Oguntuase et al. *Journal of Inequalities and Applications* (2015) 2015:17
DOI 10.1186/s13660-014-0533-z
Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Time scale Hardy-type inequalities with ?broken? exponent p

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Abstract
In this paper, some new Hardy-type inequalities involving ?broken? exponents are derived on arbitrary time scales. Our approach uses both convexity and superquadracity arguments, and the results obtained generalize, complement and provide refinements of some known results in literature.
MSC: Primary 39B82; secondary 44B20; 46C05

Keywords: Hardy-type inequalities; ?broken? exponent; ?broken? time scale; superquadracity

1 Introduction

In 1920, Hardy [1] stated (without proof) the following inequality:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \quad p>1, \tag{1.1}$$

where f is a non-negative measurable function. This result was finally proved by Hardy [2] (see also Hardy [3]) in 1925.

In 1928, Hardy [4] obtained and proved the following generalization of inequality (1.1):

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p x^\alpha\,dx \le \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)x^\alpha\,dx, \tag{1.2}$$

which holds for all measurable and non-negative functions f on $(0,\infty)$ whenever $\alpha < p-1$, $p \ge 1$.

In 1965, Godunova [5] discovered that inequality (1.1) can be proved via convexity argument, but this result was not well known in western literature. The use of convexity argument to prove Hardy-type inequalities was independently rediscovered by Imoru [6] and Kaijser *et al.* [7] in 1977 and 2002, respectively. After that a great number of papers based on this idea have been presented and applied (see [8–10]).

In a recent paper, Persson and Samko [11] used the convexity argument to prove that inequality (1.1) is equivalent to the following inequality:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p \frac{dx}{x} \le \int_0^\infty f^p(x)\frac{dx}{x}, \tag{1.3}$$

via the substitution $f(x) = g(x^{1-1/p})x^{-1/p}$. In the same paper [11] it was also shown that inequality (1.2) is equivalent to inequality (1.3) via the substitution $f(t) = g(t^{(p-1-\alpha)/p})t^{-(1+1/p)}$. It thus follows that Hardy?s initial generalization (1.2) is not actually a generalization. Furthermore, in the same paper, sufficient conditions for a variant of inequality (1.3) to hold were given, namely the following inequality:

$$\int_0^l \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p \frac{dx}{x} \le \int_0^l f^p(x)\left(1-\frac{x}{l}\right)\frac{dx}{x} \tag{1.4}$$

for $p < 0$ or $p \ge 1$. The authors established the equivalence theorem for the one-dimensional Hardy-type inequalities. In particular, it was shown that inequality (1.4) is equivalent to the following variant of (1.2):

$$\int_0^l \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p x^\alpha\,dx \le \left(\frac{p}{p-1-\alpha}\right)^p \int_0^l f^p(x)x^\alpha\left[1-\left(\frac{x}{l}\right)^{\frac{p-\alpha-1}{p}}\right]dx \tag{1.5}$$

for $p \le 1$, $\alpha < p-1$ or $p < 0$, $\alpha > p-1$ and $0 \le l \le \infty$.

A multidimensional version of this equivalence theorem concerning Hardy-type inequalities was proved by Oguntuase *et al.* [12]. For the development of the use of convexity argument in obtaining Hardy-type inequalities, we refer interested readers to the review article by Oguntuase and Persson [9] and the references cited therein. In a recent paper, Oguntuase *et al.* [13] stated and proved multidimensional Hardy-type inequalities with ?broken? exponent. In particular, the following result was established.

Theorem 1.1 *Let $b > 0$, $0 < l \le \infty$ and*

$$p(x) = \begin{cases} p_0, & 0 \le x \le b,\\ p_1, & x > b,\end{cases} \qquad \beta(x) = \begin{cases} \beta_0, & 0 \le x \le b,\\ \beta_1, & x > b,\end{cases}$$

where $p_0, p_1, \beta_0, \beta_1 \in \Re \setminus \{0\}$. If f is non-negative and measurable and $\beta(x) > 0$, then

$$\int_0^l \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^{p(x)} x^{-\beta(x)}\,dx \le \int_0^l \frac{1}{\beta(x)} f(x)^{p(x)}x^{-\beta(x)}\left(1-\left(\frac{x}{l}\right)^{\beta(x)}\right)dx + I_0, \tag{1.6}$$

where $I_0 = 0$, if $l \le b$ (so that $\beta(x) = \beta_0$ and $p(x) = p_0$) and

$$I_0 = \frac{1}{\beta_1}\left(b^{-\beta_1} - l^{-\beta_1}\right)\int_0^b f(x)^{p_1}\,dx - \frac{1}{\beta_0}\left(b^{-\beta_0} - l^{-\beta_0}\right)\int_0^b f(x)^{p_0}\,dx,$$

if $l > b$. If $0 < p(x) \le 1$, then (1.6) *holds in the reversed direction* (*for the case $l = \infty$, $1 - (\frac{x}{l})^{\beta(x)} \equiv 1$ and $l^{-\beta_0} = l^{-\beta_1} \equiv 0$*).

Remark 1.2 Observe that under suitable substitutions, all the variants (1.1)-(1.5) can be recovered from (1.6). Thus (1.6) is more general than all the other inequalities above.

In 2005, Řehák [14, Lemma 1.1] proved that if $\mathbb{T}$ is any arbitrary time scale that is unbounded above and containing a and $\alpha > 1$, then the following estimates hold:

$$\int_a^\infty \frac{\Delta s}{(\sigma(s))^\alpha} \le \int_a^\infty \frac{ds}{s^\alpha} \le \int_a^\infty \frac{\Delta s}{s^\alpha}. \tag{1.7}$$

Řehák used inequality (1.7) to establish the time scale version of the Hardy inequality as follows.

Theorem 1.3 *If $a > 0$, $p > 1$, and f is a non-negative function such that the delta integral $\int_a^\infty (f(s))^p \Delta s$ exists as a finite number, then*

$$\int_a^\infty \left(\frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} f(t)\Delta t \right)^p < \left(\frac{p}{p-1} \right)^p \int_a^\infty f^p(x)\Delta x. \tag{1.8}$$

The above result by Řehák [14] signaled the beginning of research on the time scale Hardy inequality. Since the publication of Řehák?s result on the time scale Hardy inequality, other researchers (see, for instance, [15–19] and the references cited therein) have obtained its generalization both in the one-dimensional and multidimensional settings.

The aim of this paper is to obtain one-dimensional Hardy-type inequalities on a time scale with ?broken? exponent. It is a great interest of this subject (see *e.g.*, papers [20–24] where a lot of interesting facts complementing this paper can be found).

Before we present our results, let us recall some essentials about time scales. In 1988, Hilger introduced the calculus on time scales which unifies continuous and discrete analysis. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\Re$. The two most popular examples are $\mathbb{T} = \Re$ and $\mathbb{T} = \mathbb{Z}$. We define the forward jump operator σ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and the graininess μ of the time scale $\mathbb{T}$ by $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be right-dense and right-scattered if $\sigma(t) = t$, $\sigma(t) > t$, respectively. We define $f^\sigma := f \circ \sigma$. For a function $f : \mathbb{T} \to \Re$, the delta derivative is defined by

$$f^\Delta(t) := \lim_{s \to t, \sigma(s) \neq t} \frac{f^\sigma(s) - f(t)}{\sigma(s) - t}.$$

A function $f : \mathbb{T} \to R$ is called rd-continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. Note that we have

$$\sigma(t) = t, \qquad \mu(t) = 0, \qquad f^\Delta = f', \qquad \int_a^b f(t)\Delta t = \int_a^b f(t)\,dt, \quad \text{when } \mathbb{T} = R,$$

$$\sigma(t) = t + 1, \qquad \mu(t) = 1, \qquad f^\Delta = \Delta f, \qquad \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t), \quad \text{when } \mathbb{T} = \mathbb{Z}.$$

For more understanding of the theory of time scales, we refer the interested reader to [25, 26].

We recall the following definition of the well-known binomial theorem.

Definition 1.4 ([25, Definition 1.51]) (Binomial theorem) If $\alpha, x \in \Re$, the expansion of $(1 + x)^\alpha$ defined by

$$\begin{aligned}(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-\beta+1)x^\beta}{\beta!} + \cdots \\ &= \sum_{\beta=0}^\infty \frac{(\alpha)^{(\beta)}}{\Gamma(\beta+1)} x^\beta \end{aligned} \tag{1.9}$$

is known as the binomial theorem.

Definition 1.5 ([27, Definition 2.1]) A function $\phi : [0,\infty) \to \Re$ is called superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \Re$ such that

$$\phi(y) - \phi(x) - \phi\bigl(|y-x|\bigr) \geq C_x(y-x)$$

for all y ≥ 0.

We say that ϕ is subquadratic if $-\phi$ is superquadratic.

2 Time scale Hardy-type inequalities with ?broken? exponent p via convexity

Before we state our results in this section, we shall need the following lemmas.

Lemma 2.1 ([28, Theorem 1.1]) (Fubini?s theorem on time scales) *Let* $(\Omega, \mathcal{M}, \mu_\Delta)$ *and* $(\Lambda, \mathcal{L}, \lambda_\Delta)$ *be two finite dimensional time scale measure spaces. If* $f : \Omega \times \Lambda \to \Re$ *is a* $\mu_\Delta \times \lambda_\Delta$*-integrable function and the function* $\phi(y) := \int_\Omega f(x,y)\Delta x$ *for a.e.* $y \in \Lambda$ *and* $\varphi(x) = \int_\lambda f(x,y)\Delta y$ *for a.e.* $x \in \Omega$, *then* ϕ *is* λ_Δ*-integrable on* Λ, φ *is* μ_Δ*-integrable on* Ω *and*

$$\int_\Omega \Delta x \int_\Lambda f(x,y)\Delta y = \int_\Lambda \Delta y \int_\Omega f(x,y)\Delta x. \tag{2.1}$$

Lemma 2.2 ([25, Theorem 6.17]) *Let* $a, b \in \mathbb{T}$ *and* $c, d \in \Re$. *Suppose that* $f : [a,b]_{\mathbb{T}^k} \to (c,d)$ *is rd-continuous and* $\phi : (c,d) \to \Re$ *is convex. Then*

$$\phi\left(\frac{1}{b-a}\int_a^b f(t)\Delta t\right) \leq \frac{1}{b-a}\int_a^b \phi\bigl(f(t)\bigr)\Delta t. \tag{2.2}$$

First, we give the following proposition which is an adaptation of Lemma 1.1 in [14].

Proposition 2.3 *Let* $\alpha > 1$ *and* $\mathbb{T}$ *be any arbitrary time scale that is unbounded above. Let* $a, l \in \mathbb{T}$ *be such that* $0 < a < l \leq \infty$. *Then the following estimates hold*:

$$\int_a^l \bigl(\sigma(s)\bigr)^{-\alpha}\Delta s \leq \int_a^l \frac{ds}{s^\alpha} \leq \int_a^l \frac{\Delta s}{s^\alpha}. \tag{2.3}$$

Proof Suppose $l < \infty$ and denote $[a,l]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leq t \leq l\}$. We prove only that $I \leq I^*$, where

$$I = \int_a^b \bigl(\sigma(s)\bigr)^{-\alpha}\Delta s$$

and

$$I^* = \int_a^l \frac{ds}{s^\alpha},$$

since the other inequality can be proven analogously. Suppose by contradiction that there exists a time scale $\mathbb{T}^*$ such that $a, l \in \mathbb{T}^*$ and $I > I^*$, where I^* is taken over $[a,l]_{\mathbb{T}^*}$. This implies that there exists $\epsilon > 0$ such that $I - \epsilon > I^*$.

On the other hand, by virtue of the definition of the delta Riemann integrability, there exists a time scale $\mathbb{T}_D$ containing a and satisfying

$$\mathbb{T}_D = \{t_k : 0 \leq k \leq n\} \quad \text{with } 0 < a = t_0 < t_1 < t_2 < \cdots < t_n = l$$

such that $|I^* - I_D| < \epsilon/2$, where

$$\mathbb{I}_D := \mathbb{T}_D \int_a^l \big(\sigma(s)\big)^{-\alpha} \Delta s.$$

Here, the delta integral is taken over $[a, l]_{\mathbb{T}_D}$. Thus we get

$$I^* + \epsilon \leq I_D < I^* + \epsilon/2,$$

a contradiction.

For the case $l = \infty$, the proof is given in [14]. □

Our first result in this section reads as follows.

Theorem 2.4 *Let $\beta > 0$ and $\mathbb{T}$ be any arbitrary time scale. If $f : \mathbb{T} \to \Re$ is differentiable, then the following inequality*

$$\int_b^t (s-a)^{\beta-1}\left[1 + \sum_{k=1}^{n_\beta+1} \frac{(\beta-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(s)}{s-a}\right)^k\right]\Delta s \leq \frac{1}{\beta}\left[(t-a)^\beta - (b-a)^\beta\right] \tag{2.4}$$

holds for any $a, b, t \in \mathbb{T}^k$ such that $0 \leq a < b \leq t$, where

$$n_\beta := \inf\bigl\{n \in \bigl\{\mathbb{N} \cup \{0\}\bigr\} : \beta - n \geq 0\bigr\}. \tag{2.5}$$

Proof Let $f : \mathbb{T} \to \Re$ be a function defined by

$$f(t) := \frac{1}{\beta}\left[(t-a)^\beta - (b-a)^\beta\right] \quad \forall t \in \mathbb{T}.$$

By Definition 1.4 and equation (2.5) we have that

$$\begin{aligned} f^\Delta(t) &= \frac{(t-a)^\beta}{\beta\mu(t)}\left[\left(1 + \frac{\beta\mu(t)}{(t-a)} + \frac{\beta(\beta-1)}{2!}\left(\frac{\mu(t)}{t-a}\right)^2 + \cdots\right) - 1\right] \\ &= (t-a)^{\beta-1}\left[1 + \frac{(\beta-1)}{2!}\left(\frac{\mu(t)}{t-a}\right) + \frac{(\beta-1)(\beta-2)}{3!}\left(\frac{\mu(t)}{t-a}\right)^2 + \cdots\right] \\ &= (t-a)^{\beta-1}\left[1 + \sum_{k=1}^{n_\beta-1}\frac{(\beta-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(t)}{t-a}\right)^k\right] \\ &\quad + O\left(\frac{(\beta-1)(\beta-2)\cdots(\beta-n_\beta)}{\Gamma(n_\beta+2)}\big(\mu^{n_\beta}(t)(t-a)^{\beta-(n_\beta+1)}\big)\right) \\ &\geq (t-a)^{\beta-1}\left[1 + \sum_{k=1}^{n_\beta-1}\frac{(\beta-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(t)}{t-a}\right)^k\right]. \end{aligned}$$

Integrating, we get that

$$\int_b^t (s-a)^{\beta-1}\left[1+\sum_{k=1}^{n_\beta-1}\frac{(\beta-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(s)}{s-a}\right)^k\right]\Delta s\leq\frac{1}{\beta}\left[(t-a)^\beta-(b-a)^\beta\right].\qquad\square$$

Remark 2.5 We observed that the chain rule can be applied to simplify the proof of Theorem 2.4. The techniques for doing this can be found in the papers [20–24] and the details are left to interested readers. Also, a discrete version of Theorem 2.4 can easily be obtained, and interested readers can fill this gap since this is not the main focus of this paper.

Theorem 2.6 *Let $b>0$, $\beta(x)>0$, $0<l\leq\infty$, and*

$$p(x)=\begin{cases}p_0, & 0\leq x\leq b,\\ p_1, & x>b,\end{cases}\qquad \beta(x)=\begin{cases}\beta_0, & 0\leq x\leq b,\\ \beta_1, & x>b,\end{cases}\tag{2.6}$$

where $p_0,p_1,\beta_0,\beta_1\in\Re\setminus\{0\}$. If $f:\mathbb{T}\to\Re$ is non-negative Δ-integrable and $f\in C_{rd}([a,b],\Re)$ for which

$$\int_a^l\frac{f(x)^{p(x)}}{\beta(x)}\left[1-\left(\frac{y-a}{l-a}\right)^{\beta(x)}\right](y-a)^{-\beta(x)}\Delta x<\infty,$$

then

$$\begin{aligned}&\int_a^l\left(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)}f(y)\Delta(y)\right)^{p(x)}\left(\sigma(x)-a\right)^{-\beta(x)}\Delta(x)\\&\quad\leq\int_a^l\frac{f(x)^{p(x)}}{\beta(x)}\left[1-\left(\frac{x-a}{l-a}\right)^{\beta(x)}\right](x-a)^{-\beta(x)}\Delta(x)+I_0,\end{aligned}\tag{2.7}$$

where $I_0=0$ if $l\leq b$ (so that $\beta(x)=\beta_0$ and $p(x)=p_0$) and

$$I_0=\frac{1}{\beta_1}\left[(b-a)^{-\beta_1}-(l-a)^{-\beta_1}\right]\int_a^b f(x)^{p_1}\Delta x-\frac{1}{\beta_0}\left[(b-a)^{-\beta_0}-(l-a)^{-\beta_0}\right]\int_a^b f(x)^{p_0}\Delta x.$$

Moreover, assume that $p_0\geq1$, $p_1\geq1$ or $p_0\geq1$, $p_1<0$ or $p_0<0$, $p_1\geq1$ or $p_0<0$, $p_1<0$ (for the case with negative parameters, we assume that the function f is strictly positive on the corresponding interval).

If $0<p(x)\leq1$, then (2.7) holds in the reverse direction.

Proof Let $l\leq b$. Applying Jensen?s inequality (2.2), Fubini?s Theorem2.1 and Proposition 2.3, we obtain that

$$\begin{aligned}&\int_a^l\left(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)}f(y)\Delta y\right)^{p(x)}\left(\sigma(x)-a\right)^{-\beta(x)}\Delta x\\&\quad\leq\int_a^l\left[\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)}f(y)^{p_0}\Delta y\right]\left(\sigma(x)-a\right)^{-\beta_0}\Delta x\\&\quad\leq\int_a^l\frac{f(y)^{p(y)}}{\beta(y)}(y-a)^{-\beta(y)}\left[1-\left(\frac{(y-a)}{(l-a)}\right)^{\beta(y)}\right]\Delta y.\end{aligned}$$

Next, for the case $b < l$, by applying Jensen?s inequality (2.2), Fubini?s Theorem2.1, Propositions 2.3 and 2.6, we find that

$$\begin{aligned}
&\int_a^l \left(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(y)\Delta y\right)^{p(x)} \big(\sigma(x)-a\big)^{-\beta(x)}\Delta x\\
&= \int_a^b \left(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(y)\Delta y\right)^{p_0} \big(\sigma(x)-a\big)^{-\beta_0}\Delta x\\
&\quad + \int_b^l \left(\frac{1}{(\sigma(x)-a)}\int_a^{b} f(y)\Delta y\right)^{p_1} \big(\sigma(x)-a\big)^{-\beta_1}\Delta x\\
&\quad + \int_b^l \left(\frac{1}{(\sigma(x)-a)}\int_b^{\sigma(x)} f(y)\Delta y\right)^{p_1} \big(\sigma(x)-a\big)^{-\beta_1}\Delta x\\
&\le \int_a^b \frac{f(y)^{p_0}}{\beta_0}(y-a)^{-\beta_0}\left(1-\left(\frac{y-a}{l-a}\right)^{\beta_0}\right)\Delta y\\
&\quad + \int_b^l \frac{f(y)^{p_1}}{\beta_1}(y-a)^{-\beta_1}\left(1-\left(\frac{y-a}{y-a}\right)^{\beta_1}\right)\\
&\quad + \int_a^b \left(\big((b-a)^{-\beta_1}-(l-a)^{-\beta_1}\big)\frac{f(y)^{p_1}}{\beta_1} - \big((b-a)^{-\beta_0}-(l-a)^{-\beta_0}\big)\frac{f(y)^{p_0}}{\beta_0}\right)\Delta y\\
&= \int_a^l \frac{f(y)^{p(x)}}{\beta(x)}(y-a)^{-\beta(x)}\left(1-\left(\frac{y-a}{l-a}\right)^{\beta(x)}\right)\Delta y + I_0.
\end{aligned}$$

For the proof of the case $0 < p(x) \le 1$, we first note that the functions involving exponents p_0 and p_1 are concave. Therefore the two inequalities above hold in the reverse direction so also this case is proved. □

Remark 2.7 By taking $\mathbb{T} = \Re$ and $a = 0$ in Theorem 2.6, inequality (2.7) coincides with inequality (1.6) obtained in [13].

Next, we state a dual version of Theorem 2.6, when the Hardy operator

$$H : f(x) \to \frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(y)\Delta y$$

is replaced by the dual Hardy operator

$$H^* : f(x) \to \big(\sigma(x)-a\big)\int_{\sigma(x)}^{\infty} \frac{f(t)\Delta t}{(\sigma(t)-a)(t-a)}.$$

Hence, our result in this direction reads as follows.

Theorem 2.8 *Let $b > 0$, $\beta > 0$, $0 \le l < \infty$ and*

$$p(x) = \begin{cases} p_0, & 0 \le x \le b,\\ p_1, & x > b,\end{cases} \qquad \beta(x) = \begin{cases} \beta_0, & 0 \le x \le b,\\ \beta_1, & x > b,\end{cases} \tag{2.8}$$

where $p_0, p_1, \beta_0, \beta_1 \in \Re \setminus \{0\}$. Moreover, assume that $p_0 \ge 1, p_1 \ge 1$ or $p_0 \ge 1, p_1 < 0$ or $p_0 < 0$, $p_1 \ge 1$ or $p_0 < 0, p_1 < 0$ (for the case with negative parameters, we assume that the function

f is strictly positive on the corresponding interval). If f is a non-negative delta integrable function and $f \in C_{rd}([a,b],\Re)$ for which

$$\int_l^\infty \frac{1}{\beta(t)}\big(f(t)\big)^{p(t)}(t-a)^{\beta(t)}\left(1-\left(\frac{l-a}{t-a}\right)^{\beta(t)}\right)\frac{\Delta t}{(\sigma(t)-a)(t-a)} < \infty,$$

then

$$\begin{aligned}
&\int_l^\infty \left((\sigma(x)-a)\int_{\sigma(x)}^\infty \frac{f(t)\Delta t}{(\sigma(t)-a)(t-a)}\right)^{p(x)}(x-a)^{\beta(x)} \\
&\qquad\times\left[1+\sum_{k=1}^{n_{\beta(x)}-1}\frac{(\beta(x)-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right]\frac{\Delta x}{(\sigma(x)-a)(x-a)} \\
&\quad\leq \int_l^\infty \frac{1}{\beta(x)}\big(f(x)\big)^{p(x)}(x-a)^{\beta(x)}\left(1-\left(\frac{l-a}{x-a}\right)^{\beta(x)}\right)\frac{\Delta x}{(\sigma(x)-a)(x-a)} + I_0,
\end{aligned}\tag{2.9}$$

where $I_0 = 0$ if $l \leq b$ (so that $\beta(x) = \beta_0$ and $p(x) = p_0$) and

$$\begin{aligned}
I_0 &= \frac{1}{\beta_0}\big((b-a)^{\beta_0}-(l-a)^{\beta_0}\big)\int_b^\infty \frac{(f(x))^{p_0}}{(\sigma(x)-a)(x-a)}\Delta x \\
&\quad-\frac{1}{\beta_1}\big((b-a)^{\beta_1}-(l-a)^{\beta_1}\big)\int_b^\infty \frac{(f(x))^{p_1}}{(\sigma(x)-a)(x-a)}\Delta x.
\end{aligned}$$

If $0 < p(x) \leq 1$, then (2.9) *holds in the reverse direction.*

Proof Let $l > b$. By utilizing Jensen?s inequality (2.2), Fubini?s Theorem2.1 and Lemma 2.4 and taking into account (2.8), we find that

$$\begin{aligned}
&\int_l^\infty \left((\sigma(x)-a)\int_{\sigma(x)}^\infty \frac{f(t)\Delta t}{(\sigma(t)-a)(t-a)}\right)^{p(x)}(x-a)^{\beta(x)} \\
&\qquad\times\left[1+\sum_{k=1}^{n_{\beta(x)}-1}\frac{(\beta(x)-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right]\frac{\Delta x}{(\sigma(x)-a)(x-a)} \\
&\quad\leq \int_l^\infty \int_{\sigma(x)}^\infty \frac{(f(t))^{p_1}\Delta t}{(\sigma(t)-a)(t-a)}\left[1+\sum_{k=1}^{n_{\beta_1}-1}\frac{(\beta_1-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right](x-a)^{\beta_1-1}\Delta x \\
&\quad= \int_l^\infty \frac{(f(t))^{p_1}}{(\sigma(t)-a)(t-a)}\left(\int_l^t (x-a)^{\beta_1-1}\left[1+\sum_{k=1}^{n_{\beta_1}-1}\frac{(\beta_1-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right]\Delta x\right)\Delta t \\
&\quad\leq \int_l^\infty \frac{1}{\beta(t)}\big(f(t)\big)^{p(t)}(t-a)^{\beta(t)}\left[1-\left(\frac{l-a}{t-a}\right)^{\beta(t)}\right]\frac{\Delta t}{(\sigma(t)-a)(t-a)}.
\end{aligned}$$

Next, let $l \leq b$. Then, by applying Jensen?s inequality (2.2), Fubini?s Theorem2.1 and Lemma 2.4, and taking into account (2.8), we find that

$$\begin{aligned}
&\int_l^\infty \left((\sigma(x)-a)\int_{\sigma(x)}^\infty \frac{f(t)\Delta t}{(\sigma(t)-a)(t-a)}\right)^{p(x)}(x-a)^{\beta(x)} \\
&\qquad\times\left[1+\sum_{k=1}^{n_{\beta(x)}-1}\frac{(\beta(x)-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right]\frac{\Delta x}{(\sigma(x)-a)(x-a)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_l^b \int_{\sigma(x)}^\infty \frac{(f(t))^{p_0}\Delta t}{(\sigma(t)-a)(t-a)}\left[1+\sum_{k=1}^{n_{\beta_0}-1}\frac{(\beta_0-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right](x-a)^{\beta_0-1}\Delta x \\
&\quad + \int_b^\infty \int_{\sigma(x)}^\infty \frac{(f(t))^{p_1}\Delta t}{(\sigma(t)-a)(t-a)}\left[1+\sum_{k=1}^{n_{\beta_1}-1}\frac{(\beta_1-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right](x-a)^{\beta_1-1}\Delta x \\
&= \int_l^b \frac{(f(t))^{p_0}}{(\sigma(t)-a)(t-a)}\left(\int_l^t (x-a)^{\beta_0-1}\left[1+\sum_{k=1}^{n_{\beta_0}-1}\frac{(\beta_1-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right]\Delta x\right)\Delta t \\
&\quad + \int_b^\infty \frac{(f(t))^{p_0}}{(\sigma(t)-a)(t-a)}\left(\int_l^b (x-a)^{\beta_0-1}\left[1+\sum_{k=1}^{n_{\beta_0}-1}\frac{(\beta_0-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right]\Delta x\right)\Delta t \\
&\quad + \int_b^\infty \frac{(f(t))^{p_1}}{(\sigma(t)-a)(t-a)}\left(\int_b^t (x-a)^{\beta_1-1}\left[1+\sum_{k=1}^{n_{\beta_1}-1}\frac{(\beta_1-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right]\Delta x\right)\Delta t \\
&\leq \int_l^b \frac{(f(t))^{p_0}}{(\sigma(t)-a)(t-a)}\frac{1}{\beta_0}\left[(t-a)^{\beta_0}-(l-a)^{\beta_0}\right]\Delta t \\
&\quad + \int_b^\infty \frac{(f(t))^{p_0}}{(\sigma(t)-a)(t-a)}\frac{1}{\beta_0}\left[(b-a)^{\beta_0}-(l-a)^{\beta_0}\right]\Delta t \\
&\quad + \int_b^\infty \frac{(f(t))^{p_1}}{(\sigma(t)-a)(t-a)}\frac{1}{\beta_1}\left[(t-a)^{\beta_1}-(b-a)^{\beta_1}\right]\Delta t \\
&= \int_l^\infty \frac{1}{\beta(t)}\big(f(t)\big)^{p(t)}(t-a)^{\beta(t)}\left(1-\left(\frac{b-a}{t-a}\right)^{\beta(t)}\right)\frac{\Delta t}{(\sigma(t)-a)(t-a)} + I_0. \qquad \square
\end{aligned}
$$

Remark 2.9 By taking $\mathbb{T} = \Re$ and $a = 0$ in Theorem 2.8, inequality (2.9) coincides with Theorem 2.5 in [13].

3 Time scale Hardy-type inequalities with ?broken? exponent p via superquadracity

Refined Jensen?s inequality on time scales for superquadratic functions has been recently obtained by Barić *et al.* This inequality is very useful in the proof of our results in this section.

Lemma 3.1 ([16, Theorem 2.5]) *Let $a, b \in \mathbb{T}$. Suppose that $f : [a,b]_{\mathbb{T}^k} \to [0,\infty]$ is rd-continuous and $\phi : [0,\infty] \to \Re$ is continuous and superquadratic. Then*

$$
\phi\left(\frac{1}{b-a}\int_a^b f(t)\Delta t\right) \leq \frac{1}{b-a}\int_a^b \left[\phi\big(f(s)\big) - \phi\left(\left|f(s) - \frac{1}{b-a}\int_a^b f(t)\Delta t\right|\right)\right]\Delta s. \tag{3.1}
$$

Proof For the proof, see [16]. $\square$

Our first result in this section reads as follows.

Theorem 3.2 *Let the assumptions of Theorem* 2.6 *be satisfied. Moreover, let $u \in C_{rd}([a,b], \Re)$ be a non-negative function such that the Δ-integral $\int_t^l \frac{u(x)}{(\sigma(x)-a)^{\beta(x)+1}}\Delta x < \infty$ and define the weight function υ by*

$$
v(t) := (t-a)\int_t^l \frac{u(x)}{(\sigma(x)-a)^{\beta(x)+1}}\Delta x, \quad t \in (a,b). \tag{3.2}
$$

(1) *If* Φ *is a non-negative superquadratic function on* (a,c), $0 < a < c \leq \infty$, *then*

$$\begin{aligned}&\int_a^l u(x)\Phi\biggl(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\biggr)\bigl(\sigma(x)-a\bigr)^{-\beta(x)}\Delta x\\&\quad+\int_a^l\int_t^l \Phi\biggl(\biggl|f(t)-\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\biggr|\biggr)\frac{u(x)}{(\sigma(x)-a)^{\beta(x)+1}}\Delta x\Delta t\\&\quad\leq\int_a^l \upsilon(x)\Phi\bigl(f(x)\bigr)\frac{\Delta x}{x-a}\end{aligned}\tag{3.3}$$

holds for all Δ*-integrable functions and* $f \in C_{rd}([a,b],R)$ *such that* $f(x) \in (a,c)$.

(2) *If the real-valued function* Φ *is subquadratic on* (a,c), $0 < a < c \leq \infty$, *then* (3.3) *holds in the reversed direction*.

Proof (1) Let $l \leq b$. Applying refined Jensen?s inequality (3.1), after taking into account Definition 2.8, we get that

$$\begin{aligned}&\int_a^l u(x)\Phi\biggl(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\biggr)\bigl(\sigma(x)-a\bigr)^{-\beta(x)}\Delta x\\&\quad\leq\int_a^l \frac{u(x)}{(\sigma(x)-a)^{\beta_0+1}}\int_a^{\sigma(x)}\Phi\bigl(f(t)\bigr)\Delta t\Delta x\\&\qquad-\int_a^l \frac{u(x)}{(\sigma(x)-a)^{\beta_0+1}}\int_a^{\sigma(x)}\Phi\biggl(\biggl|f(t)-\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\biggr|\biggr)\Delta t\Delta x.\end{aligned}\tag{3.4}$$

By utilizing Fubini?s Theorem2.1 and taking into account Definition 3.2 of the weight function ν, we obtain that the right-hand side of (3.4) is not greater than

$$\begin{aligned}&\int_a^l \upsilon(t)\Phi\bigl(f(t)\bigr)\frac{\Delta t}{t-a}\\&\quad-\int_a^l\int_t^l \Phi\biggl(\biggl|f(t)-\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\biggr|\biggr)\frac{u(x)}{(\sigma(x)-a)^{\beta_0+1}}\Delta x\Delta t.\end{aligned}$$

Let $l > b$. Applying again refined Jensen?s inequality (3.1), after taking into account Definition 2.8, we find that

$$\begin{aligned}&\int_a^l u(x)\Phi\biggl(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\biggr)\bigl(\sigma(x)-a\bigr)^{-\beta(x)}\Delta x\\&\quad\leq\int_a^l \frac{u(x)}{(\sigma(x)-a)^{\beta_0+1}}\int_a^{\sigma(x)}\Phi\bigl(f(t)\bigr)\Delta t\Delta x\\&\qquad-\int_a^l \frac{u(x)}{(\sigma(x)-a)^{\beta_0+1}}\int_a^{\sigma(x)}\Phi\biggl(\biggl|f(t)-\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\biggr|\biggr)\Delta t\Delta x\\&\qquad+\int_a^l \frac{u(x)}{(\sigma(x)-a)^{\beta_1+1}}\int_a^{\sigma(x)}\Phi\bigl(f(t)\bigr)\Delta t\Delta x\\&\qquad-\int_a^l \frac{u(x)}{(\sigma(x)-a)^{\beta_1+1}}\int_a^{\sigma(x)}\Phi\biggl(\biggl|f(t)-\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\biggr|\biggr)\Delta t\Delta x.\end{aligned}\tag{3.5}$$

Finally, utilizing Fubini?s Theorem2.1 and taking into account Definition 3.2 of the weight function ν, we obtain that the right-hand side of (3.5) equals

$$\begin{aligned}
&\int_a^l \upsilon(t)\Phi\big(f(t)\big)\frac{\Delta t}{t-a} \\
&\quad - \int_a^l \int_t^l \Phi\left(\left|f(t) - \frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\right|\right)\frac{u(x)}{(\sigma(x)-a)^{\beta_0+1}}\Delta x\Delta t \\
&\quad + \int_a^l \upsilon(t)\Phi\big(f(t)\big)\frac{\Delta t}{t-a} \\
&\quad - \int_a^l \int_t^l \Phi\left(\left|f(t) - \frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\right|\right)\frac{u(x)}{(\sigma(x)-a)^{\beta_1+1}}\Delta x\Delta t.
\end{aligned} \tag{3.6}$$

The proof for the case when ϕ is subquadratic is similar except that the only inequality above holds in the reverse direction. The proof is now complete. □

We now give some applications of Theorem 3.2.

Example 3.3 If we let $\beta(x) = 1$ and apply Theorem 3.2 to $(\frac{\sigma(x)-a}{x-a})u(x)$ instead of $u(x)$, then we get

$$\begin{aligned}
&\int_a^l u(x)\Phi\left(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\right)\frac{\Delta x}{x-a} \\
&\quad + \int_a^l \int_t^l \Phi\left(\left|f(t) - \frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\right|\right)\frac{u(x)}{(x-a)(\sigma(x)-a)}\Delta x\Delta t \\
&\leq \int_a^l \upsilon(x)\Phi\big(f(x)\big)\frac{\Delta x}{x-a}.
\end{aligned} \tag{3.7}$$

The sign of inequality (3.7) is reversed for the case $1 < p(x) \leq 2$.

Remark 3.4 Inequality (3.7) coincides with Theorem 2.3 in [17].

Now we will use the well-known fact that $\phi(u) = u^{p(x)}$ is superquadratic for $p(x) \geq 2$ and (subquadratic if $0 < p(x) \leq 2$) in the next example.

Example 3.5 Let u(x) = 1 and $p(x) \geq 2$. By Proposition 2.3, we get that

$$\nu(x) \leq \frac{(x-a)^{1-\beta(x)}}{\beta(x)}\left[1 - \left(\frac{x-a}{l-a}\right)^{\beta(x)}\right] \quad \text{if } l < \infty.$$

Under these conditions, inequality (3.3) yields

$$\begin{aligned}
&\int_a^l \left(\frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\right)^{p(x)} \big(\sigma(x)-a\big)^{-\beta(x)}\Delta x \\
&\quad + \int_a^l \int_t^l \left|f(t) - \frac{1}{(\sigma(x)-a)}\int_a^{\sigma(x)} f(t)\Delta t\right|^{p(x)} \frac{u(x)}{(\sigma(x)-a)^{\beta(x)+1}}\Delta x\Delta t \\
&\leq \int_a^l \frac{f^{p(x)}(x)}{\beta(x)}\left[1 - \left(\frac{x-a}{l-a}\right)^{\beta(x)}\right](x-a)^{1-\beta(x)}\Delta x.
\end{aligned} \tag{3.8}$$

The sign of inequality (3.8) is reversed for the case $1 < p(x) \leq 2$.

Remark 3.6 Since ϕ is a non-negative function, the second term on the left-hand side of inequality (3.8) is non-negative.

Theorem 3.7 *Let the assumptions of Theorem* 2.8 *be satisfied. Moreover, let* $u \in C_{rd}([a,b], R)$ *be a non-negative function such that*

$$\int_l^t \left(1 + \sum_{k=1}^{n_{\beta_1}-1} \frac{(\beta_1 - 1)^{(k)}}{\Gamma(k+2)} \left(\frac{\mu(x)}{x-a}\right)^k\right) u(x)(x-a)^{\beta(x)-1} \Delta x < \infty,$$

and define the weight function υ *by*

$$\nu(t) := \int_l^t \left(1 + \sum_{k=1}^{n_{\beta_1}-1} \frac{(\beta_1 - 1)^{(k)}}{\Gamma(k+2)} \left(\frac{\mu(x)}{x-a}\right)^k\right) u(x)(x-a)^{\beta(x)-1} \Delta x, \quad t \in (a,b). \tag{3.9}$$

(1) *If the real-valued function* Φ *is a superquadratic on* (a,c), $0 < a < c \le \infty$, *then*

$$\begin{aligned}
&\int_l^\infty u(x)\Phi\left(\left(\sigma(x) - a\right) \int_{\sigma(x)}^\infty \frac{f(t)\Delta t}{(\sigma(t)-a)(t-a)}\right)(x-a)^{\beta(x)} \\
&\quad \times \left[1 + \sum_{k=1}^{n_{\beta(x)}-1} \frac{(\beta(x) - 1)^{(k)}}{\Gamma(k+2)} \left(\frac{\mu(x)}{x-a}\right)^k\right] \frac{\Delta x}{(\sigma(x)-a)(x-a)} \\
&\quad + \int_l^\infty \int_l^t \left[1 + \sum_{k=1}^{n_{\beta_1}-1} \frac{(\beta_1 - 1)^{(k)}}{\Gamma(k+2)} \left(\frac{\mu(x)}{x-a}\right)^k\right] u(x)(x-a)^{\beta_1-1} \\
&\quad \times \Phi\left(\left|f(t) - \left(\sigma(x) - a\right) \int_{\sigma(x)}^\infty \frac{f(t)\Delta t}{(\sigma(t)-a)(t-a)}\right|\right) \Delta x \Delta t \\
&\le \int_l^\infty \upsilon(x)\Phi\left(f(x)\right) \frac{\Delta x}{(\sigma(x)-a)(x-a)}
\end{aligned} \tag{3.10}$$

holds for all Δ*-integrable functions* $f \in C_{rd}([a,b], R)$ *such that* $f(x) \in (a,c)$.

(2) *If the real-valued function* Φ *is subquadratic on* (a,c), $0 < a < c \le \infty$, *then* (3.5) *holds in the reversed direction.*

Proof Let $l > b$. Applying refined Jensen?s inequality β.1), we find that the first term on the left-hand side of inequality (3.5) is not greater than

$$\begin{aligned}
&\int_l^\infty u(x)(x-a)^{\beta_1-1} \int_{\sigma(x)}^\infty \frac{\Phi(f(t))\Delta t}{(\sigma(t)-a)(t-a)} \\
&\quad \times \left[1 + \sum_{k=1}^{n_{\beta_1}-1} \frac{(\beta_1 - 1)^{(k)}}{\Gamma(k+2)} \left(\frac{\mu(x)}{x-a}\right)^k\right] \Delta x \\
&\quad - \int_l^\infty u(x)(x-a)^{\beta_1-1} \left[1 + \sum_{k=1}^{n_{\beta_1}-1} \frac{(\beta_1 - 1)^{(k)}}{\Gamma(k+2)} \left(\frac{\mu(x)}{x-a}\right)^k\right] \\
&\quad \times \int_{\sigma(x)}^\infty \Phi\left(\left|f(t) - \left(\sigma(x) - a\right) \int_{\sigma(x)}^\infty \frac{f(t)\Delta t}{(\sigma(t)-a)(t-a)}\right|\right) \Delta t \Delta x.
\end{aligned} \tag{3.11}$$

Finally, employing Fubini?s Theorem2.1 and Proposition 2.4, we obtain that the right-hand side of (3.11) is not greater than

$$\int_l^\infty \upsilon(t)\Phi\big(f(t)\big)\frac{\Delta t}{(\sigma(t)-a)(t-a)}$$
$$-\int_l^\infty \int_l^t \left[1+\sum_{k=1}^{n_{\beta_1}-1}\frac{(\beta_1-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right]u(x)(x-a)^{\beta_1-1}$$
$$\times \Phi\left(\left|f(t)-\big(\sigma(x)-a\big)\int_{\sigma(x)}^\infty \frac{f(t)\Delta t}{(\sigma(t)-a)(t-a)}\right|\right)\Delta x\Delta t. \qquad \square$$

Now we consider Theorem 3.2 in some special cases. First we note that if we set $u(x) = 1$, then we find that

$$\nu(x) \le \frac{(x-a)^{\beta(x)}}{\beta(x)}\left[1-\left(\frac{l-a}{x-a}\right)^{\beta(x)}\right] \quad \text{if } l<\infty$$

by Proposition 2.4.

Corollary 3.8 *Let the assumptions of Theorem* 3.7 *be satisfied. If* ϕ *is non-negative superquadratic, then*

$$\int_l^\infty \Phi\left(\big(\sigma(x)-a\big)\int_{\sigma(x)}^\infty \frac{f(t)\Delta t}{(\sigma(t)-a)(t-a)}\right)(x-a)^{\beta(x)}$$
$$\times\left[1+\sum_{k=1}^{n_{\beta(x)}-1}\frac{(\beta(x)-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right]\frac{\Delta x}{(\sigma(x)-a)(x-a)}$$
$$+\int_l^\infty \int_l^t \left[1+\sum_{k=1}^{n_{\beta_1}-1}\frac{(\beta_1-1)^{(k)}}{\Gamma(k+2)}\left(\frac{\mu(x)}{x-a}\right)^k\right](x-a)^{\beta_1-1}$$
$$\times \Phi\left(\left|f(t)-\big(\sigma(x)-a\big)\int_{\sigma(x)}^\infty \frac{f(t)\Delta t}{(\sigma(t)-a)(t-a)}\right|\right)\Delta x\Delta t$$
$$\le \int_l^\infty \frac{(x-a)^{\beta(x)}}{\beta(x)}\left[1-\left(\frac{l-a}{x-a}\right)^{\beta(x)}\right]\Phi\big(f(x)\big)\frac{\Delta x}{(\sigma(x)-a)(x-a)}. \tag{3.12}$$

Example 3.9 Assume that $\mathbb{T} = \Re$, $a = 0$ and $\Phi(x) = x^p$. Then inequality (3.12) yields the inequality

$$\int_l^\infty \left(x\int_x^\infty \frac{f(t)\,dt}{t^2}\right)^{p(x)} x^{\beta(x)}\frac{dx}{x^2}$$
$$+\int_l^\infty \int_l^t (x-a)^{\beta(x)-1}\left(\left|f(t)-x\int_x^\infty \frac{f(t)\,dt}{t^2}\right|\right)^{p(x)}dx\,dt$$
$$\le \int_l^\infty \frac{1}{\beta(x)}\big(f(x)\big)^{p(x)}x^{\beta(x)}\left(1-\left(\frac{l}{x}\right)^\beta\right)\frac{dx}{x^2}. \tag{3.13}$$

Remark 3.10 Since ϕ is a non-negative function, the second term on the left-hand side of inequality (3.13) is non-negative. Hence inequality (3.13) provides a refinement of inequality (2.3) in [13] if written for $l \le b$.

Competing interests
The authors declare that they have no competing interests.

Authors? contributions
All the authors have contributed in all parts to equal extent. Also, all the authors read and approved the final manuscript.

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Acknowledgements
The authors are grateful to the careful referees for useful comments and advice that have led to the improvement of this work. The first author who is a senior associate of the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy would like to thank the Abdus Salam ICTP for support.

Received: 23 August 2014 Accepted: 17 December 2014 Published online: 16 January 2015

References
1. Hardy, GH: Note on a theorem of Hilbert. Math. Z. **6**, 314-317 (1920)
2. Hardy, GH: Notes on some points in the integral calculus, LX. An inequality between integrals. Messenger Math. **54**, 150-156 (1925)
3. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge University Press, Cambridge (1959)
4. Hardy, GH: Notes on some points in the integral calculus, LXIV. Further inequalities between integrals. Messenger Math. **57**, 12-16 (1928)
5. Godunova, EK: Inequalities with convex functions. Transl. Am. Math. Soc. **88**, 57-66 (1970). Translation from Izv. Vys?. Učebn. Zaved., Mat. 4(47), 45-53 (1965)
6. Imoru, CO: On some integral inequalities related to Hardy?s. Can. Math. Bull.**20**(3), 307-312 (1977)
7. Kaijser, S, Persson, L-E, Öberg, A: On Carleman?s and Knopp?s inequalities. J. Approx. Theory**117**, 140-151 (2002)
8. Krulic, K: Generalizations and refinement of Hardy?s inequalities. Ph.D thesis, Department of Mathematics, University of Zagreb (2010)
9. Oguntuase, JA, Persson, L-E: Hardy-type inequalities via convexity - the journey so far. Aust. J. Math. Anal. Appl. **7**(2), 18 (2011)
10. Persson, L-E, Samko, N: Some remarks and new developments concerning Hardy-type inequalities. Rend. Circ. Mat. Palermo **82**, 1-29 (2010)
11. Persson, L-E, Samko, N: What should have happened if Hardy had discovered this? J. Inequal. Appl. **2012**, 29 (2012)
12. Oguntuase, JA, Persson, L-E, Samko, N, Sonubi, A: On the equivalence between some multidimensional Hardy-type inequalities. Banach J. Math. Anal. **8**, 1-13 (2014)
13. Oguntuase, JA, Persson, L-E, Samko, N: Some Hardy-type inequalities with ?broken? exponent. J. Math. Inequal**8**(3), 405-416 (2014)
14. Řehák, P: Hardy inequality on time scales and its applications to half-linear dynamic equations. J. Inequal. Pure Appl. Math. **5**, 495-507 (2005)
15. Agarwal, RP, Bohner, M, Peterson, A: Inequalities on time scales: a survey. Math. Inequal. Appl. **4**(4), 535-557 (2001)
16. Barić, J, Bibi, R, Bohner, M, Pečarić, J: Time scales integral inequalities for superquadratic functions. J. Korean Math. Soc. **50**, 465-477 (2013)
17. Oguntuase, JA, Persson, L-E: Time scales Hardy-type inequalities via superquadracity. Ann. Funct. Anal. **5**(2), 61-73 (2014)
18. Özkan, UM, Yildirim, H: Hardy-Knoop type inequalities on time scales. Dyn. Syst. Appl. **17**(3-4), 477-486 (2008)
19. Özkan, UM, Yildirim, H: Time scale Hardy-Knoop type integral inequalities. Commun. Math. Anal. **6**(1), 36-41 (2009)
20. Agarwal, RP, Bohner, M, Saker, SH: Dynamic Littlewood-type inequalities. Proc. Am. Math. Soc. **143**(2), 667-677 (2015)
21. Saker, SH, Graef, J: A new class of dynamic inequalities of Hardy?s type on time scales. Dyn. Syst. Appl.**23**, 83-93 (2014)
22. Saker, SH, O?Regan, D, Agarwal, RP: Some dynamic inequalities of Hardy?s type on time scales. Math. Inequal. Appl**17**, 1183-1199 (2014)
23. Saker, SH, O?Regan, D, Agarwal, RP: Dynamic inequalities of Hardy and Copson types on time scales. Analysis**34**, 391-402 (2014)
24. Saker, SH, O?Regan, D, Agarwal, RP: Generalized Hardy, Copson, Leindler and Bennett inequalities on time scales. Math. Nachr. **287**(5-6), 686-698 (2014)
25. Bohner, M, Peterson, A: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
26. Bohner, M, Peterson, A (eds.): Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston (2003)
27. Abramovich, S, Jameson, G, Sinnamon, G: Refining of Jensen?s inequality. Bull. Math. Soc. Sci. Math. Roum.**47**(1-2), 3-14 (2004)
28. Bibi, R, Bohner, M, Pečarić, J, Varo?anec, S: Minkowski and Beckenbach-Dresher inequalities and functionals on time scales. J. Math. Inequal. **7**(3), 299-312 (2013)