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Some existence results for nonlinear fractional differential equations with impulsive and fractional integral boundary conditions

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Abstract
In this paper, we study the boundary value problems for a class of nonlinear fractional differential equations with impulsive and fractional integral boundary conditions. By means of standard fixed point theorems, some existence and uniqueness results are obtained. As applications, two examples are given to illustrate the results.
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1 Introduction

The subject of fractional differential equations has evolved as an interesting and popular field of research. It is mainly due to the extensive applications of fractional calculus in the mathematical modeling of physical, engineering, and biological phenomena etc. [1–4]. For some developments on the theory of fractional differential equations, we can refer to [5–25] and the references therein.

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamic, and so forth. Recently, there has been a great deal of research on the questions of existence and uniqueness of solutions for boundary value problems of fractional differential equations with integral boundary conditions. For example, Ahmad *et al.* [8] investigated the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations with three-point integral boundary conditions given by

$$\begin{cases} {}^cD^\alpha x(t) = f(t,x(t)), & t \in [0,1], 1 < \alpha \leq 2, \\ x(0) = 0, \quad x(1) = a\int_0^\eta x(s)\,ds, & 0 < \eta < 1, \end{cases}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α, f is a given continuous function, and $a \in \mathbb{R}$ with $a\eta^2 \neq 2$.

In [13], Guezane-Lakoud and Khaldi discussed the fractional differential equations with fractional integral boundary conditions as the following form:

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t), {}^cD^\beta x(t)), & t \in [0,1], 1 < \alpha \le 2, 0 < \beta < 1, \\ x(0) = 0, \qquad I^\beta x(1) = x'(1), \end{cases}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, I^β the Riemann-Liouville fractional integral of order β, f is a given continuous function.

For the case of nonlinear impulsive fractional differential equations with integral boundary conditions, Ahmad and Sivasundaram [9] studied the existence of solutions for the following equation:

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t)), & t \in J = [0,1], t \neq t_k, k = 1,2,\ldots,m, \\ \Delta x(t_k) = I_k(x(t_k^-)), \qquad \Delta x'(t_k) = J_k(x(t_k^-)), & k = 1,2,\ldots,m, \\ ax(0) + bx'(0) = \int_0^1 q_1(x(s))\,ds, \qquad ax(1) + bx'(1) = \int_0^1 q_2(x(s))\,ds, \end{cases}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (1,2)$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $I_k, J_k \in C(\mathbb{R}, \mathbb{R})$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$, $x(t_k^-) = \lim_{\epsilon \to 0^-} x(t_k + \epsilon)$, $k = 1,2,\ldots,m$, $\Delta x'(t_k)$ has a similar meaning for $x'(t_k)$, $q_1, q_2 : \mathbb{R} \to \mathbb{R}$ and $a > 0$, $b \ge 0$.

Motivated by the above mentioned papers, in this article, we will consider the following impulsive problem:

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t)), & t \in J = [0,1], t \neq t_k, k = 1,2,\ldots,m, \\ \Delta x(t_k) = I_k(x(t_k^-)), \qquad \Delta x'(t_k) = J_k(x(t_k^-)), & k = 1,2,\ldots,m, \\ x(0) = 0, \qquad aI^\gamma x(1) + bx'(1) = c, \end{cases} \tag{1}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (1,2)$, I^γ the Riemann-Liouville fractional integral of order γ, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $I_k, J_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1,2,\ldots,m$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$, $x(t_k^-) = \lim_{\epsilon \to 0^-} x(t_k + \epsilon)$ representing the right and left limits of $x(t)$ at $t = t_k$, $\Delta x'(t_k)$ has a similar meaning for $x'(t_k)$, a, b, c are real constants and $a \neq -b\Gamma(\gamma + 2)$.

The paper is organized as follows: in Section 2 we present the notations, definitions and give some preliminary results that we need in the sequel, Section 3 is dedicated to the existence results of problem (1), in the final Section 4, two examples are given to illustrate the results.

2 Preliminaries

Definition 2.1 The Riemann-Liouville fractional integral of order q for a function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}}\,ds, \quad q > 0,$$

provided the integral exists.

Definition 2.2 For a function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^{c}D^{q}f(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}f^{(n)}(s)\,ds, \quad n-1<q<n, n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Lemma 2.1 ([23]) *Let $\alpha > 0$, then the differential equation*

$$^{c}D^{\alpha}h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$ and

$$I^{\alpha\,c}D^{\alpha}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$, $n = [\alpha]+1$.

For the sake of convenience, we introduce the following notation.

Let $J_0 = [0,t_1]$, $J_1 = (t_1,t_2],\dots,J_{m-1} = (t_{m-1},t_m]$, $J_m = (t_m,1]$, $J = [0,1]$, $J' := J\backslash\{t_1,t_2,\dots,t_m\}$ and $PC(J,\mathbb{R}) = \{u : J \to \mathbb{R} | u \in C(J_k,\mathbb{R}), k = 0,1,2,\dots,m, u(t_k^+)$ and $u(t_k^-)$ exist, $k = 1,2,\dots,m$, and $u(t_k^-) = u(t_k)\}$. Obviously, $PC(J,\mathbb{R})$ is a Banach space with the norm $\|u\| = \sup_{t\in J}|u(t)|$.

Lemma 2.2 *For any $y \in PC(J,\mathbb{R})$, the unique solution of the impulsive boundary value problem*

$$\begin{cases} ^{c}D^{\alpha}x(t) = y(t), & t \in J, t \neq t_k, k = 1,2,\dots,m, \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad \Delta x'(t_k) = J_k(x(t_k^-)), & k = 1,2,\dots,m, \\ x(0) = 0, \quad aI^{\gamma}x(1) + bx'(1) = c \end{cases} \tag{2}$$

is given by

$$x(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + \frac{(c-\Lambda-W)t}{\frac{a}{\Gamma(\gamma+2)}+b} \\ \quad - \sum_{i=1}^m J_i(x(t_i^-))t, \quad t \in J_0; \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + I_1(x(t_1^-)) - t_1 J_1(x(t_1^-)) + \frac{(c-\Lambda-W)t}{\frac{a}{\Gamma(\gamma+2)}+b} \\ \quad - \sum_{i=2}^m J_i(x(t_i^-))t, \quad t \in J_1; \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + \sum_{i=1}^k I_i(x(t_i^-)) - \sum_{i=1}^k t_i J_i(x(t_i^-)) \\ \quad + \frac{(c-\Lambda-W)t}{\frac{a}{\Gamma(\gamma+2)}+b} - \sum_{i=k+1}^m J_i(x(t_i^-))t, \quad t \in J_k, k = 2,3,\dots,m, \end{cases} \tag{3}$$

where

$$\Lambda = a\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}y(s)\,ds + b\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)\,ds,$$
$$W = \frac{a(\sum_{i=1}^m I_i(x(t_i^-)) - \sum_{i=1}^m t_i J_i(x(t_i^-)))}{\Gamma(\gamma+1)}.$$

Proof For $1 < \alpha < 2$, by Lemma 2.1, we know that a general solution of the equation ${}^cD^\alpha x(t) = y(t)$ on each interval J_k ($k = 0, 1, 2, \dots, m$) is given by

$$x(t) = I^\alpha y(t) + d_k + e_k t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + d_k + e_k t, \quad t \in J_k,$$

where $d_k, e_k \in \mathbb{R}$ are arbitrary constants.

Since $x(0) = 0$, $aI^\gamma x(1) + bx'(1) = c$,

$$x'(t) = I^{\alpha-1} y(t) + e_k = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\,ds + e_k, \quad t \in J_k,$$

we have $d_0 = 0$, $c = (\frac{a}{\Gamma(\gamma+2)} + b)e_m + \frac{ad_m}{\Gamma(\gamma+1)} + \Lambda$. By using the impulsive conditions in (2), we obtain, for $k = 1, 2, \dots, m$,

$$\begin{aligned} &d_k - d_{k-1} + (e_k - e_{k-1})t_k = I_k\bigl(x\bigl(t_k^-\bigr)\bigr), \\ &e_k - e_{k-1} = J_k\bigl(x\bigl(t_k^-\bigr)\bigr). \end{aligned}$$

Now we can derive the values of d_k, e_k,

$$d_k = \sum_{i=1}^k I_i\bigl(x\bigl(t_i^-\bigr)\bigr) - \sum_{i=1}^k t_i J_i\bigl(x\bigl(t_i^-\bigr)\bigr)$$

for $k = 1, 2, \dots, m$ and

$$\begin{aligned} &e_m = \frac{c - \Lambda - W}{\frac{a}{\Gamma(\gamma+2)} + b}, \\ &e_k = e_m - \sum_{i=k+1}^m J_i\bigl(x\bigl(t_i^-\bigr)\bigr), \quad \text{for } k = 0, 1, 2, \dots, m-1. \end{aligned}$$

Hence for $k = 1, 2, \dots, m$, we have

$$d_k + e_k t = \sum_{i=1}^k I_i\bigl(x\bigl(t_i^-\bigr)\bigr) - \sum_{i=1}^k t_i J_i\bigl(x\bigl(t_i^-\bigr)\bigr) + \frac{(c - \Lambda - W)t}{\frac{a}{\Gamma(\gamma+2)} + b} - \sum_{i=k+1}^m J_i\bigl(x\bigl(t_i^-\bigr)\bigr)t.$$

This completes the proof. □

The following are two fixed point theorems which will be used in the sequel.

Theorem 2.1 (Nonlinear alternative of Leray-Schauder type [26]) *Let X be a Banach space, C a nonempty convex subset of X, U a nonempty open subset of C with $0 \in U$. Suppose that $P : \overline{U} \to C$ is a continuous and compact map. Then either* (a) *P has a fixed point in $\overline{U}$, or* (b) *there exist a $x \in \partial U$* (*the boundary of U*) *and $\lambda \in (0,1)$ with $x = \lambda P(x)$.*

Theorem 2.2 (Schaefer fixed point theorem [26]) *Let X be a normed space, P a continuous mapping of X into X which is compact on each bounded subset B of X. Then either* (1) *the equation $x = \lambda Px$ has a solution for $\lambda = 1$, or* (2) *the set of all such solutions x is unbounded for* $0 < \lambda < 1$.

3 Main results

This section deals with the existence and uniqueness of solutions for problem (1). In view of Lemma 2.2, we define an operator $F : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$\begin{aligned}(Fx)(t) = {}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,x(s)\bigr)\,ds + \sum_{i=1}^k I_i\bigl(x\bigl(t_i^-\bigr)\bigr) - \sum_{i=1}^k t_i J_i\bigl(x\bigl(t_i^-\bigr)\bigr) \\ & + \frac{(c-\Lambda_x - W_x)t}{\frac{a}{\Gamma(\gamma+2)}+b} - \sum_{i=k+1}^m J_i\bigl(x\bigl(t_i^-\bigr)\bigr)t, \quad t \in J_k, k = 0,1,2,\dots,m, \end{aligned} \tag{4}$$

with

$$\begin{aligned}\Lambda_x &= a\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f\bigl(s,x(s)\bigr)\,ds + b\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\bigl(s,x(s)\bigr)\,ds, \\ W_x &= \frac{a(\sum_{i=1}^m I_i(x(t_i^-)) - \sum_{i=1}^m t_i J_i(x(t_i^-)))}{\Gamma(\gamma+1)}. \end{aligned}$$

Here Λ_x, W_x mean that Λ, W defined in Lemma 2.2 are related to $x \in PC(J,\mathbb{R})$. It is obvious that F is well defined because of the continuity of f, I_k and J_k. Observe that problem (1) has solutions if and only if the operator F has fixed points.

Let $L^\infty(J,\mathbb{R}^+)$ be the essentially bounded function space from J to $\mathbb{R}^+$ and $m(t)$ an element of $L^\infty(J,\mathbb{R}^+)$, we denote the sup-norm of m by $\|m\| = \sup_{t\in J}|m(t)|$. Now, we are in a position to present our main results.

Theorem 3.1 *Assume that there exist $h \in L^\infty(J,\mathbb{R}^+)$ and positive constants L, L^* such that, for $t \in J$, $x, y \in \mathbb{R}$, $k = 1,2,\dots,m$,*

$$\bigl|f(t,x) - f(t,y)\bigr| \le h(t)|x-y|, \tag{5}$$

$$\bigl|I_k(x) - I_k(y)\bigr| \le L|x-y|, \qquad \bigl|J_k(x) - J_k(y)\bigr| \le L^*|x-y|. \tag{6}$$

Moreover,

$$\begin{aligned} &\|h\|\left(\frac{1}{\Gamma(\alpha+1)} + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\alpha+\gamma+1)} + \frac{|b|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\alpha)}\right) \\ &\quad + L^*\left(m + \sum_{i=1}^m t_i + \frac{|a|\sum_{i=1}^m t_i}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+1)}\right) \\ &\quad + mL\left(1 + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+1)}\right) < 1. \end{aligned} \tag{7}$$

Then BVP (1) *has a unique solution on J.*

Proof Denote $\|h\| = \sup_{t\in J}|h(t)|$ and $\mathcal{N}(x,y) = f(s,x(s)) - f(s,y(s))$. For any $x, y \in PC(J,\mathbb{R})$ and each $t \in J$, we have

$$\begin{aligned} &\bigl|(Fx)(t) - (Fy)(t)\bigr| \\ &\quad \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|\mathcal{N}(x,y)\bigr|\,ds + \sum_{i=1}^m \bigl|I_i\bigl(x\bigl(t_i^-\bigr)\bigr) - I_i\bigl(y\bigl(t_i^-\bigr)\bigr)\bigr| \end{aligned}$$

$$+ \sum_{i=1}^{m} t_i \left| J_i\left(x\left(t_i^-\right)\right) - J_i\left(y\left(t_i^-\right)\right) \right| + \frac{|\Lambda_x - \Lambda_y| + |W_x - W_y|}{\left|\frac{a}{\Gamma(\gamma+2)} + b\right|}$$
$$+ \sum_{i=1}^{m} \left| J_i\left(x\left(t_i^-\right)\right) - J_i\left(y\left(t_i^-\right)\right) \right|.$$

Since

$$\begin{aligned} |\Lambda_x - \Lambda_y| &\le |a| \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} \left|\mathcal{N}(x,y)\right| ds + |b| \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left|\mathcal{N}(x,y)\right| ds \\ &\le \frac{|a|\|h\|}{\Gamma(\alpha+\gamma+1)} \|x-y\| + \frac{|b|\|h\|}{\Gamma(\alpha)} \|x-y\|, \end{aligned}$$

and

$$\begin{aligned} |W_x - W_y| &\le \frac{|a|}{\Gamma(\gamma+1)} \sum_{i=1}^{m} \left| I_i\left(x\left(t_i^-\right)\right) - I_i\left(y\left(t_i^-\right)\right) \right| + \frac{|a|}{\Gamma(\gamma+1)} \sum_{i=1}^{m} t_i \left| J_i\left(x\left(t_i^-\right)\right) - J_i\left(y\left(t_i^-\right)\right) \right| \\ &\le \frac{|a|mL}{\Gamma(\gamma+1)} \|x-y\| + \frac{|a|L^*}{\Gamma(\gamma+1)} \sum_{i=1}^{m} t_i \|x-y\|, \end{aligned}$$

we can deduce that

$$\begin{aligned} \|Fx - Fy\| \le \Bigg[&\|h\| \left(\frac{1}{\Gamma(\alpha+1)} + \frac{|a|}{\left|\frac{a}{\Gamma(\gamma+2)} + b\right| \Gamma(\alpha+\gamma+1)} + \frac{|b|}{\left|\frac{a}{\Gamma(\gamma+2)} + b\right| \Gamma(\alpha)} \right) \\ &+ mL \left(1 + \frac{|a|}{\left|\frac{a}{\Gamma(\gamma+2)} + b\right| \Gamma(\gamma+1)} \right) \\ &+ L^* \left(m + \sum_{i=1}^{m} t_i + \frac{|a| \sum_{i=1}^{m} t_i}{\left|\frac{a}{\Gamma(\gamma+2)} + b\right| \Gamma(\gamma+1)} \right) \Bigg] \|x-y\|. \end{aligned}$$

Therefore, by (7), the operator F is a contraction mapping on $PC(J,\mathbb{R})$. Then it follows from Banach's fixed point theorem that problem (1) has a unique solution on J. This completes the proof. □

Lemma 3.1 *The operator* $F : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ *defined by* (4) *is completely continuous.*

Proof Since f, I_k and J_k are continuous, it is easy to show that F is continuous on $PC(J,\mathbb{R})$.

Let $B \subseteq PC(J,\mathbb{R})$ be bounded, then there exist three positive constants N_i, $i = 1,2,3$, such that $|f(t,x(t))| \le N_1$, $|I_k(x(t_k^-))| \le N_2$ and $|J_k(x(t_k^-))| \le N_3$ for all $t \in J$, $x \in B$, $k = 1,2,\dots,m$. Thus, for $x \in B$ and $t \in J$, we have

$$\left|(Fx)(t)\right| \le \frac{N_1}{\Gamma(\alpha+1)} + mN_2 + N_3 \left(\sum_{i=1}^{m} t_i + m \right) + \frac{(|c| + |\Lambda_x| + |W_x|)}{\left|\frac{a}{\Gamma(\gamma+2)} + b\right|},$$

$$|\Lambda_x| \le \frac{|a|N_1}{\Gamma(\alpha+\gamma+1)} + \frac{|b|N_1}{\Gamma(\alpha)}, \tag{8}$$

$$|W_x| \le \frac{|a|(mN_2 + \sum_{i=1}^{m} t_i N_3)}{\Gamma(\gamma+2)}. \tag{9}$$

This means that for all $x \in B$ and $t \in J$,

$$\begin{aligned}\bigl|(Fx)(t)\bigr| &\leq \frac{N_1}{\Gamma(\alpha+1)} + \frac{N_1}{|\frac{a}{\Gamma(\gamma+2)}+b|}\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha)}\right) \\ &\quad + mN_2\left(1 + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+2)}\right) + \frac{|c|}{|\frac{a}{\Gamma(\gamma+2)}+b|} \\ &\quad + N_3\left(\left(\sum_{i=1}^m t_i + m\right) + \frac{|a|\sum_{i=1}^m t_i}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+2)}\right),\end{aligned}$$

which shows that the operator F is uniformly bounded on B.

On the other hand, let $x \in B$ and for any $t_1, t_2 \in J_k$, $k = 0, 1, 2, \ldots, m$, with $t_1 < t_2$, we have

$$\begin{aligned}&\bigl|(Fx)(t_2) - (Fx)(t_1)\bigr| \\ &\quad \leq \left|\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,x(s)\bigr)\,ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,x(s)\bigr)\,ds\right| \\ &\qquad + \frac{(|c| + |\Lambda_x| + |W_x|)}{|\frac{a}{\Gamma(\gamma+2)}+b|}(t_2-t_1) + mL^*(t_2-t_1) \\ &\quad \leq \frac{N_1(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)} + \left(\frac{|c| + |\Lambda_x| + |W_x|}{|\frac{a}{\Gamma(\gamma+2)}+b|} + mL^*\right)(t_2-t_1).\end{aligned}$$

By (8), (9), and the above inequality, we can deduce that

$$\bigl\|(Fx)(t_2) - (Fx)(t_1)\bigr\| \to 0 \quad \text{as } t_2 \to t_1.$$

This implies that F is equicontinuous on the interval J_k. Hence by PC-type Arzela-Ascoli Theorem (see [27]), the operator $F : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is completely continuous. □

Theorem 3.2 *Assume that*: (a) *there exist* $h \in L^\infty(J,\mathbb{R}^+)$ *and* $\varphi : [0,\infty) \to (0,\infty)$ *continuous, nondecreasing such that* $|f(t,x)| \leq h(t)\varphi(|x|)$ *for* $(t,x) \in J \times \mathbb{R}$; (b) *there exist* $\psi, \psi^* : [0,\infty) \to (0,\infty)$ *continuous, nondecreasing such that* $|I_k(x)| \leq \psi(|x|)$, $|J_k(x)| \leq \psi^*(|x|)$ *for all* $x \in \mathbb{R}$ *and* $k = 1, 2, \ldots, m$; (c) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{P\varphi(M) + Q\psi(M) + R\psi^*(M) + H} > 1, \tag{10}$$

where

$$\begin{aligned}P &= \frac{\|h\|}{\Gamma(\alpha+1)} + \frac{\|h\|}{|\frac{a}{\Gamma(\gamma+2)}+b|}\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha)}\right), \\ Q &= m + \frac{m|a|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+2)}, \\ R &= \left(\sum_{i=1}^m t_i + m + \frac{|a|\sum_{i=1}^m t_i}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+2)}\right), \qquad H = \frac{|c|}{|\frac{a}{\Gamma(\gamma+2)}+b|}.\end{aligned}$$

Then BVP (1) *has at least one solution*.

Proof We will show that the operator F defined by (4) satisfies the assumptions of the nonlinear alternative of Leray-Schauder type.

By Lemma 3.1, we know that the operator $F : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is continuous and completely continuous.

Let $x \in PC(J,\mathbb{R})$ be such that $x(t) = \lambda(Fx)(t)$ for some $\lambda \in (0,1)$. Then using the computations in proving that F maps bounded sets into bounded sets in Lemma 3.1, we obtain

$$\begin{aligned} \left|x(t)\right| \le{}& \frac{\|h\|\varphi(\|x\|)}{\Gamma(\alpha+1)} + \frac{\|h\|\varphi(\|x\|)}{|\frac{a}{\Gamma(\gamma+2)}+b|}\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha)}\right) \\ &+ m\psi\left(\|x\|\right)\left(1 + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+2)}\right) + \frac{|c|}{|\frac{a}{\Gamma(\gamma+2)}+b|} \\ &+ \psi^*\left(\|x\|\right)\left(\left(\sum_{i=1}^m t_i + m\right) + \frac{|a|\sum_{i=1}^m t_i}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+2)}\right) \\ ={}& P\varphi\left(\|x\|\right) + Q\psi\left(\|x\|\right) + R\psi^*\left(\|x\|\right) + H. \end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{P\|h\|_{L^\infty}\varphi(\|x\|) + Q\psi(\|x\|) + R\psi^*(\|x\|) + H} \le 1.$$

Then in view of condition (10), there exists M such that $\|x\| \neq M$. Let us set

$$U = \left\{x \in PC(J,\mathbb{R}) : \|x\| < M\right\}.$$

The operator $F : \overline{U} \to PC(J,\mathbb{R})$ is continuous and compact. From the choice of the set U, there is no $x \in \partial U$ such that $x = \lambda Fx$ for some $\lambda \in (0,1)$. Therefore by the nonlinear alternative of Leray-Schauder type, we deduce that F has a fixed point x in $\overline{U}$ which is a solution of the problem (1). The proof is completed. □

Theorem 3.3 *Assume that there exist $h \in L^\infty(J,\mathbb{R}^+)$ and positive constants H_1, H_2 such that, for $t \in J$, $x \in \mathbb{R}$, $k = 1,2,\ldots,m$,*

$$\left|f(t,x)\right| \le h(t), \qquad \left|I_k(x)\right| \le H_1, \qquad \left|I_k^*(x)\right| \le H_2.$$

Then the BVP (1) *has at least one solution on J.*

Proof Lemma 3.1 tells us that the operator $F : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by (4) is continuous and compact on each bounded subset B of $PC(J,\mathbb{R})$.

Now, we show that the set $V = \{v \in PC(J,\mathbb{R}) : v = \lambda Fv, 0 < \lambda < 1\}$ is bounded. Let $x \in V$, then $x = \lambda Fx$ for some $0 < \lambda < 1$. For each $t \in J$, by using a discussion similar to the one in Theorem 3.2, we have

$$\begin{aligned} \left|x(t)\right| &= \left|\lambda(Fx)(t)\right| \\ &\le \|h\|\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\frac{a}{\Gamma(\gamma+2)}+b|}\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha)}\right)\right] \end{aligned}$$

$$+ mH_1\left(1 + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\gamma + 2)}\right) + \frac{|c|}{|\frac{a}{\Gamma(\gamma+2)} + b|}$$
$$+ H_2\left(\left(\sum_{i=1}^{m} t_i + m\right) + \frac{|a|\sum_{i=1}^{m} t_i}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\gamma + 2)}\right).$$

This implies that there exists some $M > 0$ such that $\|x\| \le M$ for all $x \in V$, *i.e.* V is bounded. Thus, by Theorem 2.2, the operator F has at least one fixed point. Hence the problem (1) has at least one solution. The proof is completed. □

4 Examples

In this section, we give two examples to illustrate the main results.

Example 1 Consider the boundary value problem

$$\begin{cases} {}^cD^{\frac{7}{4}}x(t) = \frac{\sin t}{(t+6)^2}(x(t) + \arctan x(t)), & t \in [0,1], t \neq \frac{1}{2}, \\ \Delta x(\frac{1}{2}) = \frac{|x(\frac{1}{2}^-)|}{10+|x(\frac{1}{2}^-)|}, \qquad \Delta x'(\frac{1}{2}) = \frac{|x(\frac{1}{2}^-)|}{20+|x(\frac{1}{2}^-)|}, \\ x(0) = 0, \qquad I^{\frac{5}{3}}x(1) + \frac{1}{2}x'(1) = 2. \end{cases} \tag{11}$$

Here $\alpha = \frac{7}{4}$, $\gamma = \frac{5}{3}$, $m = 1$ $a = 1$ $b = \frac{1}{2}$ and $c = 2$. Clearly, we can take $h(t) = \frac{2\sin t}{(t+6)^2}$, $L = \frac{1}{10}$ and $L^* = \frac{1}{20}$ such that the relations (5) and (6) hold. Moreover,

$$\|h\|\left(\frac{1}{\Gamma(\alpha+1)} + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\alpha+\gamma+1)} + \frac{|b|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\alpha)}\right)$$
$$+ L^*\left(m + \sum_{i=1}^{m} t_i + \frac{|a|\sum_{i=1}^{m} t_i}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\gamma+1)}\right)$$
$$+ mL\left(1 + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\gamma+1)}\right)$$
$$\approx \frac{1}{18} \times 0.9128 + \frac{1}{20} \times 1.6112 + 0.1223 = 0.2536 < 1.$$

Thus, all the assumptions of Theorem 3.1 are satisfied. Hence, by the conclusion of Theorem 3.1, the impulsive fractional BVP (11) has a unique solution on $[0,1]$.

Example 2 Consider the following fractional differential equation:

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = 6t^3 + e^{-|x(t)|} + \sin x(t), & t \in [0,1], t \neq \frac{1}{4}, \\ \Delta x(\frac{1}{4}) = \frac{3|x(\frac{1}{4}^-)|}{(1+|x(\frac{1}{4}^-)|)}, \qquad \Delta x'(\frac{1}{4}) = 2\cos x(\frac{1}{4}^-) + 3, \\ x(0) = 0, \qquad I^{\frac{5}{4}}x(1) - \frac{1}{2}x'(1) = -3. \end{cases} \tag{12}$$

In the context of this problem, we have

$$\left|f(t,x)\right| = \left|6t^3 + e^{-|x|} + \sin x\right| \le 8, \quad t \in [0,1], x \in \mathbb{R},$$
$$\left|I_k(x)\right| \le 3, \qquad \left|I_k^*(x)\right| \le 5, \quad x \in \mathbb{R}.$$

Put $h(t) \equiv 8$, $H_1 = 3$ and $H_2 = 5$. Then from Theorem 3.3, the impulsive fractional BVP (12) has at least one solution on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the manuscript. Both authors have read and approved the final version of the manuscript.

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