgrounds are locally homogeneous. Together with the results of [4], and using that the homogeneity theorem survives dimensional reduction, this establishes the validity of the homogeneity theorem for all (pure, Poincaré) supergravity theories which have been constructed thus far. Details will appear in a forthcoming paper.

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A Summary of spinorial results

A.1 The Clifford module and its inner products

Our Clifford algebra conventions follow [12]. We define $\mathrm{C}\ell(s,t)$ to be the Clifford algebra associated with the real vector space $\mathbb{R}^{s+t}$ with inner product given by the matrix

$$\eta = \begin{pmatrix} \mathbb{1}_s & 0 \\ 0 & -\mathbb{1}_t \end{pmatrix}\,, \tag{A.1}$$

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where $\mathbb{1}_p$ is the $p \times p$ identity matrix. This means that $C\ell(s,t)$ is the associative unital algebra generated by Γ_a, $a = 1, \dots, s+t$, subject to the relations

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = -2\eta_{ab} \mathbb{1} \ . \tag{A.2}$$

(Notice the sign!) In this paper we are interested in $C\ell(5,1)$.

As a real associative algebra, $C\ell(5,1)$ is isomorphic to the algebra $\mathbb{H}(4)$ of 4×4 quaternionic matrices. This means that it has a unique irreducible representation, $\mathfrak{M}$, which is quaternionic and of dimension 4. We prefer, however, to work over the complex numbers, so that we will represent Γ_a as complex 8×8 matrices leaving invariant a quaternionic structure. The resulting 8-dimensional complex representation is the complex vector space P obtained from the right quaternionic vector space $\mathfrak{M}$ via restriction of scalars to $\mathbb{C}$. We call P the *pinor representation* of $C\ell(5,1)$.

There are two natural involutions of $C\ell(5,1)$, each one realisable as the adjoint relative to a quaternionic inner product on $\mathfrak{M}$. The two inner products are denoted $\langle -, - \rangle_\pm$ and defined by

$$\langle \Gamma_a \varepsilon_1, \varepsilon_2 \rangle_\pm = \pm \langle \varepsilon_1, \Gamma_a \varepsilon_2 \rangle \ , \tag{A.3}$$

where $\langle -, - \rangle_+$ is $\mathbb{H}$-hermitian, and $\langle -, - \rangle_-$ is $\mathbb{H}$-skewhermitian. These quaternionic inner products induce inner products on the pinor representation P. This is done by decomposing $\langle -, - \rangle_\pm$, which are $\mathbb{H}$-valued, into $\mathbb{C}$-valued inner products:

$$\langle -, - \rangle_+ = h_+(-,-) + j\omega_+(-,-) \, , \tag{A.4}$$

where h_+ is $\mathbb{C}$-hermitian and ω_+ is $\mathbb{C}$-symplectic; and

$$\langle -, - \rangle_- = ih_-(-,-) + jg_-(-,-) \, , \tag{A.5}$$

where h_- is $\mathbb{C}$-hermitian and g_- is $\mathbb{C}$-symmetric. In either case, one determines the other: $\omega_+(\varepsilon_1, \varepsilon_2) = h_+(\varepsilon_1 j, \varepsilon_2)$ and $g_-(\varepsilon_1, \varepsilon_2) = ih_-(\varepsilon_1 j, \varepsilon_2)$.

A.2 The spinor representations

The spin group $\mathrm{Spin}(5,1) \cong \mathrm{SL}(2,\mathbb{H})$ is contained in $C\ell(5,1)$ and hence the irreducible Clifford module $\mathfrak{M}$ decomposes under $\mathrm{Spin}(5,1)$ into the direct sum of two irreducible spinor modules $\mathfrak{S}_\pm$, labelled by their chirality, i.e., the eigenvalue of the volume element $\Gamma_7 = \Gamma^{012345}$ in $C\ell(5,1)$, which obeys $\Gamma_7^2 = \mathbb{1}$. The volume element Γ_7 is not in the centre of $C\ell(5,1)$, but it commutes with $\mathrm{Spin}(5,1)$, whence its eigenspaces $\mathfrak{S}_\pm$ are preserved by $\mathrm{Spin}(5,1)$. These are the positive- and negative-chirality spinor representations of $\mathrm{Spin}(5,1)$. They are quaternionic and of dimension two, but we will again restrict scalars to obtain four-dimensional complex representations $S_\pm$ with an invariant quaternionic structure. This means that under $\mathrm{Spin}(5,1)$, $P = S_+ \oplus S_-$. There is no $\mathrm{Spin}(5,1)$-invariant inner product on $S_\pm$ (or $\mathfrak{S}_\pm$), but of course there is on their direct sum, relative to which $S_\pm$ are isotropic subspaces. This means that $S_- = S_+^*$.

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A.3 The R-symmetry representations

The R-symmetry group of the $d = 6$ (p, q) supersymmetry algebra is $\mathrm{USp}(2p) \times \mathrm{USp}(2q)$, whence $\mathrm{USp}(2) \cong \mathrm{Sp}(1)$ for the $(1, 0)$ theory and $\mathrm{USp}(4) \cong \mathrm{Sp}(2)$ for the $(2, 0)$ theory. The spinor parameters in the supergravity theory transform according to the fundamental representations of these groups, which are quaternionic representations $\mathfrak{S}_1 \cong \mathbb{H}$ for the $(1, 0)$ theory and $\mathfrak{S}_2 \cong \mathbb{H}^2$ for the $(2, 0)$ theory. Restricting scalars to $\mathbb{C}$ we arrive at complex representations S_1, of dimension two, and S_2 of dimension four, with invariant quaternionic structures, respectively.

The representations S_1 and S_2 have $\mathbb{C}$-hermitian inner products invariant under $\mathrm{USp}(2)$ and $\mathrm{USp}(4)$, respectively. However the gravitino connection in the $(2, 0)$ theory uses explicitly an equivariant bilinear map $\mathbb{V} \times S_2 \to S_2$, where $\mathbb{V}$ is the real 5-dimensional representation of $\mathrm{USp}(4) \cong \mathrm{Spin}(5)$. There are precisely two such maps, corresponding to the Clifford actions of $C\ell(\mathbb{V}) \cong C\ell(0, 5)$ on either of its two irreducible Clifford modules. This means that S_2 is to be thought of not just as a spinor representation of $\mathrm{Spin}(5)$, but actually as one of the two pinor representations of $C\ell(0, 5)$.

As a real associative algebra, $C\ell(0, 5)$ is isomorphic to two copies of the algebra $\mathbb{H}(2)$ of 2×2 quaternionic matrices. Therefore it has two inequivalent irreducible representations, which are quaternionic of dimension 2 or, after restricting scalars, complex of dimension 4 with an invariant quaternionic structure. Let us call these latter complex representations S_2 and S_2'. The action of $C\ell(0, 5)$ is via 4×4 complex matrices γ_i, satisfying $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}\mathbb{1}$. The two representations are distinguished by the action of the volume element $\gamma_6 = \gamma_{12345}$, which is central in $C\ell(5)$, satisfies $\gamma_6^2 = 1$ and acts like the identity on S_2. The $\mathrm{Spin}(5)$-invariant inner product on S_2 is such that

$$\langle \gamma_i \varepsilon_1, \varepsilon_2 \rangle = + \langle \varepsilon_1, \gamma_i \varepsilon_2 \rangle \ . \tag{A.6}$$

Indeed, with the opposite sign the volume element would be skewsymmetric making S_2 isotropic.

A.4 The underlying real spinorial representations

In the six-dimensional supergravity theories, the spinor parameters of the supersymmetry transformations take values in a real representation whose complexification is the tensor product of the chiral spinor representation of $\mathrm{Spin}(5, 1)$ and the fundamental representation of the R-symmetry group. As discussed above, these representations are complex of quaternionic type and hence their tensor product (over $\mathbb{C}$) is a complex representation of real type and thus the complexification of a real representation. In this section of the appendix we provide the details.

For brevity, we will consider the more general case of a tensor product $V \otimes_{\mathbb{C}} W$ of two complex representations of quaternionic type. This means that V and W have invariant quaternionic structures J_V and J_W, respectively. They are complex *anti*linear maps which square to $-\mathbb{1}$. Their tensor product $c = J_V \otimes J_W$ is a complex antilinear map squaring to $\mathbb{1}$ —i.e., a conjugation. The eigenspace of c with eigenvalue 1 is a real subrepresentation U of $V \otimes_{\mathbb{C}} W$ and indeed $V \otimes_{\mathbb{C}} W = U \otimes_{\mathbb{R}} \mathbb{C} = U \oplus iU$. The complex inner products

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on V and W (induced from the quaternionic inner products on the original quaternionic representations) determine real inner products on U.

Let us apply this now to the cases of interest. As we have seen, the spinor representations $S_\pm$ are isotropic, so if we want an inner product we must work with the pinor representation $P = S_+ \oplus S_-$. As we saw in section A.1, on P we have one of two possible pairs of inner products: one pair consisting of a $\mathbb{C}$-hermitian and a $\mathbb{C}$-symplectic inner product, and another pair consisting of a $\mathbb{C}$-skewhermitian and a $\mathbb{C}$-symmetric inner product. We will choose the latter, in order for the Lie bracket on the Killing spinors to be symmetric and hence have a chance of generating a Lie superalgebra with nontrivial odd subspace. This means that we choose the inner product $\langle -,-\rangle_-$ on P.

On $P \otimes_{\mathbb{C}} S_1$ we therefore have a $\mathbb{C}$-symplectic structure, consisting of the tensor product of the $\mathbb{C}$-symmetric and $\mathbb{C}$-symplectic inner products on P and S_1, respectively. This restricts to a real symplectic inner product on the underlying real subrepresentation $\mathcal{S}$ of $P \otimes_{\mathbb{C}} S_1$. We will denote it by $\langle -,-\rangle$ and simply notice that the subspaces $\mathcal{S}_\pm$, defined as the underlying real representations of $S_\pm \otimes_{\mathbb{C}} S_1$, are lagrangian subspaces.

For the $(2,0)$ theory, we again pick the $\mathbb{C}$-symmetric inner product on P and the $\mathbb{C}$-symplectic inner product on S_2, so that on $P \otimes_{\mathbb{C}} S_2$ we have a $\mathbb{C}$-symplectic structure, restricting to a real symplectic inner product on the underlying real representation denoted $\mathscr{S}$ of $P \otimes_{\mathbb{C}} S_2$. We will again denote it by $\langle -,-\rangle$ and again notice that $\mathscr{S} = \mathscr{S}_+ \oplus \mathscr{S}_-$, where the lagrangian subspaces $\mathscr{S}_\pm$ are now defined as the underlying real representations of $S_\pm \otimes_{\mathbb{C}} S_2$.

A.5 Explicit matrix realisation

An essential ingredient in the proof of the homogeneity theorem is the fact that the Dirac current of a spinor is a causal vector. Whereas for the $(1,0)$-theory, this fact admits a rather elegant proof, for the $(2,0)$-theory we have only managed to show this by calculating using an explicit matrix realisation. For completeness, and because it may be useful in the future, we record here the necessary formulae. We let $\mathbb{1}_n$ denote the $n \times n$ identity matrix and σ_i the (hermitian) Pauli spin matrices with $\sigma_1\sigma_2 = i\sigma_3$, et cetera. An explicit realisation for the generators Γ_a of $C\ell(5,1)$ is given by the following matrices:

$$\begin{aligned} \Gamma_0 &= \mathbb{1} \otimes \mathbb{1} \otimes \sigma_3 & \Gamma_3 &= i\sigma_1 \otimes \sigma_3 \otimes \sigma_1 \\ \Gamma_1 &= -i\mathbb{1} \otimes \sigma_1 \otimes \sigma_1 & \Gamma_4 &= i\sigma_3 \otimes \sigma_3 \otimes \sigma_1 \\ \Gamma_2 &= -i\mathbb{1} \otimes \sigma_2 \otimes \sigma_1 & \Gamma_5 &= i\mathbb{1} \otimes \mathbb{1} \otimes \sigma_2 \,, \end{aligned} \tag{A.7}$$

with $\mathbb{1} = \mathbb{1}_2$.

The invariant quaternionic structure is given by the composition $J = m_J \circ \chi$, where χ is complex conjugation and m_J is a matrix which obeys $\Gamma_a m_J = m_J \overline{\Gamma_a}$ (invariance) and in addition $m_J \overline{m_J} = -\mathbb{1}_8$. Invariance says that m_J commutes with the real Γ_a, namely $\Gamma_{0,2,5}$, and anticommutes with the imaginary Γ_a, namely $\Gamma_{1,3,4}$. Thus we can take $m_J = \Gamma_{025}$, which is real and obeys $\Gamma_{025}^2 = -\mathbb{1}_8$.

The $\mathbb{H}$-skewhermitian inner product $\langle -,-\rangle_-$ decomposes into $ih_- + jg_-$, where ih_- is $\mathbb{C}$-skewhermitian and g_- is $\mathbb{C}$-symmetric. In this explicit realisation, ih_- is determined by a matrix B such that

$$ih_-(\varepsilon_1, \varepsilon_2) = \varepsilon_1^\dagger B \varepsilon_2\,, \tag{A.8}$$

and the defining property (A.3) becomes

$$\Gamma_a^\dagger B = -B\Gamma_a\,, \tag{A.9}$$

which says that B anticommutes with Γ_0 and commutes with the rest. In other words, B must be proportional to Γ_{12345}, which in this realisation is given by

$$\Gamma_{12345} = i\sigma_2 \otimes \sigma_3 \otimes \sigma_2\,, \tag{A.10}$$

which is symmetric and imaginary, hence skewhermitian, as expected. We define $B := -\Gamma_{12345}$, where the sign is for later convenience.

The symmetric inner product g_- is given by a matrix C such that

$$g_-(\varepsilon_1, \varepsilon_2) = \varepsilon_1^T C \varepsilon_2\,, \tag{A.11}$$

where now

$$\Gamma_a^T C = -C\Gamma_a\,, \tag{A.12}$$

which says that C commutes with the skewsymmetric Γ_a, namely $\Gamma_{2,5}$, and anticommutes with $\Gamma_{0,1,3,4}$. This means that C must be proportional to Γ_{0134}, which in this realisation is given by

$$\Gamma_{0134} = i\sigma_2 \otimes \sigma_1 \otimes \sigma_2\,, \tag{A.13}$$

which is imaginary and symmetric. We will define $C := \Gamma_{0134}$.

For the $(2,0)$-theory we will also need an explicit realisation of $\mathrm{C}\ell(0,5)$, conveniently given by the following 4×4 matrices

$$\gamma_1 = \sigma_1 \otimes \sigma_2 \quad \gamma_2 = \sigma_2 \otimes \sigma_2 \quad \gamma_3 = -\sigma_3 \otimes \sigma_2 \quad \gamma_4 = \mathbb{1} \otimes \sigma_3 \quad \gamma_5 = \mathbb{1} \otimes \sigma_1\,, \tag{A.14}$$

with $\mathbb{1} = \mathbb{1}_2$ again.

The invariant quaternionic structure j is given by the composition $m_j \circ \chi$, with χ again complex conjugation and m_j a matrix satisfying $m_j \overline{m_j} = -\mathbb{1}_4$ and $m_j \gamma_i = \overline{\gamma_i} m_j$. We can therefore take $m_j = \gamma_{245}$.

The $\mathbb{H}$-hermitian invariant inner product $\langle -, - \rangle_+$ decomposes into $h_+ + j\omega_+$, where h_+ is $\mathbb{C}$-hermitian and ω_+ is $\mathbb{C}$-symplectic. In this realisation, h_+ is defined in terms of a matrix b by

$$h_+(\varepsilon_1, \varepsilon_2) = \varepsilon_1^\dagger b \varepsilon_2\,, \tag{A.15}$$

where $\gamma_i^\dagger b = b\gamma_i$. Since all $\gamma_i^\dagger = \gamma_i$ for all i, we can choose $b = \mathbb{1}_4$ without loss of generality. The $\mathbb{C}$-symplectic inner product ω_+ is given in terms of a matrix c by

$$\omega_+(\varepsilon_1, \varepsilon_2) = \varepsilon_1^T c \varepsilon_2\,, \tag{A.16}$$

where $\gamma_i^T c = c\gamma_i$. Thus c must commute with $\gamma_{2,4,5}$ and anticommute with $\gamma_{1,3}$, whence we can take $c = \gamma_{245}$ which is real and symplectic.

In the tensor product representation $S_+ \otimes S_2$, the conjugation $\mathcal{C} = J \otimes j$ is given explicitly by

$$\mathcal{C} = (\Gamma_{025} \otimes \gamma_{245}) \circ \chi \,, \tag{A.17}$$

so that an element ε of $\mathscr{S}$ obeys

$$\overline{\varepsilon} = (\Gamma_{025} \otimes \gamma_{245})\, \varepsilon \ . \tag{A.18}$$

Let $\varepsilon_{1,2} \in \mathscr{S}$. Then it is an easy calculation to show that the sesquilinear and bilinear inner products agree, as expected. This is nothing but the fact that for a Majorana spinor, the Dirac conjugate agrees with the Majorana conjugate; explicitly,

$$\varepsilon_1^\dagger (B \otimes b)\varepsilon_2 = \varepsilon_1^T (\Gamma_{025} \otimes \gamma_{245})^T (B \otimes b)\varepsilon_2 = \varepsilon_1^T (C \otimes c)\varepsilon_2 \ . \tag{A.19}$$

A.6 Vectors, forms and their Clifford action

As a vector space, the Clifford algebra is isomorphic (as a $\mathbb{Z}_2$-graded vector space) to the exterior algebra. When we globalise, the Clifford bundle $C\ell(TM)$ is isomorphic as a $\mathbb{Z}_2$-graded vector bundle to the bundle of differential forms $\Omega^*(M)$. This means that differential forms can act on spinors. If $\theta \in \Omega^k(M)$ is a differential form of rank k and $\varepsilon \in C^\infty(M; \mathcal{S})$ is a spinor field, then we will denote $\theta \cdot \varepsilon$ the spinor field obtained by Clifford acting with θ on ε. Explicitly,

$$\theta \cdot \varepsilon = \frac{1}{k!} \theta_{a_1 \ldots a_k} \Gamma^{a_1 \ldots a_k} \varepsilon \ . \tag{A.20}$$

Similarly, if $X \in C^\infty(M; TM)$ is a vector field, we can define its Clifford action $X \cdot \varepsilon$ on a spinor field as the Clifford action of its dual 1-form $X^\flat$.

Let $\nu \in \Omega^6(M)$ denote the volume form. Its Clifford action is via $\Gamma_7 = \Gamma^{012345}$. Then if $\theta \in \Omega^k(M)$ and ε is any spinor field,

$$\Gamma_7 \theta \cdot \varepsilon = -(\star\theta) \cdot \varepsilon \,, \tag{A.21}$$

where $\star\theta$ is the Hodge dual. A very useful consequence of this calculation is the following.

Lemma 2. *Let $H \in \Omega^3_-(M)$ be an anti-selfdual 3-form and let $\varepsilon \in C^\infty(M; \mathcal{S}_+)$ be a positive-chirality spinor field. Then $H \cdot \varepsilon = 0$.*

Proof. Let Γ_7 denote the volume element in the Clifford algebra, so that the volume form acts via Γ_7. Since ε has positive chirality, $\Gamma_7 \varepsilon = \varepsilon$. If $H \in \Omega^3(M)$ is any 3-form, then $H\Gamma_7 = -\Gamma_7 H$, whence on the one hand

$$\Gamma_7 H \cdot \varepsilon = -H \cdot \Gamma_7 \varepsilon = -H \cdot \varepsilon$$

and on the other hand, for H anti-selfdual

$$\Gamma_7 H \cdot \varepsilon = -(\star H) \cdot \varepsilon = H \cdot \varepsilon \ . \qquad \square$$

Also useful are the following identities, where $\theta \in \Omega^k(M)$ and X is any vector field:

$$\begin{aligned} X^\flat \cdot \theta - (-1)^k \theta \cdot X^\flat &= -2\iota_X \theta \\ X^\flat \cdot \theta + (-1)^k \theta \cdot X^\flat &= +2X^\flat \wedge \theta \ . \end{aligned} \tag{A.22}$$

Two more useful consequences of the Clifford relations are

$$\Gamma^a \Gamma_b \Gamma_a = 4\Gamma_b \qquad \text{and} \qquad \Gamma^a \Gamma_{bcd} \Gamma_a = 0 \ . \tag{A.23}$$

A.7 Fierz formulae

In this section we derive two important Fierz formulae.

A.7.1 The (1,0) Fierz formula

Let us first of all consider the $(1,0)$ theory. Let $\varepsilon \in \mathcal{S}_+$. By choosing a complex basis e_A, $A = 1,2$, for the fundamental two-dimensional representation S_1 of USp(2) relative to which the invariant complex symplectic form is given by the Levi-Civita symbol ϵ_{AB}, we may decompose $\varepsilon \in \mathcal{S}_+$ as $\varepsilon = \varepsilon^A e_A$, where each $\varepsilon^A \in S_+$ is a chiral spinor of $\mathrm{Spin}(5,1)$. In addition, the ε^A satisfy a reality condition whose explicit form we will not need. The real symplectic inner product on $\mathcal{S}_+ \oplus \mathcal{S}_-$ is such that if $\varepsilon, \eta \in \mathcal{S}_+ \oplus \mathcal{S}_-$, then

$$\langle \varepsilon, \eta \rangle = \epsilon_{AB} \left(\varepsilon^A, \eta^B \right) , \tag{A.24}$$

where $(-,-)$ is the symmetric inner product on $S_+ \oplus S_-$, which we had denoted g_- in section A.1.

Now let $\psi_{1,2} \in S_+$ and consider the complex linear map $\psi_1 \otimes \psi_2^\flat : S_- \to S_+$ defined by

$$\left(\psi_1 \otimes \psi_2^\flat \right) (\psi_3) = (\psi_2, \psi_3)\, \psi_1 \ . \tag{A.25}$$

This can be extended to an endomorphism of $P = S_+ \oplus S_-$ by declaring it to be zero on S_+ and hence it defines an element of the Clifford algebra $C\ell(5,1)$, which is the endomorphism algebra of P. Since the map reverses chirality, it lives in $C\ell(5,1)^{\mathrm{odd}}$, whence it is a linear combination of products of an odd number of Γ_a and since it annihilates S_+, it takes the form

$$\psi_1 \otimes \psi_2^\flat = \left(c^a \Gamma_a + \frac{1}{6} c^{abc} \Gamma_{abc} \right) \Pi_- \ , \tag{A.26}$$

for some c^a and c^{abc} to be determined and where $\Pi_- = \frac{1}{2}(\mathbb{1} - \Gamma_7)$ is the projector onto negative chirality spinors.

It is a simple matter of taking the trace of $\left(\psi_1 \otimes \psi_2^\flat\right)\Gamma_b$ and $\left(\psi_1 \otimes \psi_2^\flat\right)\Gamma_{abc}$ to determine that

$$c^a = \frac{1}{4} (\psi_1, \Gamma^a \psi_2) \qquad \text{and} \qquad c^{abc} = \frac{1}{4} \left(\psi_1, \Gamma^{abc} \psi_2 \right) , \tag{A.27}$$

whence we arrive at the Fierz identity

$$\psi_1 \otimes \psi_2^\flat = \frac{1}{4} (\psi_1, \Gamma^a \psi_2)\, \Gamma_a \Pi_- + \frac{1}{24} \left(\psi_1, \Gamma^{abc} \psi_2 \right) \Gamma_{abc} \Pi_- \ . \tag{A.28}$$

If now $\varepsilon_{1,2} \in \mathcal{S}_+$ and we apply the above Fierz formula to the linear map $\varepsilon_1^A \otimes (\varepsilon_2^B)^\flat : S_- \to S_+$, we arrive at

$$\varepsilon_1^A \otimes \left(\varepsilon_2^B\right)^\flat = \frac{1}{4} \left(\varepsilon_1^A, \Gamma^a \varepsilon_2^B \right) \Gamma_a \Pi_- + \frac{1}{24} \left(\varepsilon_1^A, \Gamma^{abc} \varepsilon_2^B \right) \Gamma_{abc} \Pi_- \ . \tag{A.29}$$

A simple consequence of this Fierz identity is the following result.

Lemma 3. *Let $\varepsilon \in \mathcal{S}_+$. Then for all $A, B, C = 1, 2$,*

$$\left(\varepsilon^A, \Gamma^a \varepsilon^B \right) \Gamma_a \varepsilon^C = 0 \ .$$

Proof. An immediate consequence of the Fierz identity (A.29) and equation (A.23) is that

$$X^{ABC} := \left(\varepsilon^A, \Gamma^a \varepsilon^B\right) \Gamma_a \varepsilon^C$$

is invariant under cyclic permutations of its indices: $X^{ABC} = X^{BCA} = X^{CAB}$. It also follows from the fact that Γ^a is skewsymmetric relative to the symmetric inner product $(-,-)$, that

$$X^{ABC} = -X^{BAC} \; .$$

In other words, X^{ABC} is totally skewsymmetric, but since $A, B, C = 1, 2$, it has to vanish. □

A.7.2 The (2,0) Fierz formula

Every $\varepsilon \in \mathscr{S}_+$ defines a linear map $\varepsilon \otimes \varepsilon^\flat : \mathscr{S}_- \to \mathscr{S}_+$ by

$$\left(\varepsilon \otimes \varepsilon^\flat\right)(\varepsilon') = \left\langle \varepsilon, \varepsilon' \right\rangle \varepsilon \,, \tag{A.30}$$

with $\langle -,- \rangle$ the symplectic inner product on $\mathscr{S} = \mathscr{S}_+ \oplus \mathscr{S}_-$. The linear map $\varepsilon \otimes \varepsilon^\flat$ extends to an endomorphism of $\mathscr{S}$ which is trivial on $\mathscr{S}_+$ and hence can be expressed as an element of $C\ell(5,1) \otimes C\ell(0,5)$. Symmetry and chirality imply that

$$\varepsilon \otimes \varepsilon^\flat = c^a \Gamma_a \Pi_- + c^{a\,i} \Gamma_a \gamma_i \Pi_- + \frac{1}{12} c^{abc\,ij} \Gamma_{abc} \gamma_{ij} \Pi_- \,, \tag{A.31}$$

for some coefficients c^a, $c^{a\,i}$ and $c^{abc\,ij}$ which must be determined. Taking traces and remembering that γ_i are 4×4 matrices, we find that

$$\varepsilon \otimes \varepsilon^\flat = -\frac{1}{16} \left(\langle \varepsilon, \Gamma^a \varepsilon \rangle \Gamma_a + \left\langle \varepsilon, \Gamma^a \gamma^i \varepsilon \right\rangle \Gamma_a \gamma_i + \frac{1}{24} \left\langle \varepsilon, \Gamma^{abc} \gamma^{ij} \varepsilon \right\rangle \Gamma_{abc} \gamma_{ij} \right) \Pi_- \; . \tag{A.32}$$

A consequence of this Fierz identity is that if $V^\mu = \langle \varepsilon, \Gamma^\mu \varepsilon \rangle$ and $\theta^i_\mu = \left\langle \varepsilon, \Gamma_\mu \gamma^i \varepsilon \right\rangle$, then

$$5 V^\mu \Gamma_\mu \varepsilon + \theta^i_\mu \Gamma^\mu \gamma_i \varepsilon = 0 \; ; \tag{A.33}$$

although in contrast with the $(1,0)$ case, V_ε does not Clifford annihilate ε. In particular, V_ε is not necessarily null, but only causal.

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