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Spin in Relativistic Quantum Theory

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Abstract We discuss the role of spin in Poincaré invariant formulations of quantum mechanics.

1 Introduction

In this paper, we discuss the role of spin in relativistic few-body models. The new feature with spin in relativistic quantum mechanics is that sequences of rotationless Lorentz transformations that map rest frames to rest frames can generate rotations. To get a well-defined spin observable one needs to define it in one preferred frame and then use specific Lorentz transformations to relate the spin in the preferred frame to any other frame. Different choices of the preferred frame and the specific Lorentz transformations lead to an infinite number of possible observables that have all of the properties of a spin. This paper provides a general discussion of the relation between the spin of a system and the spin of its elementary constituents in relativistic few-body systems.

In Sect. 2, we discuss the Poincaré group, which is the group relating inertial frames in special relativity. Unitary representations of the Poincaré group preserve quantum probabilities in all inertial frames, and define the relativistic dynamics. In Sect. 3, we construct a large set of abstract operators out of the Poincaré generators, and determine their commutation relations and transformation properties. In Sect. 4, we identify complete sets of commuting observables, including a large class of spin operators. We use the Poincaré commutation relations to determine the eigenvalue spectrum of these operators. Representations of the physical Hilbert space are constructed as square integrable functions of these commuting observables over their spectra. The transformation properties of these operators are used to construct unitary representations of the Poincaré group and its infinitesimal generators on this space. This construction gives irreducible representations of the Poincaré group.

The commuting observables introduced in Sect. 4 include a large class of spin observables. All of these operators are functions of the Poincaré generators, are Hermetian, satisfy $SU(2)$ commutation relations, commute with the four momentum, and have a square that is the spin Casimir operator of the Poincaré group.

During the preparation of this paper, Walter Glöckle passed away. We dedicate this paper to Walter, who was a great friend and collaborator.

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In Sect. 5, we discuss the most important examples of spin operators and how they are related. These are the helicity, canonical, and light-front spins. In Sect. 6, we discuss the problem of adding angular momenta; specifically how single-particle spins and orbital angular momenta are added in relativistic systems to obtain the total spin of the system. This is the problem of constructing Clebsch–Gordan coefficients for the Poincaré group. We consider the implications doing this using different spin and orbital observables. We show that the coupling is, up to some overall rotations on the initial and final spins, independent of the spin observables used in the coupling. In Sect. 7, we discuss the relation between two and four component spinors and their relation to field theory. We show in general, how boosts transform Poincaré covariant spinors to Lorentz covariant spinors, and how this role is played by the u and v Dirac spinors in the spin $1/2$ case. In Sect. 8, we argue that there is no loss of generality in working with models with a non-interacting spin (Bakamjian–Thomas models) by showing that any model is related to a Bakamjian–Thomas model [1] by an S-matrix preserving unitary transformation. In Sect. 9, we consider aspects of the relativistic three-nucleon problem. We show how relativistic invariance can be realized by requiring invariance with respect to rotations in the rest frame of the three-nucleon system. We also argue that S-matrix cluster properties is an important additional constraint on the treatment of the spin. Finally in Sect. 10, we discuss the relation between the different types of spins and experimental observables.

2 The Poincaré Group

The Poincaré group is the group of space–time transformations that relate different inertial frames in the theory of special relativity. In a relativistically invariant quantum theory the Poincaré group is a symmetry group of the theory [2].

The Poincaré group is the group of point transformations that preserve the proper time, τ_{ab}, or proper distance, d_{ab}, between any two events with space–time coordinates x_a^μ and x_b^μ,

$$-\tau_{ab}^2 = d_{ab}^2 = (x_a - x_b)^\mu (x_a - x_b)^\nu \eta_{\mu\nu} = (x_a - x_b)^2, \tag{1}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric with signature $(-,+,+,+)$ and repeated 4-vector indices are assumed to be summed from 0 to 3.

The most general point transformation, $x^\mu \to x'^\mu = f^\mu(x)$, satisfying (1) has the form

$$x^\mu \to x^{\mu'} = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{2}$$

where a^μ is a constant 4-vector, x^μ is x_a^μ or x_b^μ and the Lorentz transformation, $\Lambda^\mu{}_\nu$, is a constant matrix satisfying

$$\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}. \tag{3}$$

The full Poincaré group includes Lorentz transformations, $\Lambda^\mu{}_\nu$, that are not continuously connected to the identity. These transformations involve discrete space reflections, time reversals, or both. Since time reversal and space reflection are not symmetries of the weak interactions, the symmetry group associated with special relativity is the subgroup of Poincaré transformations that is continuously connected to the identity. This subgroup contains the active transformations that can be experimentally realized. In this paper, the term Poincaré group refers to this subgroup.

It is sometimes useful to represent Poincaré transformations using the group of complex 2×2 matrices with unit determinant [3], $SL(2,\mathbb{C})$. In this representation, real four vectors are represented by 2×2 Hermetian matrices. A basis for the 2×2 Hermetian matrices (over the real numbers) are the identity and the three Pauli spin matrices

$$\sigma_\mu := (I, \sigma_1, \sigma_2, \sigma_3). \tag{4}$$

There is a 1–1 correspondence between real four vectors and 2×2 Hermetian matrices given by

$$\mathsf{X} := x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \qquad x^\mu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu \mathsf{X}). \tag{5}$$

The determinant of X is the square of the proper time of the vector

$$\tau^2 = \det(\mathsf{X}) = -\eta_{\mu\nu}x^\mu x^\nu. \tag{6}$$

The most general linear transformation that preserves both the Hermiticity and the determinant of X has the form

$$\mathsf{X} \to \mathsf{X}' = \Lambda \mathsf{X} \Lambda^\dagger \tag{7}$$

where Λ are complex 2×2 matrices with $\det(\Lambda) = 1$. We have used the notation Λ for these 2×2 matrices because they are related to the 4×4 Lorentz transformation $\Lambda^\mu{}_\nu$ by

$$\Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\sigma_\mu \Lambda \sigma_\nu \Lambda^\dagger). \tag{8}$$

This is a 2 to 1 correspondence because both Λ and $-\Lambda$ result in the same $\Lambda^\mu{}_\nu$ in (8). This relation between $SL(2, \mathbb{C})$ and the Lorentz group is the same 2 to 1 correspondence that one has in relating $SU(2)$ rotations to $SO(3)$ rotations. It emerges when the $SL(2, \mathbb{C})$ matrices are restricted to the $SU(2)$ subgroup. In the 2×2 matrix representation, a Poincaré transformation has the form

$$\mathsf{X} \to \mathsf{X}' = \Lambda \mathsf{X} \Lambda^\dagger + \mathsf{A} \qquad \mathsf{A} = \mathsf{A}^\dagger \tag{9}$$

where

$$\mathsf{A} = a^\mu \sigma_\mu \qquad a^\mu = \frac{1}{2}\mathrm{Tr}(\mathsf{A}\sigma_\mu). \tag{10}$$

Elements of the Poincaré group are pairs $(\Lambda^\mu{}_\nu, a^\mu)$ or equivalently (Λ, A). The group product is

$$(\Lambda_2^\mu{}_\nu, a_2^\mu)(\Lambda_1^\mu{}_\nu, a_1^\mu) = (\Lambda_2^\mu{}_\alpha \Lambda_1^\alpha{}_\nu, \Lambda_2^\mu{}_\nu a_1^\nu + a_2^\mu) \tag{11}$$

or equivalently

$$(\Lambda_2, \mathsf{A}_2)(\Lambda_1, \mathsf{A}_1) = (\Lambda_2 \Lambda_1, \Lambda_2 \mathsf{A}_1 \Lambda_2^\dagger + \mathsf{A}_2). \tag{12}$$

The identity is $(\eta^\mu{}_\nu, 0)$ or $(I, 0)$, and the inverse is

$$((\Lambda^{-1})^\mu{}_\nu, -(\Lambda^{-1})^\mu{}_\nu a^\nu) \tag{13}$$

or

$$(\Lambda^{-1}, -\Lambda^{-1}\mathsf{A}\Lambda^{\dagger -1}). \tag{14}$$

The most general 2×2 matrix with unit determinant has the form

$$\Lambda(\mathbf{z}) = e^{\mathbf{z}\cdot\boldsymbol{\sigma}} = \cosh(z) + \frac{1}{z}\sinh(z)\mathbf{z}\cdot\boldsymbol{\sigma} \ , \tag{15}$$

where $\mathbf{z}$ is a complex 3-vector and $z = \sqrt{\sum_k z_k^2}$ is a complex scalar. The branch of the square root does not matter because both $\frac{1}{z}\sinh(z)$ and $\cosh(z)$ are even in z. This representation can be understood by noting that the σ_μ are a basis (over the complex numbers) for all complex matrices and $\det(e^{z^\mu\sigma_\mu}) = e^{z^\mu \mathrm{Tr}(\sigma_\mu)} = e^{2z^0}$ which is 1 for $z^0 = 0$.

It is easy to see that the matrices $\Lambda(\mathbf{z})$ correspond to Lorentz transformations that are continuously connected to the identity because

$$\Lambda(\lambda\mathbf{z}) = e^{\lambda\mathbf{z}\cdot\boldsymbol{\sigma}} \tag{16}$$

is a Lorentz transformation for all λ and it continuously approaches the identity as λ varies between 1 and 0. The $SO(3, 1)$ Lorentz transformation constructed by using (16) in (8) is also continuously connected to the identity.

When $\mathbf{z} = \boldsymbol{\rho}/2$ is a real vector, $\Lambda(\boldsymbol{\rho}/2)$ is a positive matrix (Hermetian with positive eigenvalues). It corresponds to a rotationless Lorentz transformation in the direction $\hat{\boldsymbol{\rho}}$ with rapidity ρ:

$$e^{\mathbf{z}\cdot\boldsymbol{\sigma}} \to e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} = \cosh(\rho/2) + \hat{\boldsymbol{\rho}}\cdot\boldsymbol{\sigma}\sinh(\rho/2). \tag{17}$$

Using Eq. (17) in (8) leads to the four-vector form of this transformation

$$\begin{aligned}\mathbf{x}' &= \mathbf{x} + \hat{\boldsymbol{\rho}}\sinh(\rho)x^0 + \hat{\boldsymbol{\rho}}(\cosh(\rho)-1)(\hat{\boldsymbol{\rho}}\cdot\mathbf{x})\\ x^{0'} &= \cosh(\rho)x^0 + \sinh(\rho)(\hat{\boldsymbol{\rho}}\cdot\mathbf{x})\\ \rho &= \sqrt{\boldsymbol{\rho}^2}.\end{aligned} \tag{18}$$

If we make the identifications $\mathbf{p}/m = \hat{\boldsymbol{\rho}}\sinh(\rho) = \gamma\boldsymbol{\beta}$ and $p^0/m = \cosh(\rho) = \gamma$ this Lorentz transformation can also be parameterized by the transformed momentum of a mass m particle initially at rest as

$$(\Lambda_c(p))^\mu{}_\nu = \begin{pmatrix} \frac{p^0}{m} & \frac{\mathbf{p}}{m} \\ \frac{\mathbf{p}}{m} & \delta_{ij} + \frac{p_i p_j}{m(m+p^0)} \end{pmatrix}. \tag{19}$$

Equations (17–19) are different ways of parameterizing the same Lorentz transformation. We refer to this transformation as a rotationless or canonical Lorentz boost, hence the subscript c.

When $\mathbf{z} = i\boldsymbol{\theta}/2$ is an imaginary vector, $\Lambda(\mathbf{z})$ is unitary and corresponds to a rotation about the $\hat{\boldsymbol{\theta}}$ axis through an angle θ

$$e^{\mathbf{z}\cdot\boldsymbol{\sigma}} \to e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} = \cos(\theta/2) + i\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}\sin(\theta/2). \tag{20}$$

Again using Eq.(20) in (8) leads to

$$\mathbf{x}' = \cos(\theta)\mathbf{x} + \sin(\theta)(\mathbf{x}\times\hat{\boldsymbol{\theta}}) + (1-\cos(\theta))\hat{\boldsymbol{\theta}}(\hat{\boldsymbol{\theta}}\cdot\mathbf{x}). \tag{21}$$

A rotation about any axis by 2π transforms $\Lambda(\mathbf{z})$ to $-\Lambda(\mathbf{z})$ which corresponds to the same $\Lambda^\mu{}_\nu$ in (8).

Since any $SL(2,\mathbb{C})$ matrix A has a polar decomposition:

$$A = PU \qquad P = (AA^\dagger)^{1/2} \qquad U = (AA^\dagger)^{-1/2}A \tag{22}$$

or

$$A = U'P' \qquad P' = (A^\dagger A)^{1/2} \qquad U' = A(A^\dagger A)^{-1/2} \tag{23}$$

into the product of a positive Hermetian matrix P and a unitary matrix U, every Lorentz transformation can be decomposed into the product of a canonical boost and a rotation, in either order. The boost and rotation are the matrices $P(P')$ and $U(U')$, respectively, in (22–23).

In what follows we use the notation $\mathcal{P}$ to refer to both the group of Poincaré transformations connected to the identity and the group inhomogeneous $SL(2,\mathbb{C})$. $\mathcal{P}$ is a ten parameter group; six parameters are needed to fix the complex 3-vector $\mathbf{z}$ in $\Lambda(\mathbf{z})$, and four additional parameters are needed to fix a^μ.

One important property of the group $SL(2,\mathbb{C})$ that is relevant for the treatment of spin in Lorentz covariant theories is that $\Lambda(\mathbf{z})$ and $\Lambda(\mathbf{z})^*$ are inequivalent representations of $SL(2,\mathbb{C})$, which means that there are no constant matrices C satisfying

$$C\Lambda(\mathbf{z})C^{-1} = \Lambda(\mathbf{z})^* \tag{24}$$

for all $\mathbf{z}$. This is distinct from the subgroup $SU(2)$ (rotations) where for $\Lambda = R \in SU(2)$

$$\sigma_2 R\sigma_2 = R^*. \tag{25}$$

This observation is related to the appearance of four-component spinors in Lorentz covariant theories. The reason for this is that

$$\sigma_2 \mathsf{X}^*\sigma_2 = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix} \tag{26}$$

represents space reflection in the 2×2 matrix representation (5). Because space reflection (26) involves both a similarity transformation and a complex conjugation, the space reflected vector transforms under an inequivalent complex conjugate representation of $SL(2,\mathbb{C})$. In order to realize space reflection as a linear transformation in Lorentz covariant theories it is necessary to double the dimension of the representation space by including the direct sum of a space that transforms with the complex conjugate representation of $SL(2,\mathbb{C})$.

These considerations do not apply to Poincaré covariant representations because the little group, $SU(2)$, is equivalent (25) to its conjugate representation. This will be discussed in Sect. 6.

To show (25) note

$$\sigma_2 R \sigma_2 = \sigma_2 e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} \sigma_2 = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\sigma_2\boldsymbol{\sigma}\sigma_2} = e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}^*} = R^*. \tag{27}$$

To prove that no matrix C satisfying (24) exists, assume by contradiction that there is such a matrix. Let $\mathbf{z} = \mathbf{x}$ be real and then let $\mathbf{z} = i\mathbf{y}$ be imaginary. Differentiating both sides of Eq. (24) with respect to x_i and then y_i and setting $z_i = x_i = 0$ or $z_i = y_i = 0$ gives

$$C\sigma_i C^{-1} = \sigma_i^* \qquad \text{and} \qquad C\sigma_i C^{-1} = -\sigma_i^*. \tag{28}$$

Adding these equations gives $C\sigma_i C^{-1} = 0$ which is impossible for product of three invertible matrices, contradicting the assumed existence of such a C.

3 Operators

Wigner showed that the Poincaré symmetry in a quantum theory is realized by a unitary ray representation, $U(\Lambda, a)$, of the Poincaré group. Bargmann [4] showed that the ray representation can be replaced by a single-valued unitary representation of inhomogeneous $SL(2, \mathbb{C})$, satisfying

$$U(\Lambda_2, \mathsf{A}_2)U(\Lambda_1, \mathsf{A}_1) = U(\Lambda_2\Lambda_1, \Lambda_2\mathsf{A}_1\Lambda_2^\dagger + \mathsf{A}_2). \tag{29}$$

The infinitesimal generators of this representation form a source of an irreducible set of operators on the model Hilbert space.

Because the Poincaré group is a ten parameter group there are ten independent unitary one-parameter groups associated with space translations (3), time translations (1), rotations (3), and rotationless Lorentz transformations (3). The ten parameters can be chosen as the space–time translation parameters a^μ, three angles of rotation and the rapidities ρ in three independent directions. One can see from the inhomogeneous $SL(2, \mathbb{C})$ representation (12) that these define one-parameter groups:

$$(I, A_2)(I, A_1) = (I, A_1 + A_2) \tag{30}$$

$$(e^{i\frac{\theta_2}{2}\hat{\mathbf{e}}\cdot\boldsymbol{\sigma}}, 0)(e^{i\frac{\theta_1}{2}\hat{\mathbf{e}}\cdot\boldsymbol{\sigma}}, 0) = (e^{i\frac{\theta_2+\theta_1}{2}\hat{\mathbf{e}}\cdot\boldsymbol{\sigma}}, 0) \tag{31}$$

$$(e^{\frac{\rho_2}{2}\hat{\mathbf{e}}\cdot\boldsymbol{\sigma}}, 0)(e^{\frac{\rho_1}{2}\hat{\mathbf{e}}\cdot\boldsymbol{\sigma}}, 0) = (e^{\frac{\rho_2+\rho_1}{2}\hat{\mathbf{e}}\cdot\boldsymbol{\sigma}}, 0). \tag{32}$$

The unitary representation of these one-parameter groups have self-adjoint infinitesimal generators $\mathbf{P}$, H, $\mathbf{J}$, and $\mathbf{K}$, that can be obtained by differentiating with respect to the appropriate parameter. Equivalently, the one-parameter groups can be expressed directly as exponentials of these generators:

$$U[I, (0, \lambda\hat{\mathbf{a}})] = e^{-i\lambda\hat{\mathbf{a}}\cdot\mathbf{P}} \tag{33}$$

$$U[I, (a^0, \mathbf{0})] = e^{i\mathbf{a}^0\cdot H} \tag{34}$$

$$U[\Lambda(i\lambda\hat{\boldsymbol{\theta}}/2), 0] = e^{i\lambda\hat{\boldsymbol{\theta}}\cdot\mathbf{J}} \tag{35}$$

$$U[\Lambda(\lambda\hat{\boldsymbol{\rho}}/2), 0] = e^{i\lambda\hat{\boldsymbol{\rho}}\cdot\mathbf{K}}. \tag{36}$$

These generators are the linear momentum operators, $\mathbf{P}$, Hamiltonian, H, angular momentum operators, $\mathbf{J}$, and rotationless Lorentz boost generators, $\mathbf{K}$. These designations follow from the commutation relations which show that both the linear momentum $\mathbf{P}$ and angular momentum $\mathbf{J}$ commute with H and are thus conserved.

The commutation relations and transformation properties of the generators follow from the group representation property. To construct the commutator of the generators of the one-parameter groups $g_2(\lambda_2)$ and $g_1(\lambda_1)$ with parameters λ_2 and λ_1 use the group representation property to express the product as:

$$U^\dagger[g_1(\lambda_1)]U[g_2(\lambda_2)]U[g_1(\lambda_1)] = U[g_1(-\lambda_1)g_2(\lambda_2)g_1(\lambda_1)]\,. \tag{37}$$

Taking the second derivative $\frac{\partial^2}{\partial\lambda_1\partial\lambda_2}$ of this expression and setting $\lambda_1 = \lambda_2 = 0$ gives the commutator of the generators of the unitary one parameter groups $U[g_2(\lambda_2)]$ and $U[g_1(\lambda_1)]$. For example, to calculate the commutator of P^i with K^j use the group representation property to express the product of the three transformations as a single transformation:

$$U(\Lambda, 0)U(I, (0, \mathbf{a}))U(\Lambda^{-1}, 0) = U(I, \Lambda a). \tag{38}$$

The commutator between P^i and K^j can be determined by considering infinitesimal transformations

$$U(\Lambda, 0) \to e^{i\boldsymbol{\rho}\cdot\mathbf{K}} = I + i\boldsymbol{\rho}\cdot\mathbf{K} + \cdots \tag{39}$$
$$U(I, (0, \mathbf{a})) \to e^{-i\mathbf{a}\cdot\mathbf{P}} = I - i\mathbf{a}\cdot\mathbf{P} + \cdots \tag{40}$$
$$U(\Lambda^{-1}, 0) \to e^{-i\boldsymbol{\rho}\cdot\mathbf{K}} = I - i\boldsymbol{\rho}\cdot\mathbf{K} + \cdots \tag{41}$$
$$\Lambda a = (\sinh(\rho)(\hat{\boldsymbol{\rho}}\cdot\mathbf{a}), \mathbf{a} + \hat{\boldsymbol{\rho}}(\cosh(\rho) - 1)(\hat{\boldsymbol{\rho}}\cdot\mathbf{a})) \to (0, \mathbf{a}) + (\rho(\hat{\boldsymbol{\rho}}\cdot\mathbf{a}), \mathbf{0}) + \cdots . \tag{42}$$

To compute the commutator between P^i and K^j expand both sides of (38) using (39–42) to leading order in $\boldsymbol{\rho}$ and $\mathbf{a}$

$$\begin{aligned} U(\Lambda, 0)U(I, (0, \mathbf{a}))U(\Lambda^{-1}, 0) &= I - i\mathbf{a}\cdot\mathbf{P} - (i)^2 a^i \rho^j (K^j P^i - P^i K^j) + \cdots \\ &= U(I, \Lambda a) = I - i\mathbf{a}\cdot\mathbf{P} + i\rho^j \delta_{ji} a^i H + \cdots . \end{aligned} \tag{43}$$

Equating the coefficient of $a^i \rho^j$ gives

$$[K^j, P^i] = i\delta_{ij} H. \tag{44}$$

The 45 commutation relations involving all ten generators can be computed using different pairs of unitary one-parameter groups.

The group representation property (29) also implies transformation properties of the infinitesimal generators. For example if we set $\Lambda' = I$ the group representation properties give

$$\begin{aligned} U(\Lambda, a)U(\Lambda', a')U(\Lambda^{-1}, -\Lambda^{-1}a) &= U(\Lambda\Lambda'\Lambda^{-1}, \Lambda a'\Lambda^\dagger - \Lambda\Lambda'\Lambda^{-1}\Lambda'^\dagger a + a) \\ U(\Lambda, a)U(I, a')U(\Lambda^{-1}, -\Lambda^{-1}a) &= U(I, \Lambda a'\Lambda^\dagger - a + a). \end{aligned} \tag{45}$$

The parameter a' only appears in the translation generators on both sides of (45). Differentiating with respect to a'_μ and setting $a'_\mu = 0$ gives

$$U(\Lambda, a)(H, \mathbf{P})^\mu U^\dagger(\Lambda, a) = (\Lambda^{-1})^\mu{}_\nu (H, \mathbf{P})^\nu = (H, \mathbf{P})^\nu \Lambda_\nu{}^\mu. \tag{46}$$

It shows that the generators H and $\mathbf{P}$ transform like components of a four-vector under Lorentz transformations. This four vector is the four momentum:

$$P^\mu = (H, \mathbf{P}). \tag{47}$$

Similarly, letting $U(\Lambda', a') \to U(\Lambda', 0)$ in the first line of (45) and differentiating with respect to angle or rapidity shows that the six Lorentz generators transform as an antisymmetric tensor operator

$$J^{\mu\nu} = \begin{pmatrix} 0 & -K^x & -K^y & -K^z \\ K^x & 0 & J^z & -J^y \\ K^y & -J^z & 0 & J^x \\ K^z & J^y & -J^x & 0 \end{pmatrix}. \tag{48}$$

The transformation properties of these operators can be compactly summarized by the covariant forms of the transformation laws

$$U(\Lambda, a)P^\mu U^\dagger(\Lambda, a) = P^\nu \Lambda_\nu{}^\mu \tag{49}$$
$$U(\Lambda, a)J^{\mu\nu} U^\dagger(\Lambda, a) = (J^{\alpha\beta} - a^\alpha P^\beta + a^\beta P^\alpha)\Lambda_\alpha{}^\mu \Lambda_\beta{}^\nu \tag{50}$$

Table 1 The little groups for each of the standard vectors

Class	Standard vector	Little group
$P^2 = -M^2 < 0$; $P^0 > 0$	$p_s^\mu = (M, 0, 0, 0)$	SO(3)
$P^2 = -M^2 < 0$; $P^0 < 0$	$p_s^\mu = (-M, 0, 0, 0)$	SO(3)
$P^2 = 0$; $P^0 > 0$	$p_s^\mu = (1, 0, 0, 1)$	E(2)
$P^2 = 0$; $P^0 < 0$	$p_s^\mu = (-1, 0, 0, 1)$	E(2)
$P^2 = -M^2 = N^2 > 0$	$p_s^\mu = (0, 0, 0, N)$	SO(2,1)
$P^\mu = 0$	$p_s^\mu = (0, 0, 0, 0)$	SO(3,1)

and the commutation relations for the infinitesimal generators

$$\left[J^{\mu\nu}, J^{\alpha\beta}\right] = i(\eta^{\mu\alpha}J^{\nu\beta} - \eta^{\nu\alpha}J^{\mu\beta} + \eta^{\nu\beta}J^{\mu\alpha} - \eta^{\mu\beta}J^{\nu\alpha}) \tag{51}$$

$$\left[P^\mu, J^{\alpha\beta}\right] = i(\eta^{\mu\beta}P^\alpha - \eta^{\mu\alpha}P^\beta) \tag{52}$$

$$\left[P^\mu, P^\nu\right] = 0. \tag{53}$$

The spin is associated with another four vector that is a quadratic polynomial in the generators, called the Pauli–Lubanski vector [5], defined by

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu J_{\alpha\beta}. \tag{54}$$

The commutation relations

$$\left[W^\mu, W^\nu\right] = i\epsilon^{\mu\nu\alpha\beta}W_\alpha P_\beta \tag{55}$$

$$\left[W^\mu, P^\nu\right] = 0 \tag{56}$$

follow from (51–53).

The Poincaré Lie-algebra has two independent polynomial invariants [2] which are the square of the invariant mass (rest energy) of the system,

$$M^2 = -\eta_{\mu\nu}P^\mu P^\nu, \tag{57}$$

and the square of the Pauli–Lubanski vector

$$W^2 = \eta_{\mu\nu}W^\mu W^\nu. \tag{58}$$

When $M^2 \neq 0$ the spin is related to the invariants M^2 and W^2 by

$$\mathbf{j}^2 = W^2/M^2. \tag{59}$$

For massive systems ($M > 0$) the invariant W^2 is replaced by the spin $\mathbf{j}^2$ of the system. The operators are invariant because they commute with all of the Poincaré generators.

4 Spin and Irreducible Representations

Wigner [2] classified the unitary irreducible representations of the Poincaré group. His classification was based on the observation that Lorentz transformations can be used to transform an arbitrary four vector to one of six standard forms. This divides the set of four vectors into six disjoint Lorentz invariant equivalence classes. These familiar equivalence classes are time-like positive time, time-like negative time, light-like positive time, light-like negative time, space-like, and zero. Every four vector is a member of one of these six classes. Standard vectors, which are arbitrary but fixed vectors in each class, are given in Table 1. For each standard vector there is a little group, which is the subgroup of the Lorentz group that leaves that standard vector invariant. The little groups for each of the standard vectors are given in Table 1.

In Table 1 $SO(3)$ is the group of rotations in three dimensions, $E(2)$ is the Euclidean group in two dimensions, $SO(2,1)$ is the Lorentz group in $2+1$ dimensions, and $SO(3,1)$ is the Lorentz group in $3+1$ dimensions. Irreducible representations of the little groups are used as labels for the irreducible representation of the Poincaré group. The treatment of each of the six little groups is different and is not relevant to our treatment of particle spins. The interested reader is referred to Wigner's original paper [2].

For particles the relevant four vector is the particle's four momentum, which is a time-like positive-energy four vector for massive particles or a light-like positive-energy four vector for massless particles.

For a particle of mass $m > 0$ the most natural choice for the standard vector p_s is the rest four momentum $p_s = p_0 = (m, \mathbf{0})$. The little group for p_0 is the rotation group. If a particle at rest is observed in a rotated frame, the particle remains at rest but the spin of the particle will be rotated relative to the spin observed in the original frame. The particle's spin degrees of freedom are associated with irreducible representations of $SO(3)$, the little group that leaves $p_s = p_0$ unchanged.

The treatment of spin in relativistic quantum mechanics is slightly more complicated than it is in non-relativistic quantum mechanics. The relevant complication is because the commutator of two different rotationless Lorentz boost generators, $[K^k, K^l] = -i\epsilon^{klm} J^m$, gives a rotation generator. This means the sequences of rotationless Lorentz boosts can generate rotations. If we define the spin of a particle to be the spin measured in the particle's rest frame, then its spin seen by an observer in any other frame will depend on both the momentum of the particle in the transformed frame and the specific Lorentz transformation relating the two frames. To get an unambiguous definition of a spin observable it is necessary to specify both the frame where the spin is defined (or measured) and a set of standard Lorentz transformations relating a frame where the particle has momentum $\mathbf{p}$ to the frame where the spin is defined (or measured). The result is that there are an infinite number of possible choices of spin observables in relativistic quantum mechanics. Some common spin observables are the canonical spin, the light-front spin, and the helicity. While all of the spins that we will consider satisfy $SU(2)$ commutation relations, the most useful choices are characterized by different simplifying properties. The different spin observables are related by momentum-dependent rotations.

In this section, we discuss the general structure of spin operators in Poincaré invariant quantum mechanics. We will define spin operators as operator-valued functions of the infinitesimal generators. We begin by assuming that we are given a fixed standard vector p_s with $p_s^2 = -m^2$ and $p_s^0 > 0$. The standard vector does not have to be the rest vector, p_0. We also assume that we are given a parameterized set of Lorentz transformations, $\Lambda_s(p)$, that transform the standard vector, p_s, to any other four vector p with $p^2 = -m^2$,

$$\Lambda_s(p)^{\mu}{}_{\nu} p_s^{\nu} = p^{\mu}. \tag{60}$$

The choice of $\Lambda_s(p)$ and p_s are arbitrary, subject to the constraints $p_s^2 = -m^2$, $p_s^0 > 0$ and (60).

For example, one possible choice is $p_s = p_0$ and $\Lambda_s(p) = \Lambda_c(p)$, the rotationless Lorentz transformation (19) with rapidity $\boldsymbol{\rho} = \hat{\mathbf{p}} \sinh^{-1}(\frac{|\mathbf{p}|}{m})$, that transforms p_0 to p.

The next step is to make $\Lambda_s(p)$ into a *Lorentz transformation valued operator* by replacing p in the expression for $\Lambda_s(p)$ by the four-momentum operator $\underline{P}$ (we use underlines to indicate operators in this section). For example, for $\Lambda_s(p) = \Lambda_c(p)$ given in (19), the matrix of operators becomes

$$\Lambda_c^{\mu}{}_{\nu}(\underline{P}) := \begin{pmatrix} \underline{H}/\underline{M} & \underline{P}_x/\underline{M} & \underline{P}_y/\underline{M} & \underline{P}_z/\underline{M} \\ \underline{P}_x/\underline{M} & 1 + \frac{\underline{P}_x \underline{P}_x}{\underline{M}(\underline{M}+\underline{H})} & \frac{\underline{P}_x \underline{P}_y}{\underline{M}(\underline{M}+\underline{H})} & \frac{\underline{P}_x \underline{P}_z}{\underline{M}(\underline{M}+\underline{H})} \\ \underline{P}_y/\underline{M} & \frac{\underline{P}_y \underline{P}_x}{\underline{M}(\underline{M}+\underline{H})} & 1 + \frac{\underline{P}_y \underline{P}_y}{\underline{M}(\underline{M}+\underline{H})} & \frac{\underline{P}_y \underline{P}_z}{\underline{M}(\underline{M}+\underline{H})} \\ \underline{P}_z/\underline{M} & \frac{\underline{P}_z \underline{P}_x}{\underline{M}(\underline{M}+\underline{H})} & \frac{\underline{P}_z \underline{P}_y}{\underline{M}(\underline{M}+\underline{H})} & 1 + \frac{\underline{P}_z \underline{P}_z}{\underline{M}(\underline{M}+\underline{H})} \end{pmatrix} \tag{61}$$

More generally we define

$$\Lambda_{0s}(\underline{P}) = \Lambda_s(\underline{P}) \Lambda_s^{-1}(p_0). \tag{62}$$

Here the first transformation, $\Lambda_s^{-1}(p_0)$, is a constant matrix that transforms the constant 4-vector p_0 to the constant standard 4-vector p_s. The second transform is a matrix of *operators* that maps p_s to the operator $\underline{P}$. The first matrix is the identity when $p_s = p_0$. The combined transformation (62) is still a boost valued operator that transforms p_0 to P. The reason for discussing this more general case of spins with $p_s \neq p_0$ is that a similar type of spin arises naturally in composite systems when spins are coupled in the many-body problem [see (152)]. In the many-body case it is natural to choose p_s to be the momentum of the particle in the rest frame of the system rather than in the rest momentum of the particle. In the many-body case the constant transformation $\Lambda_s^{-1}(p_0)$ in (62) is replaced by an operator-valued transformation that transforms a particle at rest to its momentum in the rest frame of the system. This transformation is operator-valued because the momentum of the particle in the system's rest frame is an independent variable. These spin observables have the advantage that they can be added with ordinary $SU(2)$ Clebsch–Gordan coefficients. This will be illustrated in Sect. 6.

Given the operator $\Lambda_{0s}^{-1}(\underline{P})$ we define the s-spin operator

$$(0, \underline{\mathbf{j}}_s) = \frac{1}{\underline{m}} \Lambda_{0s}^{-1}(\underline{P})^{\mu}{}_{\nu} \underline{W}^{\nu} = -\frac{1}{2\underline{m}} \Lambda_{0s}^{-1}(\underline{P})^{\mu}{}_{\nu} \epsilon^{\nu\alpha\beta\gamma} \underline{P}_{\alpha} \underline{J}_{\beta\gamma}. \tag{63}$$

Note that all three components of $\underline{\mathbf{j}}$ are well-defined Hermetian operators because the mass, all components of the four momentum, and the Pauli–Lubanski vector are Hermetian and commute (53, 56). The definition (63) of $\underline{\mathbf{j}}_s$ depends on both the choice of a standard vector p_s and a standard boost $\Lambda_s(p)$.

The most familiar choices of s are associated with canonical spin, helicity, and light-front spin. For the canonical spin the standard boost $\Lambda_s(p) = \Lambda_c(p)$ is given by (19). For the helicity the standard boost is $\Lambda_h(p) = \Lambda_c(p)R(\hat{\mathbf{p}} \leftarrow \hat{\mathbf{z}}) = R(\hat{\mathbf{p}} \leftarrow \hat{\mathbf{z}})\Lambda_c(\hat{\mathbf{z}}p)$, where $R(\hat{\mathbf{p}} \leftarrow \hat{\mathbf{z}})$ is a rotation about the axis $\hat{\mathbf{z}} \times \hat{\mathbf{p}}$ through an angle $\cos^{-1}(\hat{\mathbf{z}} \cdot \hat{\mathbf{p}})$. For the light-front spin the $SL(2,\mathbb{C})$ representation of the standard boost is

$$\Lambda_f(p) = \begin{pmatrix} a & 0 \\ b+ic & 1/a \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{p^0+p^3}{m}} & 0 \\ \frac{p_1+ip_2}{\sqrt{m(p^0+p^3)}} & \sqrt{\frac{m}{p^0+p^3}} \end{pmatrix}. \tag{64}$$

Each of these choices has simplifying features that are advantageous for certain problems. These examples will be discussed in more detail in the next section.

Note that if the operator $\mathbf{j}_s$ is applied to an eigenstate of the four momentum with eigenvalue $p_0^{\mu} = (m, 0, 0, 0)$ then on this state the operator $\Lambda_{0s}(\underline{P})$ becomes the identity and $\underline{P}_{\alpha}$ becomes p_0. It follows that $\mathbf{j}_s$ becomes $j_s^i = \frac{1}{2}\epsilon^{ijk} J_{jk}$ which is the total angular momentum. This is consistent with the interpretation of the spin as the angular momentum in the particle's rest frame. In the relativistic case different spin operators are distinguished by the transformation used to get to the particles rest frame. These spins are normally identified in the particle's rest frame.

The spin (63) looks like it should be a set of four operators that transform as the components of a four vector because it has the form of a four-vector operator, W^{μ}, multiplied by the product of a Lorentz transformation $\Lambda_{0s}^{-1}(\underline{P})^{\mu}{}_{\nu}$ and a scalar, $1/\underline{m}$. Because of this it may be surprising that it has no zero-component. The reason that the spin is *not* a four-vector operator is because $\Lambda_s(\underline{P})$ is a Lorentz transform valued *operator* rather than a Lorentz transformation, so it corresponds to a different Lorentz transformation for each value of p. To see explicitly how the zero component vanishes assume that $\mathbf{j}_s$ acts on an eigenstate of the four momentum. Then the four-momentum $\underline{P}$ operators are replaced by the components of the four-momentum eigenvalue, p, including the $\underline{P}$ in the definition of the Pauli–Lubanski vector:

$$(0, \underline{\mathbf{j}}_s) \to \frac{1}{m} (\Lambda_{0s}^{-1}(p))^{\mu}{}_{\nu} \underline{W}^{\nu} = -\frac{1}{2m} \Lambda_{0s}^{-1}(p)^{\mu}{}_{\nu} \epsilon^{\nu\alpha\beta\gamma} p_{\alpha} \underline{J}_{\beta\gamma} \,. \tag{65}$$

Using the shorthand notation $\Lambda = \Lambda_{0s}^{-1}(p)^{\mu}{}_{\nu}$, along with the fact the $\epsilon^{\rho\alpha\beta\gamma}$ is a constant tensor,

$$\epsilon^{\rho\alpha\beta\gamma} = \epsilon^{\rho'\alpha'\beta'\gamma'} \Lambda_{\rho'}{}^{\rho} \Lambda_{\alpha'}{}^{\alpha} \Lambda_{\beta'}{}^{\beta} \Lambda_{\gamma'}{}^{\gamma} = (\Lambda^{-1})^{\rho}{}_{\rho'} \epsilon^{\rho'\alpha'\beta'\gamma'} \Lambda_{\alpha'}{}^{\alpha} \Lambda_{\beta'}{}^{\beta} \Lambda_{\gamma'}{}^{\gamma} \tag{66}$$

or equivalently

$$\Lambda^{\mu}{}_{\rho} \epsilon^{\rho\alpha\beta\gamma} = \epsilon^{\mu\rho\eta\chi} \Lambda_{\rho}{}^{\alpha} \Lambda_{\eta}{}^{\beta} \Lambda_{\chi}{}^{\gamma} \tag{67}$$

gives

$$\Lambda^{\mu}{}_{\rho} \epsilon^{\rho\alpha\beta\gamma} \underline{P}_{\alpha} \underline{J}_{\beta\gamma} = \epsilon^{\mu\rho\eta\chi} \Lambda_{\rho}{}^{\alpha} \Lambda_{\eta}{}^{\beta} \Lambda_{\chi}{}^{\gamma} \underline{P}_{\alpha} \underline{J}_{\beta\gamma}. \tag{68}$$

Since $\Lambda_{\rho}{}^{\alpha} p_{\alpha} = -m\delta_{\rho 0} = (-m, 0, 0, 0)$ it follows that

$$\underline{\Lambda}^{\mu}{}_{\rho} \epsilon^{\rho\alpha\beta\gamma} \underline{P}_{\alpha} \underline{J}_{\beta\gamma} = \underline{m} \epsilon^{\mu 0\eta\chi} \underline{\Lambda}_{\eta}{}^{\beta} \underline{\Lambda}_{\chi}{}^{\gamma} \underline{J}_{\beta\gamma} = \underline{m}(0, \underline{\mathbf{j}}_s), \tag{69}$$

which always has a 0 time component. The index $\alpha = 0$ arises because the only non-vanishing component of the transformed four momentum is the zero component. From this expression we obtain the following equivalent formula for the s-spin operator:

$$\underline{j}_s^i = \frac{1}{2} \epsilon_{ijk} \Lambda_{0s}^{-1}(\underline{P})^{j}{}_{\mu} \Lambda_{0s}^{-1}(\underline{P})^{k}{}_{\nu} \underline{J}^{\mu\nu}. \tag{70}$$

This spin observable is interpreted as the angular momentum measured in the particle's rest frame if the particle is transformed to the rest frame using the boost $\Lambda_{0s}^{-1}(\underline{P})$. The index s on the spin operator indicates that it is one of many possible spin operators that are functions of the Poincaré generators. Spin operators associated with different choices, s, t, of boosts are related by

$$(0, \underline{\mathbf{j}}_t)^\mu = \Lambda_{0t}^{-1}(\underline{P})^\mu{}_\nu \Lambda_{0s}(\underline{P})^\nu{}_\rho (0, \underline{\mathbf{j}}_s)^\rho = R_{ts}(\underline{P})^\mu{}_\nu (0, \underline{\mathbf{j}}_s)^\nu \tag{71}$$

where $R_{ts}(\underline{P}) := \Lambda_{0t}^{-1}(\underline{P})\Lambda_{0s}(\underline{P})$ is a rotation valued function of the momentum operators. We refer to rotations that relate different spin observables as generalized Melosh rotations. The original Melosh rotation [6] is the corresponding rotation that relates the light-front and canonical spins.

In what follows we no longer use an underscore to indicate operators. It follows from (63) that independent of how the individual components of $\mathbf{j}_s$ are constructed they satisfy $SU(2)$ commutation relations because, using (55), we find

$$[\mathbf{j}_s^l, \mathbf{j}_s^m] = \frac{1}{m^2} \Lambda_{0s}^{-1}(P)^l{}_\mu \Lambda_{0s}^{-1}(P)^m{}_\nu [W^\mu, W^\nu] = \frac{1}{m^2} \Lambda_{0s}^{-1}(P)^l{}_\mu \Lambda_{0s}^{-1}(P)^m{}_\nu i\epsilon^{\mu\nu\alpha\beta} W_\alpha P_\beta. \tag{72}$$

Again, because $\epsilon^{\mu\nu\alpha\beta}$ is a constant tensor this commutator is equal to

$$\begin{aligned}[\mathbf{j}_s^l, \mathbf{j}_s^m] &= \frac{i}{m^2}\epsilon^{lm\alpha\beta} \Lambda_{0s}^{-1}(P)_\alpha{}^\mu \Lambda_{0s}^{-1}(P)_\beta{}^\nu W_\mu P_\nu = -i\frac{m}{m^2}\epsilon^{lm\alpha 0} \Lambda_{0s}^{-1}(P)_\alpha{}^\nu W_\nu \\ &= \frac{i}{m}\epsilon^{lmn} \Lambda_{0s}^{-1}(P)_n{}^\nu W_\nu = i\epsilon^{lmn}\mathbf{j}_s^n. \end{aligned} \tag{73}$$

We also have

$$\mathbf{j}_s^2 = \eta_{\alpha\beta}\frac{1}{m^2}\Lambda_{0s}^{-1}(p)^\alpha{}_\mu \Lambda_{0s}^{-1}(p)^\beta{}_\nu W^\mu W^\nu = \frac{1}{m^2}\eta_{\alpha\beta} W^\alpha W^\beta = \mathbf{j}^2. \tag{74}$$

Thus, no matter which choice of p_s and $\Lambda_{0s}^{-1}(p)$ are used to define the spin, the components are always Hermetian functions of the Poincaré generators, commute with the four momentum, satisfy $SU(2)$ commutation relations, and the square is always the invariant $\mathbf{j}^2 = W^2/M^2$.

There are an infinite number of possible spins depending on how one chooses Λ_{0s} and p_s. Which one is measured in an actual experiment is determined by how the different spins couple to a classical electromagnetic field. If this is known for one type of spin it is easy to determine the corresponding relations for any other type of spin. This will be discussed in Sect. 8.

We are now in a position to construct the irreducible representation spaces that we use to describe the states of massive particles. In addition to the mass and the square of the spin, three independent components of the four momentum and one component of any spin vector, for example $\hat{\mathbf{z}} \cdot \mathbf{j}_s$, define a maximal set of commuting Hermetian functions of the generators. The simultaneous measurement of these quantities also determine the state of a particle of that mass and spin. Once the spin j^2 is fixed, the spectrum of both j^2 and $\hat{\mathbf{z}} \cdot \mathbf{j}_s$ is fixed by the $SU(2)$ commutation relations. The $SU(2)$ commutation relations imply that the eigenvalues μ of $\hat{\mathbf{z}} \cdot \mathbf{j}_s$ range from $-j$ to j in integer steps while the eigenvalues of j^2 are $j(j+1)$ for j integer or half integer. The spectrum of the three space components of the linear momentum are fixed to be $(-\infty, \infty)$ by the covariance relation (46). The subscript s indicates that μ is an eigenvalue of $\hat{\mathbf{z}} \cdot \mathbf{j}_s$. Since $\mathbf{j}^2 = \mathbf{j}_s^2$, the total spin does not depend on the choice of $\mathbf{j}_s$.

For fixed mass m and spin j we define the mass m spin j irreducible representation space to be the space of square integrable functions

$$\psi_j(\mathbf{p}, \mu) = {}_s\langle (m, j)\mathbf{p}, \mu|\psi\rangle \tag{75}$$

with inner product

$$\langle\psi|\phi\rangle = \sum_{\mu=-j}^{j} \int d\mathbf{p}\psi_j^*(\mathbf{p}, \mu)\phi(\mathbf{p}, \mu). \tag{76}$$

The irreducible basis vectors for this space, $|(m, j)\mathbf{p}, \mu\rangle_s$, are the simultaneous eigenstates of M, $\mathbf{j}^2$, $\mathbf{p}$, and $\hat{\mathbf{z}} \cdot \mathbf{j}_s$. We use the subscript s on the basis vectors to emphasize that the magnetic quantum number μ is an eigenvalue of $\mathbf{j}_s \cdot \hat{\mathbf{z}}$ and $\mathbf{j}_s$ defined in (75) depends on the choice of p_s and $\Lambda_s(p)$.

To show that this Hilbert space is an irreducible representation space for the Poincaré group we first calculate the unitary representation of the Poincaré group on this space.

We begin by considering the action of the little group on the basis vectors $|(m, j)\mathbf{p}_s, \mu\rangle_s$ when p is the standard vector $p = p_s$.

When $\mathbf{p}_s \neq \mathbf{0}$ the representation of the rotation group that leaves p_s invariant is related to the standard $SO(3)$ representations by a constant boost that acts as a similarity transform:

$$R_s = \Lambda_s^{-1}(p_0) R \Lambda_s(p_0). \tag{77}$$

We consider the action of $U(R_s, 0)$ on vectors of the form $|(m, j)\mathbf{p}_s, \mu\rangle_s$. Because R_s is an element of the little group, it will not change p_s. The result of this operator will be a linear combination of states with the same $\mathbf{p}_s$, but different magnetic quantum numbers. Formally

$$U(R_s, 0)|(m, j)\mathbf{p}_s, \mu\rangle_s = \sum_{\nu=-j}^{j} |(m, j)\mathbf{p}_s, \nu\rangle_{s\,s}\langle(m, j)\nu|U(R_s, 0)|(m, j)\mu\rangle_{p_s} \tag{78}$$

where

$$\begin{aligned} {}_s\langle(m, j)\nu|U(R_s, 0)|(m, j)\mu\rangle_{p_s} &:= \int {}_s\langle(m, j)\mathbf{p}, \nu|U(R_s, 0)|(m, j)\mathbf{p}_s, \mu\rangle_s d\mathbf{p} \\ &= \int {}_s\langle(m, j)\mathbf{p}, \nu|U(\Lambda_s^{-1}(p_0), 0)U(R, 0)U(\Lambda_s(p_0), 0)|(m, j)\mathbf{p}_s, \mu\rangle_s d\mathbf{p} \\ &= \int \frac{m}{\omega_m(\mathbf{p}_s)} \delta(\boldsymbol{\Lambda}_s(p_0)p - \boldsymbol{\Lambda}_s(p_0)p_s) D^j_{\nu\mu}(R) d\mathbf{p} = \int \delta(\mathbf{p} - \mathbf{p}_s) D^j_{\nu\mu}(R) d\mathbf{p} \\ &= D^j_{\nu\mu}(R) \end{aligned} \tag{79}$$

with $\omega_m(\mathbf{p}_s) = \sqrt{\mathbf{p}_s^2 + m^2}$. $D^j_{\nu\mu}(R)$ are the ordinary finite dimensional unitary irreducible representations [7] of the rotation group

$$D^j_{\mu'\mu}(R) = \sum_\nu \frac{[(j + \mu')!(j - \mu')!(j + \mu)!(j - \mu)!]^{1/2}}{(j + \mu' - \nu)!\nu!(\nu - \mu' + \mu)!(j - \mu - \nu)!} \times R_{11}^{j+\mu'-\nu} R_{12}^{\nu} R_{21}^{\nu-\mu'+\mu} R_{22}^{j-\mu-\nu}\,, \tag{80}$$

where R_{ij} are the $SU(2)$ matrix elements

$$R = e^{i\frac{\boldsymbol{\theta}}{2}\cdot\boldsymbol{\sigma}} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}. \tag{81}$$

Equation (78) can also be understood by considering the transformation properties of $\mathbf{j}_s$ under rotations. Using the definition of the s-spin (63) gives

$$U^\dagger(R_s, 0)(0, \mathbf{j_s})U(R_s, 0)|(m, j)\mathbf{p}_s, \mu\rangle_s = \frac{1}{m}(\Lambda_s(p_0)\Lambda_s^{-1}(R_s p_s)R_s W)^\nu|(m, j)\mathbf{p}_s, \mu\rangle_s. \tag{82}$$

Using the identities (77) and $\Lambda_s(R_s p_s) = \Lambda_s(p_s) = I$, (82) becomes

$$\begin{aligned} &= \frac{1}{m}(\Lambda_s(p_0)\Lambda_s^{-1}(p_0)R\Lambda_s(p_0)W)^\nu|(m, j)\mathbf{p}_s, \mu\rangle_s = R\frac{1}{m}(\Lambda_s(p_0)\Lambda_s^{-1}(P)W)^\nu|(m, j)\mathbf{p}_s, \mu\rangle_s \\ &= (0, R\mathbf{j}_s)|(m, j)\mathbf{p}_s, \mu\rangle_s \end{aligned} \tag{83}$$

where we have replaced $I = \Lambda_s^{-1}(p_s) = \Lambda_s^{-1}(P)$ because the operator P acts on an eigenstate with eigenvalue p_s. Equation (83) shows that $\mathbf{j}_s$ transforms like an ordinary three-vector, $U^\dagger(R_s)\mathbf{j}_s U(R_s) = R\mathbf{j}_s$, under rotations when applied to $|(m, j)\mathbf{p}_s, \mu\rangle_s$, which is also consistent with (79).

Equations (78) and (79) lead to the following transformation properties for states with the standard momentum with respect to the little group

$$U(R_s, 0)|(m, j)\mathbf{p}_s, \mu\rangle_s = \sum_{\nu=-j}^{j} |(m, j)\mathbf{p}_s, \nu\rangle_s D^j_{\nu\mu}(R). \tag{84}$$

We can also calculate the action of spacetime translations on these standard vectors using (33)

$$U(I,a)|(m,j)\mathbf{p}_s,\mu\rangle_s = e^{-ip_s\cdot a}|(m,j)\mathbf{p}_s,\mu\rangle_s. \tag{85}$$

The last step needed to construct irreducible representations is to compute the action of $U(\Lambda_s(p),0)$ on the standard states. First we show that $U(\Lambda_s(p),0)|(m,j)\mathbf{p}_s,\mu\rangle_s$ is an eigenstate of P^μ with eigenvalue p^μ. To show this use (47) to get

$$\begin{aligned} P^\mu U(\Lambda_s(p),0)|(m,j)\mathbf{p}_s,\mu\rangle_s &= U(\Lambda_s(p),0)U^\dagger(\Lambda_s(p),0)P^\mu U(\Lambda_s(p),0)|(m,j)\mathbf{p}_s,\mu\rangle_s \\ &= U(\Lambda_s(p),0)\Lambda_s(p)^\mu{}_\nu p_s^\nu|(m,j)\mathbf{p}_s,\mu\rangle_s = p^\mu U(\Lambda_s(p),0)|(m,j)\mathbf{p}_s,\mu\rangle_s, \end{aligned} \tag{86}$$

which is the desired result.

Next we show that $U(\Lambda_s(p),0)|(m,j)\mathbf{p}_s,\mu\rangle_s$ is an eigenstate of $\hat{\mathbf{z}}\cdot\mathbf{j}_s$ with eigenvalue μ. Using (60) we get

$$\begin{aligned} \hat{\mathbf{z}}\cdot\mathbf{j}_s U(\Lambda_s(p),0)|(m,j)\mathbf{p}_s,\mu\rangle_s &= U(\Lambda_s(p),0)U^\dagger(\Lambda_s(p),0)\hat{\mathbf{z}}\cdot\mathbf{j}_s U(\Lambda_s(p),0)|(m,j)\mathbf{p}_s,\mu\rangle_s \\ &= U(\Lambda_s(p),0)\hat{\mathbf{z}}\cdot\left(\frac{1}{m}\Lambda_s(p_0)\Lambda_s^{-1}(\Lambda_s(p)p_s)\Lambda_s(p)W\right)^\nu|(m,j)\mathbf{p}_s,\mu\rangle_s \\ &= U(\Lambda_s(p),0)\hat{\mathbf{z}}\cdot\left(\frac{1}{m}\Lambda_s(p_0)\Lambda_s^{-1}(p)\Lambda_s(p)W\right)^\nu|(m,j)\mathbf{p}_s,\mu\rangle_s \\ &= U(\Lambda_s(p),0)\hat{\mathbf{z}}\cdot\left(\frac{1}{m}\Lambda_s(p_0)W\right)^\nu|(m,j)\mathbf{p}_s,\mu\rangle_s. \end{aligned} \tag{87}$$

Inserting $\Lambda_s^{-1}(P)$, which is the identity on the standard basis state, (87) becomes

$$\begin{aligned} U(\Lambda_s(p),0)\hat{\mathbf{z}}\cdot\left(\frac{1}{m}\Lambda_s(p_0)\Lambda_s^{-1}(P)W\right)^\nu|(m,j)\mathbf{p}_s,\mu\rangle_s &= U(\Lambda_s(p),0)\hat{\mathbf{z}}\cdot\mathbf{j}_s|(m,j)\mathbf{p}_s,\mu\rangle_s \\ &= \mu U(\Lambda_s(p),0)|(m,j)\mathbf{p}_s,\mu\rangle_s\ , \end{aligned} \tag{88}$$

which is the desired result.

It follows from (86) and (88) that $U(\Lambda_s(p),0)|(m,j)\mathbf{p}_s,\mu\rangle_s$ is a simultaneous eigenstate of $\mathbf{j}^2$, $\hat{\mathbf{z}}\cdot\mathbf{j}_s$, $\mathbf{p}$, and m. Thus it is proportional to $|(m,j)\mathbf{p},\mu\rangle_s$. The constant factor is fixed up to phase by the requirement that $U(\Lambda_s(p),0)$ is unitary. If we normalize the states to give Dirac delta functions in the momentum variables and choose the phase so that the constant factor is real and positive then the normalization constant is fixed by

$$\delta(\mathbf{p}'-\mathbf{p}) = {}_s\langle(m,j)\mathbf{p}',\mu|U^\dagger(\Lambda,0)U(\Lambda,0)|(m,j)\mathbf{p},\mu\rangle_s = |c|^2\delta(\boldsymbol{\Lambda}p'-\boldsymbol{\Lambda}p) = |c|^2\left|\frac{\partial\mathbf{p}}{\partial\boldsymbol{\Lambda}p}\right|\delta(\mathbf{p}'-\mathbf{p}) \tag{89}$$

which gives

$$c = \left|\frac{\partial\boldsymbol{\Lambda}p}{\partial\mathbf{p}}\right|^{1/2} = \left|\frac{\omega_m(\boldsymbol{\Lambda}p)}{\omega_m(\mathbf{p})}\right|^{1/2}. \tag{90}$$

Thus unitarity and an assumed delta function normalization imply the transformation property

$$U(\Lambda_s(p),0)|(m,j)\mathbf{p}_s,\mu\rangle_s = |(m,j)\mathbf{p},\mu\rangle_s\sqrt{\frac{\omega_m(\mathbf{p})}{\omega_m(\mathbf{p}_s)}}. \tag{91}$$

We see that on these states the little group rotates the spin operator leaving p_s invariant, while $\Lambda_s(p)$ changes the momentum from the standard value to any other value without changing the z-component of the s-spin.

Using the elementary transformations (84, 85) and (91), we can construct the action of an arbitrary Poincaré transformation on any basis state. To do this note that for any p any Lorentz transformation can be decomposed into the following product

$$\Lambda = \Lambda_s(\Lambda p)\Lambda_s^{-1}(\Lambda p)\Lambda\Lambda_s(p)\Lambda_s^{-1}(p)\ , \tag{92}$$

where the s-spin Wigner rotation,

$$\Lambda_s^{-1}(\Lambda p)\Lambda\Lambda_s(p) := R_{ws}(\Lambda, p) \tag{93}$$

is an element of the little group associated with p_s since it maps p_s to p_s. Thus

$$\begin{aligned} U(\Lambda, a)|(m, j)\mathbf{p}, \mu\rangle_s &= U(\Lambda, 0)U(I, \Lambda^{-1}a)|(m, j)\mathbf{p}, \mu\rangle_s = e^{-i\Lambda^{-1}a\cdot p}U(\Lambda, 0)|(m, j)\mathbf{p}, \mu\rangle_s \\ &= e^{-i(\Lambda^{-1}a)\cdot p}U(\Lambda_s(\Lambda p), 0)U(R_{ws}(\Lambda, p), 0)U(\Lambda_s^{-1}(p), 0)|(m, j)\mathbf{p}, \mu\rangle_s \\ &= e^{-ia\cdot\Lambda p}U(\Lambda_s(\Lambda p), 0)U(R_{ws}(\Lambda, p), 0)|(m, j)\mathbf{p}_s, \mu\rangle_s\sqrt{\frac{\omega_m(\mathbf{p}_s)}{\omega_m(\mathbf{p})}}. \end{aligned} \tag{94}$$

Using

$$|(m, j)\mathbf{p}_s, \mu\rangle_s = U(\Lambda_s^{-1}(p_0), 0)|(m, j)\mathbf{p}_0, \mu\rangle_s\sqrt{\frac{\omega_m(\mathbf{p}_0)}{\omega_m(\mathbf{p}_s)}} \tag{95}$$

and the fact that $\Lambda_s(p_0)R_{ws}(\Lambda, p)\Lambda_s^{-1}(p_0)$ is a rotation, (94) becomes:

$$\begin{aligned} &\sum_{\nu=-j}^{j} e^{-ia\cdot\Lambda p}U(\Lambda_s(\Lambda p), 0)|(m, j)\mathbf{p}_s, \nu\rangle_s D^j_{\nu_s\mu_s}[\Lambda_s(p_0)R_{ws}(\Lambda, p)\Lambda_s^{-1}(p_0)]\sqrt{\frac{\omega_m(\mathbf{p}_s)}{\omega_m(\mathbf{p})}} \\ &= \sum_{\nu=-j}^{j} e^{-ia\cdot\Lambda p}|(m, j)\boldsymbol{\Lambda}p, \nu\rangle_s D^j_{\nu\mu}[\Lambda_s(p_0)R_{ws}(\Lambda, p)\Lambda_s^{-1}(p_0)]\sqrt{\frac{\omega_m(\mathbf{p}_s)}{\omega_m(\mathbf{p})}}\sqrt{\frac{\omega_m(\boldsymbol{\Lambda}p)}{\omega_m(\mathbf{p}_s)}} \\ &= \sum_{\nu=-j}^{j} e^{-ia\cdot\Lambda p}|(m, j)\boldsymbol{\Lambda}p, \nu\rangle_s D^j_{\nu\mu}[\Lambda_s(p_0)R_{ws}(\Lambda, p)\Lambda_s^{-1}(p_0)]\sqrt{\frac{\omega_m(\boldsymbol{\Lambda}p)}{\omega_m(\mathbf{p})}}. \end{aligned} \tag{96}$$

Thus the general form of any finite Poincaré transformation in this representation of the Hilbert space is

$$U(\Lambda, a)|(m, j)\mathbf{p}, \mu\rangle_s = \sum_{\nu=-j}^{j} e^{-ia\cdot\Lambda p}|(m, j)\boldsymbol{\Lambda}p, \nu\rangle_s D^j_{\nu\mu}[\Lambda_s(p_0)R_{ws}(\Lambda, p)\Lambda_s^{-1}(p_0)]\sqrt{\frac{\omega_m(\boldsymbol{\Lambda}p)}{\omega_m(\mathbf{p})}}. \tag{97}$$

By construction it is apparent that it is possible to start from the highest weight, $\mu = j$, spin state with standard momentum $p = p_s$ and generate all of the basis vectors in the Hilbert space using only Poincaré transformations. This establishes the irreducibility of this representation.

It is useful to introduce for the Wigner functions of the Poincaré group a notation that is similar to the notation used for Wigner functions of the rotation group:

$$\begin{aligned} \mathcal{D}^{m,j}_{s:\mathbf{p}'\mu';\mathbf{p}\mu}[\Lambda, a] &:= {}_s\langle(m, j)\mathbf{p}', \mu'|U(\Lambda, a)|(m, j)\mathbf{p}, \mu\rangle_s \\ &= e^{-ia\cdot\Lambda p}\delta(\mathbf{p}' - \boldsymbol{\Lambda}p)\mathcal{D}^j_{\mu'\mu}[\Lambda_s(p_0)R_{ws}(\Lambda, p)\Lambda_s^{-1}(p_0)]\sqrt{\frac{\omega_m(\boldsymbol{\Lambda}p)}{\omega_m(\mathbf{p})}}. \end{aligned} \tag{98}$$

Note that the Poincaré group Wigner functions are basis dependent.

Using this notation (97) can be written as

$$U(\Lambda, a)|(m, j)\mathbf{p}, \mu\rangle_s = \sum_{\nu=-j}^{j}\int d\mathbf{p}'|(m, j)\mathbf{p}', \nu\rangle_s\mathcal{D}^{m,j}_{s:\mathbf{p}'\nu;\mathbf{p}\mu}[\Lambda, a]. \tag{99}$$

A consequence of definition (98) is that these Poincaré group Wigner functions are explicit unitary representations of the Poincaré group. They satisfy the group representation property

$$\int d\mathbf{p}''\sum_{\mu''=-j}^{j}\mathcal{D}^{m,j}_{s:\mathbf{p}'\mu';\mathbf{p}''\mu''}[\Lambda_2, a_2]\mathcal{D}^{m,j}_{s:\mathbf{p}''\mu'';\mathbf{p}\mu}[\Lambda_1, a_1] = \mathcal{D}^{m,j}_{s:\mathbf{p}'\mu';\mathbf{p}\mu}[\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2] \tag{100}$$

and unitarity

$$\int d\mathbf{p}'' \sum_{\mu''=-j}^{j} \mathcal{D}^{m,j*}_{s:\mathbf{p}''\mu'';\mathbf{p}'\mu'}[\Lambda, a]\mathcal{D}^{m,j}_{s:\mathbf{p}''\mu'';\mathbf{p}\mu}[\Lambda, a] = \delta(\mathbf{p}' - \mathbf{p})\delta_{\mu'\mu}. \tag{101}$$

In dealing with electromagnetic interactions where the coupling of the spin to a magnetic field is known for one type of spin, say the s-spin, and the dynamics is given in a basis with a different type of spin, say the t-spin, the transformation from a basis associated with the standard vector p_s and standard boost $\Lambda_s(p)$ to the basis associated with the pair, p_t and $\Lambda_t(p)$, is needed. The corresponding spin operators are

$$(0, \underline{\mathbf{j}}_s) = \frac{1}{\underline{m}}\Lambda^{-1}_{0s}(\underline{P})^{\mu}{}_{\nu}\underline{W}^{\nu} \tag{102}$$

and

$$(0, \underline{\mathbf{j}}_t) = \frac{1}{\underline{m}}\Lambda^{-1}_{0t}(\underline{P})^{\mu}{}_{\nu}\underline{W}^{\nu}\ , \tag{103}$$

where

$$\Lambda^{-1}_{0s}(\underline{P}) = \Lambda_s(p_0)\Lambda^{-1}_{s}(\underline{P}) \tag{104}$$

and

$$\Lambda^{-1}_{0t}(\underline{P}) = \Lambda_t(p_0)\Lambda^{-1}_{t}(\underline{P}). \tag{105}$$

When $p = p_0$ (i.e. $\mathbf{p} = \mathbf{0}$) we have the identity $\Lambda^{-1}_{0s}(p_0) = \Lambda^{-1}_{0t}(p_0) = I$. This means that

$$\mathbf{j}_s|(m, j)\mathbf{p}_0, \mu\rangle_s = \mathbf{j}_t|(m, j)\mathbf{p}_0, \mu\rangle_t = (\mathbf{W}/m)|(m, j)\mathbf{p}_0, \mu\rangle. \tag{106}$$

This is because the spin operators (102) and (103) are defined so they are identical when they are applied to zero-momentum, $\mathbf{p} = \mathbf{p}_0 = \mathbf{0}$:

$$|(m, j)\mathbf{0}, \mu\rangle_s = |(m, j)\mathbf{0}, \mu\rangle_t. \tag{107}$$

It follows that

$$\begin{aligned}
|(m, j)\mathbf{p}, \mu\rangle_s &= U(\Lambda_s(p), 0)|(m, j)\mathbf{p}_s, \mu\rangle_s\sqrt{\frac{\omega_m(\mathbf{p}_s)}{\omega_m(\mathbf{p})}} \\
&= U(\Lambda_s(p), 0)U(\Lambda_s^{-1}(p_0), 0)|(m, j)\mathbf{0}, \mu\rangle_s\sqrt{\frac{\omega_m(\mathbf{0})}{\omega_m(\mathbf{p})}} \\
&= U(\Lambda_s(p), 0)U(\Lambda_s^{-1}(p_0), 0)|(m, j)\mathbf{0}, \mu\rangle_t\sqrt{\frac{\omega_m(\mathbf{0})}{\omega_m(\mathbf{p})}} \\
&= U(\Lambda_s(p), 0)U(\Lambda_s^{-1}(p_0), 0)U(\Lambda_t(p_0), 0)|(m, j)\mathbf{p}_t, \mu\rangle_t\sqrt{\frac{\omega_m(\mathbf{p}_t)}{\omega_m(\mathbf{p})}} \\
&= U(\Lambda_s(p), 0)U(\Lambda_s^{-1}(p_0), 0)U(\Lambda_t(p_0), 0)U(\Lambda_t^{-1}(p), 0)|(m, j)\mathbf{p}, \mu\rangle_t \\
&= U(\Lambda_s(p)\Lambda_s^{-1}(p_0)\Lambda_t(p_0)\Lambda_t^{-1}(p), 0)|(m, j)\mathbf{p}, \mu\rangle_t \\
&= \sum_{\mu'=-j}^{j}\int |(m, j)\mathbf{p}', \mu'\rangle_t d\mathbf{p}'\mathcal{D}^{mj}_{t:\mathbf{p}'\mu';\mathbf{p};\mu}[\Lambda_s(p)\Lambda_s^{-1}(p_0)\Lambda_t(p_0)\Lambda_t^{-1}(p), 0] \\
&= \sum_{\mu'=-j}^{j} |(m, j)\mathbf{p}, \mu'\rangle_t D^{j}_{\mu'\mu}[\Lambda^{-1}_{0t}(p)\Lambda_{0s}(p)]\ ,
\end{aligned} \tag{108}$$

where for the last step we used (98) and $\Lambda_s(p)\Lambda_s^{-1}(p_0)\Lambda_t(p_0)\Lambda_t^{-1}(p)p = p$. This shows these bases differ by a momentum-dependent generalized Melosh rotation (71).

The relation (108) is

$$|(m, j)\mathbf{p}, \mu\rangle_s = \sum_{\mu'=-j}^{j} |(m, j)\mathbf{p}, \mu'\rangle_t D^j_{\mu'\mu}[\Lambda_{0t}^{-1}(p)\Lambda_{0s}(p)] \, . \tag{109}$$

This section illustrated the general structure of positive-mass positive-energy irreducible representations of the Poincaré group. We started with a complete set of commuting Hermetian operators constructed as functions of the Poincaré generators. The spectrum of all of the commuting observables was fixed once the spectrum of m and $\mathbf{j}^2$ was fixed. A representation of the Hilbert space was defined by square integrable functions of the eigenvalues of these commuting observables over their spectra. On each space associated with a fixed mass and spin we constructed an irreducible unitary representation of the Poincaré group.

In this construction, we found that there are many different observables that behave like spins. They are all non-linear functions of the Poincaré generators satisfying $SU(2)$ commutation relations. It is also possible to change the choice of independent continuous variables. Different choices of commuting continuous variables (linear momentum, four velocity, light-front components of the momentum) along with the appropriate choice of spin variable are relevant in dynamical models based on Dirac's forms of dynamics [8].

The construction of both the irreducible representation and the representation space can be done for many-body systems in the same way that it was done for single particles. The idea is to use the elementary transformations (84, 85) and (91), with eigenstates of commuting observables constructed from the many-body generators. To do this it is necessary to decompose states with total $p = p_s$ into irreducible representations of the little group. The main difference is that the mass will generally have a continuous spectrum and there may be multiple copies of representations of given mass and spin.

5 Examples

In this section, we discuss the three most common spin observables and discuss the properties that distinguish each of them.

In the previous section, we introduced a large number of different types of observables which we identified as spins. Each of these were functions of the Poincaré generators satisfying $SU(2)$ commutation relations and commuting with the linear momentum. All have the same square, which is W^2/M^2, the ratio of the two Casimir operators for the Poincaré group. Each of the spins used in applications has some particular property that makes them useful. In general, different types of spin are characterized by the choice of boost used to relate the spin of a particle (system) with momentum $\mathbf{p}$ to the spin in a standard frame. Specifically the magnetic quantum number remains unchanged when *this* boost is applied to a standard frame eigenstate of the z-component of spin and momentum (91). For the examples in this section we assume that the standard vector is the rest vector, p_0.

5.1 Canonical Spin

The boost used to define the canonical spin is the rotationless boost (19). Under rotations

$$U(R, 0)|\mathbf{p}, \mu\rangle = \sum_{\nu=-j}^{j} |R\mathbf{p}, \nu\rangle D^j_{\nu\mu}[\Lambda_c^{-1}(Rp)R\Lambda_c(p)]. \tag{110}$$

The special property of the canonical boost is that the Wigner rotation (93) of any rotation R is R:

$$\Lambda_c^{-1}(Rp)R\Lambda_c(p) = R. \tag{111}$$

Using this in (110) gives

$$U(R, 0)|\mathbf{p}, \mu\rangle = \sum_{\nu=-j}^{j} |R\mathbf{p}, \nu\rangle D^j_{\nu\mu}[R]. \tag{112}$$

where the argument of the D function is independent of $\mathbf{p}$. This is useful when applied to a system of particles with different momenta. Under rotations all of the particles transform with the same rotation, independent of their individual momenta. This allows the spins to be coupled with ordinary $SU(2)$ Clebsch–Gordan coefficients. For the other types of spins the arguments of the D-functions involve Wigner rotations with different values of p. In order to couple the spins it is normally necessary first to convert them to canonical spins so all spins rotate the same way.

The identity (111) is most easily proved in the $SU(2)$ representation, (17). In this representation $\Lambda_c(p) = e^{\frac{1}{2}\mathbf{z}\cdot\boldsymbol{\sigma}}$ with $\mathbf{z} = \hat{\mathbf{p}} \sinh^{-1}(|\mathbf{p}|/m)$. For this proof we let boldface $\mathbf{R}$ denote a three-dimensional rotation and R denote the corresponding $SU(2)$ rotation. It follows that

$$R\Lambda_c(p)R^\dagger = Re^{\mathbf{z}\cdot\boldsymbol{\sigma}}R^\dagger = e^{\mathbf{z}\cdot(R\boldsymbol{\sigma}R^\dagger)} = e^{\mathbf{z}\cdot(\mathbf{R}^{-1}\boldsymbol{\sigma})} = e^{\mathbf{R}\mathbf{z}\cdot\boldsymbol{\sigma}} = \Lambda_c(\mathbf{R}p) \;, \tag{113}$$

where we have used (7) for rotations ($\Lambda \to R$):

$$x^\mu(\mathbf{R}^{-1}\sigma)_\mu = (\mathbf{R}x)^\mu\sigma_\mu = R(x^\mu\sigma_\mu)R^\dagger = x^\mu R(\sigma_\mu)R^\dagger. \tag{114}$$

Equation (113) and $R^\dagger = R^{-1}$ imply the desired result (111).

5.2 Helicity

The helicity [9] is the operator $\hat{\mathbf{p}}\cdot\mathbf{j}_c$ where $\mathbf{j}_c$ is the canonical spin. To relate this to the formalism derived in Sect. 4 we let $R(\hat{\mathbf{z}} \to \hat{\mathbf{p}})$ denote the rotation about an axis perpendicular to the plane containing $\hat{\mathbf{z}}$ and $\hat{\mathbf{p}}$ that rotates $\hat{\mathbf{z}}$ into the direction $\hat{\mathbf{p}}$. The helicity boost is defined by

$$\Lambda_h(p) = \Lambda_c(p)R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}) = R(\hat{\mathbf{z}} \to \hat{\mathbf{p}})\Lambda_c(p_z) \;, \tag{115}$$

where p_z is the 4-vector with 3-magnitude $|\mathbf{p}|$ in the z direction.

Helicity eigenstates are related to canonical spin eigenstates by

$$|\mathbf{p}, \mu\rangle_h := U(R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}))||\mathbf{p}|\hat{\mathbf{z}}, \mu\rangle_c = \sum_{\nu=-j}^{j} |\mathbf{p}, \nu\rangle_c D^j_{\nu\mu}[R(\hat{\mathbf{z}} \to \hat{\mathbf{p}})]. \tag{116}$$

This equation shows that the generalized Melosh rotation (71, 109) relating the canonical and helicity spins is $R(\hat{\mathbf{z}} \to \hat{\mathbf{p}})$.

The helicity spin, $\mathbf{j}_h$, defined using the helicity boost in (63) satisfies

$$\hat{\mathbf{z}}\cdot\mathbf{j}_h = \hat{\mathbf{p}}\cdot\mathbf{j}_c, \tag{117}$$

which means that the z-component of the helicity spin is the helicity.

The helicity-spin Wigner rotation (93) is

$$R_{wh}(\Lambda, p) = R^{-1}(\hat{\mathbf{z}} \to \hat{\mathbf{p}})\Lambda_c^{-1}(\Lambda p)\Lambda\Lambda_c(p)R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}) \;, \tag{118}$$

which is a rotation about the z axis. Thus

$$U(\Lambda, 0)|\mathbf{p}, \mu\rangle_h := |\boldsymbol{\Lambda}p, \mu\rangle_h e^{i\mu\phi}\sqrt{\frac{\omega_m(\boldsymbol{\Lambda}p)}{\omega_m(\mathbf{p})}} \;, \tag{119}$$

where ϕ is the angle of rotation of the Wigner rotation. Thus the helicity eigenvalue is Lorentz invariant.

The identification of $\hat{\mathbf{z}}\cdot\mathbf{j}_h$ with $\hat{\mathbf{p}}\cdot\mathbf{j}_c$ follows from the calculation

$$\begin{aligned}
\hat{\mathbf{p}}\cdot\mathbf{j}_c|\mathbf{p}', \mu\rangle_h &= \hat{\mathbf{p}}'\cdot\mathbf{j}_c U(R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}'))||\mathbf{p}'|\hat{\mathbf{z}}, \mu\rangle_c \\
&= \hat{\mathbf{p}}'\cdot U(R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}'))U(R^{-1}(\hat{\mathbf{z}} \to \hat{\mathbf{p}}'))\mathbf{j}_c U(R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}'))||\mathbf{p}'|\hat{\mathbf{z}}, \mu\rangle_c \\
&= U(R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}'))(\hat{\mathbf{p}}'\cdot R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}')\mathbf{j}_c)||\mathbf{p}'|\hat{\mathbf{z}}, \mu\rangle_c \\
&= U(R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}'))(R^{-1}(\hat{\mathbf{z}} \to \hat{\mathbf{p}}')\hat{\mathbf{p}}')\cdot\mathbf{j}_c||\mathbf{p}'|\hat{\mathbf{z}}, \mu\rangle_c \\
&= U(R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}'))|\mathbf{p}|\hat{\mathbf{z}}\cdot\mathbf{j}_c||\mathbf{p}'|\hat{\mathbf{z}}, \mu\rangle_c \\
&= \mu U(R(\hat{\mathbf{z}} \to \hat{\mathbf{p}}'))||\mathbf{p}'|\hat{\mathbf{z}}, \mu\rangle_c = \mu|\mathbf{p}', \mu\rangle_h \\
&= \hat{\mathbf{z}}\cdot\mathbf{j}_h|\mathbf{p}', \mu\rangle_h \;.
\end{aligned} \tag{120}$$

5.3 Light-Front Spin

In the $SL(2,\mathbb{C})$ representation the light-front boosts are represented by the three-parameter *subgroup* of lower triangular matrices with real entries on the diagonal. Considering the transformation properties (7) for the four momentum

$$\begin{pmatrix} a & 0 \\ b+ic & 1/a \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} a & b-ic \\ 0 & 1/a \end{pmatrix} = \begin{pmatrix} ma^2 & ma(b-ic) \\ ma(b+ic) & m(b^2+c^2)/a^2 \end{pmatrix} \tag{121}$$

we can identify the parameters of the light-front boost as follows

$$ma^2 = p^0 + p^3 \qquad a = \sqrt{\frac{p^0+p^3}{m}} \tag{122}$$

$$ma(b-ic) = p^1 - ip^2 \qquad b - ic = \frac{p^1 - ip^2}{ma}\ . \tag{123}$$

Because the light-front boosts form a subgroup, any sequence of light-front boosts is the unique light-front boost parameterized by the final momentum of the sequence. This means that the Wigner rotation (93) of a light-front boost is the *identity* so the light-front spins remain unchanged under the three parameter group of light-front boosts.

Unlike helicities, the light-front spin is not invariant with respect to rotations.

6 Adding Spins

Multiparticle systems can be described by tensor products of single-particle systems. The Hilbert space is the tensor product,

$$\mathcal{H} = \otimes_i \mathcal{H}_{m_i j_i}, \tag{124}$$

where $\mathcal{H}_{m_i j_i}$ are the mass m_i spin j_i single-particle irreducible representation spaces constructed in Sect. 4.

There is a natural representation $U_0(\Lambda, a)$ of the Poincaré group on this space which is the tensor product of the irreducible representations constructed in Sect. 4

$$U_0(\Lambda, a) = \otimes_i U_{m_i j_i}(\Lambda, a). \tag{125}$$

Dynamically this representation describes a system of free particles. In this representation the infinitesimal generators are sums of the single-particle generators. In the s-spin basis this representation has the explicit form

$$\begin{aligned} &U_0(\Lambda,a)|(m_1,j_1)\mathbf{p}_1,\mu_1\rangle_s \otimes \cdots \otimes |(m_N,j_N)\mathbf{p}_N,\mu_N\rangle_s \\ &= \sum_{\mu'_1\cdots\mu'_N} \int d\mathbf{p}'_1 \cdots d\mathbf{p}'_N |(m_1,j_1)\mathbf{p}'_1,\mu'_1\rangle_s \otimes \cdots \otimes |(m_N,j_N)\mathbf{p}'_N,\mu'_N\rangle_s \\ &\quad \times \mathcal{D}^{m_1,j_1}_{s:\mathbf{p}'_1\mu'_1;\mathbf{p}_1\mu_1}[\Lambda,a] \cdots \mathcal{D}^{m_N,j_N}_{s:\mathbf{p}'_N\mu'_N;\mathbf{p}_N\mu_N}[\Lambda,a] \\ &= e^{-ia\cdot\sum_i \Lambda p_i} \sum_{\nu_1\cdots\nu_N} |(m_1,j_1)\boldsymbol{\Lambda}p_1,\nu_1\rangle_s \otimes \cdots |(m_N,j_N)\boldsymbol{\Lambda}p_N,\nu_N\rangle_s \\ &\quad \times \prod_i D^j_{\nu_i\mu_i}[\Lambda_s(p_0)R_{ws}(\Lambda,p_i)\Lambda_s^{-1}(p_0)]\sqrt{\frac{\omega_{m_i}(\boldsymbol{\Lambda}p_i)}{\omega_{m_i}(\mathbf{p}_i)}}. \end{aligned} \tag{126}$$

Just like in the case of ordinary rotations, the tensor product of irreducible representations of the Poincaré group is reducible. Poincaré group Clebsch–Gordan coefficients are coefficients of a unitary transformation that transforms the tensor product into irreducible blocks labeled by many-body mass and spin eigenvalues. For non-interacting systems the many-body mass is just the invariant mass of the many-body system. The structure of the Clebsch–Gordan coefficients depends on the choice of basis used to define vectors in the irreducible blocks. They are derived below.

We start by evaluating the coefficients of the unitary transformation that transforms a tensor product of two irreducible representations into a superposition of irreducible representations.

Basis states for the tensor product state are simultaneous eigenstates of the mass, spin, linear momentum and magnetic quantum number for each particle

$$|(m_1, j_1)\mathbf{p}_1, \mu_1\rangle_s \otimes |(m_2, j_2)\mathbf{p}_2, \mu_2\rangle_s. \tag{127}$$

As in the previous section the subscript s indicates choice of spin operator.

Infinitesimal generators for the combined system are sums of generators for the individual constituent particles. The four momentum of the combined system is $P = p_1 + p_2$. This is a sum of timelike positive-time vectors so it is a timelike positive-time vector. Following what was done for single-particles we look at representations of the little group for a standard momentum vector. For many-body systems we choose the standard vector to be the zero of the total three-momentum vector.

Tensor product eigenstates with the standard vector for the two-particle system $P = p_s = p_0 = (M, \mathbf{0})$ have the form

$$|(m_1, j_1)\mathbf{k}, \mu_1\rangle_s \otimes |(m_2, j_2) - \mathbf{k}, \mu_2\rangle_s \tag{128}$$

where $\mathbf{p}_1 = -\mathbf{p}_2 := \mathbf{k}$. We also define

$$k_1 = (\omega_{m_1}(\mathbf{k}), \mathbf{k}) \qquad k_2 = (\omega_{m_2}(\mathbf{k}), -\mathbf{k}). \tag{129}$$

where $\omega_{m_i}(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m_i^2}$.

It is useful to decompose the vector $\mathbf{k}$ into orbital angular momentum components using spherical harmonics:

$$|(m_1, j_1, m_2, j_2)k, l, \mu_l, \mu_1, \mu_2\rangle_s := \int |(m_1, j_1)\mathbf{k}, \mu_1\rangle_s \otimes |(m_1, j_2) - \mathbf{k}, \mu_2\rangle_s \, d\hat{\mathbf{k}} \, Y^l_{\mu_l}(\hat{\mathbf{k}}). \tag{130}$$

To construct irreducible representations consider the transformation properties of (130) under rotations (the little group associated with p_0).

Let $U(R) = U_1(R, 0) \otimes U_2(R, 0)$. Applying this operator to (130) gives

$$\begin{aligned} U(R,0)|(m_1, j_1, m_2, j_2)k, l, \mu_l, \mu_1, \mu_2\rangle_s &:= U(R,0) \int |(m_1, j_1)\mathbf{k}, \mu_1\rangle_s \otimes |(m_2, j_2) - \mathbf{k}, \mu_2\rangle_s d\hat{\mathbf{k}} Y^l_{\mu_l}(\hat{\mathbf{k}}) \\ &= \sum_{\mu_1'\mu_2'} \int |(m_1, j_1)R\mathbf{k}, \mu_1'\rangle_s \otimes |(m_2, j_2) - R\mathbf{k}, \mu_2'\rangle_s \, d\hat{\mathbf{k}} \, Y^l_{\mu_l}(\hat{\mathbf{k}}) \\ &\quad \times D^{j_1}_{\mu_1'\mu_1}[\Lambda_s^{-1}(Rk_1)R\Lambda_s(k_1)] D^{j_2}_{\mu_2'\mu_2}[\Lambda_s^{-1}(Rk_2)R\Lambda_s(k_2)]. \end{aligned} \tag{131}$$

Changing variables $\mathbf{k} \to R^{-1}\mathbf{k}$ (131) becomes

$$\begin{aligned} &= \sum_{\mu_1'\mu_2'} \int |(m_1, j_1)\mathbf{k}, \mu_1'\rangle_s \otimes |(m_2, j_2) - \mathbf{k}, \mu_2'\rangle_s \, d\hat{\mathbf{k}} \, Y^l_{\mu_l}(R^{-1}\hat{\mathbf{k}}) \\ &\quad \times D^{j_1}_{\mu_1'\mu_1}[\Lambda_s^{-1}(k_1)R\Lambda_s(R^{-1}k_1)] D^{j_2}_{\mu_2'\mu_2}[\Lambda_s^{-1}(k_2)R\Lambda_s(R^{-1}k_2)]. \end{aligned} \tag{132}$$

Noting that

$$Y^l_{\mu_l}(R^{-1}\hat{\mathbf{k}}) = \langle R^{-1}\hat{\mathbf{k}}|l, \mu_l\rangle = \langle \hat{\mathbf{k}}|U(R)|l, \mu_l\rangle = \sum_{m_l'} \langle \hat{\mathbf{k}}|l, \mu_l'\rangle\langle l, \mu_l'|U(R)|l, \mu_l\rangle = \sum_{m_l'} Y^l_{\mu_l'}(\hat{\mathbf{k}}) D^l_{\mu_l'\mu_l}[R] \tag{133}$$

Eq. (132) becomes

$$\begin{aligned} &= \sum_{\mu_1'\mu_2'\mu_l'} \int |(m_1, j_1)\mathbf{k}, \mu_1'\rangle_s \otimes |(m_2, j_2) - \mathbf{k}, \mu_2'\rangle_s \, d\hat{\mathbf{k}} \, Y^l_{\mu_l'}(\hat{\mathbf{k}}) \\ &\quad \times D^{j_1}_{\mu_1'\mu_1}[\Lambda_s^{-1}(k_1)R\Lambda_s(R^{-1}k_1)] D^{j_2}_{\mu_2'\mu_2}[\Lambda_s^{-1}(k_2)R\Lambda_s(R^{-1}k_2)] D^l_{\mu_l'\mu_l}[R]. \end{aligned} \tag{134}$$

The important observation is that the spins and orbital angular momenta all transform with *different* rotations so they cannot be consistently added with ordinary Clebsch–Gordan coefficients. The rotations of the spins are Wigner rotations of the rotations that appear in the orbital angular momentum. The canonical spin, that uses the rotationless boost (17–19), $\Lambda_s(p) = \Lambda_c(p)$, with $p_s = p_o$ has the unique feature, (111), that the Wigner rotations of any rotation is the rotation. This means that for canonical spins ($s = c$)

$$\begin{aligned} \Lambda_c^{-1}(Rk_1)R\Lambda_c(k_1) &= R \\ \Lambda_c^{-1}(Rk_2)R\Lambda_c(k_2) &= R \end{aligned} \tag{135}$$

or equivalently

$$\begin{aligned} \Lambda_c^{-1}(k_1)R\Lambda_c(R^{-1}k_1) &= R \\ \Lambda_c^{-1}(k_2)R\Lambda_c(R^{-1}k_2) &= R \end{aligned} \tag{136}$$

so all three $SU(2)$-Wigner functions in (134) have the same arguments, independent of $\mathbf{k}_i$. Thus, for *canonical* spin, $s = c$, we have

$$\begin{aligned} &U(R,0)|(m_1, j_1, m_2, j_2)k, l, \mu_l, \mu_1, \mu_2\rangle_c := \\ &\sum_{\mu'_1\mu'_2\mu'_l} \int |(m_1, j_1)\mathbf{k}, \mu'_1\rangle_c \otimes |(m_2, j_2) - \mathbf{k}, \mu'_2\rangle_c \, d\hat{\mathbf{k}} \, Y^l_{\mu'_l}(\hat{\mathbf{k}}) \\ &\quad \times D^{j_1}_{\mu'_1\mu_1}[R] D^{j_2}_{\mu'_2\mu_2}[R] D^{l}_{\mu'_l\mu_l}[R]. \end{aligned} \tag{137}$$

which has the property that the spins and orbital angular momenta *all* rotate with the same rotation.

Recall that we initially defined all types of spins so that they agree in the particle's rest frame (106). The state (130) is a rest state of the two-body system. Following what was done in the one-body case we assume all of the two-body s-spins agree with the canonical spin state (137) in the two-body rest frame. Since we want to treat the case of coupling any type of spins we use generalized Melosh rotations (71, 109) to express the single-particle canonical spin state (137) in terms of single-particle s-spin states:

$$|(m_1, j_1)\mathbf{k}, \mu_1\rangle_c = \sum_{\mu'_1=-j_1}^{j_1} |(m_1, j_1)\mathbf{k}, \mu'_1\rangle_s D^{j_1}_{\mu'_1\mu_1}[\Lambda_s^{-1}(k_1)\Lambda_c(k_1)] \tag{138}$$

$$|(m_2, j_2) - \mathbf{k}, \mu_2\rangle_c = \sum_{\mu'_2=-j_2}^{j_2} |(m_2, j_2) - \mathbf{k}, \mu'_2\rangle_s D^{j_2}_{\mu'_2\mu_2}[\Lambda_s^{-1}(k_2)\Lambda_c(k_2)]. \tag{139}$$

By using (138) and (139) in (137) the two-body rest canonical spin state can be expressed in terms of the single-particle s-spin states as

$$\begin{aligned} |(m_1, j_1, m_2, j_2)k, l, \mu_l, \mu_1, \mu_2\rangle_c := \sum_{\mu'_1\mu'_2} \int |(m_1, j_1)\mathbf{k}, \mu'_1\rangle_s \otimes |(m_2, j_2) - \mathbf{k}, \mu'_2\rangle_s d\hat{\mathbf{k}} Y^l_{\mu_l}(\hat{\mathbf{k}}) \\ \times D^{j_1}_{\mu'_1\mu_1}[\Lambda_s^{-1}(k_1)\Lambda_c(k_1)] D^{j_2}_{\mu'_2\mu_2}[\Lambda_s^{-1}(k_2)\Lambda_c(k_2)]. \end{aligned} \tag{140}$$

This state is identical to the state in (137) and it necessarily has the same transformation property under rotations.

Using the property

$$\sum_{\mu'_1\mu'_2} D^{j_1}_{\mu_1\mu'_1}[R] D^{j_2}_{\mu_2\mu'_2}[R] \langle j_1, \mu'_1, j_2, \mu'_2 | j_{12}, \mu_{12}\rangle = \sum_{j_{12}\mu'_{12}} \langle j_1, \mu_1, j_2, \mu_2 | j_{12}, \mu'_{12}\rangle D^{j_{12}}_{\mu'_{12}\mu_{12}}[R] \tag{141}$$

of the $SU(2)$ Clebsch–Gordan coefficients, the spins and orbital angular momenta can be coupled to a total spin that transform irreducibly under rotations. Thus, we are led to define the rest states of the system by

$$
\begin{aligned}
&|k, j(m_1, j_1, m_2, j_2, l, s_{12})\mathbf{0}, \mu\rangle_c \\
&:= \sum{}' |(m_1, j_1, m_2, j_2)k, l, \mu_l', \mu_1', \mu_2'\rangle_c \langle j_1, \mu_1', j_2, \mu_2'|s_{12}, \mu_s\rangle\langle l, m, s_{12}, \mu_s|j, \mu\rangle \\
&= \sum{}' \sum{}'' \int |(m_1, j_1)\mathbf{k}, \mu_1'\rangle_s \otimes |(m_2, j_2) - \mathbf{k}, \mu_2'\rangle_s \, d\hat{\mathbf{k}}\, Y^l_m(\hat{\mathbf{k}}) \\
&\quad \times D^{j_1}_{\mu_1'\mu_1''}[\Lambda_s^{-1}(k_1)\Lambda_c(k_1)] D^{j_2}_{\mu_2'\mu_2''}[\Lambda_s^{-1}(k_2)\Lambda_c(k_2)] \\
&\quad \times \langle j_1, \mu_1'', j_2, \mu_2''|s_{12}, \mu_s\rangle\langle l, m, s_{12}, \mu_s|j, \mu\rangle .
\end{aligned}
\tag{142}
$$

It follows from (137) and (141) that these vectors transform as spin-j irreducible representations with respect to rotations

$$
U(R, 0)|k, j(m_1, j_1, m_2, j_2, l, s_{12})\mathbf{0}, \mu\rangle_c = \sum_{\mu=-j}^{j} |k, j(m_1, j_1, m_2, j_2)\mu', l, s_{12}\rangle_c D^j_{\mu'\mu}[R] \,. \tag{143}
$$

Note that in this expression k is a function of the invariant mass

$$
M_0 = \sqrt{m_1^2 + \mathbf{k}^2} + \sqrt{m_2^2 + \mathbf{k}^2} = \omega_{m_1}(\mathbf{k}) + \omega_{m_2}(\mathbf{k}). \tag{144}
$$

This is an irreducible basis for the rest states. Following again what was done in the one-particle case, having decomposed the rest states into irreducible representations of the rotation group (little group of p_0), we define s-spin states with arbitrary momentum by applying $U(\Lambda_s(P)) = U_1(\Lambda_s(P)) \otimes U_2(\Lambda_s(P))$ to the rest states. Thus we define the s-states of total momentum P following the construction used in (91)

$$
|k, j(m_1, j_1, m_2, j_2, l, s_{12})\mathbf{P}, \mu\rangle_s := U(\Lambda_s(P), 0)|k, j(m_1, j_1, m_2, j_2, l, s_{12})\mathbf{0}, \mu\rangle_c \sqrt{\frac{M_0}{\sqrt{M_0^2 + \mathbf{P}^2}}} \,, \tag{145}
$$

where the square root factors (89–90) imply a $\delta(\mathbf{P} - \mathbf{P}')$ normalization for unitarity. To calculate the Clebsch–Gordan coefficients we express the irreducible basis state (142) in terms of tensor products of the single-particle basis states.

Using (142) in (145) gives

$$
\begin{aligned}
&|k, j(m_1, j_1, m_2, j_2, l, s_{12})\mathbf{P}, \mu\rangle_s \\
&= \sum_{\mu_1', \mu_2', \mu_1'', \mu_2'', m, \mu_s} \int U_1(\Lambda_s(P), 0)|(m_1, j_1)\mathbf{k}, \mu_1'\rangle_s \otimes U_2(\Lambda_s(P), 0)|(m_2, j_2) - \mathbf{k}, \mu_2'\rangle_s \, d\hat{\mathbf{k}} \\
&\quad \times Y^l_m(\hat{\mathbf{k}}) D^{j_1}_{\mu_1'\mu_1''}[\Lambda_s^{-1}(k_1)\Lambda_c(k_1)] D^{j_2}_{\mu_2'\mu_2''}[\Lambda_s^{-1}(k_2)\Lambda_c(k_2)] \\
&\quad \times \langle j_1, \mu_1'', j_2, \mu_2''|s_{12}, \mu_s\rangle\langle l, m, s_{12}, \mu_s|j, \mu\rangle \sqrt{\frac{M_0}{\sqrt{M_0^2 + \mathbf{P}^2}}} \\
&= \sum_{\mu_1, \mu_2, \mu_1', \mu_2', \mu_1'', \mu_2'', m, \mu_s} \int |(m_1, j_1)\mathbf{p}_1, \mu_1\rangle_s \otimes |(m_2, j_2)\mathbf{p}_2, \mu_2\rangle_s \, d\hat{\mathbf{k}}\, Y^l_m(\hat{\mathbf{k}}) \\
&\quad \times D^{j_1}_{\mu_1\mu_1'}[R_{ws}(\Lambda_s(P), k_1)] \sqrt{\frac{\omega_{m_1}(\boldsymbol{p_1})}{\omega_{m_1}(\mathbf{k})}} D^{j_2}_{\mu_2\mu_2'}[R_{ws}(\Lambda_s(P), k_2)] \sqrt{\frac{\omega_{m_2}(\boldsymbol{p_2})}{\omega_{m_2}(\mathbf{k})}} \\
&\quad \times D^{j_1}_{\mu_1'\mu_1''}[\Lambda_s^{-1}(k_1)\Lambda_c(k_1)] D^{j_2}_{\mu_2'\mu_2''}[\Lambda_s^{-1}(k_2)\Lambda_c(k_2)] \\
&\quad \times \langle j_1, \mu_1'', j_2, \mu_2''|s_{12}, \mu_s\rangle\langle l, m, s_{12}, \mu_s|j, \mu\rangle \sqrt{\frac{M_0}{\sqrt{M_0^2 + \mathbf{P}^2}}} \,,
\end{aligned}
\tag{146}
$$

where

$$p_i = p_i(P, k_i) = \Lambda_s(P)k_i. \tag{147}$$

This equation expresses a two-particle s-spin state as a linear combination of tensor products of single s-spin states. The overlap with the single-particle s-spin states gives

$$\begin{aligned}
&{}_s\langle (m_1, j_1)\mathbf{p}_1, \mu_1 (m_2, j_2)\mathbf{p}_2, \mu_2 | k, j(m_1, j_1, m_2, j_2)\mathbf{P}, \mu, l, s_{12}\rangle_s \\
&= \sum_{\mu_1', \mu_2', \mu_1'', \mu_2'', \mu_s, m} \int \delta(\mathbf{p}_1 - \mathbf{p}_1(P,k))\delta(\mathbf{p}_2 - \mathbf{p}_2(P,k))\, d\hat{\mathbf{k}}\, Y^l_m(\hat{\mathbf{k}}) \\
&\quad \times D^{j_1}_{\mu_1\mu_1'}[R_{ws}(\Lambda_s(P), k_1)] D^{j_1}_{\mu_1'\mu_1''}[\Lambda_s^{-1}(k_1)\Lambda_c(k_1)] \\
&\quad \times D^{j_2}_{\mu_2\mu_2'}[R_{ws}(\Lambda_s(P), k_2)] D^{j_2}_{\mu_2'\mu_2''}[\Lambda_s^{-1}(k_2)\Lambda_c(k_2)] \\
&\quad \times \langle j_1, \mu_1'', j_2, \mu_2'' | s_{12}, \mu_s\rangle \langle l, m, s_{12}, \mu_s | j, \mu\rangle \\
&\quad \times \sqrt{\frac{\omega_{m_1}(\boldsymbol{p}_1)}{\omega_{m_1}(\mathbf{k})}} \sqrt{\frac{\omega_{m_2}(\boldsymbol{p}_2)}{\omega_{m_2}(\mathbf{k})}} \sqrt{\frac{M_0}{\sqrt{M_0^2 + \mathbf{P}^2}}}.
\end{aligned} \tag{148}$$

This expression is one form of the Poincaré group Clebsch–Gordan coefficient *in the s-basis*. Changing variables from $\mathbf{p}_1$ and $\mathbf{p}_2$ to $\mathbf{P}$ and $\mathbf{k}$ inverts all of the Jacobians (square root factors) and eliminates the angular integral

$$\begin{aligned}
&{}_s\langle (m_1, j_1)\mathbf{p}_1, \mu_1 (m_2, j_2)\mathbf{p}_2, \mu_2 | k, j(m_1, j_1, m_2, j_2)\mathbf{P}, \mu, l, s_{12}\rangle_s \\
&= \sum_{\mu_1', \mu_2', \mu_1'', \mu_2'', \mu_s, m} \delta(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2) \frac{\delta(k - k(\mathbf{p}_2, \mathbf{p}_2))}{k^2} Y^l_m(\hat{\mathbf{k}}(\mathbf{p}_1, \mathbf{p}_2)) \\
&\quad \times D^{j_1}_{\mu_1\mu_1'}[R_{ws}(\Lambda_s(P), k_1)] D^{j_1}_{\mu_1'\mu_1''}[\Lambda_s^{-1}(k_1)\Lambda_c(k_1)] \\
&\quad \times D^{j_2}_{\mu_2\mu_2'}[R_{ws}(\Lambda_s(P), k_2)] D^{j_2}_{\mu_2'\mu_2''}[\Lambda_s^{-1}(k_2)\Lambda_c(k_2)] \\
&\quad \times \langle j_1, \mu_1'', j_2, \mu_2'' | s_{12}, \mu_s\rangle \langle l, m, s_{12}, \mu_s | j, \mu\rangle \\
&\quad \times \sqrt{\frac{\omega_{m_1}(\mathbf{k})}{\omega_{m_1}(\mathbf{p}_1)}} \sqrt{\frac{\omega_{m_2}(\mathbf{k})}{\omega_{m_2}(\mathbf{p}_2)}} \sqrt{\frac{\sqrt{M_0^2 + \mathbf{P}^2}}{M_0}}.
\end{aligned} \tag{149}$$

These are the formal expressions for the Poincaré group Clebsch–Gordan coefficients in the s-basis. This construction is based on our convention that all different types of one-body spins are identified in the one-body rest frame and the different types of many-body spins are identified in the many-body rest frame. The quantum numbers l and s_{12} are degeneracy quantum numbers that separate different irreducible representations with the same mass and spin.

The Poincaré group Clebsch–Gordan coefficients have the same relations with the Poincaré group Wigner functions as the rotation group Clebsch–Gordan coefficients have with the rotation group Wigner functions:

$$\begin{aligned}
&\int \sum_{\mu_1', \mu_2'} d\mathbf{p}_1' \mathbf{p}_2' \mathcal{D}^{m_1, j_1}_{s:\mathbf{p}_1\mu_1;\mathbf{p}_1'\mu_1'}[\Lambda, a] \mathcal{D}^{m_2, j_2}_{s:\mathbf{p}_2\mu_2;\mathbf{p}_2'\mu_2'}[\Lambda, a] \\
&\quad \times {}_s\langle (m_1, j_1)\mathbf{p}_1', \mu_1' (m_2, j_2)\mathbf{p}_2', \mu_2' | k, j(m_1, j_1, m_2, j_2)\mathbf{P}, \mu, l, s_{12}\rangle_s \\
&= \int \sum_{\mu'} d\mathbf{P}' k^2 dk\, {}_s\langle (m_1, j_1)\mathbf{p}_1, \mu_1 (m_2, j_2)\mathbf{p}_2, \mu_2 | k, j(m_1, j_1, m_2, j_2)\mathbf{P}', \mu', l, s_{12}\rangle_s \\
&\quad \times \mathcal{D}^{m(k), j}_{s:\mathbf{P}'\mu';\mathbf{P}\mu}[\Lambda, a].
\end{aligned} \tag{150}$$

If we compare (142) to (140) we see that they differ by a pair of $SU(2)$ Clebsch–Gordan coefficients. It has the same structure as a non-relativistic state where two single-particle spins are added to an orbital angular momentum to get a total spin. In the relativistic case the spins that are added in this way differ from the single particle s-spins by the rotations

$$R_{ws}(\Lambda_s(P), k_i)\Lambda_s^{-1}(k_i)\Lambda_c(k_i) = \Lambda_s^{-1}(p_i)\Lambda_s(P)\Lambda_c(k_i)\ , \tag{151}$$

which are the composition of a Melosh rotation (71, 109) from the canonical spin to the s-spin followed by a Wigner rotation (93) for the s-boost.

It is useful to identify the corresponding relativistic spin operators that can be added, using the ordinary rules of angular momentum addition, to the orbital angular momentum to get the total two-body spin. We define the single-particle s-constituent spin operator for particle i by

$$\mathbf{j}_{iss} := \Lambda_c^{-1}(k_i)\Lambda_s^{-1}(P)\Lambda_s(p_i)\mathbf{j}_{is} = \frac{1}{m_i}\Lambda_c^{-1}(k_i)\Lambda_s^{-1}(P)W_i. \tag{152}$$

These single-particle constituent spin operators are actually *many-body* operators because they depend on the total momentum of the system. In Eq. (152) the quantities k_i, p_i, P, $\mathbf{j}_s$, W, and m_i are interpreted as operators. The transformation $\Lambda_c^{-1}(k_i)\Lambda_s^{-1}(P)\Lambda_s(p_i)$ relating $\mathbf{j}_{ss}$ to $\mathbf{j}_s$ is a momentum-dependent rotation.

The transformation $\Lambda_s(P)\Lambda_c(k_i)$ is a boost from the rest frame of particle i to its final momentum by first boosting to the standard frame of the many-body system followed by a boost to the final momentum of the particle. It has the same form as the boost in (62), $\Lambda_s(P)\Lambda_s^{-1}(p_0)$, with the constant boost $\Lambda_s^{-1}(p_0)$ replaced by $\Lambda_c(k_i)$. This has the consequence that the constituent spins are defined so that the zero-momentum vector of the *many-body system* is the standard vector.

The constituent spin operators defined in (152) have the property that they can be added to the orbital angular momentum to get the total s-spin of the combined system:

$$\mathbf{j}_s = \mathbf{l} + \mathbf{j}_{1ss} + \mathbf{j}_{2ss} = \mathbf{l} + \Lambda_c^{-1}(k_1)\Lambda_s^{-1}(P)\Lambda_s(p_1)\mathbf{j}_{1s} + \Lambda_c^{-1}(k_2)\Lambda_s^{-1}(P)\Lambda_s(p_2)\mathbf{j}_{2s}\ , \tag{153}$$

where again the Lorentz transformations above are interpreted as matrices of operators. These constituent spins add like ordinary non-relativistic spins, however they differ from the corresponding single-particle spins by momentum-dependent rotations. The rotations in Eq. (153) can be factored into the product of a generalized Melosh rotation and an s-spin Wigner rotation:

$$\mathbf{j}_s = \mathbf{l} + [\Lambda_c^{-1}(k_1)\Lambda_s(k_1)][\Lambda_s^{-1}(k_1)\Lambda_s^{-1}(P)\Lambda_s(p_1)]\mathbf{j}_{1s} + [\Lambda_c^{-1}(k_2)\Lambda_s(k_2)][\Lambda_s^{-1}(k_2)\Lambda_s^{-1}(P)\Lambda_s(p_2)]\mathbf{j}_{2s}. \tag{154}$$

This illustrates how to add single particle spins in a composite relativistic system. It is more complicated than the way that they are added in non-relativistic systems.

For the canonical spins basis $s = c$ the Melosh rotations are the identity, so the combined rotations in (154) reduce to a canonical-spin Wigner rotation. For light-front spins, since the light-front boosts form a subgroup, the Wigner rotations of the light-front boosts become the identity and the combined rotations in (154) reduce to a Melosh rotation. This is the origin of introducing the Melosh rotation. For the helicity basis the Melosh rotation is $R(\hat{\mathbf{z}} \to \hat{\mathbf{p}})$ and the Wigner rotations are diagonal and only contribute a phase. For a general s-spins the Clebsch–Gordan coefficients have both generalized Melosh rotations and s-spin Wigner rotations.

It is instructive to examine the transformation properties of the constituent s-spins under Lorentz transformations. To do this first note the transformation property of $k_i := \Lambda_s^{-1}(P)p_i$ is

$$U^\dagger(\Lambda, 0)k_i U(\Lambda, 0) = \Lambda_s^{-1}(\Lambda P)\Lambda p_i = \Lambda_s^{-1}(\Lambda P)\Lambda\Lambda_s(P)\Lambda_s^{-1}(P)p_i = \Lambda_s^{-1}(\Lambda P)\Lambda\Lambda_s(P)k_i\ . \tag{155}$$

This shows that the operators k_i are *not* four-vectors; instead they Wigner rotate under Lorentz transformations. We compare this to the transformation properties of $\mathbf{j}_{ss}$ given by (152):

$$U^\dagger(\Lambda)\mathbf{j}_{ss}U(\Lambda) = \frac{1}{m_i}\Lambda_c^{-1}(\Lambda_s^{-1}(\Lambda P)\Lambda\Lambda_s(P)k_i)\Lambda_s^{-1}(\Lambda P)\Lambda W\ . \tag{156}$$

Using the property (111) of canonical boosts, (156) becomes

$$\begin{aligned}
&\frac{1}{m_i}\Lambda_s^{-1}(\Lambda P)\Lambda\Lambda_s(P)\Lambda_c^{-1}(k_i)\Lambda_s^{-1}(P)\Lambda^{-1}\Lambda_s(\Lambda P)\Lambda_s^{-1}(\Lambda P)\Lambda W \\
&\quad = \frac{1}{m_i}\Lambda_s^{-1}(\Lambda P)\Lambda\Lambda_s(P)\Lambda_c^{-1}(k_i)\Lambda_s^{-1}(P)W \\
&\quad = \Lambda_s^{-1}(\Lambda P)\Lambda\Lambda_s(P)\mathbf{j}_{ss}.
\end{aligned} \tag{157}$$

This is identical to the transformation property (155) of k_i. It is precisely because the constituent spins and relative momentum have the same transformation properties with respect to rotations that allows them to be combined. The difference between the constituent spins and the single-particle spins is that because the standard vector is the zero-momentum vector of the system, the Wigner rotations all involve the same boost $\Lambda_s(P)$ rather than the different single-particle boosts, $\Lambda_s(p_i)$.

It is interesting to compare the constituent spin defined in Eq. (152) to the spin defined in (62) and (64) for the case that p_s is the rest frame of the system:

$$\mathbf{j}_{ss} := \frac{1}{m_i}\Lambda_c^{-1}(k_i)\Lambda_s^{-1}(P)W_i \ , \tag{158}$$

$$\mathbf{j}_s = \frac{1}{m_i}\Lambda_s(p_0)\Lambda_s^{-1}(P)W_i . \tag{159}$$

We see that both correspond to single-particle spins with a *standard vector that is different than the single-particle rest vector*. The only difference is that the boost from the standard frame to the rest frame, $\Lambda_c^{-1}(k_i)$, involves another variable, while the corresponding boost $\Lambda_s(p_0)$ involves a constant.

As a final remark, when coupling spins with Poincaré group Clebsch–Gordan coefficients, the observation that the spins must be first converted to canonical spins in the system rest frame before being added means that the Clebsch–Gordan coefficients in any s-spin basis are related to the Clebsch–Gordan coefficients in the canonical spin basis by applying generalized Melosh rotations to the single-particle spins and the total spin. Thus the Clebsch–Gordan coefficients in different spin bases are related by

$$\begin{aligned}
&{}_t\langle(m_1, j_1)\mathbf{p}_1, \mu_1(m_2, j_2)\mathbf{p}_2, \mu_2|k, j(m_1, j_1, m_2, j_2)\mathbf{P}, \mu, l, s_{12}\rangle_t \\
&= \sum_{\mu_1',\mu_2',\mu'} D^{j_1}_{\mu_1\mu_1'}[\Lambda_t^{-1}(p_1)\Lambda_s(p_1)]D^{j_2}_{\mu_2\mu_2'}[\Lambda_t^{-1}(p_2)\Lambda_s(p_2)] \\
&\quad \times {}_s\langle(m_1, j_1)\mathbf{p}_1, \mu_1'(m_2, j_2)\mathbf{p}_2, \mu_2'|k, j(m_1, j_1, m_2, j_2)\mathbf{P}, \mu', l, s_{12}\rangle_s \times D^{j}_{\mu'\mu}[\Lambda_s^{-1}(P)\Lambda_t(P)].
\end{aligned} \tag{160}$$

When the spins are successively added using pairwise coupling in any basis, the intermediate state Melosh rotations *all* identically cancel. It follows, for example, if we use s-basis Poincaré group Clebsch–Gordan coefficients to successively couple products of s-basis unitary representations of the Poincaré group to a direct integral of s-base unitary representations, the results are identical to what one would get using c-basis Poincaré group Clebsch–Gordan coefficients and Melosh rotating the initial and final spin states to the s-basis. Thus *the effect of combining spins in composite systems is independent of the choice of spin basis up to the initial and final Melosh rotations*. For example, this means that the spin structures of composite systems using canonical spin or light-front spin are identical up to a trivial overall change of basis.

This means that for systems of free particles there is no loss in generality in coupling using only canonical spins (or any other type of spin).

7 Two and Four Component Spinors

In Poincaré invariant quantum mechanics spin-$\frac{1}{2}$ particles are described using two-component spinors while in the Dirac equation they are described by four-component spinors. In any experiment there are only two spin states that can be measured. In this section we discuss the relation between these two equivalent treatments of spin.

The difference between these two treatments of spin is that the two-component spinor description uses irreducible representations of the Poincaré group to describe particles while the four-component spinor description uses finite-dimensional representations of the Lorentz group.

The connection between these two representations is most easily illustrated by taking apart a Wigner rotation and absorbing the momentum-dependent boosts into the state vectors. For a spin j particle the action of the Lorentz group on an irreducible s-basis state is (97)

$$U(\Lambda, 0)|(m, j)\mathbf{p}, \mu\rangle_s = \sum_{\mu'} |(m, j)\boldsymbol{\Lambda} p, \mu'\rangle_s \sqrt{\frac{\omega_m(\boldsymbol{\Lambda} p)}{\omega_m(\mathbf{p})}} D^j_{\mu'\mu}[\Lambda_s^{-1}(\Lambda p)\Lambda\Lambda_s(p)]. \tag{161}$$

In what follows we use the fact that the $SU(2)$ Wigner functions, $D^j_{\mu'\mu}[R]$, which are $2j+1$ dimensional representations of $SU(2)$ are also $2j+1$ dimensional representations of $SL(2,\mathbb{C})$ when the $SU(2)$ matrix elements, R, are replaced by the corresponding $SL(2,\mathbb{C})$ matrix elements, Λ.

To show this first note that the group representation property for the $D^j_{\mu'\mu}[R]$ can be written as

$$0 = \sum_{\mu''=-j}^{j} D^j_{\mu\mu''}[e^{\frac{i}{2}\boldsymbol{\theta}_1\cdot\boldsymbol{\sigma}}]D^j_{\mu''\mu'}[e^{\frac{i}{2}\boldsymbol{\theta}_2\cdot\boldsymbol{\sigma}}] - D^j_{\mu\mu'}[e^{\frac{i}{2}\boldsymbol{\theta}_1\cdot\boldsymbol{\sigma}}e^{\frac{i}{2}\boldsymbol{\theta}_2\cdot\boldsymbol{\sigma}}]. \tag{162}$$

The right hand side of Eq. (162) is an entire function of the three components of the two real angles, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. This is because the $D^j_{\mu'\mu}[R]$ are homogeneous polynomials in the matrix elements of R with real coefficients (80), so they are entire functions of R, and the $SU(2)$ rotations, $R = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}$, are entire (exponential) functions of the angles. It follows that $D^j_{\mu'\mu}[e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}]$ is an entire function of the angles. Since Eq. (162) is identically zero for all real $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, by analytic continuation it is identically zero for all complex angles, $\boldsymbol{\theta}_i \to \mathbf{z}_i$. Since the most general $SL(2,\mathbb{C})$ matrix, $\Lambda = e^{\frac{\mathbf{z}}{2}\cdot\boldsymbol{\sigma}}$ (15), is an analytic continuation, $\boldsymbol{\theta} \to -i\mathbf{z}$, to a complex angle of an $SU(2)$ matrix, $R = e^{i\frac{\boldsymbol{\theta}}{2}\cdot\boldsymbol{\sigma}}$, it follows that

$$\sum_{\mu''=-j}^{j} D^j_{\mu\mu''}[e^{\frac{\mathbf{z}_1}{2}\cdot\boldsymbol{\sigma}}]D^j_{\mu''\mu'}[e^{\frac{\mathbf{z}_2}{2}\cdot\boldsymbol{\sigma}}] = D^j_{\mu\mu'}[e^{\frac{\mathbf{z}_1}{2}\cdot\boldsymbol{\sigma}}e^{\frac{\mathbf{z}_2}{2}\cdot\boldsymbol{\sigma}}]. \tag{163}$$

This shows that $D^j_{\mu'\mu}[\Lambda]$, given by (80), is a $2j+1$ dimensional representation of $SL(2,\mathbb{C})$. While these representations are irreducible, they are no longer unitary.

Using the group representation property (163) with respect to $SL(2,\mathbb{C})$ we can split up the Wigner function in (161) into a product of three distinct parts:

$$D^j_{\mu'\mu}[\Lambda_s^{-1}(\Lambda p)\Lambda\Lambda_s(p)] = \sum_{\mu_1\mu_2} D^j_{\mu'\mu_1}[\Lambda_s^{-1}(\Lambda p)]D^j_{\mu_1\mu_2}[\Lambda]D^j_{\mu_2\mu}[\Lambda_s(p)]. \tag{164}$$

If we use (164) in (161) and right multiply by the inverse of the last matrix we obtain

$$\begin{aligned}&\sum_{\mu'} U(\Lambda,0)|(m,j)\mathbf{p},\mu'\rangle_s\sqrt{\omega_m(\mathbf{p})}D^j_{\mu'\mu}[\Lambda_s^{-1}(p)]\\ &\quad= \sum_{\mu'\mu''}|(m,j)\boldsymbol{\Lambda}p\mu''\rangle_s\sqrt{\omega_m(\boldsymbol{\Lambda}p)}D^j_{\mu''\mu'}[\Lambda_s^{-1}(\Lambda p)]D^j_{\mu'\mu}[\Lambda].\end{aligned} \tag{165}$$

This is completely equivalent to (161). This leads us to define a *Lorentz covariant* basis state by

$$|(m,j)\mathbf{p},b\rangle := \sum_{\mu'}|(m,j)\mathbf{p},\mu'\rangle_s\sqrt{\omega_m(\mathbf{p})}D^j_{\mu'b}[\Lambda_s^{-1}(p)]. \tag{166}$$

Here we use the index notation b to emphasize that it is not a magnetic quantum number, even though it has $2j+1$ values. The Hilbert space resolution of the identity in this representation is

$$\begin{aligned}I &= \int\sum_{\mu=-j}^{j}|(m,j)\mathbf{p},\mu\rangle_s\, d\mathbf{p}\;{}_s\langle(m,j)\mathbf{p},\mu|\\ &= \int\sum_{b,b'=-j}^{j}|(m,j)\mathbf{p},b\rangle\frac{d\mathbf{p}}{\omega(\mathbf{p})}D^j_{bb'}[\Lambda_s(p)\Lambda_s^{\dagger}(p)]\langle(m,j)\mathbf{p},b'|\,.\end{aligned} \tag{167}$$

The matrix $D^j_{bb'}[\Lambda_s(p)\Lambda_s^{\dagger}(p)]$ looks like it depends on s, but because $\Lambda_s(p) = \Lambda_c(p)R_{cs}(p)$ (71, 109), the Melosh rotations, $R_{cs}(p)$, cancel giving

$$\Lambda_s(p)\Lambda_s^{\dagger}(p) = \Lambda_c(p)\Lambda_c(p)^{\dagger} = \Lambda_c^2(p) = \frac{1}{m}p^{\mu}\sigma_{\mu}\,, \tag{168}$$

which is a positive (has positive eigenvalues) Hermetian kernel (for timelike p) that is independent of s. Here we used the fact that a general boost can be expressed as a rotation followed by a canonical boost (22), the fact the canonical boosts are positive Hermetian $SL(2,\mathbb{C})$ matrices and the identity $\Lambda_c^2 = \cosh(\rho) I + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\sinh(\rho) = p^\mu\sigma_\mu$. Using (168) the resolution of the identity (167) can be expressed in the following manifestly covariant form

$$I = \int \sum_{bb'} |(m,j)\mathbf{p}, b\rangle d^4p\theta(p^0)\delta(p^2+m^2) D^j_{bb'}[p^\mu\sigma_\mu/m]\langle(m,j)\mathbf{p}, b'|. \tag{169}$$

Wightman [10] uses symmetric tensor products of the spin 1/2 representations of this form as representations of the irreducible representation of the Poincaré group.

This means that if we define covariant wave functions

$$\psi(p,b) := \langle(m,j)\mathbf{p}, b|\psi\rangle \tag{170}$$

the Hilbert space scalar product is

$$\langle\psi|\phi\rangle = \int \sum \psi^*(p,b) d^4p\theta(p^0)\delta(p^2+m^2) D^j_{bb'}[p^\mu\sigma_\mu/m]\phi(p,b') \tag{171}$$

and

$$U(\Lambda,0)|(m,j)\mathbf{p}, b\rangle = |(m,j)\boldsymbol{\Lambda}p, b'\rangle D^j_{b'b}[\Lambda] \tag{172}$$

is unitary with respect to the inner product (171). Here the wave functions are really equivalence classes of functions that agree on the mass shell. The presence of non-trivial scalar products is a generic feature of covariant unitary representations of the Poincaré group. Note the mass is selected by the kernel of inner product, which carries all of the dynamical information in this representation. In quantum field theory the kernel of the non-trivial scalar products are the Wightman functions (i.e. vacuum expectation values of products of fields) which also carry all of the dynamical information.

While Eqs. (167–172) contain exactly the same information as (161), the Wigner function of the little group is replaced by a *momentum-independent* $2j+1$ dimensional representation of $SL(2,\mathbb{C})$. The covariant representation has the advantage that it is independent of the choice of the standard vector or standard boost that are used in the construction of irreducible representations of the Poincaré group. The disadvantage is that the finite dimensional irreducible representations of the Lorentz group are not unitary and do not admit a linear representation of space reflection; however the norm associated with the inner product (171) with the non-trivial kernel is non-negative.

To understand origin of the spin doubling in Lorentz covariant theories recall that (25) implies $R = \sigma_2 R^*\sigma_2$ for any $SU(2)$ rotation R. Using this identity the Wigner rotation R in (161) can be replaced by $\sigma_2 R^*\sigma_2$. Making this replacement and repeating steps (164–167) we define a new covariant state

$$|(m,j)\mathbf{p}, \dot{b}\rangle := \sum_\mu |(m,j)\mathbf{p},\mu\rangle_s \sqrt{\omega_m(\mathbf{p})} D^j_{\mu\dot{b}}[\sigma_2\Lambda_s^{-1*}(p)\sigma_2] \,, \tag{173}$$

which has the transformation property

$$U(\Lambda,0)|(m,j)\mathbf{p}, \dot{b}\rangle = \sum_{\dot{b}'} |(m,j)\boldsymbol{\Lambda}p, \dot{b}'\rangle D^j_{\dot{b}'\dot{b}}[\sigma_2\Lambda^*\sigma_2]. \tag{174}$$

In $SU(2)$ $i\sigma_2$ corresponds to a rotation about the y axis by π so (174) can equivalently be written as

$$U(\Lambda,0)|(m,j)\mathbf{p}, \dot{b}\rangle = \sum_{\dot{b}'} |(m,j)\boldsymbol{\Lambda}p, \dot{b}'\rangle D^j_{\dot{b}'\dot{b}}[R_y(\pi)\Lambda^* R_y^{-1}(\pi)]. \tag{175}$$

We use the dot on the b index to distinguish the complex-conjugate representation from the original representation. The relevant observation (see the discussion following (27)) is that while the complex-conjugate representation of $SU(2)$ is equivalent (related by a constant similarity transformation) to $SU(2)$, this is no longer true for $SL(2,\mathbb{C})$. The states (166) and (173) transform under different inequivalent representations of $SL(2,\mathbb{C})$.

From a strictly mathematical point of view it is possible to use either of the two inequivalent representations, but there is also physical relation between these two representations. They are related by space reflection as discussed in (26). The difficulty arises because $D^j_{bb'}[p^\mu\sigma_\mu/m]$ appears in the kernel of the scalar product, $d^4p\theta(p^0)\delta(p^2+m^2)D^j_{bb'}[p^\mu\sigma_\mu/m]$, rather than in the wave function. This means that space reflection not only transforms the wave function, it also changes the scalar product by replacing $D^j_{bb'}[p^\mu\sigma_\mu/m]$ by $D^j_{\dot b\dot b'}[p^\mu\sigma_2\sigma^*_\mu\sigma_2/m]$ (which is also positive for timelike p).

The simplest way to allow space reflection to be represented by a linear operator in the Lorentz covariant representation is to use a Hilbert space representation where the kernel of the covariant scalar product contains direct sum of both representations:

$$d^4p\theta(p^0)\delta(p^2+m^2)D^j_{bb'}[p^\mu\sigma_\mu/m] \to d^4p\theta(p^0)\delta(p^2+m^2)\begin{pmatrix} D^j_{bb'}[p^\mu\sigma_\mu/m] & 0 \\ 0 & D^j_{\dot b\dot b'}[p^\mu\sigma_2\sigma^*_\mu\sigma_2/m]\end{pmatrix}. \tag{176}$$

Then space reflection can be represented by a linear operator. This is the origin of the 4-component treatment of spin 1/2.

The key observation is the identity

$$\Lambda\Lambda_s(p) = \Lambda_s(\Lambda p)\Lambda_s^{-1}(\Lambda p)\Lambda\Lambda_s(p) = \Lambda_s(\Lambda p)R_{ws}(\Lambda, p)\,, \tag{177}$$

which shows that p-dependent boosts, $\Lambda_s(p)$, convert Lorentz transformations, Λ, to Wigner rotations, $R_{ws}(\Lambda, p)$.

To relate this to the transformation properties of two-component spinors under $SL(2,\mathbb{C})$ we define four different types of 2 component spinors

$$\xi_a,\ \xi^a,\ \xi_{\dot a},\ \xi^{\dot a}. \tag{178}$$

These two component spinors are characterized by their $SL(2,\mathbb{C})$ transformation properties:

$$\Lambda_a{}^b = e^{i\mathbf{z}\cdot\boldsymbol{\sigma}} \tag{179}$$

$$\Lambda_{\dot a}{}^{\dot b} = (e^{i\mathbf{z}\cdot\boldsymbol{\sigma}})^* \tag{180}$$

$$\Lambda^b{}_a = \sigma_2 e^{i\mathbf{z}\cdot\boldsymbol{\sigma}}\sigma_2 = ((e^{i\mathbf{z}\cdot\boldsymbol{\sigma}})^t)^{-1} \tag{181}$$

$$\Lambda^{\dot a}{}_{\dot b} = \sigma_2(e^{i\mathbf{z}\cdot\boldsymbol{\sigma}})^*\sigma_2 = ((e^{i\mathbf{z}\cdot\boldsymbol{\sigma}})^\dagger)^{-1} \tag{182}$$

$$\xi_a \to \xi'_a = \Lambda_a{}^b\xi_b \tag{183}$$

$$\xi_{\dot a} \to \xi'_{\dot a} = \Lambda_{\dot a}{}^{\dot b}\xi_{\dot b} \tag{184}$$

$$\xi^a \to \xi^{a\prime} = \Lambda^a{}_b\xi^b \tag{185}$$

$$\xi^{\dot a} \to \xi^{\dot a\prime} = \Lambda^{\dot a}{}_{\dot b}\xi^{\dot b} \tag{186}$$

$$\xi_a = (\sigma_2)_{ab}\xi^b \quad \xi_{\dot a} = (\sigma_2)_{\dot a\dot b}\xi^{\dot b}\,. \tag{187}$$

The reason for introducing the upper- and lower-index 2-component Lorentz spinors is that the products

$$\sum_a \xi^a\chi_a \qquad \text{and} \qquad \sum_{\dot a}\xi^{\dot a}\chi_{\dot a} \tag{188}$$

are Lorentz invariant. This follows from the identity $\Lambda^{-1} = \sigma_2\Lambda^t\sigma_2$ which holds for any $SL(2,\mathbb{C})$ matrix. The proof is elementary:

$$\Lambda^{-1} = e^{-\mathbf{z}\cdot\boldsymbol{\sigma}} = e^{\mathbf{z}\cdot\sigma_2\boldsymbol{\sigma}^t\sigma_2} = \sigma_2(e^{\mathbf{z}\cdot\boldsymbol{\sigma}})^t\sigma_2 = \sigma_2\Lambda^t\sigma_2. \tag{189}$$

Using (189)

$$\sum_a \xi^a\chi_a \to \sum_a \xi'^a\chi'_a = (\Lambda^t)^{-1}\xi\cdot\Lambda\chi = \xi\cdot\Lambda^{-1}\Lambda\chi = \sum_a\xi^a\chi_a \tag{190}$$

$$\sum_{\dot a} \xi^{\dot a}\chi_{\dot a} \to \sum_{\dot a} \xi'^{\dot a}\chi'_{\dot a} = (\Lambda^\dagger)^{-1}\xi\cdot\Lambda^*\chi = \xi\cdot(\Lambda^*)^{-1}\Lambda^*\chi = \sum_{\dot a}\xi^{\dot a}\chi_{\dot a}. \tag{191}$$

The matrix σ_2 acts like a metric tensor—it can be used to raise and lower indices. The sum over an upper and lower undotted or dotted index is Lorentz invariant.

The difference with an ordinary metric is that σ_2 is antisymmetric so the invariants $\xi^a \xi_a = \xi^{\dot{a}} \xi_{\dot{a}} = 0$ always vanish. In the literature, σ_2 is sometimes replaced by the real antisymmetric matrix $\epsilon = i\sigma_2$ and its inverse $\epsilon^{-1} = -\epsilon$, which is the $SL(2,\mathbb{C})$ representation of a rotation about the y-axis by π.

To motivate the choice of the spinor representation of space reflection note the four-vector X transforms like a mixed spin tensor

$$\mathsf{X} \to \mathsf{X}' = \Lambda \mathsf{X} \Lambda^\dagger = (\Lambda \otimes \Lambda^*)\mathsf{X} \,, \tag{192}$$

which suggests the notation

$$\mathsf{X}_{a\dot{b}} \to \mathsf{X}'_{a\dot{b}} = \Lambda_a{}^c \Lambda_{\dot{c}}{}^{\dot{d}} \mathsf{X}_{c\dot{d}} \,, \tag{193}$$

where we have assumed that repeated spinor indices are summed from 1 to 2. Space reflection, given by (26), is represented by

$$\mathsf{X} \to \mathsf{X}' = \sigma_2 \mathsf{X}^* \sigma_2 = -(\sigma_2 \otimes \sigma_2)\mathsf{X}^* \,. \tag{194}$$

The Lorentz transformation properties of the reflected vector X' are

$$\mathsf{X}' \to \mathsf{X}'' = -(\sigma_2 \otimes \sigma_2)(\Lambda^* \otimes \Lambda)\mathsf{X}^* = (\sigma_2 \Lambda^* \sigma_2) \otimes (\sigma_2 \Lambda \sigma_2)(-\sigma_2 \otimes \sigma_2)\mathsf{X}^* \,. \tag{195}$$

Eq. (195) shows that the reflected four-vector X' transforms like a mixed-spin tensor with upper indices

$$X_{a\dot{b}} \to X^{\dot{a}b}. \tag{196}$$

To determine the spinor representation of space reflection we note that a positive energy light-like four vector can be represented as the tensor product of a two-spinor and its complex conjugate

$$X_{a\dot{b}} = \xi_a(\mathbf{x})\xi^*_{\dot{b}}(\mathbf{x}) \,, \tag{197}$$

where

$$\mathbf{x} = \frac{1}{2}\mathrm{Tr}(\boldsymbol{\sigma}\xi_a(\mathbf{x})\xi^*_{\dot{b}}(\mathbf{x})) \tag{198}$$

are the space components of the light-like four vector. The space reflection operator on X in (194) on this four vector is

$$X_{a\dot{b}} \to \xi_a(-\mathbf{x})\xi^*_{\dot{b}}(-\mathbf{x}) = -(\sigma_2 \xi^*(\mathbf{x})^{\dot{a}}) \otimes (\sigma_2 \xi^{**}(\mathbf{x})^b) \,. \tag{199}$$

This is consistent with the following spinor representation of space reflection

$$\xi_a(\mathbf{x}) \to \xi_a(-\mathbf{x}) = (i\sigma_2 \xi^*(\mathbf{x}))^{\dot{a}} \tag{200}$$

$$\xi_{\dot{a}}(\mathbf{x}) \to \xi_{\dot{a}}(-\mathbf{x}) = (i\sigma_2 \xi^*(\mathbf{x}))^a. \tag{201}$$

Because space reflection changes a spinor that transforms under one representation of $SL(2, \mathbf{C})$ to one that transforms under the conjugate representation it cannot be represented by a linear transformation in terms of Lorentz covariant spinors. The two different kinds of Lorentz covariant spinors are called right and left handed spinors because they are related by space reflection.

In order to represent space reflection by a linear transformation it is enough to replace a single spinor by the direct sum of a right and left handed spinor. This 4 spinor has the Lorentz transformation properties:

$$\xi \to \begin{pmatrix} \xi_a \\ \chi^{\dot{b}} \end{pmatrix} \qquad \begin{pmatrix} \xi \\ \chi \end{pmatrix} \to \begin{pmatrix} \xi' \\ \chi' \end{pmatrix} = \begin{pmatrix} \Lambda & 0 \\ 0 & (\Lambda^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} \xi \\ \chi \end{pmatrix}. \tag{202}$$

With this choice both spinors have the same transformation under $SU(2)$ rotations because $(R^\dagger)^{-1} = R$. Space reflection becomes a linear transformation that interchanges the right and left handed spinors and multiplies by $\pm i\sigma_2$ (i.e. raises or lowers the spin indices). The new components allow for a linear realization of space reflection.

We define the doubled representation of the Lorentz group and space reflection operator by

$$S(\Lambda) = \begin{pmatrix} \Lambda & 0 \\ 0 & (\Lambda^\dagger)^{-1} \end{pmatrix} \qquad \mathcal{P} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{203}$$

The doubling also occurs for higher spins. For $2(2j+1)$ component spinors the $SL(2,\mathbb{C})$ matrices Λ are replaced by $D^j_{\mu\mu'}[\Lambda]$:

$$S(\Lambda) = \begin{pmatrix} D^j[\Lambda] & 0 \\ 0 & D^j[\Lambda^\dagger]^{-1} \end{pmatrix}. \tag{204}$$

If we restrict Λ to $SU(2)$ then

$$S(\Lambda) \to S(R) = \begin{pmatrix} D^j[R] & 0 \\ 0 & D^j[(R^\dagger)^{-1}] \end{pmatrix} = \begin{pmatrix} D^j[R] & 0 \\ 0 & D^j[R] \end{pmatrix}. \tag{205}$$

This means that a four-component spin-up state is a direct sum of two identical $SU(2)$ spin up states, similarly for spin down states.

The structure of field operators is based on the dual roles played by the Lorentz group and little group of the Poincaré group.

Fields are operator densities that transform linearly with respect to a finite dimensional representation $S(\Lambda)$ of the Lorentz group.

$$U(\Lambda, a)\Psi_a(x)U^\dagger(\Lambda, a) = S(\Lambda^{-1})_{aa'}\Psi_{a'}(\Lambda x + a). \tag{206}$$

Free fields are linear in operators that create and/or annihilate particles. The operator $a_s^\dagger(\mathbf{p}, \mu)$ creates the one-particle state with s-spin, $|(m, j)\mathbf{p}, \mu\rangle_s$, out of the vacuum

$$a_s^\dagger(\mathbf{p}, \mu)|0\rangle = |(m, j)\mathbf{p}, \mu\rangle_s. \tag{207}$$

The creation operator has the same Poincaré transformation properties as the single-particle basis states; the spins transform with a representation of the little group of the Poincaré group:

$$\begin{aligned} U(\Lambda, a)a_s^\dagger(\mathbf{p}, \mu)U^\dagger(\Lambda, a) &= e^{-i\Lambda p\cdot a}a_s^\dagger(\boldsymbol{\Lambda} p, \nu)\sqrt{\frac{\omega(\boldsymbol{\Lambda} p)}{\omega(\mathbf{p})}}D^j_{\nu\mu}(\Lambda_{0s}^{-1}(\Lambda p)\Lambda\Lambda_{0s}(p)) \\ &= e^{-i\Lambda p\cdot a}\sqrt{\frac{\omega(\boldsymbol{\Lambda} p)}{\omega(\mathbf{p})}}D^j_{\mu\nu}(\Lambda_{0s}^t(p)\Lambda^t\Lambda_{0s}^{-1t}(\Lambda p))a_s^\dagger(\boldsymbol{\Lambda} p, \nu). \end{aligned} \tag{208}$$

Taking adjoints gives the transformation properties of the annihilation operator

$$\begin{aligned} U(\Lambda, a)a_s(\mathbf{p}, \mu)U^\dagger(\Lambda, a) &= e^{i\Lambda p\cdot a}a_s(\boldsymbol{\Lambda} p, \nu)\sqrt{\frac{\omega(\boldsymbol{\Lambda} p)}{\omega(\mathbf{p})}}D^{j*}_{\nu\mu}(\Lambda_{0s}^{-1}(\Lambda p)\Lambda\Lambda_{0s}(p)) \\ &= e^{i\Lambda p\cdot a}D^j_{\mu\nu}(\Lambda_{0s}^{-1}(p)\Lambda^{-1}\Lambda_{0s}(\Lambda p))a_s(\boldsymbol{\Lambda} p, \nu)\sqrt{\frac{\omega(\boldsymbol{\Lambda} p)}{\omega(\mathbf{p})}}. \end{aligned} \tag{209}$$

Note that for the equations with the Wigner functions on the left, the argument of the Wigner function in the creation operator is $\Lambda_{0s}^t(p)\Lambda^t\Lambda_{0s}^{-1t}(\Lambda p)$ while the argument of the Wigner function in the annihilation operator is $(\Lambda_{0s}^{-1}(p)\Lambda^{-1}\Lambda_{0s}(\Lambda p))$. If we use (189) we have

$$\Lambda_{0s}^t(p)\Lambda^t\Lambda_{0s}^{-1t}(\Lambda p) = \sigma_2(\Lambda_{0s}^{-1}(p)\Lambda^{-1}\Lambda_{0s}(\Lambda p)\sigma_2. \tag{210}$$

In general, the field has a representation of the form

$$\Psi_a(x) = \int d\mathbf{p}(U_s(\mathbf{p}, \mu)a_s(\mathbf{p}, \mu)e^{-i\omega(\mathbf{p})t+i\mathbf{p}\cdot\mathbf{x}} + V_s(\mathbf{p}, \mu)b_s^\dagger(\mathbf{p}, \mu)e^{i\omega(\mathbf{p})t-i\mathbf{p}\cdot\mathbf{x}}). \tag{211}$$

The structure of the complex coefficients $U_s(\mathbf{p},\mu)$ and $V_s(\mathbf{p},\mu)$ are determined by comparing the coefficients of the creation and annihilation operator in

$$U(\Lambda,a)\Psi_a(x)U^\dagger(\Lambda,a)=S(\Lambda^{-1})_{aa'}\Psi_{a'}(\Lambda x+a)=\int d\mathbf{p}[U_s(\mathbf{p},\mu)e^{i\Lambda p\cdot a}a_s(\boldsymbol{\Lambda}p,\nu)\sqrt{\frac{\omega(\boldsymbol{\Lambda}p)}{\omega(\mathbf{p})}}D^j_{\nu\nu'}(R_y(\pi))$$
$$\times D^j_{\nu'\nu''}(\Lambda^{-1}_{0s}(\Lambda p)\Lambda\Lambda_{0s}(p))D^j_{\nu''\mu}(R_y^{-1}(\pi))e^{-i\omega(\mathbf{p})t+i\mathbf{p}\cdot\mathbf{x}}$$
$$+V_s(\mathbf{p},\mu)e^{-i\Lambda p\cdot a}b_s^\dagger(\boldsymbol{\Lambda}p,\nu)\sqrt{\frac{\omega(\boldsymbol{\Lambda}p)}{\omega(\mathbf{p})}}D^j_{\nu\mu}(\Lambda^{-1}_{0s}(\Lambda p)\Lambda\Lambda_{0s}(p))e^{i\omega(\mathbf{p})t-i\mathbf{p}\cdot\mathbf{x}}]\ , \tag{212}$$

where we have used (210) in (212). Comparison of these equivalent expressions, after the variable change $p'=\Lambda p$ in equation (212), including the Jacobian from the change of variables in the momentum integral, gives

$$S(\Lambda)_{ab}U_b(\mathbf{p},\mu)\sqrt{\omega(\mathbf{p})}=U_a(\boldsymbol{\Lambda}p,\nu)\sqrt{\omega(\boldsymbol{\Lambda}p)}D^j_{\nu\mu}(\Lambda^{-1}_{0s}(\Lambda p)\Lambda\Lambda_{0s}(p)) \tag{213}$$
$$S(\Lambda)_{ab}V_b(\mathbf{p},\mu)\sqrt{\omega(\mathbf{p})}=V_a(\boldsymbol{\Lambda}p,\nu)\sqrt{\omega(\boldsymbol{\Lambda}p)}D^{j*}_{\nu\mu}(\Lambda^{-1}_{0s}(\Lambda p)\Lambda\Lambda_{0s}(p))\ . \tag{214}$$

It is useful to define new quantities

$$u_a(\mathbf{p},\mu):=U_a(\mathbf{p},\mu)\sqrt{\omega(\mathbf{p})} \tag{215}$$
$$v_a(\mathbf{p},\mu):=V_a(\mathbf{p},\nu)D^j_{\nu\mu}(R_y(\pi))\sqrt{\omega(\mathbf{p})}. \tag{216}$$

In terms of these new quantities the covariance relations take on the form

$$S(\Lambda)_{ab}u_b(\mathbf{p},\mu)=u_a(\boldsymbol{\Lambda}p,\nu)D^j_{\nu\mu}(\Lambda^{-1}_{0s}(\Lambda p)\Lambda\Lambda_{0s}(p)) \tag{217}$$
$$S(\Lambda)_{ab}v_b(\mathbf{p},\mu)=v_a(\boldsymbol{\Lambda}p,\nu)D^j_{\nu\mu}(\Lambda^{-1}_{0s}(\Lambda p)\Lambda\Lambda_{0s}(p)). \tag{218}$$

To determine the $\mathbf{p}$ dependence of $u_b(\mathbf{p},\mu)$ or $v_a(\mathbf{p},\mu)$ we set $p=p_0$, $\Lambda=\Lambda_{0s}(p)$. In this case the Wigner rotation is the identity

$$\Lambda^{-1}_{0s}(\Lambda p)\Lambda\Lambda_{0s}(p)\to\Lambda^{-1}_{0s}(\Lambda_{0s}(p)p_0)\Lambda_{0s}(p)\Lambda_{0s}(p_0)=I\ , \tag{219}$$

which gives

$$u_a(\mathbf{p},\nu)=S(\Lambda_{0s}(p))_{ab}u_b(\mathbf{0},\mu) \tag{220}$$
$$v_a(\mathbf{p},\nu)=S(\Lambda_{0s}(p))_{ab}v_b(\mathbf{0},\mu). \tag{221}$$

From these expressions we see that $u_a(\mathbf{p},\nu)$ and $v_a(\mathbf{p},\nu)$ are representations of an s-boost from the rest frame, multiplied by a constant matrix that maps the 2(2j+1) component spinors to (2j+1) component spinors. It is instructive to note the similarity with equation (177), which also uses a Lorentz boost to intertwine finite-dimensional representations of the Lorentz group with unitary representations of the rotation group. In the field theory case, when Λ is a rotation

$$S(\Lambda)=\begin{pmatrix}D^j(\Lambda) & 0\\ 0 & D^j(\Lambda^\dagger)^{-1})\end{pmatrix}\to\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}D^j(R), \tag{222}$$

both the left and right handed spinor have identical transformation properties and can be factored out. Similarly the spin operator becomes

$$\mathbf{J}=-i\frac{d}{d\lambda}\begin{pmatrix}D^j(e^{i\lambda\boldsymbol{\sigma}/2}) & 0\\ 0 & D^j(e^{i\lambda\boldsymbol{\sigma}/2})\end{pmatrix}_{\lambda=0}\to\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}\left(-i\frac{d}{d\lambda}D^j(e^{i\lambda\boldsymbol{\sigma}/2})\right)_{\lambda=0}. \tag{223}$$

Using these relations we get a standard looking representation for a free field

$$\Psi_a(x) = \int \frac{d\mathbf{p}}{\sqrt{\omega(\mathbf{p})}}(u_a(\mathbf{p},\mu)a_s(\mathbf{p},\mu)e^{-i\omega(\mathbf{p})t+i\mathbf{p}\cdot\mathbf{x}} + v_a(\mathbf{p},\mu)D^{1/2}_{\mu\nu}(R_y^{-1}(\pi))b_s^\dagger(\mathbf{p},\nu)e^{i\omega(\mathbf{p})t-i\mathbf{p}\cdot\mathbf{x}})$$
$$= \int \frac{d\mathbf{p}}{\sqrt{\omega(\mathbf{p})}}(S(\Lambda_{0s}(p))_{ab}u_b(\mathbf{0},\mu)a_s(\mathbf{p},\mu)e^{-i\omega(\mathbf{p})t+i\mathbf{p}\cdot\mathbf{x}}$$
$$+S(\Lambda_{0s}(p))_{ab}v_b(\mathbf{0},\mu)D^{1/2}_{\mu\nu}(R_y^{-1}(\pi))b_s^\dagger(\mathbf{p},\nu)e^{i\omega(\mathbf{p})t-i\mathbf{p}\cdot\mathbf{x}}). \tag{224}$$

This has the standard form of a free Dirac field operator, up to normalization, except normally the y-rotation is absorbed in the definition of the $v_b(\mathbf{0},\mu)$.

In this section we started from a Poincaré covariant description of a particle as developed in Sect. 4, absorbed the momentum-dependent boosts from the Wigner rotation into the wave function, doubled the representation of $SL(2,\mathbb{C})$ to represent space reflection linearly, constructed fields that transform covariantly under the same doubled representation of the Lorentz group and arrived at the standard form of a free Dirac field. The Dirac equation was never used in this derivation, even though the resulting free field is a solution of the Dirac equation. The Hilbert space in this case has scalar product with a momentum-dependent kernel. This same construction trivially generalizes to higher spin fields and states.

The important observation is that Poincaré covariant two-component spinors contain exactly the same information as Lorentz covariant 4-component spinors. In the field theory the boosts in (166) and (173) appear in the spinors $u_a(\mathbf{p},\mu)$ and $v_a(\mathbf{p},\mu)$ which intertwine finite dimensional representations of the Lorentz group with irreducible representations of the little group of the Poincaré group. By using the Lorentz covariant representations all of the dependence on the spin (s) representation of the particle states disappears. This is because s-dependence in the creation and annihilation operators cancels with the s-dependence in the coefficient functions $u_a(\mathbf{p},\mu)$ and $v_a(\mathbf{p},\mu)$.

8 Spin and Dynamics

The N-particle representation of the Poincaré group given in (125–126) describes the dynamics of a system of N free particles. The mass operator for this representation is the invariant mass of N free particles and the spin is the s-spin of N free particles. These are both functions of the Poincaré generators, which are sums of the one-body generators.

In a dynamical model one expects that both the mass and spin will be interaction dependent. This is because the mass and spin operators are functions of the generators, some of which are interaction dependent [8] in dynamical models. Because

$$M^2 = H^2 - \mathbf{P}^2 \tag{225}$$

it is clear that the mass operator acquires an interaction dependence through the Hamiltonian.

The s-spin (65) is a function of the mass, $\Lambda_s(P)$, and the Pauli–Lubanski vector, W^μ. Each of these terms also involves interactions, and while it is possible to satisfy the commutation relations with interactions that lead to a non-interacting spin (they are called generalized Bakamjian–Thomas models [1,11]), for systems of more than two particles this condition is not compatible with the additional requirements imposed by cluster properties of the generators. The origin of this problem is the treatment of the relative orbital angular momentum of two interacting subsystems. The dynamical and kinematic masses of these subsystems are different (in fact they are represented by non-commuting operators) which implies a dynamical dependence on the relative orbital angular momentum of these subsystems. The interaction dependence in the orbital angular momentum leads to an interaction dependence in the spin. This leads to the question of how to understand the relation between the spin of an interacting system and the spin of the constituent subsystems.

In this section, we argue that it is enough to understand how the total and single-particle spins are related in a non-interacting system. To establish this result we show that it is always possible to find a unitary transformation, A, that (1) preserves the S matrix and (2) leads to an equivalent model with a non-interacting spin. While the transformed Poincaré generators will no longer satisfy cluster properties, the transformed S-matrix must satisfy cluster properties and in this transformed model the relation between the single-particle spins and the system spin is the same as it is for a system of N free particles. Since the S-matrix is the only observable, there is no loss of generality in working with models where the spins are coupled as they are in a non-interacting relativistic system.

The same unitary transformation must be applied to operators for the equivalence to hold for matrix elements of operators, like current matrix elements. Furthermore, as we have argued in Sect. 6, in the process of adding the spins and angular momenta with Poincaré group Clebsch–Gordan coefficients, the intermediate generalized Melosh rotations (71, 109) all cancel up to the overall single-particle and system Melosh rotations, so there is no loss of generality in using the canonical (or any other type of) spin to add the single-particle spins and orbital angular momenta to get the system spins. The generalized Melosh rotations can be used to transform the system spin and single-particle spins to any other type of s-spin.

This means that in order to understand the relation between single-particle spins and the system spin in S-matrix or bound states observables in dynamical models, it is sufficient to study dynamical models where the system spin is the canonical spin of the corresponding non-interacting system. When this system is embedded in a larger system there will generally be violations of cluster properties with observable consequences.

To construct the desired unitary transformation we introduce another function of the Poincaré generators that is conjugate to the linear momentum and commutes with the canonical spin. We first consider the case of a single-particle in a canonical spin basis. In this representation the desired operator is represented by $\mathbf{X}_c = i\boldsymbol{\nabla}_p$ where the partial derivative with respect to the linear momentum is computed by holding the canonical spin constant (because different spins are related by momentum-dependent Melosh rotations (71, 109), holding different spins constant leads to different "position" operators).

In this single-particle (irreducible) representation we define the operator $\mathbf{X}_c$ by the equation

$$
{}_c\langle (m,j)\mathbf{p},\mu|\mathbf{X}_c|\psi\rangle := i\frac{\partial}{\partial \mathbf{p}}{}_c\langle (m,j)\mathbf{p},\mu|\psi\rangle. \tag{226}
$$

This looks like a non-relativistic position operator except in the relativistic case the partial derivative is computed holding the z-component μ of the canonical spin constant. In addition it has no simple transformation properties with respect to the Poincaré group. Since the single-particle representation is irreducible, the operator $\mathbf{X}_c$ is expressible as a function of the infinitesimal generators.

To determine the relation of $\mathbf{X}_c$ to the infinitesimal generators we consider the action of Lorentz transformations on states in this single-particle basis. Since both boosts and rotations change the momenta, the operator $\mathbf{X}_c$ will appear in expressions for both the boost and rotation generators:

$$
\begin{aligned}
{}_c\langle (m,j)\mathbf{p},\mu|K^i|\psi\rangle &= -i\frac{\partial}{\partial\rho}{}_c\langle (m,j)\mathbf{p},\mu|e^{iK_i\rho}|\psi\rangle_{\rho=0} \\
&= -i\frac{\partial}{\partial\rho}\left(\int\sum_{\mu'} d\mathbf{p}'\mathcal{D}^{m,j}_{c:\mathbf{p},\mu;\mathbf{p}',\mu'}[\Lambda_c(\rho\hat{\mathbf{x}}_i),0)]d\mathbf{p}'\psi(\mathbf{p}',\mu')\right)_{\rho=0}
\end{aligned} \tag{227}
$$

$$
\begin{aligned}
{}_c\langle (m,j)\mathbf{p},\mu|J^i|\psi\rangle &= -i\frac{\partial}{\partial\theta}{}_c\langle (m,j)\mathbf{p},\mu|e^{iJ_i\theta}|\psi\rangle_{\theta=0} \\
&= -i\frac{\partial}{\partial\theta}\left(\int\sum_{\mu'} d\mathbf{p}'\mathcal{D}^{m,j}_{c:\mathbf{p},\mu;\mathbf{p}',\mu'}[(R(\theta\hat{\mathbf{x}}_i,0),0)]d\mathbf{p}'\psi(\mathbf{p}',\mu')\right)_{\theta=0} .
\end{aligned} \tag{228}
$$

Specifically, using the chain rule, the derivatives with respect to the rapidity or angle can be replaced by derivatives with respect to the momentum, which in this representation is identified with the operator, $\mathbf{X}_c$. Straightforward calculations lead to the following relations between $\mathbf{X}_c$, $\mathbf{K}$ and $\mathbf{J}$:

$$
\mathbf{J} = \mathbf{j}_c + \mathbf{X}_c \times \mathbf{P} \tag{229}
$$

and

$$
\mathbf{K} = \frac{1}{2}\{H,\mathbf{X}_c\} + \frac{1}{M+H}(\mathbf{p}\times\mathbf{j}_c) . \tag{230}
$$

The second equation can be inverted to express the operator $\mathbf{X}_c$ in terms of the Poincaré generators:

$$
\mathbf{X}_c = \frac{1}{2}\left\{\frac{1}{H},\mathbf{K}\right\} - \frac{\mathbf{P}\times(H\mathbf{J}+\mathbf{P}\times\mathbf{K})}{MH(M+H)}. \tag{231}
$$

The operator $\mathbf{X}_c$ is called the Newton–Wigner [12] position operator. There are similar operators [11] that are partial derivatives with respect to momentum holding various s-spins constant. All of these "position operators" are well-defined functions of the Poincaré generators, but none of them have the physical interpretation of a position observable.

Equation (231) leads to the following expression for the canonical spin $\mathbf{j}_c$ in term of $\mathbf{P}$, $\mathbf{J}$ and $\mathbf{X}_c$.

$$\mathbf{j}_c = \mathbf{J} - \mathbf{X}_c \times \mathbf{P}. \tag{232}$$

This looks just like the standard non-relativistic expression showing that the total angular momentum is the sum of an orbital part angular momentum and a spin.

While we derived these formulas by considering properties of a single particle, because both $\mathbf{X}_c$ and $\mathbf{j}_c$ are functions of the infinitesimal generators, relations (230) and (231) between $\mathbf{X}_c$ and the Poincaré generators hold for any representation of the Poincaré group.

For a system of N non-interacting particles equation (232) is replaced by

$$\mathbf{j}_{c0} = \mathbf{J}_0 - \mathbf{X}_{c0} \times \mathbf{P}_0 \,, \tag{233}$$

where the 0 means that the operators are functions of the *non-interacting* generators, which are sums of the single-particle generators.

Next we consider an interacting system, and to be specific we assume an instant-form dynamics where both the linear, $\mathbf{P}$, and angular momentum, $\mathbf{J}$, do not have interactions. In an instant form dynamics the interactions appear in the Hamiltonian and rotationless boost generators. For a system with interactions $\mathbf{X}$ defined by (231) becomes interaction dependent due to the interactions in H, M, and $\mathbf{K}$ unless we carefully engineer the interactions to cancel in (231). This will be done in what follows. More generally (232) implies that the canonical spin of this system becomes interaction-dependent when $\mathbf{X}_c$ is interaction dependent. When the system is at rest the orbital angular momentum containing the interaction dependence disappears—however for a system satisfying cluster properties, subsystems move relative to each other and the system. In these cases the relative orbital angular momenta of the subsystems acquire an interaction dependence. This is the origin of the interaction dependence of the spins.

The desired unitary transformation is constructed from multichannel wave operators. We briefly summarize the construction of these operators; a more complete discussion can be found in [13]. Asymptotically a scattering state in a channel α looks like a number of mutually non-interacting bound clusters. Each bound cluster will have a mass, spin, total momentum, and spin projection. Relativistically these clusters transform like free particles with the mass and spin of the bound subsystem. We write these states in the form

$$|\phi_{\alpha_i,m_i,s_i}\mathbf{p}_i, \mu_i\rangle_c \,, \tag{234}$$

where α_i is a label for the ith bound cluster in scattering channel α.

The vector (234) can be considered as a mapping from the square integrable functions of $\mathbf{p}_i$ and μ_i, called $\mathcal{H}_{\alpha_i}$ to the Hilbert space for the particles in the bound cluster. Here $\mathcal{H}_{\alpha_i}$ is a mass m_i spin s_i irreducible representation space for the Poincaré group. The asymptotic states in a reaction with n_α asymptotic clusters having wave packets $f_i(\mathbf{p}_i, \mu_i)$ have the form

$$|\Psi_\alpha\rangle = \prod_i \int d\mathbf{p}_i \sum_{\mu_i} |\phi_{\alpha_i,m_i,s_i}\mathbf{p}_i, \mu_i\rangle_c f_i(\mathbf{p}_i, \mu_i). \tag{235}$$

We write (235) formally as

$$|\Psi_\alpha\rangle := \Phi_\alpha |f_\alpha\rangle, \tag{236}$$

where Φ_α is a mapping, called the channel injection operator, from the channel Hilbert space

$$\mathcal{H}_\alpha := \otimes \mathcal{H}_{\alpha_i} \tag{237}$$

to the N-particle Hilbert space.

The non-interacting dynamics of the bound clusters is given by the tensor product of the irreducible unitary representations of the Poincaré group associated with mass and spin of each cluster:

$$\langle \mathbf{p}_1, \mu_1, \ldots, \mathbf{p}_{n_\alpha}, \mu_{n_\alpha} | U_\alpha(\Lambda, a) | f_\alpha\rangle = \prod \int \sum \mathcal{D}^{m_i,j_i}_{\mu_i \mathbf{p}_i;\nu_i \mathbf{p}'_i}[\Lambda, a] d\mathbf{p}'_i \langle \mathbf{p}'_i, \nu_i | f_i\rangle \,, \tag{238}$$

which in the notation (236) becomes

$$\Phi_\alpha U_\alpha(\Lambda, a)|f_\alpha\rangle \ . \tag{239}$$

To treat multichannel scattering and bound states on the same footing we define the asymptotic Hilbert space as the direct sum of the channel spaces, including the N-body bound state channels,

$$\mathcal{H}_{as} := \oplus\mathcal{H}_\alpha . \tag{240}$$

The asymptotic injection operator that maps the asymptotic Hilbert space $\mathcal{H}_{as}$ to the N-particle Hilbert space $\mathcal{H}$ is defined by

$$\Phi = \sum_\alpha \Phi_\alpha \tag{241}$$

and the free asymptotic dynamics by

$$\Phi U_{as}(\Lambda, a) = \sum_\alpha \Phi_\alpha U_\alpha(\Lambda, a) \ . \tag{242}$$

In this notation scattering states, $|\Psi_\pm\rangle$, are defined by the strong limits

$$\lim_{t\to\pm\infty} \|U(I, (t, \mathbf{0}))|\Psi_\pm\rangle - \Phi U_{as}(I, (t, \mathbf{0}))|f\rangle\| = 0, \tag{243}$$

where $|f\rangle$ represents a wave packet in the asymptotic Hilbert space, $\mathcal{H}_{as}$.

Wave operators are mappings from the asymptotic Hilbert space to the N-particle Hilbert space defined by

$$\Omega_\pm := \lim_{t\to\pm\infty} U(I, (-t, \mathbf{0}))\Phi U_{as}(I, (t, \mathbf{0})). \tag{244}$$

The wave operators are asymptotically complete when they are unitary mappings from $\mathcal{H}_{as}$ to the N-particle Hilbert space (recall that the asymptotic space includes system bound states.) The wave operators are relativistically invariant when they satisfy

$$U(\Lambda, a)\Omega_\pm = \Omega_\pm U_{as}(\Lambda, a). \tag{245}$$

Wave operators that do not satisfy these properties are considered pathological, and in what follows we assume that the wave operators are both asymptotically complete and relativistically invariant.

The scattering operator is defined as the unitary mapping

$$S = \Omega_+^\dagger \Omega_- \tag{246}$$

on $\mathcal{H}_{as}$. In an instant-form dynamics $\mathbf{P} = \mathbf{P}_0$. It follows from (245) that

$$\mathbf{P}_0\Omega_\pm = \Omega_\pm \mathbf{P}_{as} \tag{247}$$

$$\mathbf{X}\Omega_\pm = \Omega_\pm \mathbf{X}_{as}. \tag{248}$$

The first equation means that the mixed-basis matrix elements of the wave operators in eigenstates of $\mathbf{P}_0$ and $\mathbf{P}_{as}$ have the form

$$\langle \mathbf{P}, \ldots |\Omega_\pm|\mathbf{P}_{as}, \ldots\rangle = \delta(\mathbf{P} - \mathbf{P}_{as})\langle \cdots |\hat{\Omega}_\pm(\mathbf{P})| \cdots\rangle. \tag{249}$$

Equation (248) means that if $\langle \mathbf{P}, \ldots | =_I \langle \mathbf{P}, \ldots |$ are irreducible eigenstates associated with the dynamical representation $U(\Lambda, a)$, then the reduced matrix elements ${}_I\langle \cdots |\hat{\Omega}_\pm(\mathbf{P})| \cdots\rangle$ are independent of $\mathbf{P}$.

Since in an instant form dynamics $\mathbf{P}_0$ is also the translation generator for the non-interacting system, if $\langle \mathbf{P}, \ldots | =_0 \langle \mathbf{P}, \ldots |$ are irreducible eigenstates associated with the non-interacting representation $U_0(\Lambda, a)$, then equation (247) still holds, but in this case the reduced matrix elements of the wave operators will have an explicit momentum dependence. On the other hand, since the S matrix only depends on the asymptotic momentum we have

$$\begin{aligned}\langle \mathbf{P}_{as},\ldots|S|\mathbf{P}'_{as},\ldots\rangle &= \delta(\mathbf{P}_{as}-\mathbf{P}'_{as})\langle\cdots|\hat{S}(\mathbf{P})|\cdots\rangle = \langle \mathbf{P}_{as},\ldots|\Omega_+^\dagger\Omega_-|\mathbf{P}'_{as},\ldots\rangle \\ &= \int \delta(\mathbf{P}_{as}-\mathbf{P}'_{as})\langle\cdots|\hat{\Omega}_+^\dagger|\cdots\rangle_{II}\langle\cdots\hat{\Omega}_-|\cdots\rangle \\ &= \int \delta(\mathbf{P}_{as}-\mathbf{P}'_{as})\langle\cdots|\hat{\Omega}_+^\dagger(\mathbf{P})|\cdots\rangle_{00}\langle\cdots\hat{\Omega}_-(\mathbf{P})|\cdots\rangle \,, \end{aligned} \tag{250}$$

where we used both the interacting and free-particle irreducible bases as intermediate states. The S matrix elements are independent of the choice of basis used in the N-particle Hilbert space. The third line of equation (250) implies that $\langle\cdots|\hat{S}(\mathbf{P})|\cdots\rangle$ is independent of $\mathbf{P}$:

$$\langle\cdots|\hat{\Omega}_+^\dagger(\mathbf{P})|\cdots\rangle_{00}\langle\cdots\hat{\Omega}_-(\mathbf{P})|\cdots\rangle = \langle\cdots|\hat{\Omega}_+^\dagger(\mathbf{0})|\cdots\rangle_{00}\langle\cdots\hat{\Omega}_-(\mathbf{0})|\cdots\rangle. \tag{251}$$

Given this information we define new wave operators $\bar{\Omega}_\pm$ in a free-particle irreducible basis by

$$_0\langle \mathbf{P},\ldots|\bar{\Omega}_\pm|\mathbf{P}_{as},\ldots\rangle = \delta(\mathbf{P}-\mathbf{P}_{as})_0\langle\cdots|\hat{\Omega}_\pm(\mathbf{0})|\cdots\rangle \,, \tag{252}$$

where we have set $\mathbf{P}$ to zero in the reduced matrix element in the mixed representation involving a non-interacting irreducible basis and the asymptotic basis.

These new wave operators have the following important properties

$$S = \bar{\Omega}_+^\dagger\bar{\Omega}_-^\dagger = \Omega_+^\dagger\Omega_-^\dagger \tag{253}$$

and

$$\mathbf{X}_0\bar{\Omega}_\pm^\dagger = \bar{\Omega}_\pm^\dagger\mathbf{X}_{as}. \tag{254}$$

The unitarity of the wave operators means that

$$A = \bar{\Omega}_-^\dagger\Omega_- = \bar{\Omega}_+^\dagger\Omega_+ \tag{255}$$

is an S-matrix preserving unitary operator. Using the unitary operator (255) we define the equivalent dynamical representation of the Poincaré group by

$$\bar{U}(\Lambda,a) := AU(\Lambda,a)A^\dagger. \tag{256}$$

Because the dynamics is instant form we have

$$\mathbf{P}_0A = A\mathbf{P}_0 \qquad \mathbf{J}_0A = A\mathbf{J}_0 \tag{257}$$

and by construction

$$\mathbf{X}_0A = A\mathbf{X}. \tag{258}$$

It follows that the transformed canonical spin

$$\bar{\mathbf{j}}_c = \mathbf{J}_0 - \mathbf{X}_0\times\mathbf{P}_0 = \mathbf{j}_{c0} \tag{259}$$

has no interactions. This shows that $\bar{U}(\Lambda,a)$ is a dynamical unitary representation of the Poincaré group that gives the same S-matrix and bound state observables as the original representation $U(\Lambda,a)$, and in addition has a non-interacting spin.

This is the desired result. To see that this also applies to other forms of the dynamics we note that once we have a mass operator that commutes with $\mathbf{P}_0$, $\mathbf{X}_{c0}$ and $\mathbf{j}_{c0}$, the kernel of that operator in an irreducible free particle basis is the product of three momentum conserving delta functions, a delta function in the total canonical spin, a delta function in the z-component of the total canonical spin, and a reduced kernel in the non-interacting mass and kinematically invariant variables. These kinematically invariant variables are just the degeneracy variables that appear in the various Clebsch–Gordan coefficients. Replacing in the delta functions linear momentum and canonical spin by the four-velocity and canonical spin or light-front components of the four momentum and light-front spin, give S-matrix equivalent models in each of Dirac's forms of dynamics. A similar construction can be used to prove the existence of scattering equivalent dynamical models in each of Dirac's forms of dynamics.

The conclusion of this section is that if one wants to understand the relation between the spins of single particles and spin of the system there is no loss of generality with treating the spins as non-interacting spins. This provides a justification for a number of applications of the Bakamjian–Thomas type of dynamics [14–19].

9 Few-Body Problems

Generalized Bakamjian–Thomas models are a class of relativistic quantum mechanical models of interacting particles where the spin is identical to the spin of a system of non-interacting particles. In the previous section we demonstrated that any relativistic dynamical model was related to an equivalent Bakamjian–Thomas model by an S-matrix preserving unitary transformation. While the equivalent Bakamjian–Thomas unitary representation of the Poincaré group will not asymptotically break up into tensor-product representations, the S matrix, which is unchanged from the original model, must satisfy cluster properties if the original model satisfies cluster properties. Thus, for the purpose of understanding bound-state or S-matrix observables, there is no loss of generality in using Bakamjian–Thomas models of the system. The important consequence is that in these models the relation of the total spin of a composite system to the spins of its constituent particles is identical to that relation for N non-interacting relativistic particles.

For this reason it is instructive to consider the structure of Bakamjian–Thomas few-body models. The important property of this class of models is that the two and three-body interactions must commute with the non-interacting three-body spin. This ensures that the dynamical spin has no interactions. The simplest way to realize this property is to couple the spins and orbital angular momenta using the Poincaré group Clebsch–Gordan coefficients (148–149) to construct a Poincaré irreducible free-particle basis. In this basis the interactions must be diagonal in the square of the spin and commute with and be independent of the magnetic quantum number.

By inspecting the structure of the Poincaré group Clebsch–Gordan coefficients (148–149) one can see that the spin is constructed using ordinary $SU(2)$ Clebsch–Gordan coefficients; but the angular momenta being added are the constituent spins (152) and orbital angular momenta in relative momentum operators (155) that Wigner rotate with the constituent spins (152). If the potential is expressed in a basis of eigenstates of the constituent spins, the projection of these spins on an axis, and the orbital angular momentum three vectors that Wigner rotate with the constituent spins, then all that is required is that the potential be a rotationally invariant in this basis.

In this basis the dynamical problem can be solved using standard methods that take advantage of the rotational invariance; either using standard partial-waves methods or direct 3-dimensional integration in the same manner that they are used in non-relativistic calculations [20,21].

The relevant momenta and constituent spins variables are related to the single-particle spins and momenta by boosting all of them to the rest frame of the non-interacting system, and then converting the resulting s spins to canonical spins. The relevant momentum variables (155) are

$$\mathbf{q}_i = \boldsymbol{\Lambda}_c^{-1}(P)p_i \tag{260}$$

and the relevant spins (152) are

$$\mathbf{j}_{iss} = \Lambda_c^{-1}(q_i)\Lambda_s^{-1}(P)\Lambda_s(p_i)\mathbf{j}_{is} \, . \tag{261}$$

These identifications are important for the relativistic transformation properties in the Bakamjian–Thomas representation. If one works in the three-body rest frame then the $\mathbf{q}_i$ are just the single-particle momenta and the constituent spins become the single particle constituent spins. In this frame the relativistic invariance requirement on the spins reduces to the requirement that the potentials are rotationally invariant functions of the three momenta and single-particle canonical spins. To transform out of the rest frame it is necessary to make the identifications (260) and (261).

It is interesting to note that the desired rotational covariance can be realized by treating the spins and orbital angular momenta in a purely non-relativistic manner in the rest frame, constructing two-body relative momenta using Galilean boost to two body-rest frames. It is equally possible to realize the rotational invariance by adding the spins and orbital angular momenta using successive coupling with the Poincaré group Clebsch–Gordan coefficients of Sect. 6. In both cases, if one starts with the momenta and spins in (260) and (261), these choices amount to a variable change. These choices have nothing to do with relativity—they are simply alternative variable choices that make the rotational invariance of the interactions easy to recognize.

This is consistent with the observation that the only symmetry that needs to be respected in the rest frame is the symmetry associated with the little group, which is the rotation group for positive mass systems.

However, there are other considerations that go beyond the Poincaré symmetry. Most notably are cluster properties. Cluster properties provide the justification for tests of special relativity on isolated subsystems. In the three-body problem it is natural first to treat the two-body problem using the Bakamjian–Thomas method.

A spectator particle can be included by taking the tensor product of the two-body Bakamjian–Thomas representation of the Poincaré group with the one-body irreducible representation associated with the spectator. The resulting tensor-product unitary representation of the Poincaré group does not have a kinematic spin, but it does satisfy cluster properties. A scattering equivalent three-body Bakamjian–Thomas model is obtained by considering this model in the non-interacting three-body rest frame, replacing all of the single-particle momenta and spins by the momenta (260) and constituent spins (261). This implies a specific and simple relation between the two-body Bakamjian–Thomas interactions in the two-body problem and the corresponding Bakamjian–Thomas interactions in the three-body problem. This connection is realized by embedding the two-body interactions in the three-body problem using Poincaré group Clebsch–Gordan coefficients. While similar considerations apply to larger systems, for these systems the equivalent Bakamjian–Thomas model that satisfies S-matrix cluster properties necessarily includes many-body interactions that are generated from the subsystem interactions [13].

10 Coupling to Electromagnetic Fields

Given all of the different kinds of spin operators introduced in this paper, one has to confront the question of relating theory to experiment. Normally the spin is measured by considering how it couples to a classical electromagnetic field. Formally, in the one-photon-exchange approximations this involves a coupling of the form

$$e \int d\mathbf{x} J^{\nu}(x) A_{\nu}(x) d\mathbf{x}. \tag{262}$$

The connection with the theory is through matrix elements of the current of the form

$${}_s\langle (m', j')\mathbf{p}', \mu', \ldots | J^{\nu}(x) | (m, j)\mathbf{p}, \mu, \ldots \rangle_s. \tag{263}$$

The Poincaré transformation properties of this matrix element means that it can be expressed in terms of invariants and geometric quantities that arise strictly from the transformation properties of the current and initial and final states. Once this operator is known in one basis the relations between the different bases discussed in this paper can be used to calculate the current in any other basis. It is only necessary to know the generalized Melosh rotations relating two different spin bases. Thus using (71, 109) we get

$$\begin{aligned} &{}_s\langle (m', j')\mathbf{p}', \mu', \ldots | J^{\nu}(x) | (m, j)\mathbf{p}, \mu, \ldots \rangle_s \\ &\quad = \sum D^{j}_{\mu'\mu''}[\Lambda_s^{-1}(p')\Lambda_t(p')]_t \langle (m', j')\mathbf{p}', \mu'', \ldots | J^{\nu}(x) | (m, j)\mathbf{p}, \mu''', \ldots \rangle_t \\ &\quad \times D^{j}_{\mu'''\mu}[\Lambda_t^{-1}(p)\Lambda_s(p)]. \end{aligned} \tag{264}$$

11 Summary

In this paper we presented a general discussion of the treatment of spin in relativistic few-body systems. The goal of this work was to understand the relation between the spin of a dynamical system and the spin of its elementary constituents. This is relevant for understanding scattering experiments where, for example, a polarized target breaks up into constituents and one is interested in the relation of the polarization of the target to the polarization of the constituents. Other examples involve using electromagnetic probes that interact with the currents of the individual constituent particles. Our intention is to include sufficient generality so models with different treatments of spin can be compared.

There are many good references on single-particle spins for relativistic systems, and also many references on Clebsch–Gordan coefficients for the Poincaré group [11,22,23], which can be used to add spins and orbital angular momenta in relativistic systems, but most of them focus on the canonical spins, and are relevant for a system of two free particles. This work discusses the addition of a more general class of spins along with the impact of the dynamics on the spin coupling. We also discussed the connection between two and four component spinors in this context.

The new feature of spin in relativistic quantum mechanics is that spins undergo momentum-dependent rotations under the action of Lorentz transformations. This means that there is no unambiguous way to compare the spins of particles with different momenta and the addition of spins becomes more complicated than it is in non-relativistic quantum mechanics. In Sect. 4 we pointed out that one way to define a spin operator is to use an arbitrary but fixed set of Lorentz transformations to refer the particles to a common frame where the spins can be compared. We constructed a number of functions of the single-particle Poincaré generators corresponding to different arbitrary but fixed Lorentz transformations and showed that the resulting spin operators all satisfied $SU(2)$ commutation relations. We also showed that the different choices of spin operators were related by momentum-dependent rotations, which we called generalized Melosh rotations. The exercise is not academic - at least three different kinds of spins are commonly used in applications. These include the canonical spin, the light-front spin, and the helicity spin and they are all related by different generalized Melosh rotations.

We then showed that the canonical spin played a special role in adding spins. This is because only for the canonical spins are Wigner rotations of rotations equal to the original rotation. This means that particles with different momenta have identical rotational properties when rotated which allowed them to be added using ordinary $SU(2)$ Clebsch–Gordan coefficients. The coupling coefficient for other types of spins are constructed by first using generalized Melosh rotations to convert to canonical spins. Next the canonical spins are added using $SU(2)$ Clebsch Gordan coefficients, and finally the resulting canonical spin is converted back to the initial type of spin using another generalized Melosh rotation. The different generalized Melosh rotations used in this construction involve different momenta (i.e. the momentum of each particle and the total momentum of the subsystem). We also remarked that in the process of successive pairwise coupling all of the intermediate generalized Melosh rotations cancel. All that remains are the generalized Melosh transformations on the single-particle spins and the final total spin. This led us to point out that there is no loss of generality in performing all of the spin additions using canonical spins. The resulting coupling coefficients can then be converted to coupling coefficients for any other type of spin using the appropriate generalized Melosh rotation. An important observation resulting from this construction is that there are a number of intermediate spins that couple to the final total spin using ordinary SU(2) Clebsch–Gordan coefficients. We called these spins constituent spins. It is important to note that the constituent spins are actually many-body operators that are related to the true single-particle spins by dynamical rotations (both Winger rotations and generalized Melosh rotations). The angles of these rotations depend on the momentum distribution of the composite system as well as on the total momentum of the system.

All of this discussion assumed that all the spins are associated with a non-interacting systems of particles. For interacting systems the internal orbital angular momenta associated with subsystems depend on the mass eigenvalues of the subsystems, rather than the invariant mass of the constituents in each subsystem. This would suggest that modifications are required to couple the particle spins and internal orbital angular momenta in interacting systems. In Sect. 8 we argued that this was not the case. We showed that is was always possible to find an S-matrix preserving unitary transformation that removes the interactions from the spin at the expense of modifying the internal momentum distribution of the wave function. In general, we do not know the momentum distribution of the wave function without doing a full dynamical calculation, however it follows that there is no loss of generality in coupling the spins, treating them all as kinematic quantities. Quantitative predictions will be sensitive to the momentum distribution in the wave functions due to the presence of dynamical rotations in the spin and orbital angular momentum coupling coefficients.

We also considered the choices of vectors that should be used to describe the internal relative orbital angular momenta for systems of particles. The most important requirement is that they must be defined by boosting the single particle momenta to a common frame—normally the rest frame of the non-interacting system. The resulting vectors are no longer 4-vectors, but they have the desirable property that they all can be Melosh rotated (if necessary) so they undergo the same Wigner rotations as the constituent spins. This allows them to be coupled with the constituent spins using ordinary Clebsch–Gordan coefficients to get the total spin. When everything is expressed in terms of these momentum vectors and the corresponding constituent spins the coupling proceeds as in the non-relativistic case.

In relativistic quantum theory both two and four component spinors arise in applications. In Sect. 7 we pointed out that two component spinors arise by considering positive mass-positive energy irreducible representations of the Poincaré group while four-component spinors are associated with finite-dimensional representations of the Lorentz group. We demonstrated the relation between these two groups by taking apart a Wigner rotation, thus removing the momentum-dependent boosts. The resulting spin no longer depends on the choice of boost, but because the $SL(2,\mathbb{C})$ representation of the boosts and their complex conjugates are inequivalent, and both representations are related by space reflection, it is natural to use a doubled representation when space reflection is an important symmetry. In making contact with the particle spins the boosts must be reintroduced - this choice appears in both the Dirac spinors and the creation and annihilation operators.

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