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First-order nonlinear differential equations with state-dependent impulses

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Abstract

The paper deals with the state-dependent impulsive problem

$$z'(t) = f(t, z(t)) \quad \text{for a.e. } t \in [a,b],$$
$$z(\tau+) - z(\tau) = \mathcal{J}(\tau, z(\tau)), \qquad \gamma(z(\tau)) = \tau,$$
$$\ell(z) = c_0,$$

where $[a,b] \subset \mathbb{R}$, $c_0 \in \mathbb{R}$, f fulfils the Carathéodory conditions on $[a,b] \times \mathbb{R}$, the impulse function $\mathcal{J}$ is continuous on $[a,b] \times \mathbb{R}$, the barrier function γ has a continuous first derivative on some subset of $\mathbb{R}$ and ℓ is a linear bounded functional which is defined on the Banach space of left-continuous regulated functions on $[a,b]$ equipped with the sup-norm. The functional ℓ is represented by means of the Kurzweil-Stieltjes integral and covers all linear boundary conditions for solutions of first-order differential equations subject to state-dependent impulse conditions. Here, sufficient and effective conditions guaranteeing the solvability of the above problem are presented for the first time.

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1 Introduction

The investigation of impulsive differential equations has a long history; see, *e.g.*, the monographs [1–3]. Most papers dealing with impulsive differential equations subject to boundary conditions focus their attention on *impulses at fixed moments*. But this is a very particular case of a more complicated case with *state-dependent impulses*. Boundary value problems with state-dependent impulses, where difficulties with an operator representation appear (*cf.* Remark 6.2), are substantially less developed. We refer to the papers [4–6] and [7] which are devoted to periodic problems, and for problems with other boundary conditions, see [8, 9] or [10–12].

Here, in our paper, we present an approach leading to a new existence principle for impulsive boundary value problems. This approach is applicable to each linear boundary condition which is considered with some first-order differential equation subject to state-dependent impulses. The important step is a proof of a transversality (Remark 2.3 and Lemmas 5.1 and 5.2), which makes possible a construction of a continuous operator (Section 6) whose fixed point leads to a solution of our original impulsive problem (Section 7).

Notation

Let $M \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $[a,b] \subset \mathbb{R}$.

- $\mathbb{C}(M)$ is the set of real functions continuous on M.
- $\mathbb{AC}(M)$ is the set of real functions absolutely continuous on M.
- $\mathbb{L}^1[a,b]$ is the set of real functions Lebesgue integrable on $[a,b]$.
- $\mathbb{L}^\infty[a,b]$ is the set of real functions essentially bounded on $[a,b]$.
- $\mathbb{BV}[a,b]$ is the set of real functions with bounded variation on $[a,b]$.
- $\mathbb{G}_L[a,b]$ is the set of real left-continuous regulated functions on $[a,b]$, that is, $z \in \mathbb{G}_L[a,b]$ if and only if $z\colon [a,b] \to \mathbb{R}$, and for each $\tau_1 \in (a,b]$ and each $\tau_2 \in [a,b)$,

$$z(\tau_1) = z(\tau_1-) = \lim_{t\to\tau_1-} z(t), \qquad z(\tau_2+) = \lim_{t\to\tau_2+} z(t) \in \mathbb{R}. \tag{1.1}$$

- $\mathrm{Car}([a,b] \times M)$ is the set of functions $f\colon [a,b] \times M \to \mathbb{R}$ such that
  (i) $f(\cdot,x)\colon [a,b] \to \mathbb{R}$ is measurable for all $x \in M$,
  (ii) $f(t,\cdot)\colon M \to \mathbb{R}$ is continuous for a.e. $t \in [a,b]$,
  (iii) for each compact set $Q \subset M$, there exists $m_Q \in \mathbb{L}^1[a,b]$ satisfying

$$\big|f(t,x)\big| \le m_Q(t) \quad \text{for a.e. } t \in [a,b] \text{ and each } x \in Q.$$

- The set $\mathbb{L}^\infty[a,b]$ equipped with the norm

$$\|z\|_\infty = \sup\operatorname{ess}\big\{\big|z(t)\big| : t \in [a,b]\big\} \quad \text{for } z \in \mathbb{L}^\infty[a,b] \tag{1.2}$$

  is a Banach space.
- Since $\mathbb{C}[a,b] \subset \mathbb{G}_L[a,b] \subset \mathbb{L}^\infty[a,b]$, we equip the sets $\mathbb{C}[a,b]$ and $\mathbb{G}_L[a,b]$ with the norm $\|\cdot\|_\infty$ and get also Banach spaces (*cf.* [13]). Then (1.2) can be written as

$$\|z\|_\infty = \sup\big\{\big|z(t)\big| : t \in [a,b]\big\} \quad \text{for } z \in \mathbb{G}_L[a,b] \tag{1.3}$$

  and

$$\|z\|_\infty = \max\big\{\big|z(t)\big| : t \in [a,b]\big\} \quad \text{for } z \in \mathbb{C}[a,b]. \tag{1.4}$$

- $\mathbb{W}^{1,\infty}[a,b]$ is the Banach space of functions $z\colon [a,b] \to \mathbb{R}$ such that $z \in \mathbb{AC}[a,b]$ and $z' \in \mathbb{L}^\infty[a,b]$, where the norm $\|\cdot\|_{1,\infty}$ is given by

$$\|z\|_{1,\infty} = \|z\|_\infty + \big\|z'\big\|_\infty \quad \text{for } z \in \mathbb{W}^{1,\infty}[a,b]. \tag{1.5}$$

- χ_A is the characteristic function of a set A, where $A \subset \mathbb{R}$.

2 Formulation of problem

We investigate the solvability of the nonlinear differential equation

$$z'(t) = f\big(t, z(t)\big) \tag{2.1}$$

subject to the state-dependent impulse condition

$$z(\tau+) - z(\tau) = \mathcal{J}\big(\tau, z(\tau)\big), \qquad \gamma\big(z(\tau)\big) = \tau, \tag{2.2}$$

and the general linear boundary condition

$$\ell(z) = c_0. \tag{2.3}$$

Here we assume that

$$\begin{cases} f \in \mathrm{Car}([a,b] \times \mathbb{R}), & \mathcal{J} \in \mathbb{C}([a,b] \times \mathbb{R}), \quad [a,b] \subset \mathbb{R}, \\ K \in (0,\infty), & \gamma \in \mathbb{C}^1[-K,K], \qquad c_0 \in \mathbb{R}, \end{cases} \tag{2.4}$$

and $\ell \colon \mathbb{G}_L[a,b] \to \mathbb{R}$ is a linear bounded functional.

Definition 2.1 A function $z \colon [a,b] \to \mathbb{R}$ is a *solution* of problem (2.1), (2.2) if

- there exists a unique $\tau \in (a,b)$ such that $\gamma(z(\tau)) = \tau$;
- the restrictions $z|_{[a,\tau]}$ and $z|_{(\tau,b]}$ are absolutely continuous;
- $z(\tau+) = z(\tau) + \mathcal{J}(\tau, z(\tau))$;
- z satisfies equation (2.1) for a.e. $t \in [a,b]$.

Definition 2.2 A graph of a function $\gamma \colon [-K,K] \to \mathbb{R}$ is called a *barrier* γ.

Remark 2.3 Let $\mathcal{S}$ be the set of all solutions of problem (2.1), (2.2). According to Definition 2.1, each function $z \in \mathcal{S}$ satisfies a *transversality property*, which means that the graph of z crosses a barrier γ at a unique point $\tau \in (a,b)$, where the impulse $\mathcal{J}$ acts on z. After that (for $t \in (\tau,b]$) the graph of z lies on the right of the barrier γ. This transversality property follows from *transversality conditions* (*cf.* (4.5), (4.6)) and it is proved in Section 5.

Assume that $z_1, z_2 \in \mathcal{S}$ and $z_1 \neq z_2$. Then there exists a unique $\tau_i \in (a,b)$ such that $\gamma(z_i(\tau_i)) = \tau_i$ for $i = 1,2$ and $\tau_1 \neq \tau_2$ can occur. Therefore different functions from $\mathcal{S}$ can have their discontinuities at different points from (a,b). Our aim in this paper is to prove the existence of a solution of problem (2.1), (2.2) satisfying the general linear boundary condition (2.3). To do this, we need a suitable linear space containing $\mathcal{S}$. Due to state-dependent impulses, the Banach space of piece-wise continuous functions on $[a,b]$ with the sup-norm cannot be used here. Therefore we choose the Banach space $\mathbb{G}_L[a,b]$. Clearly, by (1.1), $\mathcal{S} \subset \mathbb{G}_L[a,b]$. The operator ℓ in the general linear boundary condition (2.3) can be written uniquely in the form

$$\ell(z) = kz(a) + {}_{(\mathrm{KS})}\int_a^b \nu(t)\,\mathrm{d}\big[z(t)\big], \tag{2.5}$$

where $k \in \mathbb{R}$, $\nu \in \mathbb{BV}[a,b]$ and ${}_{(\mathrm{KS})}\int_a^b$ is the Kurzweil-Stieltjes integral (*cf.* [14], Theorem 3.8). Representation (2.5) is correct on $\mathcal{S}$, because for each $z \in \mathbb{G}_L[a,b]$ the integral ${}_{(\mathrm{KS})}\int_a^b \nu(t)\,\mathrm{d}[z(t)]$ exists. Its definition and properties can be found in [15] (see Perron-Stieltjes integral based on the work of Kurzweil).

Definition 2.4 A function $z \colon [a,b] \to \mathbb{R}$ is a *solution* of problem (2.1)-(2.3) if z is a solution of problem (2.1), (2.2) and fulfils (2.3).

3 Green's function

For further investigation, we will need a linear homogeneous problem corresponding to problem (2.1)-(2.3). Such problem has the form

$$z'(t) = 0, \tag{3.1}$$

$$\ell(z) = 0, \tag{3.2}$$

because the impulse in (2.2) disappears if $\mathcal{J} \equiv 0$. We will also work with the non-homogeneous equation

$$z'(t) = q(t), \tag{3.3}$$

where $q \in \mathbb{L}^1[a, b]$.

Definition 3.1 A *solution* of problem (3.3), (3.2) is a function $z \in \mathbb{AC}[a, b]$ satisfying equation (3.3) for a.e. $t \in [a, b]$ and fulfilling condition (3.2).

Remark 3.2 If x is a solution of problem (3.3), (3.2), then x belongs to $\mathbb{AC}[a, b]$, and consequently condition (3.2) can be written in the form (*cf.* (2.5))

$$\ell(x) = kx(a) + \int_a^b v(t)x'(t)\,\mathrm{d}t = 0, \tag{3.4}$$

where $k \in \mathbb{R}$, $v \in \mathbb{BV}$ and the Lebesgue integral $\int_a^b v(t)x'(t)\,\mathrm{d}t$ is used.

Definition 3.3 A function $G\colon [a, b] \times [a, b] \to \mathbb{R}$ is the *Green's function* of problem (3.1), (3.2) if

(i) for any $s \in (a, b)$, the restrictions $G(\cdot, s)|_{[a,s)}$, $G(\cdot, s)|_{(s,b]}$ are solutions of equation (3.1) and $G(s+, s) - G(s, s) = 1$, where $G(s, s) = G(s-, s)$;

(ii) $G(t, \cdot) \in \mathbb{BV}[a, b]$ for any $t \in [a, b]$;

(iii) for any $q \in \mathbb{L}^1[a, b]$, the function

$$x(t) = \int_a^b G(t, s)q(s)\,\mathrm{d}s \tag{3.5}$$

fulfils condition (3.4).

Lemma 3.4 *Let* ℓ *be from* (2.5) *with* $k \in \mathbb{R}$ *and* $v \in \mathbb{BV}[a, b]$.

(i) $k \neq 0$ *if and only if there exists the Green's function* G *of problem* (3.1), (3.2) *which has the form*

$$G(t, s) = \begin{cases} -\frac{v(s)}{k} & \text{for } a \le t \le s \le b, \\ 1 - \frac{v(s)}{k} & \text{for } a \le s < t \le b. \end{cases} \tag{3.6}$$

(ii) $k \neq 0$ *if and only if there exists a unique solution* x *of problem* (3.3), (3.4), *which has a form of* (3.5) *with* G *from* (3.6).

Proof Clearly, G given by (3.6) fulfils (i) and (ii) of Definition 3.3 if and only if $k \neq 0$. A general solution of equation (3.3) is $x(t) = c + \int_a^t q(s)\,\mathrm{d}s$, where $c \in \mathbb{R}$. By (3.4),

$$\ell(x) = kc + \int_a^b v(t)q(t)\,\mathrm{d}t = 0.$$

The equation

$$kc = -\int_a^b v(t)q(t)\,\mathrm{d}t$$

has a unique solution c if and only if $k \neq 0$. Then a unique solution x of problem (3.3), (3.4) is written as

$$\begin{aligned} x(t) &= -\frac{1}{k}\int_a^b v(s)q(s)\,\mathrm{d}s + \int_a^t q(s)\,\mathrm{d}s \\ &= \int_a^t \left(1 - \frac{v(s)}{k}\right) q(s)\,\mathrm{d}s + \int_t^b \left(-\frac{v(s)}{k}\right) q(s)\,\mathrm{d}s, \quad t \in [a,b]. \end{aligned}$$

□

Lemma 3.5 *Let G be the Green's function of problem* (3.1), (3.2), *where ℓ is from* (2.5) *and $k \neq 0$. Then, for each $s \in [a,b)$, the function $G(\cdot,s)$ belongs to $\mathbb{G}_L[a,b]$ and*

$$\ell\big(G(\cdot,s)\big) = 0, \quad s \in [a,b). \tag{3.7}$$

Proof Choose $s \in [a,b)$. By (3.6),

$$G(t,s) = \chi_{(s,b]}(t) - \frac{v(s)}{k} \quad \text{for } t \in [a,b].$$

Consequently, the function $G(\cdot,s)$ belongs to $\mathbb{G}_L[a,b]$. This yields that the integral $_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}[G(t,s)]$ exists for each $v \in \mathbb{BV}[a,b]$. Note that since $G(\cdot,s)$ is not continuous on $[a,b]$, formula (3.4) cannot be used for $G(\cdot,s)$ in place of x. Instead, we use the properties of the Kurzweil-Stieltjes integral which justify the following computation

$$\begin{aligned} {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\big[G(t,s)\big] &= {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\left[\chi_{(s,b]}(t) - \frac{v(s)}{k}\right] \\ &= {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\big[\chi_{(s,b]}(t)\big] - {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\left[\frac{v(s)}{k}\right] = v(s). \end{aligned}$$

Hence, by (2.5), we get

$$\ell\big(G(\cdot,s)\big) = kG(a,s) + {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\big[G(t,s)\big] = k\left(\frac{-v(s)}{k}\right) + v(s) = 0.$$

□

Example 3.6 Consider a solution x of problem (3.3), (3.2), where ℓ has a form of the two-point boundary condition

$$\ell(x) = \alpha x(a) + \beta x(b) = 0, \quad \alpha, \beta \in \mathbb{R}. \tag{3.8}$$

We will show that ℓ can be expressed in a form of (3.4). If $\alpha + \beta \neq 0$, then k and ν can be found from the equality

$$\alpha x(a) + \beta x(b) = kx(a) + \int_a^b \nu(t)x'(t)\,\mathrm{d}t.$$

Assuming that $\nu(t) \equiv \nu_0 \in \mathbb{R}$, we get

$$\alpha x(a) + \beta x(b) = kx(a) + \nu_0\bigl(x(b) - x(a)\bigr),$$

and hence $k = \alpha + \beta$, $\nu_0 = \beta$. In addition, if $\alpha + \beta \neq 0$, then (*cf.* (3.6))

$$G(t,s) = \begin{cases} -\frac{\beta}{\alpha+\beta} & \text{for } a \le t \le s \le b, \\ 1 - \frac{\beta}{\alpha+\beta} & \text{for } a \le s < t \le b. \end{cases}$$

Example 3.7 Consider a solution x of problem (3.3), (3.2), where ℓ has a form of the multi-point boundary condition

$$\ell(x) = \sum_{i=0}^{n} \alpha_i x(t_i), \quad \alpha_i \in \mathbb{R}, i = 0, 1, \dots, n, n \in \mathbb{N}. \tag{3.9}$$

Here $a = t_0 < t_1 < \cdots < t_n = b$. If $\sum_{i=0}^{n} \alpha_i \neq 0$, then k and ν of (3.4) can be found from the equality

$$\sum_{i=0}^{n} \alpha_i x(t_i) = kx(a) + \int_a^b \nu(t)x'(t)\,\mathrm{d}t. \tag{3.10}$$

Assume that ν is a piece-wise constant right-continuous function on $[a,b]$, that is,

$$\nu(s) = \nu_i \quad \text{for } s \in [t_i, t_{i+1}), i = 0, \dots, n-2,$$
$$\nu(s) = \nu_{n-1} \quad \text{for } s \in [t_{n-1}, b],$$

where $\nu_i \in \mathbb{R}$, $i = 0, \dots, n-1$. By (3.10), we get

$$\begin{aligned} \sum_{i=0}^{n} \alpha_i x(t_i) &= kx(a) + \sum_{i=0}^{n-1} \nu_i \int_{t_i}^{t_{i+1}} x'(t)\,\mathrm{d}t \\ &= kx(a) + \nu_0\bigl(x(t_1) - x(a)\bigr) + \nu_1\bigl(x(t_2) - x(t_1)\bigr) + \cdots + \nu_{n-1}\bigl(x(b) - x(t_{n-1})\bigr). \end{aligned}$$

Consequently,

$$\nu_i = \sum_{j=i+1}^{n} \alpha_j, \quad i = 0, \dots, n-1, \qquad k = \sum_{j=0}^{n} \alpha_j.$$

To summarize, if $\sum_{j=0}^{n} \alpha_j \neq 0$, then

$$\nu(s) = \sum_{j=i+1}^{n} \alpha_j \quad \text{for } s \in [t_i, t_{i+1}), i = 0, \dots, n-2,$$
$$\nu(s) = \alpha_n \quad \text{for } s \in [t_{n-1}, b],$$

and further (*cf.* (3.6))

$$G(t,s) = \begin{cases} -\frac{\nu(s)}{\sum_{j=0}^{n}\alpha_j} & \text{for } a \le t \le s \le b, \\ 1 - \frac{\nu(s)}{\sum_{j=0}^{n}\alpha_j} & \text{for } a \le s < t \le b. \end{cases}$$

Example 3.8 Consider a solution x of problem (3.3), (3.2), where ℓ has a form of the integral condition

$$\ell(x) = x(b) - \int_a^b h(\xi)x(\xi)\,\mathrm{d}\xi,$$

where $h \in \mathbb{L}^1[a,b]$. If $\int_a^b h(\xi)\,\mathrm{d}\xi \neq 1$, then k and ν of (3.4) can be found from the equality

$$x(b) - \int_a^b h(\xi)x(\xi)\,\mathrm{d}\xi = kx(a) + \int_a^b \nu(t)x'(t)\,\mathrm{d}t. \tag{3.11}$$

Let us put

$$\nu(s) = \int_a^s h(\xi)\,\mathrm{d}\xi + \nu(a).$$

Then

$$\int_a^b \nu(\xi)x'(\xi)\,\mathrm{d}\xi = -\int_a^b h(\xi)x(\xi)\,\mathrm{d}\xi + \nu(b)x(b) - \nu(a)x(a)$$

and (3.11) gives $\nu(a) = k$, $\int_a^b h(\xi)\,\mathrm{d}\xi + k = 1$. Consequently,

$$k = 1 - \int_a^b h(\xi)\,\mathrm{d}\xi, \qquad \nu(s) = 1 - \int_s^b h(\xi)\,\mathrm{d}\xi, \quad s \in [a,b].$$

Similarly, if

$$\ell(x) = x(a) - \int_a^b h(\xi)x(\xi)\,\mathrm{d}\xi,$$

and $\int_a^b h(\xi)\,\mathrm{d}\xi \neq 1$, we derive

$$k = 1 - \int_a^b h(\xi)\,\mathrm{d}\xi, \qquad \nu(s) = -\int_s^b h(\xi)\,\mathrm{d}\xi, \quad s \in [a,b].$$

In both cases, G is written as

$$G(t,s) = \begin{cases} -\frac{\nu(s)}{1-\int_a^b h(\xi)\,\mathrm{d}\xi} & \text{for } a \le t \le s \le b, \\ 1 - \frac{\nu(s)}{1-\int_a^b h(\xi)\,\mathrm{d}\xi} & \text{for } a \le s < t \le b. \end{cases}$$

4 Assumptions

An existence result for problem (2.1)-(2.3) will be proved in the next sections under the basic assumption (2.4) and the following additional assumptions imposed on f, ℓ, $\mathcal{J}$ and γ.

(i) Boundedness of f

$$\begin{cases} \text{There exists } h \in \mathbb{L}^\infty[a,b] \text{ such that} \\ |f(t,x)| \le h(t) \quad \text{for a.e. } t \in [a,b] \text{ and all } x \in \mathbb{R}. \end{cases} \tag{4.1}$$

(ii) Boundedness of $\mathcal{J}$

$$\begin{cases} \text{There exists } J_0 \in (0,\infty) \text{ such that} \\ |\mathcal{J}(t,x)| \le J_0 \quad \text{for } t \in [a,b], x \in \mathbb{R}. \end{cases} \tag{4.2}$$

(iii) Boundedness of γ

$$\begin{cases} \text{There exist } a_1, b_1 \in (a,b) \text{ such that} \\ a_1 \le \gamma(x) \le b_1 \quad \text{for } x \in [-K,K]. \end{cases} \tag{4.3}$$

(iv) Properties of ℓ

$$\ell \text{ fulfils (2.5)}, \quad \text{where } k \in \mathbb{R}, k \ne 0, v \in \mathbb{BV}[a,b] \cap \mathbb{C}[a_1,b_1]. \tag{4.4}$$

(v) Transversality conditions

$$\left|\gamma'(x)\right| < \frac{1}{\|h\|_\infty} \quad \text{for } x \in [-K,K], \tag{4.5}$$

$$\begin{cases} \text{either} \quad \mathcal{J}(t,x) \ge 0, \quad \gamma'(x) \le 0 \quad \text{for } t \in [a_1,b_1], x \in [-K,K], \\ \text{or} \quad \mathcal{J}(t,x) \le 0, \quad \gamma'(x) \ge 0 \quad \text{for } t \in [a_1,b_1], x \in [-K,K], \end{cases} \tag{4.6}$$

where h is from (4.1) and a_1, b_1 are from (4.3).

(vi) $\mathbb{L}^\infty$-continuity of f

$$\begin{cases} \text{For any } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that} \\ |x-y| < \delta \Rightarrow \|f(\cdot,x) - f(\cdot,y)\|_\infty < \varepsilon, \quad x,y \in [-K,K]. \end{cases} \tag{4.7}$$

Remark 4.1

(a) Boundedness of f and $\mathcal{J}$ can be replaced by more general conditions, for example, growth or sign ones, if the method of *a priori* estimates is used. See, *e.g.*, [16, 17].

(b) Continuity of v on $[a_1,b_1]$ is necessary for the construction of a continuous operator in Section 6. Note that then we need $t_1,\dots,t_{n-1} \notin [a_1,b_1]$ in Example 3.7.

(c) Clearly, if f is continuous on $[a,b] \times [-K,K]$, then f fulfils (4.7).

(d) Let there exist $p \in \mathbb{N}$, $\psi \in \mathbb{L}^\infty[a,b]$ and $g_i \in \mathbb{C}(\mathbb{R})$, $i = 1,\dots,p$, such that

$$\left|f(t,x) - f(t,y)\right| \le \psi(t) \sum_{i=1}^{p} \left|g_i(x) - g_i(y)\right|$$

for a.e. $t \in [a,b]$ and all $x, y \in [-K,K]$. Then f fulfils (4.7). An example of such a function f is

$$f(t,x) = \sum_{i=1}^{p} f_i(t) g_i(x) + f_0(t),$$

where $f_j \in \mathbb{L}^\infty[a,b]$, $j = 0,1,\dots,p$, $g_i \in \mathbb{C}[-K,K]$, $i = 1,\dots,p$.

5 Transversality

Consider $K \in (0,\infty)$, $h \in \mathbb{L}^\infty[a,b]$ and define a set $\mathcal{B}$ by

$$\mathcal{B} = \left\{ u \in \mathbb{W}^{1,\infty}[a,b] \colon \|u\|_\infty < K, \left\|u'\right\|_\infty < \|h\|_\infty \right\}. \tag{5.1}$$

The following two lemmas for functions from $\mathcal{B}$ are the modifications of lemmas in [10] and provide the transversality (*cf.* Remark 2.3) which will be essential for operator constructions in Section 6.

Lemma 5.1 *Let γ satisfy* (2.4), (4.3) *and* (4.5). *Then, for each $u \in \overline{\mathcal{B}}$, there exists a unique $\tau \in (a,b)$ such that*

$$\tau = \gamma\bigl(u(\tau)\bigr). \tag{5.2}$$

In addition $\tau \in [a_1, b_1]$.

Proof Let us take an arbitrary $u \in \overline{\mathcal{B}}$ and denote

$$\sigma(t) = \gamma\bigl(u(t)\bigr) - t, \quad t \in [a,b].$$

Then, by (2.4) and (5.1), we see that $\sigma \in \mathbb{AC}[a,b]$ and

$$\sigma'(t) = \gamma'\bigl(u(t)\bigr)u'(t) - 1 \quad \text{for a.e. } t \in [a,b].$$

Since $u(a), u(b) \in [-K,K]$, condition (4.3) gives

$$\sigma(a) = \gamma\bigl(u(a)\bigr) - a \geq a_1 - a > 0,$$
$$\sigma(b) = \gamma\bigl(u(b)\bigr) - b \leq b_1 - b < 0.$$

Consequently, there exists at least one zero of σ in (a,b). Let $\tau \in (a,b)$ be a zero of σ. By virtue of (4.5) and (5.1), we get, for $t \in [a,b]$, $t \neq \tau$,

$$\begin{aligned} \operatorname{sign}(t-\tau)\sigma(t) &= \operatorname{sign}(t-\tau)\int_\tau^t \sigma'(s)\,\mathrm{d}s = \operatorname{sign}(t-\tau)\int_\tau^t \bigl(\gamma'\bigl(u(s)\bigr)u'(s) - 1\bigr)\,\mathrm{d}s \\ &\leq \operatorname{sign}(t-\tau)\int_\tau^t \bigl(\bigl|\gamma'\bigl(u(s)\bigr)\bigr| \cdot \left\|u'\right\|_\infty - 1\bigr)\,\mathrm{d}s \\ &< \operatorname{sign}(t-\tau)\int_\tau^t \left(\frac{1}{\|h\|_\infty}\|h\|_\infty - 1\right)\mathrm{d}s = 0. \end{aligned}$$

That is,

$$\sigma > 0 \quad \text{on } [a, \tau), \qquad \sigma < 0 \quad \text{on } (\tau, b]. \tag{5.3}$$

Hence τ is a unique zero of σ, and (4.3) yields $\tau \in [a_1, b_1]$. □

Due to Lemma 5.1, we can define a functional $\mathcal{P}: \overline{\mathcal{B}} \to [a_1, b_1]$ by

$$\mathcal{P}u = \tau, \tag{5.4}$$

where τ fulfils (5.2).

Lemma 5.2 *Let γ satisfy* (2.4), (4.3) *and* (4.5). *Then the functional $\mathcal{P}$ is continuous.*

Proof Let us choose a sequence $\{u_n\}_{n=1}^{\infty} \subset \overline{\mathcal{B}}$ which is convergent in $\mathbb{W}^{1,\infty}[a, b]$. Then

$$u_n \in \mathbb{W}^{1,\infty}[a, b], \qquad \|u_n\|_\infty \le K, \qquad \|u_n'\|_\infty \le \|h\|_\infty, \quad n \in \mathbb{N}, \tag{5.5}$$

and there exists $u \in \mathbb{W}^{1,\infty}[a, b]$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{1,\infty} = 0. \tag{5.6}$$

So, by virtue of (1.5) and (5.5),

$$\|u\|_\infty \le \lim_{n\to\infty} \|u - u_n\|_\infty + \lim_{n\to\infty} \|u_n\|_\infty \le K,$$
$$\|u'\|_\infty \le \lim_{n\to\infty} \|u' - u_n'\|_\infty + \lim_{n\to\infty} \|u_n'\|_\infty \le \|h\|_\infty.$$

We see that $u \in \overline{\mathcal{B}}$. For $n \in \mathbb{N}$, define

$$\sigma_n(t) = \gamma\bigl(u_n(t)\bigr) - t, \qquad \sigma(t) = \gamma\bigl(u(t)\bigr) - t, \quad t \in [a, b].$$

By Lemma 5.1,

$$\sigma_n(\tau_n) = 0, \qquad \sigma(\tau) = 0, \quad \text{where } \tau_n = \mathcal{P}u_n, \tau = \mathcal{P}u, n \in \mathbb{N}. \tag{5.7}$$

We need to prove that

$$\lim_{n\to\infty} \tau_n = \tau. \tag{5.8}$$

Conditions (2.4), (1.5) and (5.6) yield

$$\lim_{n\to\infty} \sigma_n = \sigma \quad \text{in } \mathbb{C}[a, b]. \tag{5.9}$$

Let us take an arbitrary $\varepsilon > 0$. By (5.3) and (5.9) we can find $\xi \in (\tau - \varepsilon, \tau)$, $\eta \in (\tau, \tau + \varepsilon)$ and $n_0 \in \mathbb{N}$ such that $\sigma_n(\xi) > 0$, $\sigma_n(\eta) < 0$ for each $n \ge n_0$. By Lemma 5.1 and the continuity of σ_n, we see that $\tau_n \in (\xi, \eta) \subset (\tau - \varepsilon, \tau + \varepsilon)$ for $n \ge n_0$, and (5.8) follows. □

6 Fixed point problem

In this section we assume that

$$\text{conditions (2.4), (4.1)-(4.7) are fulfilled,} \tag{6.1}$$

and we construct a fixed point problem whose solvability leads to a solution of problem (2.1)-(2.3). To this aim, having the set $\mathcal{B}$ from (5.1), we define a set Ω by

$$\Omega = \mathcal{B} \times \mathcal{B} \subset \mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b], \tag{6.2}$$

and for $u = (u_1, u_2) \in \Omega$, we define a function $f_u\colon [a,b] \to \mathbb{R}$ as follows. We set, for a.e. $t \in [a,b]$,

$$f_u(t) = \begin{cases} f(t, u_1(t)) & \text{if } t \in [a, \mathcal{P}u_1], \\ f(t, u_2(t)) & \text{if } t \in (\mathcal{P}u_1, b], \end{cases} \tag{6.3}$$

where $\mathcal{P}$ is defined by (5.4) and the point $\mathcal{P}u_1 \in [a_1, b_1]$ is uniquely determined due to Lemma 5.1. By (4.1)

$$f_u \in \mathbb{L}^\infty[a,b], \qquad \|f_u\|_\infty \le \|h\|_\infty. \tag{6.4}$$

Now, we can define an operator $\mathcal{F}\colon \overline{\Omega} \to \mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b]$ by $\mathcal{F}(u_1, u_2) = (x_1, x_2)$, where

$$x_1(t) = \begin{cases} \int_a^b G(t,s) f_u(s)\,\mathrm{d}s + \frac{c_0}{k} \\ \quad - \frac{\nu(\mathcal{P}u_1)}{k} \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) & \text{if } t \le \mathcal{P}_{u_1}, \\ \int_a^b G(t,s) f(s, u_1(s))\,\mathrm{d}s + \frac{c_0}{k} \\ \quad - \frac{\nu(\mathcal{P}u_1)}{k} \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) + \mathcal{A}_1 u & \text{if } t > \mathcal{P}_{u_1}, \end{cases} \tag{6.5}$$

$$x_2(t) = \begin{cases} \int_a^b G(t,s) f(s, u_2(s))\,\mathrm{d}s + \frac{c_0}{k} \\ \quad + (1 - \frac{\nu(\mathcal{P}u_1)}{k}) \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) + \mathcal{A}_2 u & \text{if } t \le \mathcal{P}_{u_1}, \\ \int_a^b G(t,s) f_u(s)\,\mathrm{d}s + \frac{c_0}{k} \\ \quad + (1 - \frac{\nu(\mathcal{P}u_1)}{k}) \mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) & \text{if } t > \mathcal{P}_{u_1}. \end{cases} \tag{6.6}$$

Here the functionals $\mathcal{A}_1\colon \overline{\Omega} \to \mathbb{R}$ and $\mathcal{A}_2\colon \overline{\Omega} \to \mathbb{R}$ are defined such that the functions x_1 and x_2 are continuous at the point $\mathcal{P}u_1$. Therefore

$$\begin{cases} \mathcal{A}_1 u = \int_a^b G(\mathcal{P}u_1, s) f_u(s)\,\mathrm{d}s - \int_a^b G(\mathcal{P}u_1, s) f(s, u_1(s))\,\mathrm{d}s, \\ \mathcal{A}_2 u = \int_a^b G(\mathcal{P}u_1, s) f_u(s)\,\mathrm{d}s - \int_a^b G(\mathcal{P}u_1, s) f(s, u_2(s))\,\mathrm{d}s. \end{cases} \tag{6.7}$$

Differentiating (6.5) and using (3.6) and (6.3), we get

$$x_i'(t) = f\bigl(t, u_i(t)\bigr) \quad \text{for a.e. } t \in [a,b], i = 1, 2. \tag{6.8}$$

This together with (4.1) yields

$$\|x_i'\|_\infty \le \|h\|_\infty, \quad i = 1, 2. \tag{6.9}$$

Since $v \in \mathbb{BV}[a,b]$ (*cf.* (4.4)), we see that (6.4)-(6.6), (3.6), (4.1) and (4.2) give

$$\begin{aligned} \|x_i\|_\infty \le{}& 3\left(1 + \frac{\|v\|_\infty}{|k|}\right)(b-a)\|h\|_\infty + \frac{|c_0|}{|k|} \\ &+ \left(1 + \frac{\|v\|_\infty}{|k|}\right)J_0, \quad i = 1, 2. \end{aligned} \tag{6.10}$$

Due to (6.8)-(6.10), we see that $x_i \in \mathbb{W}^{1,\infty}[a,b]$, $i = 1, 2$, and the operator $\mathcal{F}$ is defined well.

Lemma 6.1 *Assume that* (6.1) *holds and that* Ω *and* $\mathcal{F}$ *are given by* (6.2) *and* (6.5), (6.6), *respectively. Then the operator* $\mathcal{F}$ *is compact on* $\overline{\Omega}$.

Proof

Step 1. We show that $\mathcal{F}$ is continuous on $\overline{\Omega}$. Choose a sequence

$$\{u^{[n]}\}_{n=1}^\infty = \{(u_1^{[n]}, u_2^{[n]})\}_{n=1}^\infty \subset \overline{\Omega}$$

which is convergent in $\mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b]$, that is, (*cf.* (1.5)) there exists $u = (u_1, u_2) \in \overline{\Omega}$ such that

$$\lim_{n\to\infty} \|u_1^{[n]} - u_1\|_{1,\infty} = 0, \qquad \lim_{n\to\infty} \|u_2^{[n]} - u_2\|_{1,\infty} = 0. \tag{6.11}$$

Lemma 5.1 and Lemma 5.2 yield

$$\mathcal{P}u_1, \mathcal{P}u_1^{[n]} \in [a_1, b_1], \quad n \in \mathbb{N}, \qquad \lim_{n\to\infty} \mathcal{P}u_1^{[n]} = \mathcal{P}u_1, \tag{6.12}$$

where $\mathcal{P}$ is defined by (5.4). Denote

$$x = (x_1, x_2) = \mathcal{F}(u_1, u_2), \qquad x^{[n]} = (x_1^{[n]}, x_2^{[n]}) = \mathcal{F}(u_1^{[n]}, u_2^{[n]}), \quad n \in \mathbb{N}. \tag{6.13}$$

We will prove that

$$\lim_{n\to\infty} \|x_1^{[n]} - x_1\|_{1,\infty} = 0, \qquad \lim_{n\to\infty} \|x_2^{[n]} - x_2\|_{1,\infty} = 0. \tag{6.14}$$

By (4.7), (6.8), (6.11) and (6.13),

$$\lim_{n\to\infty} \|(x_i^{[n]})' - x_i'\|_\infty = \lim_{n\to\infty} \|f(\cdot, u_i^{[n]}(\cdot)) - f(\cdot, u_i(\cdot))\|_\infty = 0, \quad i = 1, 2. \tag{6.15}$$

Using (4.1), we get

$$\lim_{n\to\infty} \left| \int_\tau^{\tau_n} |f(s, u_1^{[n]}(s)) - f(s, u_2^{[n]}(s))| \, ds \right| \le 2 \lim_{n\to\infty} \left| \int_\tau^{\tau_n} h(s) \, ds \right| = 0. \tag{6.16}$$

Since

$$\begin{aligned}\int_a^b \big(f_{u^{[n]}}(s) - f_u(s)\big)\,\mathrm{d}s &= \int_a^\tau \big(f\big(s, u_1^{[n]}(s)\big) - f\big(s, u_1(s)\big)\big)\,\mathrm{d}s \\ &\quad + \int_\tau^b \big(f\big(s, u_2^{[n]}(s)\big) - f\big(s, u_2(s)\big)\big)\,\mathrm{d}s \\ &\quad + \int_\tau^{\tau_n} \big(f\big(s, u_1^{[n]}(s)\big) - f\big(s, u_2^{[n]}(s)\big)\big)\,\mathrm{d}s,\end{aligned}$$

the Lebesgue dominated convergence theorem and (6.16) give

$$\lim_{n\to\infty} \int_a^b \big|f_{u^{[n]}}(s) - f_u(s)\big|\,\mathrm{d}s = 0. \tag{6.17}$$

Using (6.13) and (6.5), we get

$$\begin{aligned}\big|x_1^{[n]}(a) - x_1(a)\big| &\le \int_a^b \big|G(a,s)\big| \cdot \big|f_{u^{[n]}}(s) - f_u(s)\big|\,\mathrm{d}s \\ &\quad + \left|\frac{\nu(\mathcal{P}u_1^{[n]})}{k}\mathcal{J}\big(\mathcal{P}u_1^{[n]}, u_1^{[n]}\big(\mathcal{P}u_1^{[n]}\big)\big) - \frac{\nu(\mathcal{P}u_1)}{k}\mathcal{J}\big(\mathcal{P}u_1, u_1(\mathcal{P}u_1)\big)\right|.\end{aligned}$$

The continuity and boundedness of $\mathcal{P}$, $\mathcal{J}$ and ν (*cf.* Lemma 5.2, (2.4), (4.2), (4.4) and (6.12)) imply

$$\begin{aligned}&\lim_{n\to\infty}\left|\frac{\nu(\mathcal{P}u_1^{[n]})}{k}\mathcal{J}\big(\mathcal{P}u_1^{[n]}, u_1^{[n]}\big(\mathcal{P}u_1^{[n]}\big)\big) - \frac{\nu(\mathcal{P}u_1)}{k}\mathcal{J}\big(\mathcal{P}u_1, u_1(\mathcal{P}u_1)\big)\right| \\ &\quad\le \frac{\|\nu\|_\infty}{|k|}\lim_{n\to\infty}\big|\mathcal{J}\big(\mathcal{P}u_1^{[n]}, u_1^{[n]}\big(\mathcal{P}u_1^{[n]}\big)\big) - \mathcal{J}\big(\mathcal{P}u_1, u_1(\mathcal{P}u_1)\big)\big| \\ &\qquad + \frac{J_0}{|k|}\lim_{n\to\infty}\big|\nu\big(\mathcal{P}u_1^{[n]}\big) - \nu(\mathcal{P}u_1)\big| = 0,\end{aligned}$$

wherefrom, by the boundedness of G and (6.17),

$$\lim_{n\to\infty}\big|x_1^{[n]}(a) - x_1(a)\big| = 0. \tag{6.18}$$

Using (6.13) and integrating (6.8), we get

$$x_1(t) = x_1(a) + \int_a^t f\big(s, u_1(s)\big)\,\mathrm{d}s, \qquad x_1^{[n]}(t) = x_1^{[n]}(a) + \int_a^t f\big(s, u_1^{[n]}(s)\big)\,\mathrm{d}s,$$

and, due to (6.15) and (6.18), we arrive at

$$\lim_{n\to\infty}\big\|x_1^{[n]} - x_1\big\|_\infty = 0. \tag{6.19}$$

Similarly, we derive

$$\lim_{n\to\infty}\big|x_2^{[n]}(b) - x_2(b)\big| = 0, \qquad \lim_{n\to\infty}\big\|x_2^{[n]} - x_2\big\|_\infty = 0. \tag{6.20}$$

Properties (6.15), (6.19) and (6.20) yield (6.14).

Step 2. We show that the set $\mathcal{F}(\overline{\Omega})$ is relatively compact in $\mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b]$. Choose an arbitrary sequence

$$\{(x_1^{[n]}, x_2^{[n]})\}_{n=1}^{\infty} \subset \mathcal{F}(\overline{\Omega}) \subset \mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b].$$

We need to prove that there exists a convergent subsequence. Clearly, there exists $\{(u_1^{[n]}, u_2^{[n]})\}_{n=1}^{\infty} \subset \overline{\Omega}$ such that

$$\mathcal{F}(u_1^{[n]}, u_2^{[n]}) = (x_1^{[n]}, x_2^{[n]}), \quad n \in \mathbb{N}.$$

Choose $i \in \{1,2\}$. By (5.1) and (6.2), it holds

$$\{u_i^{[n]}\}_{n=1}^{\infty} \subset \mathbb{W}^{1,\infty}[a,b], \qquad \|u_i^{[n]}\|_{\infty} \le K,$$
$$|u_i^{[n]}(t_1) - u_i^{[n]}(t_2)| = \left|\int_{t_1}^{t_2} (u_i^{[n]})'(s)\,\mathrm{d}s\right| \le \|h\|_{\infty}|t_1 - t_2|$$

for $t_1, t_2 \in [a,b]$, $n \in \mathbb{N}$. Therefore, the Arzelà-Ascoli theorem yields that there exists a subsequence

$$\{(u_1^{[m]}, u_2^{[m]})\}_{m=1}^{\infty} \subset \{(u_1^{[n]}, u_2^{[n]})\}_{n=1}^{\infty}$$

which converges in $\mathbb{C}[a,b] \times \mathbb{C}[a,b]$. Consequently, for each $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for each $m \in \mathbb{N}$,

$$m \ge m_0 \quad \Rightarrow \quad \|u_i^{[m_0]} - u_i^{[m]}\|_{\infty} < \varepsilon, \quad i = 1,2.$$

Similarly as in Step 1, we prove (*cf.* (6.15), (6.19), (6.20))

$$\|(x_i^{[m_0]})' - (x_i^{[m]})'\|_{\infty} < \varepsilon, \qquad \|x_i^{[m_0]} - x_i^{[m]}\|_{\infty} < \varepsilon, \quad i = 1,2,$$

which gives by (1.5) that $\{(x_1^{[m]}, x_2^{[m]})\}_{m=1}^{\infty}$ is convergent in $\mathbb{W}^{1,\infty}[a,b] \times \mathbb{W}^{1,\infty}[a,b]$. □

Remark 6.2 If there exists $\tau_0 \in [a_1, b_1]$ such that $\gamma(x) = \tau_0$ for $x \in [-K,K]$, then problem (2.1)-(2.3) has an impulse at fixed time τ_0 and a standard operator $\mathcal{F}_0$, acting on the space of piece-wise continuous functions on $[a,b]$ and having the form

$$(\mathcal{F}_0 z)(t) = \int_a^b G(t,s) f(s, z(s))\,\mathrm{d}s + \frac{c_0}{k} + G(t,\tau_0)\mathcal{J}(\tau_0, z(\tau_0)), \quad t \in [a,b], \tag{6.21}$$

can be used instead of the operator $\mathcal{F}$ from (6.5), (6.6). But this is not possible if γ is not constant on $[-K,K]$. The reason is that then an impulse is realized at a state-dependent point $\tau = \gamma(z(\tau))$, and $\mathcal{F}_0$ with τ instead of τ_0 should be investigated on the space $\mathbb{G}_L[a,b]$. But if we write a state-dependent τ instead of a fixed τ_0 in (6.21), $\mathcal{F}_0$ loses its continuity on $\mathbb{G}_L[a,b]$, which we show in the next example.

Example 6.3 Let $a = 0$, $b = 2$ and ℓ be from (2.5) with $k \in \mathbb{R}$, $k \neq 0$ and $v \in \mathbb{C}[0,2]$. Consider the functions

$$u(t) = 1, \qquad u_n(t) = 1 - \frac{1}{n}, \quad t \in [0,2], n \in \mathbb{N}.$$

Clearly, $u_n \to u$ uniformly on $[0,2]$ and hence

$$\lim_{n\to\infty} \|u_n - u\|_\infty = 0.$$

For $n \in \mathbb{N}$, denote $x_n = \mathcal{F}_0 u_n$ and $x = \mathcal{F}_0 u$. Assume that the barrier γ is given by the linear function $\gamma(x) = x$ on $\mathbb{R}$ and the impulse function $\mathcal{J}(t,x) = 1$ for $t \in [0,2]$, $x \in \mathbb{R}$. Then

$$\tau = \gamma\bigl(u(\tau)\bigr) = u(\tau) = 1,$$
$$\tau_n = \gamma\bigl(u_n(\tau_n)\bigr) = u_n(\tau_n) = 1 - \frac{1}{n}, \quad n \in \mathbb{N},$$

and, according to (6.21), we have for $t \in [0,2]$

$$x_n(t) = \int_0^2 G(t,s) f\left(s, 1 - \frac{1}{n}\right) \mathrm{d}s + \frac{c_0}{k} + G\left(t, 1 - \frac{1}{n}\right), \quad n \in \mathbb{N},$$
$$x(t) = \int_0^2 G(t,s) f(s,1)\, \mathrm{d}s + \frac{c_0}{k} + G(t,1).$$

Consequently,

$$\begin{aligned}\lim_{n\to\infty} \bigl(x_n(1) - x(1)\bigr) &= \lim_{n\to\infty} \int_0^2 G(1,s)\left(f\left(s, 1-\frac{1}{n}\right) - f(s,1)\right) \mathrm{d}s \\ &\quad + \lim_{n\to\infty} \left(G\left(1, 1-\frac{1}{n}\right) - G(1,1)\right) \\ &= 1 - \frac{v(1)}{k} - \left(-\frac{v(1)}{k}\right) = 1\end{aligned}$$

due to (3.6). Hence $x_n(1) \nrightarrow x(1)$ and we have also $\|x_n - x\|_\infty \nrightarrow 0$, and $\mathcal{F}_0$ is not continuous on $\mathbb{G}_L[0,2]$.

Lemma 6.1 results in the following theorem.

Theorem 6.4 *Assume that* (6.1) *holds and that the set* Ω *is given by* (6.2), *where*

$$K \geq \left(1 + \frac{\|v\|_\infty}{|k|}\right)\bigl(3(b-a)\|h\|_\infty + J_0\bigr) + \frac{|c_0|}{|k|}. \tag{6.22}$$

Further, let the operator $\mathcal{F}$ *be given by* (6.5), (6.6). *Then* $\mathcal{F}$ *has a fixed point in* $\overline{\Omega}$.

Proof By Lemma 6.1, $\mathcal{F}$ is compact on $\overline{\Omega}$. Due to (5.1), (6.2), (6.5), (6.6), (6.10) and (6.22),

$$\mathcal{F}(\overline{\Omega}) \subset \overline{\Omega}.$$

Therefore, the Schauder fixed point theorem yields a fixed point of $\mathcal{F}$ in $\overline{\Omega}$. $\square$

7 Main result

The main result, which is contained in Theorem 7.1, guarantees the solvability of problem (2.1)-(2.3) provided the data functions f, $\mathcal{J}$ and γ are bounded (*cf.* (4.1)-(4.3)). As it is mentioned in Remark 4.1, Theorem 7.1 serves as an existence principle which, in combination with the method of *a priori* estimates, can lead to more general existence results for unbounded f and $\mathcal{J}$ and concrete boundary conditions.

Theorem 7.1 *Assume that* (6.1) *and* (6.22) *hold. Then there exists a solution z of problem* (2.1)-(2.3) *such that*

$$\|z\|_\infty \le K. \tag{7.1}$$

Proof By Theorem 6.4, there exists $u = (u_1, u_2) \in \overline{\Omega}$ which is a fixed point of the operator $\mathcal{F}$ defined in (6.5) and (6.6). This means that

$$u_1(t) = \begin{cases} \int_a^b G(t,s) f_u(s)\,\mathrm{d}s + \frac{c_0}{k} \\ \quad - \frac{\nu(\mathcal{P}u_1)}{k}\mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) & \text{if } t \le \mathcal{P}_{u_1}, \\ \int_a^b G(t,s) f(s, u_1(s))\,\mathrm{d}s + \frac{c_0}{k} \\ \quad - \frac{\nu(\mathcal{P}u_1)}{k}\mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) + \mathcal{A}_1 u & \text{if } t > \mathcal{P}_{u_1}, \end{cases} \tag{7.2}$$

$$u_2(t) = \begin{cases} \int_a^b G(t,s) f(s, u_2(s))\,\mathrm{d}s + \frac{c_0}{k} \\ \quad + (1 - \frac{\nu(\mathcal{P}u_1)}{k})\mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) + \mathcal{A}_2 u & \text{if } t \le \mathcal{P}_{u_1}, \\ \int_a^b G(t,s) f_u(s)\,\mathrm{d}s + \frac{c_0}{k} \\ \quad + (1 - \frac{\nu(\mathcal{P}u_1)}{k})\mathcal{J}(\mathcal{P}u_1, u_1(\mathcal{P}u_1)) & \text{if } t > \mathcal{P}_{u_1}, \end{cases} \tag{7.3}$$

where $G, \mathcal{P}, f_u, \mathcal{A}_1, \mathcal{A}_2$ are given by (3.6), (5.4), (6.3), (6.7), respectively. Recall that $\mathcal{P}u_1$ is a unique point in (a,b) satisfying

$$\mathcal{P}u_1 = \tau_1 \in [a_1, b_1], \quad \text{where } \tau_1 = \gamma\bigl(u_1(\tau_1)\bigr). \tag{7.4}$$

For $t \in [a,b]$, define a function z by

$$z(t) = \begin{cases} u_1(t) & \text{if } t \in [a, \tau_1], \\ u_2(t) & \text{if } t \in (\tau_1, b]. \end{cases} \tag{7.5}$$

Differentiating (7.2), (7.3) and using (3.6) and (6.3), we get $u_i'(t) = f(t, u_i(t))$ for a.e. $t \in [a,b]$, $i = 1, 2$, and consequently

$$z'(t) = f\bigl(t, z(t)\bigr) \quad \text{for a.e. } t \in [a,b].$$

By virtue of (7.2)-(7.5), we have

$$z(\tau_1+) - z(\tau_1) = u_2(\tau_1) - u_1(\tau_1) = \mathcal{J}\bigl(\tau_1, u_1(\tau_1)\bigr) = \mathcal{J}\bigl(\tau_1, z(\tau_1)\bigr). \tag{7.6}$$

Let us show that τ_1 is a unique solution of the equation

$$t = \gamma\bigl(z(t)\bigr) \tag{7.7}$$

in $[a,b]$. According to (7.4) and (7.5), it suffices to prove

$$t \neq \gamma\big(u_2(t)\big), \quad t \in (\tau_1, b]. \tag{7.8}$$

Since $(u_1, u_2) \in \overline{\Omega}$, we have (*cf.* (5.1) and (6.2))

$$\|u_i\|_\infty \le K, \qquad \|u_i'\|_\infty \le \|h\|_\infty, \quad i = 1, 2.$$

Assume that the first condition in (4.6) is fulfilled. Then $\mathcal{J}(\tau_1, x) \ge 0$, $\gamma'(x) \le 0$ for $x \in [-K, K]$. Put

$$\sigma(t) = \gamma\big(u_2(t)\big) - t, \quad t \in [a,b].$$

By (7.6), $u_2(\tau_1) - u_1(\tau_1) = \mathcal{J}(\tau_1, u_1(\tau_1)) \ge 0$, and since γ is non-increasing, we have

$$\sigma(\tau_1) = \gamma\big(u_2(\tau_1)\big) - \tau_1 \le \gamma\big(u_1(\tau_1)\big) - \tau_1 = 0$$

due to (7.4). Using (4.5), we derive for $t \in (\tau_1, b]$

$$\begin{aligned} \sigma(t) &= \int_{\tau_1}^t \big(\gamma'\big(u_2(s)\big)u_2'(s) - 1\big)\,\mathrm{d}s \le \int_{\tau_1}^t \big(\big|\gamma'\big(u_2(s)\big)\big| \cdot \|u_2'\|_\infty - 1\big)\,\mathrm{d}s \\ &< \int_{\tau_1}^t \left(\frac{1}{\|h\|_\infty}\|h\|_\infty - 1\right)\mathrm{d}s = 0. \end{aligned}$$

So, (7.8) is valid. If the second condition in (4.6) is fulfilled, we use the dual arguments.

Finally, let us check that $\ell(z) = c_0$. By (7.2)-(7.6) and (3.6), we have

$$z(t) = \int_a^b G(t,s)f\big(s, z(s)\big)\,\mathrm{d}s + \frac{c_0}{k} + G(t,\tau_1)\mathcal{J}\big(\tau_1, z(\tau_1)\big). \tag{7.9}$$

Put

$$x(t) = \int_a^b G(t,s)f\big(s, z(s)\big)\,\mathrm{d}s. \tag{7.10}$$

Then, according to (iii) of Definition 3.3 and Remark 3.2, we get $\ell(x) = 0$. Further, using (3.7) from Lemma 3.5, we arrive at $\ell(G(\cdot, \tau_1)) = 0$. Consequently, due to (2.5), (7.9) and (7.10), $\ell(z)$ results in

$$\begin{aligned} \ell(z) &= \ell(x) + \ell\left(\frac{c_0}{k}\right) + \ell\big(G(\cdot,\tau_1)\big)\mathcal{J}\big(\tau_1, z(\tau_1)\big) \\ &= \ell\left(\frac{c_0}{k}\right) = k\frac{c_0}{k} + {}_{(\mathrm{KS})}\int_a^b v(t)\,\mathrm{d}\left[\frac{c_0}{k}\right] = c_0. \end{aligned}$$

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the manuscript and read and approved the final manuscript.

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