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Coincidence point and fixed point theorems for a new type of G-contraction multivalued mappings on a metric space endowed with a graph

Adisak Hanjing and Suthep Suantai*

*Correspondence: suthep.s@cmu.ac.th
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand

Abstract

In this paper, a new type of G-contraction multivalued mappings in a metric space endowed with a directed graph is introduced and studied. This type of mappings is more general than that of Mizoguchi and Takahashi (J. Math. Anal. Appl. 141:177-188, 1989), Berinde and Berinde (J. Math. Anal. Appl. 326:772-782, 2007), Du (Topol. Appl. 159:49-56, 2012), and Sultana and Vetrivel (J. Math. Anal. Appl. 417:336-344, 2014). A fixed point and coincidence point theorem for this type of mappings is established. Some examples illustrating our main results are also given. The main results obtained in this paper extend and generalize those in (Tiammee and Suantai in Fixed Point Theory Appl. 2014:70, 2014) and many well-known results in the literature.

MSC: 47H10; 54H25

Keywords: fixed point; multivalued mappings; Mizoguchi-Takahashi G-contraction

1 Introduction

Fixed point theory plays a very important role in nonlinear analysis and applications. It is well known that many metric fixed point theorems were motivated from the Banach contraction principle.

Theorem 1.1 *Let (X,d) be a complete metric space and $T: X \to X$ be a self-map. Assume that there exists a nonnegative number $k < 1$ such that*

$$d\bigl(T(x),T(y)\bigr) \le kd(x,y), \quad \textit{for all } x,y \in X.$$

Then T has a unique fixed point in X.

In 1969, Nadler [1] extended the Banach contraction principle for multivalued mappings.

Theorem 1.2 ([1]) *Let (X,d) be a complete metric space and let T be a mapping from X into $\mathrm{CB}(X)$. Assume that there exists $k \in [0,1)$ such that*

$$H\bigl(T(x),T(y)\bigr) \le kd(x,y), \quad \textit{for all } x,y \in X,$$

where H is the Pompeiu-Hausdorff metric on $\mathrm{CB}(X)$. *Then there exists* $z \in X$ *such that* $z \in T(z)$.

Nadler's fixed point theorem for multivalued contractive mappings has been extended in many directions (see [2–5]). Reich [6] proved the following fixed point theorem for multivalued φ-contraction mappings.

Theorem 1.3 ([6]) *Let* (X,d) *be a complete metric space and let* T *be a mapping from* X *into* $\mathrm{Comp}(X)$. *Assume that there exists a function* $\varphi : [0,\infty) \to [0,1)$ *such that* $\limsup_{r\to t^+} \varphi(r) < 1$ *for each* $t \in [0,\infty)$ *and*

$$H\bigl(T(x), T(y)\bigr) \le \varphi\bigl(d(x,y)\bigr)d(x,y), \quad \text{for all } x,y \in X.$$

Then there exists $z \in X$ *such that* $z \in T(z)$.

In 1989, Mizoguchi and Takahashi [7] relaxed the compactness assumption on T to closed and bounded subsets of X. They proved the following theorem, which is a generalization of Nadler's theorem.

Theorem 1.4 ([7]) *Let* (X,d) *be a complete metric space and let* T *be a mapping from* X *into* $\mathrm{CB}(X)$. *Assume that there exists a function* $\varphi : [0,\infty) \to [0,1)$ *such that* $\limsup_{r\to t^+} \varphi(r) < 1$ *for each* $t \in [0,\infty)$ *and*

$$H\bigl(T(x), T(y)\bigr) \le \varphi\bigl(d(x,y)\bigr)d(x,y), \quad \text{for all } x,y \in X.$$

Then there exists $z \in X$ *such that* $z \in T(z)$.

In 2007, Berinde and Berinde [4] extended Theorem 1.1 to the class of multivalued weak contractions.

Definition 1.5 ([4]) Let (X,d) be a metric space and $T : X \to \mathrm{CB}(X)$ be a multivalued mapping. T is said to be a *multivalued weak contraction* or a *multivalued* (θ,L)*-weak contraction* if there exist two constants $\theta \in (0,1)$ and $L \ge 0$ such that

$$H\bigl(T(x), T(y)\bigr) \le \theta d(x,y) + Ld\bigl(y, T(x)\bigr), \quad \text{for all } x,y \in X.$$

Definition 1.6 ([4]) Let (X,d) be a metric space and $T : X \to \mathrm{CB}(X)$ be a multivalued mapping. *T is said to be a generalized multivalued* (α,L)*-weak contraction* if there exist $L \ge 0$ and a function $\alpha : [0,\infty) \to [0,1)$ satisfying $\limsup_{r\to t^+} \alpha(r) < 1$, for each $t \in [0,\infty)$ such that

$$H\bigl(T(x), T(y)\bigr) \le \alpha\bigl(d(x,y)\bigr)d(x,y) + Ld\bigl(y, T(x)\bigr), \quad \text{for all } x,y \in X.$$

They proved that in a complete metric space, every multivalued (θ,L)-weak contraction has a fixed point. In the same paper, they also proved that every generalized multivalued (α,L)-weak contraction has a fixed point (see [4]). This result was generalized by Du [8] in 2012 as in the following theorem.

Theorem 1.7 ([8]) *Let* (X,d) *be a complete metric space and* $T : X \to \mathrm{CB}(X)$ *be a multivalued mapping,* $g : X \to X$ *be a continuous self-map and* $\alpha : [0,\infty) \to [0,1)$ *a mapping* *satisfying* $\limsup_{r\to t^+} \alpha(r) < 1$, *for each* $t \in [0,\infty)$. *Assume that*:

(a) $T(x)$ *is g-invariant* (*i.e.*, $g(T(x)) \subseteq T(x)$ *for each* $x \in X$),

(b) *there exists a function* $h : X \to [0,\infty)$ *such that*

$$H\big(T(x), T(y)\big) \le \alpha\big(d(x,y)\big)d(x,y) + h\big(g(y)\big)d\big(g(y), T(x)\big), \quad \text{for all } x, y \in X.$$

Then $\mathcal{COP}(g,T) \cap \mathcal{F}(T) \neq \emptyset$, *where* $\mathcal{COP}(g,T) = \{x \in X : g(x) \in T(x)\}$ *and* $\mathcal{F}(T) = \{x \in X : x \in T(x)\}$.

In 2008, Jachymski [9] introduced the concept of a G-contraction and proved some fixed point results of G-contractions in a complete metric space endowed with a directed graph.

Let (X,d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, *i.e.*, $\Delta = \{(x,x) : x \in X\} \subseteq E(G)$.

Definition 1.8 ([9]) We say that a mapping $f : X \to X$ is a *G-contraction* if f preserves edges of G, *i.e.*, for each $x, y \in X$,

$$(x,y) \in E(G) \quad \Rightarrow \quad \big(f(x), f(y)\big) \in E(G) \tag{1.1}$$

and there exists $\alpha \in (0,1)$ such that for each $x, y \in X$,

$$(x,y) \in E(G) \quad \Rightarrow \quad d\big(f(x), f(y)\big) \le \alpha d(x,y).$$

He showed that in the case that there are certain properties on (X,d,G) a G-contraction $f : X \to X$ has a fixed point if and only if $X_f = \{x \in X : (x, f(x)) \in E(G)\}$ is nonempty. The mapping $f : X \to X$ satisfying condition (1.1) is also called a *graph-preserving mapping*.

In 2010, Beg and Butt [5] introduced the concept of G-contraction for a multivalued mapping $T : X \to \mathrm{CB}(X)$ as follows.

Definition 1.9 ([5]) We say that a mapping $T : X \to \mathrm{CB}(X)$ is a *G-contraction* if there exists $k \in (0,1)$ such that for each $(x,y) \in E(G)$,

$$H\big(T(x), T(y)\big) \le kd(x,y),$$

and if $u \in T(x)$ and $v \in T(y)$ are such that for each $\alpha > 0$,

$$d(u,v) \le kd(x,y) + \alpha,$$

then $(u,v) \in E(G)$.

Recently, in 2015, Alfuraidan [10] pointed out that the above definition of a G-contraction is flawed and the argument behind the proof of the main result of [5] fails.

By using the idea of multivalued contraction mappings in [11, 12], Alfuraidan introduced the following concept of a G-contraction.

Definition 1.10 ([10]) A multivalued mapping $T : X \to 2^X$ is said to be a *monotone increasing G-contraction* if there exists $\alpha \in [0,1)$ such that, for any $u, v \in X$ with $(u,v) \in E(G)$ and any $U \in T(u)$, there exists $V \in T(v)$ such that $(U,V) \in E(G)$ and $d(U,V) \le \alpha d(u,v)$.

He showed that under some properties on a metric space, a monotone increasing G-contraction multivalued mappings has a fixed point (see [10], Theorem 3.1).

In 2014, Tiammee and Suantai [13] introduced the concept of graph-preserving for multivalued mappings as follows.

Definition 1.11 ([13]) Let X be a nonempty set and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$, and let $T : X \to \mathrm{CB}(X)$. Then T is said to be *graph-preserving* if

$$(x,y) \in E(G), \quad \text{then } (u,v) \in E(G), \text{for all } u \in T(x) \text{ and } v \in T(y).$$

In the same year, Sultana and Vetrivel [14] introduced a concept of a Mizoguchi-Takahashi G-contraction as follows.

Definition 1.12 ([14]) A multivalued mapping $T : X \to \mathrm{CB}(X)$ is said to be a *Mizoguchi-Takahashi G-contraction* if there exists a function $\alpha : (0,\infty) \to [0,1)$ satisfying $\limsup_{r\to t^+} \alpha(r) < 1$ for every $t \in [0,\infty)$, and for every $x, y \in X$, $x \neq y$ with $(x,y) \in E(G)$,

(i) $H(T(x), T(y)) \leq \alpha(d(x,y))d(x,y)$,

(ii) if $u \in T(x)$ and $v \in T(y)$ are such that $d(u,v) \leq d(x,y)$, then $(u,v) \in E(G)$.

They showed that if there are some properties on a metric space, a multivalued Mizoguchi-Takahashi G-contraction has a fixed point (see [14], Theorem 3).

In this paper, we introduce a new concept of a G-contraction in a metric space endowed with a directed graph which is more general than the Mizoguchi-Takahashi G-contraction for multivalued mappings. We establish some coincidence point and fixed point theorems for this type of mappings and give some examples illustrating our main results.

2 Preliminaries

Let (X,d) be a metric space, and let $\mathrm{CB}(X)$ and $\mathrm{Comp}(X)$ be the set of all nonempty closed bounded subsets of X and the set of all nonempty compact subsets of X, respectively. For each $x \in X$ and $A \subseteq X$, let $d(x,A) = \inf_{y\in A} d(x,y)$. A function $H : \mathrm{CB}(X) \times \mathrm{CB}(X) \to [0,\infty)$ defined by

$$H(A,B) = \max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\Big\}$$

is called a *Pompeiu-Hausdorff metric* on $\mathrm{CB}(X)$ induced by d on X.

Let $g : X \to X$ be a self-map and $T : X \to 2^X$ be a multivalued map. A point x in X is a *coincidence point* of g and T if $g(x) \in T(x)$. If g is the identity map on X, then $x = g(x) \in T(x)$ and we call x a *fixed point* of T. The set of all fixed points of T and the set of all coincidence points of g and T are denoted by $\mathcal{F}(T)$ and $\mathcal{COP}(g,T)$, respectively.

The following lemmas are useful for our main results.

Lemma 2.1 ([1]) *Let (X,d) be a metric space. If $A, B \in \mathrm{CB}(X)$ and $a \in A$, then, for each $\varepsilon > 0$, there exists $b \in B$ such that $d(a,b) \leq H(A,B) + \varepsilon$.*

Lemma 2.2 ([6]) *Let (X,d) be a metric space, $\{A_k\}$ be a sequence in $\mathrm{CB}(X)$ and $\{x_k\}$ be a sequence in X such that $x_k \in A_{k-1}$. Let $\alpha : [0,\infty) \to [0,1)$ be a function satisfying*

$\limsup_{r\to t^+}\alpha(r) < 1$ *for every* $t \in [0,\infty)$. *Suppose that* $\{d(x_{k-1},x_k)\}$ *is a non-increasing sequence such that*

$$H(A_{k-1},A_k) \le \alpha\bigl(d(x_{k-1},x_k)\bigr)d(x_{k-1},x_k),$$
$$d(x_k,x_{k+1}) \le H(A_{k-1},A_k) + \bigl[\alpha\bigl(d(x_{k-1},x_k)\bigr)\bigr]^{n_k},$$

where $n_1 < n_2 < \cdots$ *and* $k, n_k \in \mathbb{N}$. *Then* $\{x_k\}$ *is a Cauchy sequence in* X.

3 Main results

We first introduce a concept of weak G-contraction on a metric space endowed with a directed graph.

Definition 3.1 Let (X,d) be a metric space and let $G = (V(G),E(G))$ be a directed graph such that $V(G) = X$. Let $T : X \to \mathrm{CB}(X)$ and $g : X \to X$. Then T is said to be a *weak* G*-contraction with respect to* g if there exists a function $\alpha : (0,\infty) \to [0,1)$ satisfying $\limsup_{r\to t^+}\alpha(r) < 1$ for every $t \in [0,\infty)$ and $h : X \to [0,\infty)$ such that for every $x,y \in X$, $x \neq y$ with $(x,y) \in E(G)$,

(i) $H(T(x),T(y)) \le \alpha(d(x,y))d(x,y) + h(g(y))d(g(y),T(x))$,

(ii) if $u \in T(x)$ and $v \in T(y)$ are such that $d(u,v) \le d(x,y)$, then $(u,v) \in E(G)$.

Example 3.2 Let $X = \{\frac{1}{2^n} \mid n \in \mathbb{N}\cup\{0\}\} \cup \{0\}$, $d(x,y) = |x-y|$ for $x,y \in X$. Let $E(G) = \{(\frac{1}{2^n},0),(\frac{1}{2^{2n}},\frac{1}{2^{2n+1}}),(\frac{1}{2^{2n+1}},\frac{1}{2^{2n+2}}); n \in \mathbb{N}\cup\{0\}\} \cup \{(0,0)\}$. Let $\alpha : (0,\infty) \to [0,1)$ be defined by $\alpha(t) = \frac{1}{2}$ for all $t \in (0,\infty)$. Let $T : X \to \mathrm{CB}(X)$ be defined by

$$T(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \{\frac{1}{2^{2k+5}},\frac{1}{2^{2k+7}},\frac{1}{2^{2k+9}},\ldots\}\cup\{0\} & \text{if } x = \frac{1}{2^{2k+1}}, k \in \mathbb{N}\cup\{0\}, \\ \{1\} & \text{if } x = \frac{1}{2^{2k}}, k \in \mathbb{N}\cup\{0\}. \end{cases}$$

Let $g : X \to X$ be defined by

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{2^{k+2}} & \text{if } x = \frac{1}{2^k}, k \in \mathbb{N}, \\ 1 & \text{if } x = 1. \end{cases}$$

Let $h : X \to [0,\infty)$ be defined by

$$h(x) = \begin{cases} 1 & \text{if } x = 0, \\ 2^{k+1} & \text{if } x = \frac{1}{2^k}, k \in \mathbb{N}, \\ 0 & \text{if } x = 1. \end{cases}$$

We show that $T : X \to \mathrm{CB}(X)$ is a weak G-contraction with respect to g. Let $(x,y) \in E(G)$.

Case 1. $(x,y) = (\frac{1}{2^n},0)$ for some $n \in \mathbb{N}\cup\{0\}$.

If $n = 2k$ for some $k \in \mathbb{N}\cup\{0\}$, we have

$$\begin{aligned} H\biggl(T\biggl(\frac{1}{2^{2k}}\biggr),T(0)\biggr) &= H\bigl(\{1\},\{0\}\bigr) \\ &\le \frac{1}{2^{2k+1}} + 1 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^{2k+1}} + h(0)d\big(0, \{1\}\big) \\
&= \alpha\left(d\left(\frac{1}{2^{2k}}, 0\right)\right)d\left(\frac{1}{2^{2k}}, 0\right) + h\big(g(0)\big)d\left(g(0), T\left(\frac{1}{2^{2k}}\right)\right).
\end{aligned}$$

If $n = 2k + 1$ for some $k \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
H\left(T\left(\frac{1}{2^{2k+1}}\right), T(0)\right) &= H\left(\left\{\frac{1}{2^{2k+5}}, \frac{1}{2^{2k+7}}, \frac{1}{2^{2k+9}}, \ldots\right\} \cup \{0\}, \{0\}\right) \\
&\leq \frac{1}{2^{2k+2}} \\
&= \frac{1}{2^{2k+2}} + h(0)d\left(0, \left\{\frac{1}{2^{2k+5}}, \frac{1}{2^{2k+7}}, \frac{1}{2^{2k+9}}, \ldots\right\} \cup \{0\}\right) \\
&= \alpha\left(d\left(\frac{1}{2^{2k+1}}, 0\right)\right)d\left(\frac{1}{2^{2k+1}}, 0\right) + h\big(g(0)\big)d\left(g(0), T\left(\frac{1}{2^{2k+1}}\right)\right).
\end{aligned}$$

Since $(x, 0) \in E(G)$ for all $x \in X$ and $T(0) = \{0\}$, it implies $(u, 0) \in E(G)$ for all $u \in T(\frac{1}{2^n})$. So the condition (ii) is satisfied.

<u>Case 2</u>. $(x, y) = (\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}})$ for some $n \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
H\left(T\left(\frac{1}{2^{2n}}\right), T\left(\frac{1}{2^{2n+1}}\right)\right) &= H\left(\{1\}, \left\{\frac{1}{2^{2n+5}}, \frac{1}{2^{2n+7}}, \frac{1}{2^{2n+9}}, \ldots\right\} \cup \{0\}\right) \\
&\leq \frac{1}{2^{2n+2}} + 2^{2n+4}\left|\frac{1}{2^{2n+3}} - 1\right| \\
&= \frac{1}{2^{2n+2}} + h\left(\frac{1}{2^{2n+3}}\right)d\left(\frac{1}{2^{2n+3}}, \{1\}\right) \\
&= \alpha\left(d\left(\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\right)\right)d\left(\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\right) \\
&\quad + h\left(g\left(\frac{1}{2^{2n+1}}\right)\right)d\left(g\left(\frac{1}{2^{2n+1}}\right), T\left(\frac{1}{2^{2n}}\right)\right).
\end{aligned}$$

We see that for each $u \in T(\frac{1}{2^{2n}})$ and $v \in T(\frac{1}{2^{2n+1}})$, we have

$$d(u, v) > d\left(\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\right).$$

<u>Case 3</u>. $(x, y) = (\frac{1}{2^{2n+1}}, \frac{1}{2^{2n+2}})$ for some $n \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned}
H\left(T\left(\frac{1}{2^{2n+1}}\right), T\left(\frac{1}{2^{2n+2}}\right)\right) &= H\left(\left\{\frac{1}{2^{2n+5}}, \frac{1}{2^{2n+7}}, \frac{1}{2^{2n+9}}, \ldots\right\} \cup \{0\}, \{1\}\right) \\
&\leq \frac{1}{2^{2n+3}} + 2^{2n+5}\left|\frac{1}{2^{2n+4}} - \frac{1}{2^{2n+5}}\right| \\
&= \frac{1}{2^{2n+3}} + h\left(\frac{1}{2^{2n+4}}\right)d\left(\frac{1}{2^{2n+4}}, \left\{\frac{1}{2^{2n+5}}, \frac{1}{2^{2n+7}}, \ldots\right\} \cup \{0\}\right) \\
&= \alpha\left(d\left(\frac{1}{2^{2n+1}}, \frac{1}{2^{2n+2}}\right)\right)d\left(\frac{1}{2^{2n+1}}, \frac{1}{2^{2n+2}}\right) \\
&\quad + h\left(g\left(\frac{1}{2^{2n+2}}\right)\right)d\left(g\left(\frac{1}{2^{2n+2}}\right), T\left(\frac{1}{2^{2n+1}}\right)\right).
\end{aligned}$$

We also see that for each $u \in T(\frac{1}{2^{2n+1}})$ and $v \in T(\frac{1}{2^{2n+2}})$, we have

$$d(u,v) > d\left(\frac{1}{2^{2n+1}}, \frac{1}{2^{2n+2}}\right).$$

Therefore $T : X \to \mathrm{CB}(X)$ is a weak G-contraction with respect to g.

Theorem 3.3 *Let (X,d) be a complete metric space and let $g : X \to X$ be a continuous self-map and $T : X \to \mathrm{CB}(X)$ a weak G-contraction with respect to g. Suppose that*:

(1) *there is $x_0 \in X$ such that $(x_0, y) \in E(G)$ for some $y \in T(x_0)$,*
(2) *$T(x)$ is g-invariant (i.e., $g(T(x)) \subseteq T(x)$ for each $x \in X$),*
(3) *for any sequence $\{x_n\}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $(x_{n_k}, x) \in E(G)$ for some $k \in \mathbb{N}$.*

Then $\mathcal{COP}(g,T) \cap \mathcal{F}(T) \neq \emptyset$.

Proof Let $x_0 \in X$ and $x_1 \in T(x_0)$ such that $(x_0, x_1) \in E(G)$.

By assumption (2), we have $g(x_1) \in T(x_0)$. If $x_0 = x_1$ or $\alpha(d(x_0,x_1)) = 0$, then $x_0 \in \mathcal{COP}(g,T) \cap \mathcal{F}(T)$. Suppose $x_0 \neq x_1$ and $\alpha(d(x_0,x_1)) \neq 0$. We can choose $n_1 \in \mathbb{N}$ such that

$$\left[\alpha\left(d(x_0,x_1)\right)\right]^{n_1} < \left[1 - \alpha\left(d(x_0,x_1)\right)\right] d(x_0,x_1).$$

This implies by Lemma 2.1 that there exists $x_2 \in T(x_1)$ such that

$$\begin{aligned} d(x_1,x_2) &\leq H\left(T(x_0), T(x_1)\right) + \left[\alpha\left(d(x_0,x_1)\right)\right]^{n_1} \\ &\leq \alpha\left(d(x_0,x_1)\right) d(x_0,x_1) + h\left(g(x_1)\right) d\left(g(x_1), T(x_0)\right) + \left[\alpha\left(d(x_0,x_1)\right)\right]^{n_1} \\ &= \alpha\left(d(x_0,x_1)\right) d(x_0,x_1) + \left[\alpha\left(d(x_0,x_1)\right)\right]^{n_1} \\ &< d(x_0,x_1). \end{aligned}$$

Hence $(x_1,x_2) \in E(G)$. By assumption (2), we have $g(x_2) \in T(x_1)$. If $x_1 = x_2$ or $\alpha(d(x_1,x_2)) = 0$, then we have $x_1 \in \mathcal{COP}(g,T) \cap \mathcal{F}(T)$. Suppose $x_1 \neq x_2$ and $\alpha(d(x_1,x_2)) \neq 0$. We can choose $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that

$$\left[\alpha\left(d(x_1,x_2)\right)\right]^{n_2} < \left[1 - \alpha\left(d(x_1,x_2)\right)\right] d(x_1,x_2).$$

It follows by Lemma 2.1 that there exists $x_3 \in T(x_2)$ such that

$$\begin{aligned} d(x_2,x_3) &\leq H\left(T(x_1), T(x_2)\right) + \left[\alpha\left(d(x_1,x_2)\right)\right]^{n_2} \\ &\leq \alpha\left(d(x_1,x_2)\right) d(x_1,x_2) + h\left(g(x_2)\right) d\left(g(x_2), T(x_1)\right) + \left[\alpha\left(d(x_1,x_2)\right)\right]^{n_2} \\ &= \alpha\left(d(x_1,x_2)\right) d(x_1,x_2) + \left[\alpha\left(d(x_0,x_1)\right)\right]^{n_1} \\ &< d(x_1,x_2). \end{aligned}$$

Hence $(x_2,x_3) \in E(G)$. By assumption (2), we have $g(x_3) \in T(x_2)$.

By induction, we obtain a sequence $\{x_k\}$ in X and a sequence of positive integers $\{n_k\}_{k \in \mathbb{N}}$ satisfying the property that for each $k \in \mathbb{N}$, $x_{k+1} \in T(x_k)$, $g(x_k) \in T(x_{k-1})$, $(x_k, x_{k+1}) \in E(G)$,

$$\left[\alpha\left(d(x_{k-1},x_k)\right)\right]^{n_k} < \left[1 - \alpha\left(d(x_{k-1},x_k)\right)\right] d(x_{k-1},x_k)$$

and

$$d(x_k,x_{k+1}) \leq H\big(T(x_{k-1}),T(x_k)\big) + \big[\alpha\big(d(x_{k-1},x_k)\big)\big]^{n_k}.$$

From the above inequality, we get $d(x_k,x_{k+1}) < d(x_{k-1},x_k)$, *i.e.*, $\{d(x_k,x_{k+1})\}$ is a decreasing sequence. It follows from Lemma 2.2 that $\{x_k\}$ is a Cauchy sequence in X. Since X is complete, there is an $x \in X$ such that $x_k \to x$ as $k \to \infty$. Since g is continuous, $g(x_k) \to g(x)$ as $k \to \infty$. By assumption (3), there is a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $(x_{k_n},x) \in E(G)$ for all $n \in \mathbb{N}$. Since T is a weak G-contraction with respect to g, we have

$$\begin{aligned} d\big(x,T(x)\big) &\leq d(x,x_{k_n+1}) + d\big(x_{k_n+1},T(x)\big) \\ &\leq d(x,x_{k_n+1}) + H\big(T(x_{k_n}),T(x)\big) \\ &\leq d(x,x_{k_n+1}) + \alpha\big(d(x_{k_n},x)\big)d(x_{k_n},x) + h\big(g(x)\big)d\big(g(x),T(x_{k_n})\big) \\ &\leq d(x,x_{k_n+1}) + \alpha\big(d(x_{k_n},x)\big)d(x_{k_n},x) + h\big(g(x)\big)d\big(g(x),g(x_{k_n+1})\big). \end{aligned}$$

By taking $n \to \infty$ in the above inequality, we get $d(x,T(x)) = 0$. Since $T(x)$ is closed, we have $x \in T(x)$. By assumption (2), we get $g(x) \in T(x)$. Therefore $x \in \mathcal{COP}(g,T) \cap \mathcal{F}(T)$. □

Example 3.4 Let $X, E(G), d, \alpha, T, g$, and h be the same as in Example 3.2. Then T is a weak G-contraction with respect to g, and g is continuous. It is easy to see that the conditions (1)-(3) of Theorem 3.3 hold. Hence all conditions of Theorem 3.3 are satisfied and we see that $\mathcal{COP}(g,T) \cap \mathcal{F}(T) = \{0,1\}$.

Remark 3.5

(i) In Theorem 3.3, if we take a directed graph G with $E(G) = X \times X$, we obtain Theorem 2.2 of Du [8] immediately.

(ii) In Theorem 3.3, if we take a function $h = 0$, then we obtain the existence result which is similar to Theorem 3 of Sultana and Vetrivel [14].

(iii) In Theorem 3.3, if we take a directed graph G with $E(G) = X \times X$ and a function $h = 0$, we obtain immediately the Mizoguchi-Takahashi theorem [7].

(iv) In Theorem 3.3, if we take a directed graph G with $E(G) = X \times X$, a function g that is the identity mapping on X, and a function $h = L$, for some $L \geq 0$, we obtain the Berinde and Berinde theorem ([4], Theorem 4).

Example 3.6 Let X, d, α be the same as in Example 3.2. Let

$$\begin{aligned} E(G) = \bigg\{ &\bigg(\frac{1}{2^n},0\bigg), \bigg(\frac{1}{2^{2n}},\frac{1}{2^{2n+1}}\bigg), \bigg(\frac{1}{2^{2n+1}},\frac{1}{2^{2n+2}}\bigg), \bigg(\frac{1}{2^{2n+2}},\frac{1}{2^{2n+1}}\bigg), \\ &\bigg(\frac{1}{2^{2n+3}},\frac{1}{2^{2n+1}}\bigg); n \in \mathbb{N} \cup \{0\} \bigg\}. \end{aligned}$$

Let $T : X \to \mathrm{CB}(X)$ be defined by

$$T(x) = \begin{cases} \{0\} & \text{if } x = 0, \\ \{\frac{1}{2^{2k}},\frac{1}{2^{2k+1}}\} & \text{if } x = \frac{1}{2^{2k+1}}, k \in \mathbb{N} \cup \{0\}, \\ \{\frac{1}{2^{2k+2}},\frac{1}{2^{2k+3}}\} & \text{if } x = \frac{1}{2^{2k}}, k \in \mathbb{N} \cup \{0\}. \end{cases}$$

Let $g : X \to X$ be defined by

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{2^{2k+1}} & \text{if } x = \frac{1}{2^{2k}}, k \in \mathbb{N} \cup \{0\}, \\ \frac{1}{2^{2k}} & \text{if } x = \frac{1}{2^{2k+1}}, k \in \mathbb{N} \cup \{0\}. \end{cases}$$

Let $L = \frac{5}{2}$ and $h : X \to [0, \frac{5}{2}]$ be defined by

$$h(x) = \begin{cases} 2 & \text{if } x = 0, \\ 2 - \frac{1}{2^{2k}} & \text{if } x = \frac{1}{2^{2k}}, k \in \mathbb{N} \cup \{0\}, \\ 5/2 & \text{if } x = \frac{1}{2^{2k+1}}, k \in \mathbb{N} \cup \{0\}. \end{cases}$$

It is easy to see that assumptions (1), (2), and (3) of Theorem 3.3 hold true and g is continuous on X. By using the same calculation as in Example 3.2, it can be shown that T is a weak G-contraction with respect to g. We note that the function h above is a bounded function on X and $0 \leq h(x) \leq \frac{5}{2}$ for all $x \in X$. Therefore all conditions of Theorem 3.3 are satisfied and we see that $\mathcal{COP}(g, T) \cap \mathcal{F}(T) = \{\frac{1}{2^{2n+1}} : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$.

The following result is also immediately obtained by Theorem 3.3, by setting $h(x) = L$ for all $x \in X$ and some $L \geq 0$.

Corollary 3.7 *Let (X, d) be a complete metric space. Let $T : X \to \mathrm{CB}(X)$ be a multivalued mapping, $g : X \to X$ be a continuous self-map, and $\alpha : (0, \infty) \to [0, 1)$ a mapping satisfying $\limsup_{r \to t^+} \alpha(r) < 1$ for every $t \in [0, \infty)$. Suppose that the following conditions hold*:

(1) *there is $x_0 \in X$ such that $(x_0, y) \in E(G)$ for some $y \in T(x_0)$*,
(2) *$T(x)$ is g-invariant (i.e., $g(T(x)) \subseteq T(x)$ for each $x \in X$)*,
(3) *for any sequence $\{x_n\}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $(x_{n_k}, x) \in E(G)$ for some $k \in \mathbb{N}$.*

If T satisfies the following two conditions:

(4) *there exists $L \geq 0$ such that*

$$H\bigl(T(x), T(y)\bigr) \leq \alpha\bigl(d(x, y)\bigr)d(x, y) + Ld\bigl(g(y), T(x)\bigr), \quad \text{for all } x, y \in X,$$

$x \neq y$ with $(x, y) \in E(G)$,
(5) *if $u \in T(x)$ and $v \in T(y)$ are such that $d(u, v) \leq d(x, y)$, then $(u, v) \in E(G)$.*

Then $\mathcal{COP}(g, T) \cap \mathcal{F}(T) \neq \emptyset$.

If we set g in Theorem 3.3 to be the identity map on X, then we obtain the following result.

Corollary 3.8 *Let (X, d) be a complete metric space. Let $T : X \to \mathrm{CB}(X)$ be a multivalued mapping, and $\alpha : (0, \infty) \to [0, 1)$ a mapping satisfying $\limsup_{r \to t^+} \alpha(r) < 1$ for every $t \in [0, \infty)$. Suppose that the following conditions hold*:

(1) *there is $x_0 \in X$ such that $(x_0, y) \in E(G)$ for some $y \in T(x_0)$*,
(2) *for any sequence $\{x_n\}_{n \in \mathbb{N}}$ in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $(x_{n_k}, x) \in E(G)$ for some $k \in \mathbb{N}$.*

If T satisfies the following two conditions:

(3) *there exists a function* $h : X \to [0,\infty)$ *such that*

$$H\big(T(x),T(y)\big) \le \alpha\big(d(x,y)\big)d(x,y) + h(y)d\big(y,T(x)\big), \quad \text{for all } x,y \in X,$$

$x \ne y$ *with* $(x,y) \in E(G)$,

(4) *if* $u \in T(x)$ *and* $v \in T(y)$ *are such that* $d(u,v) \le d(x,y)$, *then* $(u,v) \in E(G)$.

Then $\mathcal{F}(T) \ne \emptyset$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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