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Viscosity iteration method in CAT(0) spaces without the nice projection property

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Abstract

A complete CAT(0) space X is said to have the nice projection property (property $\mathcal{N}$ for short) if its metric projection onto a geodesic segment preserves points on each geodesic segment, that is, for any geodesic segment L in X and $x, y \in X$, $m \in [x, y]$ implies $P_L(m) \in [P_L(x), P_L(y)]$, where P_L denotes the metric projection from X onto L. In this paper, we prove a strong convergence theorem of a two-step viscosity iteration method for nonexpansive mappings in CAT(0) spaces without the condition on the property $\mathcal{N}$. Our result gives an affirmative answer to a problem raised by Piatek (Numer. Funct. Anal. Optim. 34:1245-1264, 2013).

Keywords: viscosity iteration method; fixed point; strong convergence; the nice projection property; CAT(0) space

1 Introduction

A mapping T on a metric space (X, ρ) is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$\rho\big(T(x), T(y)\big) \le k\rho(x, y) \quad \text{for all } x, y \in X. \tag{1}$$

If (1) is valid when $k = 1$, then T is called *nonexpansive*. A point $x \in X$ is called a *fixed point* of T if $x = T(x)$. We shall denote by $\mathrm{Fix}(T)$ the set of all fixed points of T.

One of the powerful iteration methods for finding fixed points of nonexpansive mappings was given by Moudafi [1]. More precisely, let C be a nonempty, closed, and convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, the following scheme is known as the *viscosity iteration method*:

$$\begin{aligned} &x_1 = u \in C \text{ arbitrarily chosen,} \\ &x_{n+1} = \frac{\alpha_n}{1+\alpha_n} f(x_n) + \frac{1}{1+\alpha_n} T(x_n), \end{aligned} \tag{2}$$

where $f : C \to C$ is a contraction and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying (i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and (iii) $\lim_{n\to\infty}(1/\alpha_n - 1/\alpha_{n+1}) = 0$. In [1], the author proved that the sequence $\{x_n\}$ defined by (2) converges strongly to a fixed point z of T. The point z also satisfies the following *variational inequality*:

$$\big\langle f(z) - z, z - x\big\rangle \ge 0, \quad x \in \mathrm{Fix}(T).$$

The first extension of Moudafi's result to the so-called CAT(0) space was proved by Shi and Chen [2]. They assumed that the space (X, ρ) must satisfy the property $\mathcal{P}$, *i.e.*, for $x, u, y_1, y_2 \in X$, one has

$$\rho(x, m_1)\rho(x, y_1) \leq \rho(x, m_2)\rho(x, y_2) + \rho(x, u)\rho(y_1, y_2),$$

where m_1 and m_2 are the unique nearest points of u on the segments $[x, y_1]$ and $[x, y_2]$, respectively. By using the concept of quasi-linearization introduced by Berg and Nikolaev [3], Wangkeeree and Preechasilp [4] could omit the property $\mathcal{P}$ from Shi and Chen's result as the following theorem.

Theorem A ([4], Theorem 3.4) *Let C be a nonempty, closed, and convex subset of a complete CAT(0) space X, $T : C \to C$ be a nonexpansive mapping with* $\operatorname{Fix}(T) \neq \emptyset$, *and $f : C \to C$ be a contraction with $k \in [0, 1)$. For $x_1 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(x_n), \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies the conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (iii) *either* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$. *Then $\{x_n\}$ converges strongly to $\tilde{x}$ such that $\tilde{x} = P_{\operatorname{Fix}(T)}(f(\tilde{x}))$ which is equivalent to the variational inequality*:

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \operatorname{Fix}(T).$$

Among other things, by using the geometric properties of CAT(0) spaces, Piatek [5] proved the strong convergence of a two-step viscosity iteration method as the following result.

Theorem B ([5], Theorem 4.3) *Let X be a complete CAT(0) space with the property $\mathcal{N}$. Let $T : X \to X$ be a nonexpansive mapping with* $\operatorname{Fix}(T) \neq \emptyset$ *and $f : X \to X$ be a contraction with $k \in [0, \frac{1}{2})$. Then there is a unique point $q \in \operatorname{Fix}(T)$ such that $q = P_{\operatorname{Fix}(T)}(f(q))$. Moreover, for each $u \in X$ and for each couple of sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$ satisfying* (i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* (iii) $0 < \liminf_n \beta_n \leq \limsup_n \beta_n < 1$, *the viscosity iterative sequence defined by $x_1 = u$,*

$$y_n = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(x_n),$$
$$x_{n+1} = \beta_n x_n \oplus (1 - \beta_n)y_n, \quad \forall n \geq 1,$$

converges to q.

In [5], the author provided an example of a CAT(0) space lacking property $\mathcal{N}$ and also raised the following open problem.

Problem Can we omit the property $\mathcal{N}$ in Theorem B?

In this paper, by combining the ideas of [4] and [5] intensively, we can omit the property $\mathcal{N}$ from Theorem B. This gives a complete solution to the problem mentioned above.

2 Preliminaries

Let $[0,l]$ be a closed interval in $\mathbb{R}$ and x, y be two points in a metric space (X,ρ). A *geodesic* joining x to y is a map $\xi:[0,l]\to X$ such that $\xi(0)=x$, $\xi(l)=y$, and $\rho(\xi(s),\xi(t))=|s-t|$ for all $s,t\in[0,l]$. The image of ξ is called a *geodesic segment* joining x and y which when unique is denoted by $[x,y]$. The space (X,ρ) is said to be a *geodesic space* if every two points in X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x,y\in X$. A subset C of X is said to be *convex* if every pair of points $x,y\in C$ can be joined by a geodesic in X and the image of every such geodesic is contained in C.

A *geodesic triangle* $\triangle(p,q,r)$ in a geodesic space (X,ρ) consists of three points p, q, r in X and a choice of three geodesic segments $[p,q]$, $[q,r]$, $[r,p]$ joining them. A *comparison triangle* for the geodesic triangle $\triangle(p,q,r)$ in X is a triangle $\overline{\triangle}(\bar{p},\bar{q},\bar{r})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{p},\bar{q})=\rho(p,q)$, $d_{\mathbb{R}^2}(\bar{q},\bar{r})=\rho(q,r)$, and $d_{\mathbb{R}^2}(\bar{r},\bar{p})=\rho(r,p)$. A point $\bar{u}\in[\bar{p},\bar{q}]$ is called a *comparison point* for $u\in[p,q]$ if $\rho(p,u)=d_{\mathbb{R}^2}(\bar{p},\bar{u})$. Comparison points on $[\bar{q},\bar{r}]$ and $[\bar{r},\bar{p}]$ are defined in the same way.

Definition 2.1 A geodesic triangle $\triangle(p,q,r)$ in (X,ρ) is said to satisfy the *CAT(0) inequality* if for any $u,v\in\triangle(p,q,r)$ and for their comparison points $\bar{u},\bar{v}\in\overline{\triangle}(\bar{p},\bar{q},\bar{r})$, one has

$$\rho(u,v)\le d_{\mathbb{R}^2}(\bar{u},\bar{v}).$$

A geodesic space X is said to be a *CAT(0) space* if all of its geodesic triangles satisfy the CAT(0) inequality. For other equivalent definitions and basic properties of CAT(0) spaces, we refer the reader to standard texts, such as [6, 7]. It is well known that every CAT(0) space is uniquely geodesic. Notice also that pre-Hilbert spaces, $\mathbb{R}$-trees, Euclidean buildings are examples of CAT(0) spaces (see [6, 8]). Let C be a nonempty, closed, and convex subset of a complete CAT(0) space (X,ρ). It follows from Proposition 2.4 of [6] that for each $x\in X$, there exists a unique point $x_0\in C$ such that

$$\rho(x,x_0)=\inf\bigl\{\rho(x,y):y\in C\bigr\}.$$

In this case, x_0 is called the *unique nearest point* of x in C. The *metric projection* of X onto C is the mapping $P_C:X\to C$ defined by

$$P_C(x):=\text{the unique nearest point of } x \text{ in } C.$$

Definition 2.2 A complete CAT(0) space X is said to have the *nice projection property* [9] if for any geodesic segment L in X, it is the case that $P_L(m)\in[P_L(x),P_L(y)]$ for any $x,y\in X$ and $m\in[x,y]$.

Let (X,ρ) be a CAT(0) space. For each $x,y\in X$ and $t\in[0,1]$, there exists a unique point $z\in[x,y]$ such that

$$\rho(x,z)=(1-t)\rho(x,y)\quad\text{and}\quad\rho(y,z)=t\rho(x,y).\tag{3}$$

We shall denote by $tx\oplus(1-t)y$ the unique point z satisfying (3). Now, we collect some elementary facts about CAT(0) spaces which will be used in the proof of our main theorem.

Lemma 2.3 ([10], Lemma 2.4) *Let (X,ρ) be a CAT(0) space. Then*

$$\rho\big(tx \oplus (1-t)y, z\big) \le t\rho(x,z) + (1-t)\rho(y,z)$$

for all $x,y,z \in X$ and $t \in [0,1]$.

Lemma 2.4 ([10], Lemma 2.5) *Let (X,ρ) be a CAT(0) space. Then*

$$\rho^2\big(tx \oplus (1-t)y, z\big) \le t\rho^2(x,z) + (1-t)\rho^2(y,z) - t(1-t)\rho^2(x,y)$$

for all $x,y,z \in X$ and $t \in [0,1]$.

Lemma 2.5 ([11], Lemma 3) *Let (X,ρ) be a CAT(0) space. Then*

$$\rho\big(tx \oplus (1-t)z, ty \oplus (1-t)z\big) \le t\rho(x,y)$$

for all $x,y,z \in X$ and $t \in [0,1]$.

Lemma 2.6 (*cf.* [12, 13]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a CAT(0) space (X,ρ) and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_n \beta_n \le \limsup_n \beta_n < 1$. Suppose that $x_{n+1} = \beta_n x_n \oplus (1-\beta_n)y_n$ for all $n \in \mathbb{N}$ and*

$$\limsup_{n\to\infty}\big(\rho(y_{n+1},y_n) - \rho(x_{n+1},x_n)\big) \le 0.$$

Then $\lim_{n\to\infty}\rho(x_n,y_n) = 0$.

Lemma 2.7 ([14], Lemma 2.1) *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \ge 1,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset \mathbb{R}$ such that
(i) $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(ii) $\limsup_{n\to\infty}\beta_n \le 0$ *or* $\sum_{n=1}^{\infty}|\alpha_n\beta_n| < \infty$.
Then $\{s_n\}$ converges to zero as $n \to \infty$.

We finish this section by recalling an important concept of quasi-linearization introduced by Berg and Nikolaev [3]. Let us denote a pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a *vector*. The *quasi-linearization* is a map $\langle\cdot,\cdot\rangle : (X\times X)\times(X\times X) \to \mathbb{R}$ defined by

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = \frac{1}{2}\big(\rho^2(a,d) + \rho^2(b,c) - \rho^2(a,c) - \rho^2(b,d)\big) \quad \text{for all } a,b,c,d \in X.$$

It is easy to see that $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = \langle\overrightarrow{cd},\overrightarrow{ab}\rangle$, $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle = -\langle\overrightarrow{ba},\overrightarrow{cd}\rangle$, and $\langle\overrightarrow{ax},\overrightarrow{cd}\rangle + \langle\overrightarrow{xb},\overrightarrow{cd}\rangle = \langle\overrightarrow{ab},\overrightarrow{cd}\rangle$ for all $a,b,c,d,x \in X$. We say that (X,ρ) satisfies the *Cauchy-Schwarz inequality* if

$$\big|\langle\overrightarrow{ab},\overrightarrow{cd}\rangle\big| \le \rho(a,b)\rho(c,d) \quad \text{for all } a,b,c,d \in X.$$

It is known from [3], Corollary 3, that a geodesic space X is a CAT(0) space if and only if X satisfies the Cauchy-Schwarz inequality. Some other properties of quasi-linearization are included as follows.

Lemma 2.8 ([4], Lemma 2.9) *Let X be a CAT(0) space. Then*

$$\rho^2(x,u) \leq \rho^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle$$

for all $u,x,y \in X$.

Lemma 2.9 ([4], Lemma 2.10) *Let u and v be two points in a CAT(0) space X. For each $t \in [0,1]$, we set $u_t = tu \oplus (1-t)v$. Then, for each $x,y \in X$, we have*

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ *and* $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

The following fact, which can be found in [15], is an immediate consequence of Lemma 2.4.

Lemma 2.10 *Let X be a CAT(0) space. Then*

$$\rho^2\bigl(tx \oplus (1-t)y, z\bigr) \leq t^2\rho^2(x,z) + (1-t)^2\rho^2(y,z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$$

for all $x,y,z \in X$ and $t \in [0,1]$.

3 Main theorem

Before proving our main theorem, we need one more lemma, which is proved by Wangkeeree and Preechasilp (see [4], Theorem 3.1).

Lemma 3.1 *Let C be a nonempty, closed, and convex subset of a complete CAT(0) space X, $T : C \to C$ be a nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$, and $f : C \to C$ be a contraction with $k \in [0,1)$. For each $t \in (0,1)$, let $\{z_t\}$ be given by*

$$z_t = tf(z_t) \oplus (1-t)T(z_t).$$

Then $\{z_t\}$ converges strongly to $\tilde{x}$ as $t \to 0$. Moreover, $\tilde{x} = P_{\operatorname{Fix}(T)}(f(\tilde{x}))$ and $\tilde{x}$ also satisfies the following variational inequality:

$$\bigl\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \bigr\rangle \geq 0, \quad x \in \operatorname{Fix}(T). \tag{4}$$

Now, we are ready to prove our main theorem.

Theorem 3.2 *Let C be a nonempty, closed, and convex subset of a complete CAT(0) space X, $T : C \to C$ be a nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$, and $f : C \to C$ be a contraction with $k \in [0, \frac{1}{2})$. For the arbitrary initial point $u \in C$, let $\{x_n\}$ be generated by*

$$\begin{aligned} &x_1 = u, \\ &y_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T(x_n), \\ &x_{n+1} = \beta_n x_n \oplus (1-\beta_n)y_n, \quad \forall n \geq 1, \end{aligned}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *satisfying the following conditions*:

(i) $\lim_{n\to\infty}\alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $0 < \liminf_n \beta_n \le \limsup_n \beta_n < 1$.

Then $\{x_n\}$ *converges strongly to* $\tilde{x}$ *such that* $\tilde{x} = P_{\mathrm{Fix}(T)}(f(\tilde{x}))$ *and* $\tilde{x}$ *also satisfies*

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \ge 0, \quad x \in \mathrm{Fix}(T).$$

Proof We divide the proof into three steps.

Step 1. We show that $\{x_n\}$, $\{y_n\}$, $\{T(x_n)\}$, and $\{f(x_n)\}$ are bounded sequences. Let $p \in \mathrm{Fix}(T)$. By Lemma 2.3, we have

$$\begin{aligned}
\rho(x_{n+1},p) &\le \beta_n\rho(x_n,p) + (1-\beta_n)\rho(y_n,p) \\
&\le \beta_n\rho(x_n,p) + (1-\beta_n)\bigl[\alpha_n\rho\bigl(f(x_n),p\bigr) + (1-\alpha_n)\rho\bigl(T(x_n),p\bigr)\bigr] \\
&\le \bigl[\beta_n + (1-\beta_n)(1-\alpha_n)\bigr]\rho(x_n,p) + (1-\beta_n)\alpha_n\rho\bigl(f(x_n),f(p)\bigr) \\
&\quad + (1-\beta_n)\alpha_n\rho\bigl(f(p),p\bigr) \\
&\le \bigl[1-(1-k)\alpha_n + (1-k)\alpha_n\beta_n\bigr]\rho(x_n,p) + (1-\beta_n)\alpha_n\rho\bigl(f(p),p\bigr) \\
&\le \max\left\{\rho(x_n,p), \frac{\rho(f(p),p)}{1-k}\right\}.
\end{aligned}$$

By induction, we also have

$$\rho(x_n,p) \le \max\left\{\rho(x_1,p), \frac{\rho(f(p),p)}{1-k}\right\}.$$

Hence, $\{x_n\}$ is bounded and so are $\{y_n\}$, $\{f(x_n)\}$, and $\{T(x_n)\}$.

Step 2. We show that $\lim_{n\to\infty}\rho(x_n, T(x_n)) = 0$. By applying Lemma 2.5 twice for geodesic triangles $\triangle(f(x_n), T(x_n), T(x_{n+1}))$ and $\triangle(f(x_n), f(x_{n+1}), T(x_{n+1}))$, respectively, we obtain

$$\begin{aligned}
\rho(y_n,y_{n+1}) &\le (1-\alpha_n)\rho\bigl(T(x_n),T(x_{n+1})\bigr) + |\alpha_n-\alpha_{n+1}|\rho\bigl(f(x_n),T(x_{n+1})\bigr) \\
&\quad + \alpha_{n+1}\rho\bigl(f(x_n),f(x_{n+1})\bigr) \\
&\le (1-\alpha_n)\rho(x_n,x_{n+1}) + |\alpha_n-\alpha_{n+1}|\rho\bigl(f(x_n),T(x_{n+1})\bigr) \\
&\quad + \alpha_{n+1}k\rho(x_n,x_{n+1}),
\end{aligned}$$

which implies

$$\rho(y_n,y_{n+1}) - \rho(x_n,x_{n+1}) \le (\alpha_{n+1}k-\alpha_n)\rho(x_n,x_{n+1}) + |\alpha_n-\alpha_{n+1}|\rho\bigl(f(x_n),T(x_{n+1})\bigr).$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\limsup_{n\to\infty}(\rho(y_{n+1},y_n) - \rho(x_{n+1},x_n)) \le 0$. By Lemma 2.6 we have $\lim_{n\to\infty}\rho(x_n,y_n) = 0$. Thus,

$$\begin{aligned}
\rho\bigl(x_n,T(x_n)\bigr) &\le \rho(x_n,y_n) + \rho\bigl(y_n,T(x_n)\bigr) \\
&= \rho(x_n,y_n) + \alpha_n\rho\bigl(f(x_n),T(x_n)\bigr) \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Step 3. We show that $\{x_n\}$ converges to $\tilde{x}$, which satisfies $\tilde{x} = P_{\mathrm{Fix}(T)}(f(\tilde{x}))$ and

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \mathrm{Fix}(T).$$

Let $\{z_m\}$ be a sequence in C defined by

$$z_m = \alpha_m f(z_m) \oplus (1-\alpha_m)T(z_m), \quad \forall m \in \mathbb{N}.$$

By Lemma 3.1, $\{z_m\}$ converges strongly as $m \to \infty$ to $\tilde{x}$ which satisfies (4) and $\tilde{x} = P_{\mathrm{Fix}(T)}(f(\tilde{x}))$. We claim that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \leq 0.$$

It follows from Lemma 2.9(i) that

$$\begin{aligned}
\rho^2(z_m, x_n) &= \langle \overrightarrow{z_m x_n}, \overrightarrow{z_m x_n} \rangle \\
&\leq \alpha_m \langle \overrightarrow{f(z_m)x_n}, \overrightarrow{z_m x_n} \rangle + (1-\alpha_m)\langle \overrightarrow{T(z_m)x_n}, \overrightarrow{z_m x_n} \rangle \\
&= \alpha_m \langle \overrightarrow{f(z_m)f(\tilde{x})}, \overrightarrow{z_m x_n} \rangle + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_n} \rangle + \alpha_m \langle \overrightarrow{\tilde{x}z_m}, \overrightarrow{z_m x_n} \rangle + \alpha_m \langle \overrightarrow{z_m x_n}, \overrightarrow{z_m x_n} \rangle \\
&\quad + (1-\alpha_m)\langle \overrightarrow{T(z_m)T(x_n)}, \overrightarrow{z_m x_n} \rangle + (1-\alpha_m)\langle \overrightarrow{T(x_n)x_n}, \overrightarrow{z_m x_n} \rangle \\
&\leq \alpha_m k\rho(z_m,\tilde{x})\rho(z_m,x_n) + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_n} \rangle + \alpha_m \rho(\tilde{x}, z_m)\rho(z_m,x_n) \\
&\quad + \alpha_m \rho^2(z_m,x_n) + (1-\alpha_m)\rho^2(z_m,x_n) + (1-\alpha_m)\rho\bigl(T(x_n),x_n\bigr)\rho(z_m,x_n) \\
&\leq \alpha_m(k+1)\rho(z_m,\tilde{x})M + \rho\bigl(T(x_n),x_n\bigr)M + \rho^2(z_m,x_n) + \alpha_m \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_n} \rangle,
\end{aligned}$$

for some $M > 0$. This implies

$$\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n z_m} \rangle \leq (k+1)\rho(z_m,\tilde{x})M + \frac{\rho(x_n, T(x_n))}{\alpha_m}M. \tag{5}$$

Taking the upper limit as $n \to \infty$ first and then $m \to \infty$, the inequality (5) yields

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n z_m} \rangle \leq 0. \tag{6}$$

Notice also that

$$\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle = \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n z_m} \rangle + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m\tilde{x}} \rangle \leq \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n z_m} \rangle + \rho\bigl(f(\tilde{x}),\tilde{x}\bigr)\rho(z_m,\tilde{x}).$$

This, together with (6), implies that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \leq 0.$$

Finally, we show that $x_n \to \tilde{x}$ as $n \to \infty$. It follows from Lemmas 2.4, 2.8, 2.9, and 2.10 that

$$\begin{aligned}
\rho^2(x_{n+1},\tilde{x}) &\leq \beta_n\rho^2(x_n,\tilde{x}) + (1-\beta_n)\rho^2(y_n,\tilde{x}) \\
&\leq \beta_n\rho^2(x_n,\tilde{x}) + (1-\beta_n)\bigl[\alpha_n^2\rho^2\bigl(f(x_n),\tilde{x}\bigr) + (1-\alpha_n)^2\rho^2\bigl(T(x_n),\tilde{x}\bigr)\bigr]
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle\overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{T(x_n)\tilde{x}}\rangle \\
&\le \beta_n\rho^2(x_n,\tilde{x}) + (1-\beta_n)(1-\alpha_n)^2\rho^2(x_n,\tilde{x}) \\
&\quad + \alpha_n^2(1-\beta_n)\big[\rho^2\big(x_{n+1},f(x_n)\big) + 2\langle\overrightarrow{\tilde{x}x_{n+1}}, \overrightarrow{\tilde{x}f(x_n)}\rangle\big] \\
&\quad + 2\alpha_n(1-\alpha_n)(1-\beta_n)\big[\langle\overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{T(x_n)x_n}\rangle + \langle\overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle\big] \\
&\le \big[\beta_n + (1-\beta_n)(1-\alpha_n)\big]\rho^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)\rho^2\big(x_{n+1},f(x_n)\big) \\
&\quad + 2\alpha_n^2(1-\beta_n)\big[\langle\overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\big] \\
&\quad + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle\overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{T(x_n)x_n}\rangle \\
&\quad + 2\alpha_n(1-\alpha_n)(1-\beta_n)\big[\langle\overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_n\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle\big] \\
&\le \big[\beta_n + (1-\beta_n)(1-\alpha_n)\big]\rho^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)\rho^2\big(x_{n+1},f(x_n)\big) \\
&\quad + 2\alpha_n^2(1-\beta_n)\rho\big(f(x_n),f(\tilde{x})\big)\rho(x_{n+1},\tilde{x}) + 2\alpha_n^2(1-\beta_n)\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + 2\alpha_n(1-\alpha_n)(1-\beta_n)\rho\big(f(x_n),\tilde{x}\big)\rho\big(T(x_n),x_n\big) \\
&\quad + 2\alpha_n(1-\alpha_n)(1-\beta_n)\rho\big(f(x_n),f(\tilde{x})\big)\rho(x_n,\tilde{x}) \\
&\quad + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle \\
&\le \big[\beta_n + (1-\beta_n)(1-\alpha_n)\big]\rho^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)\rho^2\big(x_{n+1},f(x_n)\big) \\
&\quad + 2k\alpha_n^2(1-\beta_n)\rho(x_n,\tilde{x})\rho(x_{n+1},\tilde{x}) + 2\alpha_n^2(1-\beta_n)\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + 2\alpha_n(1-\alpha_n)(1-\beta_n)\rho\big(f(x_n),\tilde{x}\big)\rho\big(x_n,T(x_n)\big) \\
&\quad + 2k\alpha_n(1-\alpha_n)(1-\beta_n)\rho^2(x_n,\tilde{x}) + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle \\
&\le \big[\beta_n + (1-\beta_n)(1-\alpha_n)\big]\rho^2(x_n,\tilde{x}) + \alpha_n^2(1-\beta_n)\rho^2\big(x_{n+1},f(x_n)\big) \\
&\quad + k\alpha_n^2(1-\beta_n)\big[\rho^2(x_n,\tilde{x}) + \rho^2(x_{n+1},\tilde{x})\big] + 2\alpha_n^2(1-\beta_n)\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + 2\alpha_n(1-\alpha_n)(1-\beta_n)\rho\big(f(x_n),\tilde{x}\big)\rho\big(x_n,T(x_n)\big) \\
&\quad + 2k\alpha_n(1-\alpha_n)(1-\beta_n)\rho^2(x_n,\tilde{x}) + 2\alpha_n(1-\alpha_n)(1-\beta_n)\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\rho^2(x_{n+1},\tilde{x}) &\le \left[\frac{\beta_n + (1-\beta_n)(1-\alpha_n) + 2k\alpha_n(1-\alpha_n)(1-\beta_n)}{1-k\alpha_n^2(1-\beta_n)}\right]\rho^2(x_n,\tilde{x}) \\
&\quad + \frac{k\alpha_n^2(1-\beta_n)}{1-k\alpha_n^2(1-\beta_n)}\rho^2(x_n,\tilde{x}) + \frac{\alpha_n^2(1-\beta_n)}{1-k\alpha_n^2(1-\beta_n)}\rho^2\big(x_{n+1},f(x_n)\big) \\
&\quad + \frac{2\alpha_n(1-\alpha_n)(1-\beta_n)}{1-k\alpha_n^2(1-\beta_n)}\rho\big(f(x_n),\tilde{x}\big)\rho\big(x_n,T(x_n)\big) \\
&\quad + \frac{2\alpha_n^2(1-\beta_n)}{1-k\alpha_n^2(1-\beta_n)}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \frac{2\alpha_n(1-\alpha_n)(1-\beta_n)}{1-k\alpha_n^2(1-\beta_n)}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle.
\end{aligned}
$$

Thus,

$$
\rho^2(x_{n+1},\tilde{x}) \le \big(1-\alpha_n'\big)\rho^2(x_n,\tilde{x}) + \alpha_n'\beta_n', \tag{7}
$$

where $\alpha'_n = \frac{\alpha_n(1-\beta_n)(1-k(2-\alpha_n))}{1-k\alpha_n^2(1-\beta_n)}$ and

$$\begin{aligned}\beta'_n &= \frac{k\alpha_n}{1-k(2-\alpha_n)}\rho^2(x_n,\tilde{x}) + \frac{\alpha_n}{1-k(2-\alpha_n)}\rho^2\big(x_{n+1},f(x_n)\big)\\ &\quad + \frac{2(1-\alpha_n)}{1-k(2-\alpha_n)}\rho\big(f(x_n),\tilde{x}\big)\rho\big(x_n,T(x_n)\big)\\ &\quad + \frac{2\alpha_n}{1-k(2-\alpha_n)}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle + \frac{2(1-\alpha_n)}{1-k(2-\alpha_n)}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle.\end{aligned}$$

Since $k \in [0,\frac{1}{2})$, $\alpha'_n \in (0,1)$. Applying Lemma 2.7 to the inequality (7), we can conclude that $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

4 Concluding remarks and open problems

(1) Our main theorem can be applied to CAT(κ) spaces with $\kappa \leq 0$ since any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$ (see [6]). However, the result for $\kappa > 0$ is still unknown (see [5], p.1264).

(2) Our main theorem can be viewed as an extension of Corollary 8 in [16] for a contraction f with $k \in [0,\frac{1}{2})$. It remains an open problem whether Theorem 3.2 holds for $k \in [\frac{1}{2},1)$.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

The authors read and approved the final manuscript.

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