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SHORT PAPER

Invariant Decoupling and Blocking Zeros of Positive Linear Electrical Circuits with Zero Transfer Matrices

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Abstract The invariant zeros, input-decoupling and output-decoupling zeros and blocking zeros of positive electrical circuits with zero transfer matrices are addressed. It is shown that the positive electrical circuits have no invariant zeros, input–output-decoupling zeros and blocking zeros and also the list of eigenvalues of the system matrix is the sum of the list of input-decoupling zeros and the list of output-decoupling zeros.

Keywords Positive electrical circuit · Decoupling zero · Blocking zero · Invariant zero

1 Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in the monographs [2,9]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The notion of controllability and observability and the decomposition of linear systems have been introduced by Kalman [17,18]. These notions are the basic concepts of the modern control theory [1,8,16,19–21]. They have been also extended to positive linear systems [2,9]. The positive circuits and their reachability have been investigated in [10,12] and controllability and observability of electrical circuits in [4,15].

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The reachability of linear systems is closely related to the controllability of the systems. Specially for positive linear systems, the conditions for the controllability are much stronger than for the reachability [9,15]. Tests for the reachability and controllability of standard and positive linear systems are given in [9,15]. The positivity and reachability of fractional continuous-time linear systems and electrical circuits have been addressed in [2,7,10,12,15] and the decoupling zeros of positive discrete-time linear systems and positive electrical circuits in [5,6]. Standard and positive electrical circuits with zero transfer matrices have been investigated in [14].

The positive linear systems consisting of n subsystems with different fractional orders have been analyzed in [11]. The constructability and observability of standard and positive electrical circuits have been addressed in [3].

In this paper the invariant zeros, decoupling zeros and blocking zeros of positive linear electrical circuits will be investigated. The paper is organized as follows. In Sect. 2 basic definitions and theorems concerning invariant decoupling and blocking zeros of linear systems are recalled the positivity; reachability and observability of linear systems are addressed in Sect. 3. The positive linear electrical circuits with zero transfer matrices are presented in Sect. 4. The invariant, decoupling and blocking zeros of positive electrical circuits with zero transfer matrices are analyzed in Sect. 5. Concluding remarks are given in Sect. 6.

The following notation will be used: $\Re$ is the set of real numbers, $\Re^{n \times m}$ represents the set of $n \times m$ real matrices, $\Re_+^{n \times m}$ denotes the set of $n \times m$ matrices with nonnegative and $\Re_+^n = \Re_+^{n \times 1}$, $\mathbf{C}$ is the field of complex numbers, M_n stand for the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n is the $n \times n$ identity matrix.

2 Invariant, Decoupling and Blocking Zeros of Linear Systems

Consider the linear system

$$\dot{x} = Ax + Bu, \tag{2.1a}$$
$$y = Cx, \tag{2.1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$.

The system matrix of linear system (2.1) is defined by

$$S(s) = \begin{bmatrix} I_n s - A & B \\ C & 0 \end{bmatrix} \in \Re^{(n+p) \times (n+m)}[s]. \tag{2.2}$$

Let the matrix

$$S_S(s) = \begin{bmatrix} \text{diag}\,[p_1(s) & \cdots & p_r(s)] & 0 \\ & 0 & & 0 \end{bmatrix} \in \Re^{(n+p) \times (n+m)}[s] \tag{2.3}$$

be the canonical Smith form of system matrix (2.2), where $p_1(s), \ldots, p_r(s)$ are the invariant polynomials satisfying the condition $p_i(s)|p_{i+1}(s)$ for $i = 1, \ldots, r - 1$ and $r = \text{rank } S(s)$.

The polynomial $p(s) = p_1(s) \ldots p_r(s)$ is called the invariant zero polynomial of system (2.1).

Definition 2.1 The zero of the polynomial $p(s)$ is called the invariant zero of system (2.1).

Theorem 2.1 [8] *If $m = p$ and matrix* (2.2) *has full rank then*

$$p(s) = \det S_S(s) = c \det S_S(s), \tag{2.4}$$

where $c = \det L(s) \det R(s)$ since $L(s)$ and $R(s)$ are unimodular matrices row and column operations on matrix (2.2).

Theorem 2.2 *If $m = p$ then*

$$p(s) = \det \begin{bmatrix} I_n s - A & -B \\ C & 0 \end{bmatrix} = \det [I_n s - A] \det T(s), \tag{2.5}$$

where

$$T(s) = C[I_n s - A]^{-1} B. \tag{2.6}$$

Proof It is easy to see that

$$\begin{bmatrix} I_n & 0 \\ -C[I_n s - A]^{-1} & I_p \end{bmatrix} \begin{bmatrix} I_n s - A & -B \\ C & 0 \end{bmatrix} = \begin{bmatrix} I_n s - A & -B \\ 0 & T(s) \end{bmatrix}$$

and

$$\det \left\{ \begin{bmatrix} I_n & 0 \\ -C[I_n s - A]^{-1} & I_p \end{bmatrix} \begin{bmatrix} I_n s - A & -B \\ C & 0 \end{bmatrix} \right\} = \det \begin{bmatrix} I_n s - A & -B \\ 0 & T(s) \end{bmatrix}$$

since

$$\det \begin{bmatrix} I_n & 0 \\ -C[I_n s - A]^{-1} & I_p \end{bmatrix} = 1.$$

$\square$

Consider the submatrix

$$S_1(s) = [I_n s - A \quad B] \tag{2.7}$$

of system matrix (2.2).

Definition 2.2 [8] A number $z \in \mathbf{C}$ for which

$$\text{rank } [I_n z - A \quad B] < n \tag{2.8}$$

are called the input-decoupling (i.d.) zero of system (2.1).

Let the matrix

$$S_{1S}(s) = \left[\text{diag} \left[\bar{p}_1(s) \ldots \bar{p}_n(s) \right] \quad 0 \right] \in \mathfrak{R}^{n \times (n+m)}[s] \tag{2.9}$$

be the canonical Smith form of matrix (2.7).

Note that $z \in \mathbf{C}$ is an i.d. zero of system (2.1) if and only if z is a zero of the polynomial

$$\bar{p}(s) = \bar{p}_1(s) \ldots \bar{p}_n(s). \tag{2.10}$$

Therefore, the i.d. zeros of the system are the zeros of polynomial (2.10). The system has no i.d. zeros if and only if $\bar{p}(s) = 1$, i.e., the matrix $S_1(s)$ has the canonical Smith form $[I_n \quad 0]$. The i.d. zeros represent unreachable modes of system (2.1).

The number of i.d. zeros n_1 of system (2.1) is equal to the rank defect of its controllability matrix, i.e.,

$$n_1 = n - \text{rank } R_n, \tag{2.11}$$

where

$$R_n = \left[B \quad AB \quad \cdots \quad A^{n-1}B \right]. \tag{2.12}$$

Theorem 2.3 [8] *The state vector x of system* (2.1) *for any input u(t) and zero initial state x(0) = 0 is independent of the i.d. zeros of the system.*

Consider the submatrix

$$S_2(s) = \begin{bmatrix} I_n s - A \\ C \end{bmatrix} \tag{2.13}$$

of system matrix (2.2).

Definition 2.3 [8] A number $z \in \mathbf{C}$ for which

$$\text{rank} \begin{bmatrix} I_n z - A \\ C \end{bmatrix} < n \tag{2.14}$$

are called the output-decoupling (o.d.) zero of system (2.1).

Let the matrix

$$S_{2S}(s) = \begin{bmatrix} \text{diag} \left[\hat{p}_1(s) \quad \cdots \quad \hat{p}_n(s) \right] \\ 0 \end{bmatrix} \in \mathfrak{R}^{(n+p) \times n}[s] \tag{2.15}$$

be the canonical Smith form of matrix (2.13).

Note that $z \in \mathbf{C}$ is an o.d. zero of system (2.1) if and only if z is a zero of the polynomial

$$\hat{p}(s) = \hat{p}_1(s) \ldots \hat{p}_n(s). \tag{2.16}$$

Therefore, the o.d. zeros of the system are the zeros of polynomial (2.16). The system has no o.d. zeros if and only if $\hat{p}(s) = 1$, i.e., the matrix $S_2(s)$ has the canonical Smith form $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$. The o.d. zeros represent unobservable modes of system (2.1).

The number of o.d. zeros n_2 of system (2.1) is equal to the rank defect of its observability matrix, i.e.,

$$n_2 = n - \text{rank } O_n, \tag{2.17}$$

where

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{2.18}$$

Theorem 2.4 [8] *The output y of system* (2.1) *for any input $u'(t) = Bu(t)$ and zero initial condition $x(0) = 0$ is independent of the o.d. zeros of the system.*

Definition 2.4 [8] A number $z \in \mathbf{C}$ for which both conditions (2.8) and (2.14) are satisfied are called the input–output-decoupling (i.o.d.) zero of system (2.1).

Therefore, $z \in \mathbf{C}$ is an i.o.d. zero if and only if it is both an i.d. zero and an o.d. zero of the system.

The number of i.o.d. zeros n_{io} of system (2.1) is equal to

$$n_{io} = n - \text{rank } R_n - \text{rank } O_n + \text{rank } O_n R_n. \tag{2.19}$$

Definition 2.5 [8] A number $z \in \mathbf{C}$ is called a blocking zero of system (2.1) if

$$C[I_n z - A]_{ad} B = 0, \tag{2.20}$$

where $[I_n z - A]_{ad}$ is the adjoint matrix.

Theorem 2.5 [8] *A number $z \in \mathbf{C}$ is an uncontrollable and/or unobservable mode of the system if and only if z is a blocking zero of the system.*

3 Positivity, Reachability and Observability of Electrical Circuits

Consider linear electrical circuits composed of resistors, capacitors, coils and voltage (current) sources. As the state variables [the components of the state vector $x(t)$] we choose the voltages on the capacitors and the currents in the coils. Using Kirchhoff's laws we may describe the linear circuits in transient states by the state equations

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3.1a}$$
$$y(t) = Cx(t), \tag{3.1b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$.

It is assumed that the initial conditions are zero since the system matrix $S(s)$ and the transfer matrix $T(s)$ are defined for zero initial conditions.

Definition 3.1 [9,13,15] Linear electrical circuit (3.1) is called (internally) positive if the state vector $x(t) \in \Re_+^n$ and output vector $y(t) \in \Re_+^p$, $t \geq 0$ for any initial conditions $x(0) \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 3.1 [2,9,15] *The linear electrical circuit is positive if and only if*

$$A \in M_n, \quad B \in \Re_+^{n \times m}, \quad C \in \Re_+^{p \times n}. \tag{3.2}$$

Definition 3.2 [2,9,15] Positive electrical circuit (3.1) is called reachable in time $t \in [0, t_f]$ if for every given final state $x_f \in \Re_+^n$ there exists an input $u(t) \in \Re_+^m, t \in [0, t_f]$ which steers the state of the electrical circuit from zero initial conditions $x(0) = 0$ to the final state x_f.

Definition 3.3 [9] A matrix $A \in \Re_+^{n \times n}$ is called monomial if each its row and each its column contain only one positive entry and the remaining entries are zero.

Theorem 3.2 [9,13,15] *Positive electrical circuit (3.1) is reachable if and only if the reachability matrix*

$$R_n = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \Re_+^{n \times nm} \tag{3.3}$$

contains a monomial matrix.

Definition 3.4 [9,15] Positive electrical circuit (3.1) is called observable in time $t \in [0, t_f]$ if knowing its input $u(t) \in \Re_+^m$ and its input $y(t) \in \Re_+^p$ for $t \in [0, t_f]$ it is possible to find its unique initial condition $x_0 = x(0) \in \Re_+^n$.

Theorem 3.3 [9,15] *The positive electrical circuit (3.1) is observable in time $t \in [0, t_f]$ if and only if the matrix $A \in M_n$ is diagonal and the matrix*

$$O_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \Re_+^{pn \times n} \tag{3.4}$$

contains a monomial matrix.

The transfer matrix of positive electrical circuit (3.1) is given by

$$T(s) = C[I_n s - A]^{-1} B \in \Re^{p \times m}(s), \tag{3.5}$$

where $\Re^{p \times m}(s)$ is the set of $p \times m$ rational matrices in s.

Theorem 3.4 *If for electrical circuit (3.1)*

$$T(s) = C[I_n s - A]^{-1} B = 0 \tag{3.6}$$

then

$$O_n R_n = 0 \tag{3.7}$$

where O_n and R_n are defined by (3.4) and (3.3), respectively.

Proof Note that (3.6) holds if and only if

$$L^{-1}[T(s)] = Ce^{At}B = 0 \tag{3.8}$$

where L^{-1} denotes the inverse Laplace transform.

Substitution of

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \tag{3.9}$$

into (3.8) yields

$$\sum_{k=0}^{\infty} \frac{C(At)^k B}{k!} = 0 \text{ and } CA^k B = 0 \text{ for } k = 0, 1, \ldots. \tag{3.10}$$

Using (3.3), (3.4) and (3.10) we obtain

$$O_n R_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

$$= \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \\ CAB & CA^2B & \cdots & CA^nB \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-1}B & CA^nB & \cdots & CA^{2(n-1)}B \end{bmatrix} = 0. \tag{3.11}$$

This completes the proof. □

Theorem 3.5 *Let for standard electrical circuit* (3.1) *condition* (3.6) *be satisfied. Then*

(1) *the pair* (A, B) *is unreachable if* $C \neq 0$,
(2) *the pair* (A, C) *is unobservable if* $B \neq 0$.

Proof From (3.11) we have

$$C \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0 \tag{3.12}$$

and

$$\text{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} < n \tag{3.13}$$

if $C \neq 0$. Therefore, the pair (A, B) is unreachable.

Similarly, from (3.11) we have

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} B = 0 \tag{3.14}$$

and

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} < n \tag{3.15}$$

if $B \neq 0$. Therefore, the pair (A,C) is unobservable. □

4 Linear Electrical Circuits with Zero Transfer Matrices

Following [14] the positive linear electrical circuits with zero transfer matrices will be presented.

Example 4.1 Consider the electrical circuit shown in Fig. 1 with given resistances R_1, R_2, R_3, R_4, inductance L, capacitance C and voltage source e.

Using Kirchhoff's laws we may write the equations

$$e = Ri_L + L\frac{di_L}{dt}, \quad R = R_1 + \frac{R_2 + R_3}{2},$$

$$R_4 C \frac{du_C}{dt} + u_C = 0. \tag{4.1}$$
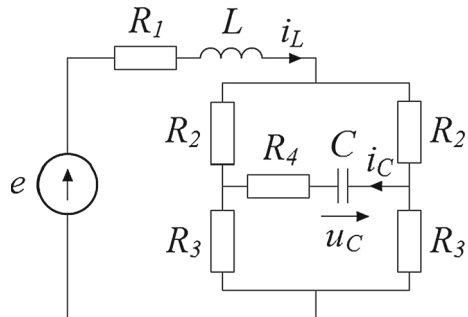
As the output y we choose

$$y = u_C. \tag{4.2}$$

Equations (4.1) and (4.2) can be rewritten in the form

$$\frac{d}{dt}\begin{bmatrix} u_C \\ i_L \end{bmatrix} = A_1 \begin{bmatrix} u_C \\ i_L \end{bmatrix} + B_1 e, \quad y = C_1 \begin{bmatrix} u_C \\ i_L \end{bmatrix}, \tag{4.3a}$$
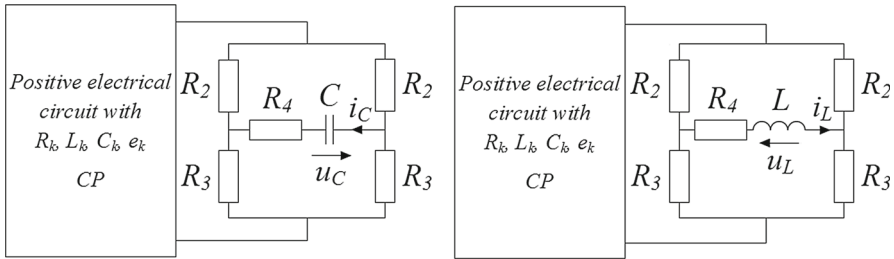
where

$$A_1 = \begin{bmatrix} -\frac{1}{R_4 C} & 0 \\ 0 & -\frac{R}{L} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}, \quad C_1 = [1 \ \ 0]. \tag{4.3b}$$

Fig. 1 Electrical circuit of Example 4.1

Fig. 2 Positive electrical circuit with zero transfer matrix

By Theorem 3.1 the electrical circuit is positive for all values of R_1, R_2, R_3, R_4, L and C since from (4.3b) we have

$$A_1 \in M_2, \quad B_1 \in \Re_+^2, \quad C_1 \in \Re_+^{1\times 2}. \tag{4.4}$$

The transfer function of the electrical circuit is

$$T(s) = C_1 \left[I_2 s - A_1 \right]^{-1} B_1 = [1 \quad 0] \begin{bmatrix} s + \frac{1}{R_4 C} & 0 \\ 0 & s + \frac{R}{L} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} = 0 \tag{4.5}$$

for all values of R_1, R_2, R_3, R_4, L and C.

Note that

$$\det[I_n s - A_1] = \begin{vmatrix} s + \frac{1}{R_4 C} & 0 \\ 0 & s + \frac{R}{L} \end{vmatrix}$$

$$= \left(s + \frac{1}{R_4 C} \right) \left(s + \frac{R}{L} \right), \quad s_1 = -\frac{1}{R_4 C}, \quad s_2 = -\frac{R}{L} \tag{4.6}$$

and the electrical circuit is stable for all nonzero values of R_1, R_2, R_3, R_4, L and C.

By Theorems 3.2 and 3.3 the positive electrical circuit with (4.3b) is unreachable and unobservable since the matrices

$$R_2 = [B_1 \quad A_1 B_1] = \begin{bmatrix} 0 & 0 \\ \frac{1}{L} & -\frac{R}{L^2} \end{bmatrix}, \quad O_2 = \begin{bmatrix} C_1 \\ C_1 A_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{R_4 C} & 0 \end{bmatrix} \tag{4.7}$$

have only one monomial column and one monomial row, respectively. From (4.7) we have

$$O_2 R_2 = \begin{bmatrix} 1 & 0 \\ -\frac{1}{R_4 C} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{1}{L} & -\frac{R}{L^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.8}$$

The outputs of the positive electrical circuits shown in Fig. 1 are zero for all values of the resistances, inductances, capacitances and all inputs.

Note that the positive electrical circuits shown in Fig. 1 are particular case of the general positive electrical circuit shown in Fig. 2 with any positive part with resistances R_k, inductances L_k, capacitances C_k and voltage sources e_k.
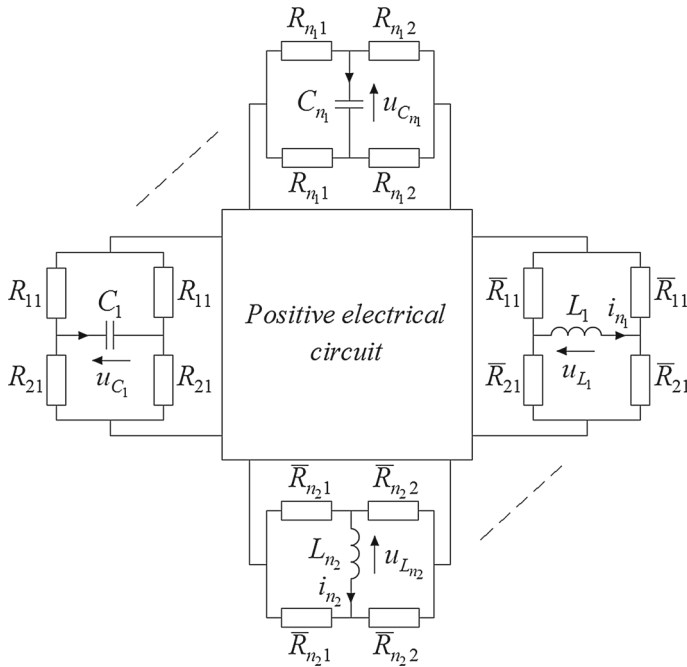
Fig. 3 Positive electrical circuit with zero transfer matrix

If the common part (CP) of the electrical circuit is not a positive electrical circuit, then the whole class of electrical circuits is not positive one with zero transfer function.

From the considerations we have the following theorem.

Theorem 4.1 *The class of electrical circuits shown in Fig. 2 is positive electrical circuits with zero transfer functions if and only if their common parts are positive electrical circuits.*

In general case the class of positive electrical circuits with zero transfer matrix can be presented in the form shown in Fig. 3 [14].

5 Invariant, Decoupling and Blocking Zeros of Positive Electrical Circuits with Zero Transfer Matrices

The following operations on polynomial matrices are called elementary row (column) operations [8]:

(1) Multiplication of the i-th row (column) by scalar (number) c. This operation will be denoted by $L(i \times c)(R(i \times c))$.
(2) Addition to the i-th row (column) of the j-th row (column) multiplied by any polynomial $b(s)$. This operation will be denoted by $L(i+j \times b(s))(R(i+j \times b(s)))$.
(3) Intercharge of the i-th and j-th rows (columns). This operations will be denoted by $L(i, j)(R(i, j))$.

Applying the elementary row and column operations to identity matrices we obtain unimodular matrices. The elementary row (column) operations are equivalent to pre-multiplication (postmultiplication) of the matrix by suitable unimodular matrices. The elementary row and column operations do not change the rank of the matrices.

First we shall consider the invariant decoupling and blocking zeros of the example of positive electrical circuits with zero transfer matrices presented in Sect. 4.

Example 5.1 (Continuation of Example 4.1) The positive electrical circuit described by (4.3) has no invariant zeros since its system matrix

$$\begin{bmatrix} I_2 s - A_1 & B_1 \\ C_1 & 0 \end{bmatrix} = \begin{bmatrix} s + \frac{1}{R_4 C} & 0 & 0 \\ 0 & s + \frac{R}{L} & \frac{1}{L} \\ 1 & 0 & 0 \end{bmatrix} \quad (5.1)$$

using the elementary operations $L\left[1 + 3 \times \left(-s - \frac{1}{R_4 C}\right)\right]$, $R[3 \times L]$, $R[2 + 3 \times \left(-s - \frac{R}{L}\right)]$, $L[1, 3]$, $R[2, 3]$ can be reduced to the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.2)$$

and the invariant zero polynomial $p(s) = 1$.

Note that

$$\text{rank} \begin{bmatrix} I_2 s - A_1 & B_1 \end{bmatrix} = \text{rank} \begin{bmatrix} s + \frac{1}{R_4 C} & 0 & 0 \\ 0 & s + \frac{R}{L} & \frac{1}{L} \end{bmatrix} = 1 \text{ for } s = -\frac{1}{R_4 C} \quad (5.3)$$

and the positive electrical circuit has one input-decoupling zero $z_{i1} = -\frac{1}{R_4 C}$. The electrical circuit has also one output-decoupling zero $z_{o1} = -\frac{R}{L}$ since

$$\text{rank} \begin{bmatrix} I_2 s - A_1 \\ C_1 \end{bmatrix} = \text{rank} \begin{bmatrix} s + \frac{1}{R_4 C} & 0 \\ 0 & s + \frac{R}{L} \\ 1 & 0 \end{bmatrix} = 1 \text{ for } s = -\frac{R}{L}. \quad (5.4)$$

By Definition 3.4 the electrical circuit has no input–output-decoupling zeros since the input-decoupling zero $\left(z_{i1} = -\frac{1}{R_4 C}\right)$ and the output-decoupling zero $\left(z_{o1} = -\frac{R}{L}\right)$ are different. The list of eigenvalues $\left\{-\frac{1}{R_4 C}, -\frac{R}{L}\right\}$ of the matrix A_1 is the sum of the list of input-decoupling zeros $\left\{-\frac{1}{R_4 C}\right\}$ and of the list of output-decoupling zeros $\left\{-\frac{R}{L}\right\}$. Note that the electrical circuit has no blocking zeros since

$$C_1 \left[I_2 s - A_1\right]_{ad} B_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s + \frac{R}{L} & 0 \\ 0 & s + \frac{1}{R_4 C} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} = 0 \quad (5.5)$$

for all values of R and L.

The list of eigenvalues of the matrix A is the sum of the list of the input-decoupling zeros and of the list of output-decoupling zeros of the positive electrical circuit.

In general case it is assumed that matrices of positive electrical circuit (3.1) satisfy the following assumption

$$A \in M_n, \quad B \in \Re_+^{n \times m}, \quad \text{rank } B = m, \quad C \in \Re_+^{p \times n}, \quad \text{rank } C = p. \tag{5.6}$$

Theorem 5.1 *The invariant zero polynomial of positive electrical circuit (3.1) with zero transfer, matrix satisfying (5.6) is equal to one, i.e., $p(s) = 1$ and the electrical circuit has no invariant zeros.*

Proof Proof is based on the elementary row and column operations. $\qquad \square$

Theorem 5.2 *Positive electrical circuit (3.1) with zero transfer matrix, satisfying (5.6), has*

$$n_i = n - \text{rank} \begin{bmatrix} I_n s - A & B \end{bmatrix} \text{ forall } s \in \mathbf{C} \tag{5.7}$$

input-decoupling zeros and

$$n_o = n - \text{rank} \begin{bmatrix} I_n s - A \\ C \end{bmatrix} \text{ forall } s \in \mathbf{C} \tag{5.8}$$

output-decoupling zeros and has not input–output-decoupling zeros ($n_i + n_o = n$).

The list of eigenvalues of the matrix A is the sum of the list of input-decoupling zeros and of the list of output-decoupling zeros of the electrical circuit.

Proof From Definition 2.2 and (2.8) it follows that the number of input-decoupling zeros of the positive electrical circuit is given by (5.7). Similarly, from Definition 2.3 and (2.14) it follows that the number of output-decoupling zeros is given by (5.8). From assumption that the transfer matrix is zero by Theorem 3.4 we have (3.11) and from (2.19) $n_{io} = 0$, i.e., $n_i + n_o = n$. $\qquad \square$

Theorem 5.3 *Positive electrical circuit (3.1) with zero transfer matrix, satisfying (5.6), has no blocking zeros.*

Proof Proof follows immediately from (2.20) and the assumption that the transfer matrix of the positive electrical circuit is zero. $\qquad \square$

6 Concluding Remarks

Using polynomial matrices and the elementary row and column operations it has been shown that the positive electrical circuits with zero transfer matrices:

(1) have no invariant zeros, input–output-decoupling zeros and blocking zeros (Theorems 5.1, 5.2),

(2) the list of eigenvalues of the system matrix A is the sum of the list of input-decoupling zeros and the list of output-decoupling zeros of the electrical circuit (Theorem 5.2).

The considerations have been illustrated by examples of positive electrical circuits with zero transfer matrices. The considerations can be extended to fractional positive linear electrical circuits and to descriptor linear electrical circuits.

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