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Multilinear Fourier multipliers on variable Lebesgue spaces

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Tianjin, 300071, China**Abstract**

In this paper, we study the properties of a bilinear multiplier space. We give a necessary condition for a continuous bounded function to be a bilinear multiplier on variable exponent Lebesgue spaces, and we prove the localization theorem of multipliers on variable exponent Lebesgue spaces. Moreover, we present a Mihlin-Hörmander type theorem for multilinear Fourier multipliers on weighted variable Lebesgue spaces and give some applications.

MSC: 42B15; 42B20; 42B25**Keywords:** bilinear multiplier space; multilinear Fourier multiplier; variable exponent space; weighted estimate

1 Introduction

Given a non-empty open set $\Omega \subset \mathbb{R}^n$, we denote by $\mathcal{P}(\Omega)$ the set of exponent functions $p(x)$ such that

$$1 \leq p_- \leq p_+ < \infty,$$

where $p_-(\Omega) := \operatorname{ess\,inf}\{p(x) : x \in \Omega\}$ and $p_+(\Omega) := \operatorname{ess\,sup}\{p(x) : x \in \Omega\}$.

Let $\mathcal{P}^0(\Omega)$ be the set of exponent functions $p(x)$ such that

$$0 < p_- \leq p_+ < \infty.$$

Given a measurable function f on Ω for $1 \leq p(\cdot) \leq \infty$, we define the modular functional associated with $p(\cdot)$ by

$$\rho_{p(\cdot), \Omega}(f) = \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} dx + \|f(x)\|_{L^\infty(\Omega_\infty)},$$

where Ω_∞ denotes the set of points in Ω on which $p(x) = \infty$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined to be the set of Lebesgue measurable functions f on Ω satisfying $\rho_{p(\cdot), \Omega}(f/\lambda) < \infty$ for some $\lambda > 0$. The norm of f in the space is defined by

$$\|f\|_{L^{p(\cdot)}} = \inf \{ \lambda > 0 : \rho_{p(\cdot), \Omega}(f/\lambda) \leq 1 \}.$$

In the case that $p(\cdot) \in \mathcal{P}^0(\Omega)$, it is defined to be the set of all functions f satisfying $|f(x)|^{p_0} \in L^{q(\cdot)}(\Omega)$, $q(x) = p(x)/p_0 \in \mathcal{P}(\Omega)$ for some $0 < p_0 < p_-$ (see [1]). A quasi-norm in the space is defined by

$$\|f\|_{p(\cdot),\Omega} = \| |f|^{p_0} \|_{q(\cdot),\Omega}^{\frac{1}{p_0}}.$$

We refer to [2] for an introduction to variable exponent Lebesgue spaces.

Similarly, for $p(\cdot) \in \mathcal{P}^0(\Omega)$ and a weight function w , the weighted variable exponent Lebesgue space $L^{p(\cdot)}(\Omega, w)$ (see [3]) is defined to be the set of Lebesgue measurable functions f on Ω that satisfies

$$\|f\|_{L^{p(\cdot)}(\Omega, w)} := \inf \left\{ \lambda > 0 : \int_{\Omega} |f(x)/\lambda|^{p(x)} w(x) dx \leq 1 \right\} < \infty.$$

In this paper, we study some properties of the space of bilinear Fourier multipliers and the Mihlin-Hörmander type theorem for multilinear Fourier multipliers on weighted variable Lebesgue spaces. Specifically, let m satisfy certain conditions. We discuss the N -linear Fourier multiplier operator T_m defined by

$$\begin{aligned} T_m(f_1, \dots, f_N)(x) \\ = \int_{\mathbb{R}^{Nn}} e^{2\pi i(\xi_1 + \dots + \xi_N, x)} m(\xi_1, \dots, \xi_N) \hat{f}_1(\xi_1), \dots, \hat{f}_N(\xi_N) d\xi_1 \cdots d\xi_N \end{aligned}$$

for $x \in \mathbb{R}^n, f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$ [4].

The multilinear Fourier multipliers have been studied for a long time. In [4], Coifman and Meyer proved that T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N, p < \infty$ with $\frac{1}{p_1} + \dots + \frac{1}{p_N} = \frac{1}{p}$ and $m \in C^s(\mathbb{R}^{Nn} \setminus \{0\})$ satisfying

$$|\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_N}^{\alpha_N} m(\xi_1, \dots, \xi_N)| \leq C_{\alpha_1, \dots, \alpha_N} (|\xi_1| + \dots + |\xi_N|)^{-(|\alpha_1| + \dots + |\alpha_N|)} \tag{1.1}$$

for all $|\alpha_1| + \dots + |\alpha_N| \leq s$, where $N \geq 2$ is an integer and s is a sufficiently large integer.

Tomita [5] gave a Hörmander type theorem for multilinear multipliers. Specifically, T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_N, p < \infty$ with $\frac{1}{p_1} + \dots + \frac{1}{p_N} = \frac{1}{p}$ and $s = \frac{Nn}{2} + 1$ in (1.1). Furthermore, Grafakos and Si studied the case $p \leq 1$ in [6]. The boundedness of multilinear Calderón-Zygmund operators with multiple weights was achieved by Grafakos *et al.* [7].

Under the Hörmander conditions, Fujita and Tomita [8] obtained some weighted estimates of T_m for classical A_p weights. Then Li *et al.* [9] got some weighted results of multilinear multipliers by considering the end-point cases, using weighted Carleson measure theory and employing multilinear interpolation theory. In [10], Chen and Lu proved a Hörmander type multilinear theorem on weighted Lebesgue spaces when the Fourier multipliers were only assumed with limited smoothness. In [11], the boundedness of T_m with multiple weights satisfying condition (1.1) was given by Bui and Duong. In [12], Li and Sun got some weighted estimates of T_m with multiple weights under the Hörmander conditions in terms of the Sobolev regularity. Huang and Xu [13] obtained the boundedness of multilinear Calderón-Zygmund operators on variable exponent Lebesgue spaces.

In this paper, we study the weighted estimates of T_m with nearly the same conditions as in [12], but on variable exponent Lebesgue spaces.

The theory of bilinear multipliers was first studied by Coifman and Meyer [14]. They considered the ones with smooth symbols. Then, Muscalu *et al.* achieved some new results for non-smooth symbols in [15].

The study of bilinear multipliers has experienced a big progress since Lacey and Thiele [16, 17] proved that $m(\xi, \nu) = \text{sign}(\xi + \alpha\nu)$ are (p_1, p_2, p_3) -multipliers for each triple (p_1, p_2, p_3) such that $1 < p_1, p_2 \leq \infty$, $p_3 > 2/3$ and each $\alpha \in \mathbb{R} \setminus \{0, 1\}$. In [18], Kulak and Gürkanlı first studied some properties of the bilinear multiplier space. In [19], Fan and Sato proved the DeLeeuw type theorems for the transference of multilinear operators on Lebesgue and Hardy spaces from \mathbb{R}^n to \mathbb{T}^n . In [20], Blasco gave the transference theorems from \mathbb{R}^n to \mathbb{Z}^n . We also refer to [21, 22] for details.

We first give some definitions.

Definition 1.1 ([18]) Let $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\Omega)$, $p_3(\cdot) \in \mathcal{P}^0(\Omega)$, and $m(\xi, \eta)$ be a bounded function on \mathbb{R}^{2n} . Define

$$B_m(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) m(\xi, \eta) e^{2\pi i(\xi + \eta, x)} d\xi d\eta$$

for all f and $g \in \mathcal{S}(\mathbb{R}^n)$.

We call m a bilinear multiplier on \mathbb{R}^{2n} of type $(p_1(\cdot), p_2(\cdot), p_3(\cdot))$ if there exists some $C > 0$ such that $\|B_m(f, g)\|_{p_3(\cdot)} \leq C \|f\|_{p_1(\cdot)} \|g\|_{p_2(\cdot)}$ for all f and $g \in \mathcal{S}(\mathbb{R}^n)$, i.e., B_m extends to a bounded bilinear operator from $L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n)$ to $L^{p_3(\cdot)}(\mathbb{R}^n)$.

We write $\text{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot))$ for the space of bilinear multipliers of type $(p_1(\cdot), p_2(\cdot), p_3(\cdot))$. Let $\|m\|_{(p_1(\cdot), p_2(\cdot), p_3(\cdot))} = \|B_m\|$.

A similar function space is defined in the following.

Definition 1.2 Given a function M on \mathbb{R}^n , let $m(\xi, \eta) = M(\xi - \eta)$. We say that

$$M \in \widetilde{\text{BM}}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot))$$

if $B_M(f, g)(x) = \int_{\mathbb{R}^{2n}} \hat{f}(\xi) \hat{g}(\eta) M(\xi - \eta) e^{2\pi i(\xi + \eta, x)} d\xi d\eta$ for all f and $g \in \mathcal{S}(\mathbb{R}^n)$ can be extended to a bounded bilinear operator from $L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n)$ to $L^{p_3(\cdot)}(\mathbb{R}^n)$.

Definition 1.3 ([2]) A function $p: \Omega \rightarrow \mathbb{R}^1$ is said to belong to the class $LH_0(\Omega)$ if

$$|p(x) - p(y)| \leq \frac{C}{-\ln(|x - y|)}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \Omega,$$

where $C > 0$ is independent of x or y .

We simply write LH_0 instead of $LH_0(\mathbb{R}^n)$ if there is no confusion. We also use $C(\mathbb{R}^n)$ to represent the collection of all continuous functions on \mathbb{R}^n . By C etc., we denote various positive constants which may have different values even in the same line.

2 Some results on the space $\text{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot))$

Some properties of the bilinear multiplier space on variable spaces were given by Kulak and Gürkanlı [18]. Here we give some other properties.

First, we introduce the standard singular kernel.

Definition 2.1 ([2]) Given a function $K \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, it is called a standard singular kernel if there exists a constant $C > 0$ such that:

1. $|K(x)| \leq \frac{C}{|x|^n}, x \neq 0$;
2. $|\nabla K(x)| \leq \frac{C}{|x|^{n+1}}, x \neq 0$;
3. for $0 < r < R$, $|\int_{\{r < |x| < R\}} K(x) dx| \leq C$;
4. $\lim_{\varepsilon \rightarrow 0} \int_{\{\varepsilon < |x| < 1\}} K(x) dx$ exists.

Theorem 2.2 (Localization) *Suppose that*

$$m \in \text{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot)),$$

Q is a rectangle in \mathbb{R}^{2n} and that the Hardy-Littlewood maximal operator \mathcal{M} is bounded on $L^{p_i(\cdot)}(\mathbb{R}^n)$, where $1 < (p_i)_- \leq (p_i)_+ < \infty, i = 1, 2$. Then

$$m\chi_Q \in \text{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot))$$

and $\|m\chi_Q\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)} \leq C\|m\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)}$, where C is independent of Q .

Let $\text{BM}(\mathbb{R}^n)(p(\cdot), p(\cdot))$ denote the space of multipliers which correspond to bounded operators from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$.

To prove Theorem 2.2, we need the following results in the theory of variable Lebesgue spaces.

Lemma 2.3 ([2, Theorem 5.39]) *Let T be a singular integral operator with a standard singular kernel K . Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p_- \leq p_+ < \infty$, if the Hardy-Littlewood maximal operator \mathcal{M} is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then for all functions f that are bounded and have compact support, $\|Tf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}$, and T extends to a bounded operator on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Theorem 2.4 *Suppose that $m_1 \in \text{BM}(\mathbb{R}^n)(s_1(\cdot), p_1(\cdot))$, $m_2 \in \text{BM}(\mathbb{R}^n)(s_2(\cdot), p_2(\cdot))$ and $m \in \text{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot))$. Then we have*

$$m_1(\xi)m(\xi, \eta)m_2(\eta) \in \text{BM}(\mathbb{R}^{2n})(s_1(\cdot), s_2(\cdot), p_3(\cdot)).$$

Proof For any f and $g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned} B_{m_1 m m_2}(f, g)(x) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) m_1(\xi) m(\xi, \eta) m_2(\eta) e^{2\pi i(\xi + \eta, x)} d\xi d\eta \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ((T_{m_1} f))^\wedge(\xi) ((T_{m_2} g))^\wedge(\eta) m(\xi, \eta) e^{2\pi i(\xi + \eta, x)} d\xi d\eta \\ &= B_m(T_{m_1} f, T_{m_2} g)(x). \end{aligned}$$

Therefore,

$$\begin{aligned} \|B_{m_1 m_2}(f, g)\|_{p_3(\cdot)} &\leq \|B_m\| \|T_{m_1} f\|_{p_1(\cdot)} \|T_{m_2} g\|_{p_2(\cdot)} \\ &\leq \|B_m\| \|m_1\|_{s_1(\cdot), p_1(\cdot)} \|m_2\|_{s_2(\cdot), p_2(\cdot)} \|f\|_{s_1(\cdot)} \|g\|_{s_2(\cdot)}. \end{aligned}$$

Then we get the result. □

The following is an explicit example.

Example 2.5 Suppose that $\frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)} = \frac{1}{p_3(\cdot)}$, $m_1 \in \text{BM}(\mathbb{R}^n)(p_1(\cdot), p_1(\cdot))$ and $m_2 \in \text{BM}(\mathbb{R}^n)(p_2(\cdot), p_2(\cdot))$, where $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $p_3(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$. Then

$$m(\xi, \eta) = m_1(\xi)m_2(\eta) \in \text{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot)).$$

Proof For any f and $g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned} B_1(f, g)(x) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i(\xi+\eta, x)} d\xi d\eta \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i(\xi, x)} \hat{g}(\eta)e^{2\pi i(\eta, x)} d\xi d\eta \\ &= f(x)g(x). \end{aligned}$$

By Hölder's inequality [2], we have

$$\|B_1(f, g)(x)\|_{p_3(\cdot)} = \|f(x)g(x)\|_{p_3(\cdot)} \leq C\|f\|_{p_1(\cdot)}\|g\|_{p_2(\cdot)}.$$

Thus $1 \in \text{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot))$. By Theorem 2.4, we have

$$m(\xi, \eta) = m_1(\xi)m_2(\eta) \in \text{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot)). \quad \square$$

Proof of Theorem 2.2 We only consider the case $n = 1$. Other cases can be proved similarly. Suppose that $Q = [a, b] \times [c, d]$. Then, for any f and $g \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} B_{m\chi_Q}(f, g)(x) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)m(\xi, \eta)\chi_Q(\xi, \eta)e^{2\pi i(\xi+\eta, x)} d\xi d\eta \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\chi_{[a, b]}(\xi)\hat{g}(\eta)\chi_{[c, d]}(\eta)m(\xi, \eta)e^{2\pi i(\xi+\eta, x)} d\xi d\eta \\ &= B_m(\hat{f}\chi_{[a, b]})^\vee, (\hat{g}\chi_{[c, d]})^\vee(x). \end{aligned}$$

Note that by (3.9) of [23], we have $(\hat{f}\chi_{[a, b]})^\vee = \frac{i}{2}(M^a H M^{-a} - M^b H M^{-b})f$, where M^a denotes the operator $M^a f(x) = e^{2\pi i a x} f(x)$ and H denotes the Hilbert transform operator. Since the Hilbert transform has a standard singular kernel, by Lemma 2.3 we have

$$\begin{aligned} \|(\hat{f}\chi_{[a, b]})^\vee\|_{p_1(\cdot)} &= \frac{1}{2} \| (M^a H M^{-a} f - M^b H M^{-b} f) \|_{p_1(\cdot)} \\ &\leq \frac{1}{2} \| H M^{-a} f \|_{p_1(\cdot)} + \frac{1}{2} \| H M^{-b} f \|_{p_1(\cdot)} \\ &\leq C\|f\|_{p_1(\cdot)}. \end{aligned}$$

So

$$\chi_{[a,b]} \in \text{BM}(\mathbb{R}^n)(p_1(\cdot), p_1(\cdot)).$$

Similarly we can prove that

$$\chi_{[c,d]} \in \text{BM}(\mathbb{R}^n)(p_2(\cdot), p_2(\cdot)).$$

Hence by Theorem 2.4, we get

$$m\chi_Q \in \text{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot)),$$

and $\|m\chi_Q\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)} \leq C\|m\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)}$. □

Next we show that the space $\widetilde{\text{BM}}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot))$ is invariant under certain operators.

Theorem 2.6 *Given $p_3(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\phi \in L^1(\mathbb{R}^n)$, if*

$$M \in \widetilde{\text{BM}}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot)),$$

then

$$\phi * M \in \widetilde{\text{BM}}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot)),$$

and $\|\phi * M\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)} \leq C\|\phi\|_1 \|M\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)}$.

Proof For any f and $g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned} B_{\phi * M}(f, g)(x) &= \int_{\mathbb{R}^{2n}} \hat{f}(\xi) \hat{g}(\eta) \left(\int_{\mathbb{R}^n} M(\xi - \eta - u) \phi(u) du \right) e^{2\pi i(\xi + \eta, x)} d\xi d\eta \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^{2n}} \widehat{M^{-u}f}(\xi) \hat{g}(\eta) M(\xi - \eta) e^{2\pi i(\xi + \eta, x)} d\xi d\eta e^{2\pi i(u, x)} \phi(u) du. \end{aligned}$$

By Minkowski's inequality,

$$\begin{aligned} \|B_{\phi * M}(f, g)(x)\|_{p_3(\cdot)} &\leq C \int_{\mathbb{R}^n} \|B_M(M^{-u}f, g)(x)\|_{p_3(\cdot)} |\phi(u)| du \\ &\leq C \|M\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)} \|\phi\|_1 \|f\|_{p_1(\cdot)} \|g\|_{p_2(\cdot)}. \end{aligned}$$
□

Theorem 2.7 *Suppose that $p_3 \geq 1$, $M \in \widetilde{\text{BM}}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3)$ and $\phi \in L^1(\mathbb{R}^n)$. Then*

$$\begin{aligned} m(\xi, \eta) &:= M(\xi - \eta) \hat{\phi}(\xi + \eta) \\ &\in \text{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3), \end{aligned}$$

and $\|m\|_{p_1(\cdot), p_2(\cdot), p_3} \leq \|\phi\|_1 \|M\|_{p_1(\cdot), p_2(\cdot), p_3}$.

Proof For any f and $g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned} B_m(f, g)(x) &= \int_{\mathbb{R}^{2n}} \hat{f}(\xi) \hat{g}(\eta) M(\xi - \eta) \left(\int_{\mathbb{R}^n} \phi(y) e^{-2\pi i(\xi + \eta, y)} dy \right) e^{2\pi i(\xi + \eta, x)} d\xi d\eta \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{2n}} \hat{f}(\xi) \hat{g}(\eta) M(\xi - \eta) e^{2\pi i(\xi + \eta, x - y)} d\xi d\eta \right) \phi(y) dy \\ &= \phi * B_M(f, g)(x). \end{aligned}$$

By Young's inequality, we have

$$\begin{aligned} \|B_m(f, g)\|_{p_3} &\leq \|\phi\|_1 \|B_M(f, g)\|_{p_3} \\ &= \|\phi\|_1 \|B_M\| \|f\|_{p_1(\cdot)} \|g\|_{p_2(\cdot)}. \end{aligned}$$

Thus, we get the conclusion. □

Finally, we consider the necessary condition of this kind of multipliers. The bilinear classical counterpart was obtained by Hörmander [24, Theorem 3.1] and Blasco [25]. The multilinear classical one was proved by Grafakos and Torres, see [26, Proposition 5] and [27, Proposition 2.1]. And the one for multipliers on Lorentz spaces was given by Villarroja [28, Proposition 3.1]. Some of their proofs used the translation-invariant property of the classical spaces, which is, however, no longer valid on $L^{p(\cdot)}$. In the following, we prove the variable version of the necessary condition.

Theorem 2.8 (Necessary condition) *Suppose that there is a non-zero continuous bounded function M such that $M \in \widetilde{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot))$. Then*

$$\frac{1}{(p_3)_+} \leq \frac{1}{(p_1)_-} + \frac{1}{(p_2)_-}.$$

To prove the theorem, we need the following results.

Proposition 2.9 ([2, Corollary 2.22]) *Fix Ω and $1 \leq p(\cdot) \leq \infty$. If $\|f\|_{p(\cdot)} \leq 1$, then $\rho(f) \leq \|f\|_{p(\cdot)}$; if $\|f\|_{p(\cdot)} > 1$, then $\rho(f) \geq \|f\|_{p(\cdot)}$.*

Proposition 2.10 ([2, Corollary 2.23]) *Given Ω and $1 \leq p(\cdot) \leq \infty$, suppose $|\Omega_\infty| = 0$. If $\|f\|_{p(\cdot)} > 1$, then*

$$\rho(f)^{1/p_+} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_-}.$$

If $0 < \|f\|_{p(\cdot)} \leq 1$, then

$$\rho(f)^{1/p_-} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_+}.$$

Lemma 2.11 *Let $M \in \widetilde{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot))$. If $\frac{1}{q} = \frac{1}{(p_1)_-} + \frac{1}{(p_2)_-} - \frac{1}{(p_3)_+}$, then there exists some $C > 0$ such that*

$$\left| \lambda^n \int_{\mathbb{R}^n} e^{-\lambda^2 \xi^2} M(\xi) d\xi \right| \leq C \|M\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)} \lambda^{\frac{n}{q}},$$

when λ is sufficiently large.

Proof Let $\lambda > 0$. Define G_λ by $\widehat{G}_\lambda(\xi) = e^{-2\lambda^2\xi^2}$. By a simple change of variable, one gets that

$$\begin{aligned} B_M(G_\lambda, G_\lambda)(x) &= \int_{\mathbb{R}^{2n}} e^{-2\lambda^2\xi^2} e^{-2\lambda^2\eta^2} M(\xi - \eta) e^{2\pi i(\xi + \eta, x)} d\xi d\eta \\ &= \frac{1}{2} \int_{\mathbb{R}^{2n}} e^{-\lambda^2 v^2} e^{-\lambda^2 \mu^2} M(v) e^{2\pi i(\mu, x)} d\mu dv \\ &= \frac{C}{\lambda^n} e^{-\pi^2 |\frac{x}{\lambda}|^2} \int_{\mathbb{R}^n} e^{-\lambda^2 v^2} M(v) dv, \end{aligned} \tag{2.1}$$

where we use the fact that $G_\lambda(x) = (e^{-2\lambda^2\xi^2})^\vee = \frac{C}{\lambda^n} e^{-\frac{\pi^2}{2} |\frac{x}{\lambda}|^2}$ [29, Example 2.2.9].

Observe that

$$\begin{aligned} \rho_{p_i(\cdot)}\left(e^{-\frac{\pi^2}{2} |\frac{x}{\lambda}|^2}\right) &= \int_{\mathbb{R}^n} e^{-\frac{\pi^2}{2} |u|^2} p_i(x) dx = \lambda^n \int_{\mathbb{R}^n} e^{-\frac{\pi^2}{2} |u|^2} p_i(\lambda u) du \\ &\leq \lambda^n \int_{\mathbb{R}^n} e^{-\frac{\pi^2}{2} |u|^2} (p_i)_- du = C_{(p_i)_-} \lambda^n, \end{aligned}$$

where $i = 1, 2$.

Similarly we have

$$\rho_{p_i(\cdot)}\left(e^{-\frac{\pi^2}{2} |\frac{x}{\lambda}|^2}\right) \geq C_{(p_i)_+} \lambda^n, \quad i = 1, 2.$$

By Proposition 2.9, we get $\|e^{-\frac{\pi^2}{2} |\frac{x}{\lambda}|^2}\|_{p_i(\cdot)} > 1$, when λ is sufficiently large. Thus by Proposition 2.10, we have

$$\rho_{p_i(\cdot)}\left(e^{-\frac{\pi^2}{2} |\frac{x}{\lambda}|^2}\right)^{\frac{1}{(p_i)_+}} \leq \|e^{-\frac{\pi^2}{2} |\frac{x}{\lambda}|^2}\|_{p_i(\cdot)} \leq \rho_{p_i(\cdot)}\left(e^{-\frac{\pi^2}{2} |\frac{x}{\lambda}|^2}\right)^{\frac{1}{(p_i)_-}}.$$

So

$$C_{(p_i)_+} \lambda^{n/(p_i)_+ - n} \leq \|G_\lambda\|_{p_i(\cdot)} \leq C_{(p_i)_-} \lambda^{n/(p_i)_- - n}, \tag{2.2}$$

where $i = 1, 2$.

Similarly we can get

$$C_{(p_3)_+} \lambda^{n/(p_3)_+ - n} \leq \left\| \frac{1}{\lambda^n} e^{-\pi^2 |\frac{x}{\lambda}|^2} \right\|_{p_3(\cdot)} \leq C_{(p_3)_-} \lambda^{n/(p_3)_- - n}. \tag{2.3}$$

All the inequalities above are established when the λ is sufficiently large.

By the assumption, we have

$$\|B_M(G_\lambda, G_\lambda)\|_{p_3(\cdot)} \leq \|M\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)} \|G_\lambda\|_{p_1(\cdot)} \|G_\lambda\|_{p_2(\cdot)}. \tag{2.4}$$

Now combining (2.1), (2.2), (2.3) and (2.4), we get

$$\begin{aligned} C_{(p_3)_+} \lambda^{\frac{n}{(p_3)_+} - n} \left| \int_{\mathbb{R}^n} e^{-\lambda^2 \xi^2} M(\xi) d\xi \right| &\leq \left\| \frac{C}{\lambda^n} e^{-\pi^2 |\frac{x}{\lambda}|^2} \right\|_{p_3(\cdot)} \left| \int_{\mathbb{R}^n} e^{-\lambda^2 v^2} M(v) dv \right| \\ &\leq \|B_M(G_\lambda, G_\lambda)\|_{p_3(\cdot)} \end{aligned}$$

$$\begin{aligned} &\leq \|M\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)} \|G_\lambda\|_{p_1(\cdot)} \|G_\lambda\|_{p_2(\cdot)} \\ &\leq C \|M\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)} \lambda^{\frac{n}{(p_1)^-} - n} \lambda^{\frac{n}{(p_2)^-} - n}. \end{aligned}$$

Hence

$$\left| \lambda^n \int_{\mathbb{R}^n} e^{-\lambda^2 \xi^2} M(\xi) d\xi \right| \leq C \|M\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)} \lambda^{\frac{n}{q}},$$

when λ is sufficiently large. □

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8 Assume that $\frac{1}{(p_1)^-} + \frac{1}{(p_2)^-} < \frac{1}{(p_3)^+}$. By a simple calculation, we obtain that

$$\begin{aligned} B_M(M^y f, M^{-y} g)(x) &= \int_{\mathbb{R}^{2n}} (M^y f)^\wedge(\xi) (M^{-y} g)^\wedge(\eta) M(\xi - \eta) e^{2\pi i(\xi + \eta, x)} d\xi d\eta \\ &= \int_{\mathbb{R}^{2n}} T_y \hat{f}(\xi) T_{-y} \hat{g}(\eta) M(\xi - \eta) e^{2\pi i(\xi + \eta, x)} d\xi d\eta \\ &= \int_{\mathbb{R}^{2n}} \hat{f}(\xi) \hat{g}(\eta) M(\xi - \eta + 2y) e^{2\pi i(\xi + \eta, x)} d\xi d\eta \\ &= B_{T_{-2y} M}(f, g)(x), \end{aligned}$$

where $T_{-2y} M = M(x + 2y)$. Thus $T_{-2y} M \in \widetilde{BM}(\mathbb{R}^{2n})(p_1(\cdot), p_2(\cdot), p_3(\cdot))$. Applying Lemma 2.11 to $T_{-2y} M$, we get

$$\left| \lambda^n \int_{\mathbb{R}^n} e^{-\lambda^2 \xi^2} M(\xi + 2y) d\xi \right| \leq C \|M\|_{p_1(\cdot), p_2(\cdot), p_3(\cdot)} \lambda^{\frac{n}{q}}.$$

Observe that $\frac{1}{q} = \frac{1}{(p_1)^-} + \frac{1}{(p_2)^-} - \frac{1}{(p_3)^+} < 0$ and M is continuous. By letting $\lambda \rightarrow \infty$, we have

$$\lim_{\lambda \rightarrow \infty} \left| \lambda^n \int_{\mathbb{R}^n} e^{-\lambda^2 \xi^2} M(\xi + 2y) d\xi \right| = \pi^{\frac{n}{2}} |M(2y)| = 0.$$

Since y is arbitrary, we have $M = 0$. This is a contradiction. Thus

$$\frac{1}{(p_3)^+} \leq \frac{1}{(p_1)^-} + \frac{1}{(p_2)^-}. \quad \square$$

3 The Mihlin-Hörmander type estimate for multilinear multipliers on weighted variable exponent Lebesgue spaces

Roughly speaking, in the linear case, by adding the condition that the Hardy-Littlewood maximal operator is bounded on weighted variable spaces, the results of multipliers on weighted variable spaces can be derived from the weighted multiplier theorem on classical Lebesgue spaces and the extrapolation theorem on weighted variable spaces. See, for example, [3, Theorem 4.5, Theorem 4.7], [30] and [31].

However, in the multilinear case, the method faces some challenges. One problem is that we have no multilinear extrapolation theorem on spaces with variable exponents yet, though the counterpart on classical Lebesgue spaces appeared early, see [32].

We give another way to get the Mihlin-Hörmander conditions for multilinear Fourier multipliers on weighted variable spaces.

First we use Q to denote a cube in \mathbb{R}^n . Recall that the Hardy-Littlewood maximal operator is defined by

$$\mathcal{M}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy.$$

And the sharp maximal function is defined by

$$M^\#(f)(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy.$$

For $\delta > 0$, we also define

$$M_\delta(f) = \mathcal{M}(|f|^\delta)^{1/\delta} \quad \text{and} \quad M_\delta^\#(f) = M^\#(|f|^\delta)^{1/\delta}.$$

For $\vec{f} = (f_1, \dots, f_N)$ and $p \geq 1$, we define

$$\mathcal{M}_p(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^N \left(\frac{1}{|Q|} \int_Q |f_i(y_i)|^p \, dy_i \right)^{1/p}.$$

Definition 3.1 ([33]) Given $\vec{p} = (p_1, \dots, p_N)$ with $1 \leq p_1, \dots, p_N < \infty$ and $1/p_1 + \dots + 1/p_N = 1/p$. Let $\vec{w} = (w_1, \dots, w_N)$. Set

$$v_{\vec{w}} = \prod_{i=1}^N w_i^{p/p_i}.$$

We say that \vec{w} satisfies the $A_{\vec{p}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q v_{\vec{w}}(x) \, dx \right)^{1/p} \prod_{i=1}^N \left(\frac{1}{|Q|} \int_Q w_i(x)^{1-p'_i} \, dx \right)^{1/p'_i} < \infty.$$

When $p_i = 1$, then $(\frac{1}{|Q|} \int_Q w_i(x)^{1-p'_i} \, dx)^{1/p'_i}$ is understood as $(\inf_Q w_i)^{-1}$.

We now give a Mihlin-Hörmander type theorem for multilinear Fourier multipliers on weighted variable exponent Lebesgue spaces.

Theorem 3.2 *Suppose that $Nn/2 < s \leq Nn$, $m \in L^\infty(\mathbb{R}^{Nn})$ and*

$$\sup_{R>0} \|m(R\xi) \chi_{\{1<|\xi|<2\}}\|_{H^s(\mathbb{R}^{Nn})} < \infty.$$

Set $r_0 := Nn/s$, a series of variable indexes $p_1(x), \dots, p_N(x) \in \mathcal{P}(\mathbb{R}^n)$, and $p(x) \in \mathcal{P}^0(\mathbb{R}^n)$, such that $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \dots + \frac{1}{p_N(x)} = \frac{1}{p(x)}$, where $(p_j)_- > r_0$, $j = 1, 2, \dots, N$. Suppose that there are $0 < q < p_-$, $r_0 < q_j < (p_j)_-$ such that the Hardy-Littlewood maximal operator \mathcal{M} is bounded on $L^{\vec{p}(\cdot)}((w_1 \dots w_N)^{-q\vec{p}'(\cdot)})$ and $L^{\vec{p}'(\cdot)}(w_j^{-q_j\vec{p}'(\cdot)})$, where $\tilde{p}(x) = \frac{p(x)}{q}$, $\tilde{p}_j(x) = \frac{p_j(x)}{q_j}$, $j = 1, 2, \dots, N$. Then there exists some $C > 0$ such that

$$\|T_m(\vec{f})\|_{L^{p(\cdot)}(w_1^{p(\cdot)} \dots w_N^{p(\cdot)})} \leq C \prod_{i=1}^N \|f_i\|_{L^{p_i(\cdot)}(w_i^{p_i(\cdot)})}.$$

Before proving the theorem, we present some preliminary results. The following inequality is a classical result of Fefferman and Stein [34].

Proposition 3.3 ([34]) *Let $0 < \delta < p < \infty$ and $w \in A_\infty$. Then there exists some constants $C_{n,p,\delta,w} > 0$ such that*

$$\int_{\mathbb{R}^n} (M_\delta f)(x)^p w(x) dx \leq C_{n,p,\delta,w} \int_{\mathbb{R}^n} (M_\delta^\# f)(x)^p w(x) dx.$$

The next result comes from Lemma 2.6 in [12]. For our purpose, we restate it in the proper way.

Proposition 3.4 ([12]) *Let $1 < r < \min\{\frac{s}{s-1}, \frac{2s}{Nn}\}$ such that $p_0 := rr_0 < q_j, j = 1, \dots, N$. If $0 < \delta < p_0/N$, then under the assumption of Theorem 3.2, there exists some $C > 0$ such that for all $\vec{f} \in L^{t_1}(\mathbb{R}^n) \times \dots \times L^{t_N}(\mathbb{R}^n)$, $p_0 \leq t_1, \dots, t_N < \infty$, we have*

$$M_\delta^\#(T_m \vec{f}) \leq C \mathcal{M}_{p_0}(\vec{f}).$$

Proposition 3.5 ([3]) *Let X be a metric measure space and Ω be an open set in X . Assume that for some p_0 and q_0 satisfying*

$$0 < p_0 \leq q_0 < \infty, \quad p_0 < p_- \quad \text{and} \quad \frac{1}{p_0} - \frac{1}{p_+} < \frac{1}{q_0},$$

and for every weight $w \in A_1(\Omega)$, there holds the inequality

$$\left(\int_{\Omega} f^{q_0}(x) w(x) d\mu(x) \right)^{\frac{1}{q_0}} \leq c_0 \left(\int_{\Omega} g^{p_0}(x) [w(x)]^{\frac{p_0}{q_0}} d\mu(x) \right)^{\frac{1}{p_0}}$$

for all (f, g) in a given family \mathcal{F} . Let the variable exponent $q(x)$ be defined by

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \left(\frac{1}{p_0} - \frac{1}{q_0} \right).$$

Let the exponent $p(x)$ and the weight ϱ satisfy that $p \in \mathcal{P}^0(\Omega)$ and \mathcal{M} is bounded on $L^{\vec{q}(\cdot)}(\Omega, \varrho^{-q_0 \vec{q}(\cdot)})$.

Then, for all $(f, g) \in \mathcal{F}$ with $f \in L^{q(\cdot)}(\Omega, \varrho^{q(\cdot)})$, the inequality

$$\|f\|_{L^{q(\cdot)}(\Omega, \varrho^{q(\cdot)})} \leq C \|g\|_{L^{p(\cdot)}(\Omega, \varrho^{p(\cdot)})}$$

is valid with a constant $C > 0$.

Remark 3.6 Note that the condition $p \in \mathcal{P}(\Omega)$ in the extrapolation theorem of [3] can be released to $p \in \mathcal{P}^0(\Omega)$ with nearly no modification to the proof.

Proposition 3.7 ([11, Proposition 2.3]) *Let $p_0 \geq 1$ and $p_i > p_0$ for $i = 1, \dots, N$ and $1/p_1 + \dots + 1/p_N = 1/p$. Then the inequality*

$$\|\mathcal{M}_{p_0}(\vec{f})\|_{L^p(w_{\vec{w}})} \leq C \prod_{i=1}^N \|f_i\|_{L^{p_i}(w_i)}$$

holds if and only if $\vec{w} \in A_{\vec{P}/p_0}$, where $\vec{P}/p_0 = (p_1/p_0, \dots, p_N/p_0)$.

Remark 3.8 When $N = 1$, the conclusion above is valid. Specifically, let $p_0 \geq 1$ and $p > p_0$, then $\|\mathcal{M}_{p_0} f\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$ holds if and only if $w \in A_{p/p_0}$.

We are now ready to prove Theorem 3.2

Proof of Theorem 3.2 For any $f_j \in \mathcal{S}(\mathbb{R}^n)$, $j = 1, \dots, N$, and $v \in A_\infty$, by Proposition 3.3 and Proposition 3.4, we have

$$\begin{aligned} \|\mathbb{T}_m(\vec{f})\|_{L^q(v)} &\leq \|M_\delta(\mathbb{T}_m(\vec{f}))\|_{L^q(v)} \\ &\leq C_{n,q,\delta,v} \|M_\delta^\#(\mathbb{T}_m(\vec{f}))\|_{L^q(v)} \\ &\leq C \|\mathcal{M}_{p_0}(\vec{f})\|_{L^q(v)}, \end{aligned} \tag{3.1}$$

where p_0 is defined as in Proposition 3.4.

Since the maximal operator \mathcal{M} is bounded on $L^{\vec{p}(\cdot)}((w_1 \cdots w_N)^{-q\vec{p}(\cdot)})$, by Proposition 3.5, we have

$$\|\mathbb{T}_m(\vec{f})\|_{L^{p(\cdot)}(w_1^{p(\cdot)} \cdots w_N^{p(\cdot)})} \leq C \|\mathcal{M}_{p_0}(\vec{f})\|_{L^{p(\cdot)}(w_1^{p(\cdot)} \cdots w_N^{p(\cdot)})}. \tag{3.2}$$

By Hölder's inequality,

$$\begin{aligned} \|\mathcal{M}_{p_0}(\vec{f})\|_{L^{p(\cdot)}(w_1^{p(\cdot)} \cdots w_N^{p(\cdot)})} &= \|\mathcal{M}_{p_0}(\vec{f})w_1 \cdots w_N\|_{L^{p(\cdot)}} \leq \left\| \prod_{i=1}^N \{\mathcal{M}_{p_0}(f_i)w_i\} \right\|_{L^{p(\cdot)}} \\ &\leq C \|\mathcal{M}_{p_0}(f_1)w_1\|_{L^{p_1(\cdot)}} \cdots \|\mathcal{M}_{p_0}(f_N)w_N\|_{L^{p_N(\cdot)}}, \end{aligned} \tag{3.3}$$

where

$$\mathcal{M}_{p_0}(f_i) := \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f_i(y_i)|^{p_0} dy_i \right)^{\frac{1}{p_0}}, \quad i = 1, \dots, N.$$

Since $p_0 < q_j$, we can choose $u_j > 1$ such that $p_0 u_j = q_j$. Thus by Proposition 3.7, we get that

$$\|\mathcal{M}_{p_0}(f)\|_{L^{q_j}(w)} \leq C\|f\|_{L^{q_j}(w)}$$

is valid for all $w \in A_{u_j}$, $f \in L^{q_j}(w)$. Using the boundedness of \mathcal{M} again, we see from Proposition 3.5 that

$$\|\mathcal{M}_{p_0}(f_j)\|_{L^{p_j(\cdot)}(w_j^{p_j(\cdot)})} \leq C\|f_j\|_{L^{p_j(\cdot)}(w_j^{p_j(\cdot)})}, \quad j = 1, \dots, N.$$

It follows from (3.3) that

$$\|\mathcal{M}_{p_0}(\vec{f})\|_{L^{p(\cdot)}(w_1^{p(\cdot)} \cdots w_N^{p(\cdot)})} \leq C\|f_1\|_{L^{p_1(\cdot)}(w_1^{p_1(\cdot)})} \cdots \|f_N\|_{L^{p_N(\cdot)}(w_N^{p_N(\cdot)})}.$$

By (3.2), we obtain the desired conclusion as follows:

$$\|\mathbb{T}_m(\vec{f})\|_{L^{p(\cdot)}(w_1^{p(\cdot)} \cdots w_N^{p(\cdot)})} \leq C\|f_1\|_{L^{p_1(\cdot)}(w_1^{p_1(\cdot)})} \cdots \|f_N\|_{L^{p_N(\cdot)}(w_N^{p_N(\cdot)})}. \quad \square$$

As an application of Theorem 3.2, we now consider the case when weight functions are defined by

$$w_j(x) = [1 + |x - x_0|]^{\beta_\infty^j} \prod_{k=1}^l |x - x_k|^{\beta_k^j}, \quad j = 1, \dots, N, \tag{3.4}$$

where x_k are fixed points in \mathbb{R}^n , $k = 1, \dots, l$.

Corollary 3.9 *Suppose that $Nn/2 < s \leq Nn$, $m \in L^\infty(\mathbb{R}^{Nn})$ and*

$$\sup_{R>0} \|m(R\xi)\chi_{\{1<|\xi|<2\}}\|_{H^s(\mathbb{R}^{Nn})} < \infty.$$

Let the variable exponents $p_1(x), \dots, p_N(x)$ and $p(x)$ satisfy that $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \dots + \frac{1}{p_N(x)} = \frac{1}{p(x)}$, where $1 < p_- \leq p_+ < \infty$, $r_0 := Nn/s < (p_j)_- \leq (p_j)_+ < \infty$, and $p_j \in LH_0(\mathbb{R}^n)$. Suppose that there exists some $R > 0$ and $x_0 \in \mathbb{R}^n$ such that $p_j(x) \equiv (p_j)_\infty = \text{const}$ for $x \in \mathbb{R}^n \setminus B(x_0, R)$, $j = 1, \dots, N$, and that

$$-\frac{n}{p_j(x_k)} < \beta_k^j < \min\left\{\frac{n}{p'_j(x_k)}, \frac{n}{Np'_j(x_k)}\right\}, \quad k = 1, \dots, l,$$

$$-\frac{n}{(p_j)_\infty} < \beta_\infty^j + \sum_{k=1}^l \beta_k^j < \min\left\{\frac{n}{(p_j)'_\infty}, \frac{n}{Np'_j_\infty}\right\}$$

for $j = 1, \dots, N$. Then T_m is bounded from $L^{p_1(\cdot)}(w_1^{p_1(\cdot)}) \times \dots \times L^{p_N(\cdot)}(w_N^{p_N(\cdot)})$ to $L^{p(\cdot)}(w_1^{p(\cdot)} \dots w_N^{p(\cdot)})$.

To prove Corollary 3.9, we need to define a class of weight functions, which is a special case of [3, Definition 2.7].

Definition 3.10 ([3]) *Let $p(\cdot) \in C(\mathbb{R}^n)$ and there exists $R > 0$ and $x_0 \in \mathbb{R}^n$ such that $p(x) \equiv p_\infty = \text{const}$ for all $x \in \mathbb{R}^n \setminus B(x_0, R)$. A weight function w of the form*

$$w = [1 + |x - x_0|]^{\beta_\infty} \prod_{k=1}^l |x - x_k|^{\beta_k}$$

is said to belong to the class $V_{p(\cdot)}(\mathbb{R}^n, \Pi)$ if

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)}, \quad k = 1, \dots, l$$

and

$$-\frac{n}{p_\infty} < \beta_\infty + \sum_{k=1}^l \beta_k < \frac{n}{p'_\infty}.$$

We first give some lemmas that are needed to prove Corollary 3.9.

Lemma 3.11 ([2, Proposition 2.3]) *Given a domain Ω , if $p_+ < \infty$, then $p(\cdot) \in LH_0(\Omega)$ is equivalent to assuming $r(\cdot) = 1/p(\cdot) \in LH_0(\Omega)$.*

Lemma 3.12 ([3, Remark 2.10]) *For every $p_0 \in (1, p_-)$, there hold the implications*

$$\varrho \in V_{p(\cdot)}(\Omega, \Pi) \implies \varrho^{-p_0} \in V_{(\tilde{p})'(\cdot)}(\Omega, \Pi),$$

where $\tilde{p}(\cdot) = \frac{p(\cdot)}{p_0}$.

Lemma 3.13 ([3, Theorem 2.10]) *Suppose that Ω is an unbounded open set of \mathbb{R}^n . Let $p(\cdot) \in LH_0$ satisfy $1 < p_- \leq p_+ < \infty$, and let there exists some $R > 0$ and $x_0 \in \mathbb{R}^n$ such that $p(x) \equiv p_\infty = \text{const}$ for $x \in \Omega \setminus B(x_0, R)$. If $\varrho \in V_{p(\cdot)}(\Omega, \Pi)$, then \mathcal{M} is bounded on the space $L^{p(\cdot)}(\Omega, \varrho^{p(\cdot)})$.*

Then we have the following lemma.

Lemma 3.14 *Let $p(\cdot) \in LH_0$ satisfy $1 < p_- \leq p_+ < \infty$. Suppose that there exists some $R > 0$ and $x_0 \in \mathbb{R}^n$ such that $p(x) \equiv p_\infty = \text{const}$ for $x \in \mathbb{R}^n \setminus B(x_0, R)$. If $\varrho \in V_{p(\cdot)}(\mathbb{R}^n, \Pi)$, then \mathcal{M} is bounded on the space $L^{(\tilde{p})'(\cdot)}(\varrho^{-q_0(\tilde{p})'(\cdot)})$ for all $q_0 \in (1, p_-)$, where $\tilde{p}(\cdot) = \frac{p(\cdot)}{q_0}$.*

Proof If $p(\cdot) \in LH_0$, then $\tilde{p}(\cdot) \in LH_0$. By Lemma 3.11, we have $(\tilde{p})'(\cdot) \in LH_0$. And since $\varrho \in V_{p(\cdot)}(\mathbb{R}^n, \Pi)$, by Lemma 3.12 we know $\varrho^{-q_0} \in V_{(\tilde{p})'(\cdot)}(\mathbb{R}^n, \Pi)$. Then it follows from Lemma 3.13 that \mathcal{M} is bounded on $L^{(\tilde{p})'(\cdot)}(\varrho^{-q_0(\tilde{p})'(\cdot)})$. \square

Now we are ready to prove Corollary 3.9.

Proof of Corollary 3.9 Fix some $1 < q < p_-$. Let $q_j, \tilde{p}(x)$ and $\tilde{p}_j(x)$ be defined as in Theorem 3.2. By the assumption, we have

$$-\frac{n}{p_j(x_k)} < \beta_k^j < \frac{n}{p_j'(x_k)}, \quad k = 1, \dots, l,$$

$$-\frac{n}{(p_j)_\infty} < \beta_\infty^j + \sum_{k=1}^l \beta_k^j < \frac{n}{(p_j)_\infty}.$$

So $w_j \in V_{p_j(\cdot)}(\mathbb{R}^n, \Pi)$. By Lemma 3.14, \mathcal{M} is bounded on $L^{(\tilde{p}_j)'(\cdot)}(w_j^{-q_j(\tilde{p}_j)'(\cdot)})$. Again, by the assumption, we get

$$-\sum_{j=1}^N \frac{n}{p_j(x_k)} < \sum_{j=1}^N \beta_k^j < \frac{n}{p'(x_k)}, \quad k = 1, \dots, l, \tag{3.5}$$

$$-\sum_{j=1}^N \frac{n}{(p_j)_\infty} < \sum_{j=1}^N \beta_\infty^j + \sum_{k=1}^l \sum_{j=1}^N \beta_k^j < \frac{n}{p'_\infty}. \tag{3.6}$$

Note that the left-hand sides of (3.5) and (3.6) are equal to $-\frac{n}{p(x_k)}$ and $-\frac{n}{p_\infty}$, respectively. So $w_1 \cdots w_N \in V_{p(\cdot)}(\mathbb{R}^n, \Pi)$.

By Lemma 3.11, we know $\frac{1}{p_j(\cdot)} \in LH_0$. Therefore, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \cdots + \frac{1}{p_N(\cdot)} \in LH_0$. Thus $p(\cdot) \in LH_0$. Now by Lemma 3.14, \mathcal{M} is bounded on $L^{(\tilde{p})'(\cdot)}((w_1 \cdots w_N)^{-q(\tilde{p})'(\cdot)})$. By Theorem 3.2, there exists some $C > 0$ such that

$$\|T_m(\vec{f})\|_{L^{p(\cdot)}(w_1^{p(\cdot)} \cdots w_N^{p(\cdot)})} \leq C \prod_{j=1}^N \|f_j\|_{L^{p_j(\cdot)}(w_j^{p_j(\cdot)})}. \quad \square$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors completed the paper together. They also read and approved the final manuscript.

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